A Generalization of the Social Coalitional Equilibrium Structure

Ken Urai[†], Kohei Shiozawa[‡], Hiromi Murakami[‡] and Weiye Chen[‡‡]

Abstract

We generalize the notion of Ichiishi (1981)'s social coalitional equilibrium to the multiple coalition structures, so that different industries having independent coalition-deviation opportunities and their industrial organizations are simultaneously determined. The result will bring about a direct extension of the standard Arrow-Debreu private ownership economy and an answer to the firm formation problem including the determination of share holdings rates.

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1 Introduction

The social coalitional equilibrium (SCE) by Ichiishi (1981) is a significant concept giving us a unified perspective on economic (abstract market equilibrium) settings and cooperative game-theoretic arguments finding stable coalition structures in a society. He utilized his social coalitional equilibrium concept to characterize the formation of firms as a hybrid cooperative nature in non-cooperative market settings (see Ichiishi (1993)).

From a general equilibrium framework, however, Ichiishi's hybrid equilibrium concept has a serious restriction that an *admissible coalition structure* is a partition of the set of agents. His characterization of the firms, therefore, is typically the case that each agent cannot be an owner of two or more firms, like the labor-owned company in Ichiishi (1977). Needless to say, in the real world, it is clearly not sufficient to restrict agents' coalitional structures to the class of partitions. Many kinds of coalitions exist for different purposes and benefits, and an agent will be allowed to belong simultaneously to several types of coalitions having different purposes. The formation of firms should also be characterized under such settings.

On the other hand, as a coalition production equilibrium (CPE) foundation of the general equilibrium model (Arrow-Debreu private ownership economy), Boehm (1974) gives a firm formation model without restricting the firm coalition structure as a partition of the agents. Unfortunately,

[†] Graduate School of Economics, Osaka University, E-mail: urai@econ.osaka-u.ac.jp
[‡] Faculty of Economics, Takasaki City University of Economics, E-mail: shiozawa@tcue.ac.jp
[‡] School of Buisiness Administration, Kwansei Gakuin University, E-mail: hmurakami@kwansei.ac.jp
[‡‡] Graduate School of Economics, Osaka University, E-mail: chenweiye198966@yahoo.co.jp

Boehm's model fails to treat the relation between the coalition technology and the coalition resources (as an investment) for the technology, which is nothing but the problem that Ichiishi (1977) pointed out and he emphasized the labor resources to describe his labor-owned firm formation structure. Ichiishi's approach, including his succeeding social coalitional equilibrium arguments in Ichiishi (1993), therefore, provides an important progress on the firm formation problem and it would be strongly desirable to extend his SCE framework so that we can treat a situation where multiple coalition structures exist, i.e., multiple kinds of resources are invested for multiple purposes, and multiple kinds of firm formations or multiple industrial organization structures are determined simultaneously as an equilibrium.

In this paper, we generalize the concept of Ichiishi's social coalitional equilibrium so that we can incorporate multiple *admissible coalition structures*. The generalization will enable us to generalize Boehm's CPE framework to determine each firm's *share holdings rates* as a result of multiple stability conditions for independent investment purposes together with their coalition deviation possibilities. In section 2, we generalize Ichiishi's social coalitional equilibrium concept. Section 3 is devoted to confirm the meaning and validity of our *balancedness condition* that plays an essential role for our existence result. The proof of our existence of equilibrium theorem is treated in section 4.

We use R as the set of real numbers. For finite set A, denote by $\sharp A$ the number of elements of A. We write R^K in the meaning of $R^{\sharp K}$, $\sharp K$-dimensional vector space. The order relations on R^K, $\geqq$ and $>$, are defined respectively as $(x_k)_{k\in K} \geqq (y_k)_{k\in K}$ iff $x_k \geqq y_k$ for all k, and $(x_k)_{k\in K} > (y_k)_{k\in K}$ iff $(x_k)_{k\in K} \geqq (y_k)_{k\in K}$ and $(x_k)_{k\in K} \neq (y_k)_{k\in K}$. We also define relation $\gg$ as $(x_k)_{k\in K} \gg (y_k)_{k\in K}$ iff $x_k > y_k$ for all k. By R^K_+ and R^K_{++}, we represent the sets $\{x \in R^K \mid x \geqq 0\}$ and $\{x \in R^K \mid x \gg 0\}$, respectively. For n-dimensional Euclidean space R^n, notation $e^1 = (1, 0, \ldots, 0), e^2 = (0, 1, 0, \ldots, 0), \ldots, e^n = (0, \ldots, 0, 1)$ will be used to represent the standard base elements.

2 Generalized Social Coalitional Equilibrium

In this section, we extend *social coalitional equilibrium* (Ichiishi, 1981) and its framework. First, to treat the *multiple cooperate opportunities*, $t = 1, \ldots, \lambda$, we generalize Ichiishi's single coalition structure model to the case where the multiple coalition structures are formed. Second, to treat messages as given parameters for each agent, the cooperative game is parametrized by an element of a set (message space).

2.1 SCE under Multiple Coalition Structures

Let $N = \{1. \ldots, n\}$ be a non-empty finite index set of all agents and $\mathfrak{N}$ be the set of all non-empty subsets of N (or, all *coalitions*). We suppose that there are λ kinds of *cooperate opportunities* (or *coalition types*) and denote $\Lambda = \{1, 2, \ldots, \lambda\}$. For each coalition type $t \in \Lambda$, agents are going to form a coalition, and a *coalition structure* is identified with a sequence of λ partitions of N, $\mathfrak{T} = (\mathfrak{T}_1, \ldots, \mathfrak{T}_\lambda)$, where $\mathfrak{T}_t$ is a partition of N. In the following, the set of all *admissible* coalition structures which consists of all the possible sequence of λ partitions, $\mathfrak{T}$, is fixed and defined as a non-empty finite set. The finiteness of $\mathfrak{T}$ derived from the finiteness of $\Lambda = \{1, \ldots, \lambda\}$ and the independency of λ types of cooperate opportunities are two important assumptions on our model in describing the multiple coalition structures.

Each agent $i \in N$ has a strategy set, X_i, a subset of a certain Euclidean space R^k. Denote by X_S the product $\prod_{i \in S} X_i$ for each $S \in \mathfrak{N}$. We also denote by $x_S = (x_i)_{i \in S} \in X_S$. In the following, without any additional notation, we do not distinguish x and x_N where x_N is an element of $X_N = \prod_{i \in N} X_i$.

Suppose that for each social coalition structure $\mathfrak{T} = (\mathfrak{T}_1, \dots, \mathfrak{T}_\lambda) \in \mathfrak{T}$, coalition type $t \in \Lambda$, and coalition $S \in \mathfrak{N}$, there is a correspondence,

$$K^{\mathfrak{T},t,S} : X_N \to X_S, \tag{1}$$

a *feasible-strategy constraint correspondence* of coalition S for coalition type t under coalition structure $\mathfrak{T}$. Adding to the constraint correspondences, we also assume that each agent i in coalition S for project type t has a preference, $\precsim_i$, on their strategy set, X_i, that can be represented by a *utility function*

$$u_i^{\mathfrak{T},t,S} : X_N \times X_S \to R_+. \tag{2}$$

Now, a *society* is described as the following list:

$$\big((X_i)_{i \in N}, (K^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i \in S})_{(\mathfrak{T},t,S) \in \mathfrak{T} \times \Lambda \times \mathfrak{N}}\big).$$

A *social coalitional equilibrium* (SCE) is a pair $(x^*, \mathfrak{T}^*)$ of strategy profile $x_N^* \in X_N$ and admissible coalition structure $\mathfrak{T}^* = (\mathfrak{T}^{*1}, \dots, \mathfrak{T}^{*\lambda}) \in \mathfrak{T}$ satisfying the following two conditions:

(SCE1: Feasibility) For each $t \in \Lambda$ and $S \in \mathfrak{T}_t^*$, $x_S^* \in K^{\mathfrak{T}^*,t,S}(x^*)$.

(SCE2: Stability) There are no $s \in \Lambda$, $D \in \mathfrak{N}$, and $y_D \in K^{\mathfrak{T}^*,s,D}(x^*)$ such that

$$u_i^{\mathfrak{T}^*,s,D}(x^*, y_D) > u_i^{\mathfrak{T}^*,s,S(i)}(x^*, x_{S(i)}^*)$$

for all $i \in D$ where $S(i)$ is the unique coalition such that $i \in S \in \mathfrak{T}_s^*$.

In the above, we have defined SCE as the concept based on the multiple coalition structures, $t = 1, \dots, \lambda$. If we assume that for each coalition type t and coalition S, feasible-strategy constraint correspondences does not depend on the coalition structure, i.e., $K^{\mathfrak{T},t,S}$ does not depend on $\mathfrak{T}$ for each $t \in \Lambda$, and if we consider the special case $\lambda = 1$, then our framework coincides with the setting of Ichiishi (1981).

2.2 Generalized SCE with Parameters

In this paper, we further generalize the above SCE framework as a social coalitional equilibrium model with parameters. Suppose that there is an additional information or message structure that parametrically defines an SCE setting. Let $X_0 \subset R^\ell$, $\ell \geq 1$, be a set of the parameters and an element $x_0 \in X_0$ parametrically defines an SCE setting through the feasible-strategy constraint correspondences of two kinds, $\bar{K}$ and $\hat{K}$, for each $(\mathfrak{T}, t, S)$ as follows:

(Constraint for Budget: $\bar{K}$): $\bar{K}^{\mathfrak{T},t,S} : X_0 \times X_N \to X_S$.

(Constraint for Deviation: $\hat{K}$): $\hat{K}^{\mathfrak{T},t,S} : X_0 \times X_N \to X_S$.

Based on these parametrized constraint correspondences, condition (SCE1) and (SCE2) are generalized for each parameter $x_0 \in X_0$ as follows:

(GSCE1: Parametrized $\bar{K}$ Feasibility under x_0) For each $t \in \Lambda$, and $S \in \mathfrak{T}_t^*$, we have $x_S^* \in \bar{K}^{\mathfrak{T}^*,t,S}(x_0, x^*)$.

(GSCE2: Parametrized $\hat{K}$ Stability under x_0) There are no $s \in \Lambda$, $D \in \mathbf{N}$, and $y_D \in \hat{K}^{\mathfrak{T}^*,s,D}(x_0, x^*)$ such that

$$u_i^{\mathfrak{T}^*,s,D}(x^*, y_D) > u_i^{\mathfrak{T}^*,s,S(i)}(x^*, x_{S(i)}^*)$$

for all $i \in D$, where $S(i)$ is the unique coalition such that $i \in S \in \mathfrak{T}_k^*$.

It is also assumed that parameters are restricted by a correspondence, $G_0 : X_0 \times X_N \to X_0$. Hence, the *generalized sense of society* is the list:

$$\big(X_0, (X_i)_{i\in N}, G_0, \big(\bar{K}^{\mathfrak{T},t,S}, \hat{K}^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i\in S}\big)_{(\mathfrak{T},t,S)\in\mathfrak{T}\times\Lambda\times\mathbf{N}}\big).$$

A *generalized social coalitional equilibrium* (GSCE) is a triplet, $(x_0^*, x^*, \mathfrak{T}^*)$, of parameter x_0^*, strategy profile $x_N^* \in X_N$, and admissible coalition structure $\mathfrak{T}^* = (\mathfrak{T}_1^*, \ldots, \mathfrak{T}_\lambda^*) \in \mathfrak{T}$, satisfying (GSCE1) under x_0^*, (GSCE2) under x_0^*, and the following (GSCE3):

(GSCE3: Fixed Point Parameter) $x_0^* \in X_0$ satisfies $x_0^* \in G_0(x_0^*, x^*)$.

For the generalized social coalitional equilibrium model, we have the following equilibrium existence theorem. This is an extension of the SCE existence lemma of Ichiishi and Quinzii (1983). The proof and a rigorous predication for condition (v) will be given in section 4.

Proposition 1. *For society* $\big(X_0, (X_i)_{i\in N}, G_0, \big(\bar{K}^{\mathfrak{T},t,S}, \hat{K}^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i\in S}\big)_{(\mathfrak{T},t,S)\in\mathfrak{T}\times\Lambda\times\mathbf{N}}\big)$, *social coalitional equilibrium* $(x_0^*, x^*, \mathfrak{T}^*) \in X_0 \times X_N \times \mathfrak{T}$ *exists if the following conditions are satisfied:*

(i) X_0 *and* X_i, $i \in N$, *are non-empty, compact, and convex subsets of a certain Euclidean space.*

(ii) For each $S \in \mathbf{N}$, $t \in \Lambda$, *and* $\mathfrak{T} \in \mathfrak{T}$, $\bar{K}^{\mathfrak{T},t,S} : X_N \to X_S$ *and* $\hat{K}^{\mathfrak{T},t,S} : X_N \to X_S$ *are continuous correspondences that are closed and non-empty valued.*

(iii) For each $i \in N$, $S \in \mathbf{N}$, $t \in \Lambda$ *and* $\mathfrak{T} \in \mathfrak{T}$, $u_i^{\mathfrak{T},t,S} : X_N \times X_S \to R_+$ *is a continuous function.*

(iv) The society is balanced. (Correspondences $\bar{K}$ *and* $\hat{K}$ *satisfy the balancedness condition described in section 3.)*

(v) For each $x \in X_N$ *and* $c \in R^N$, *socially feasible upper-contour set at* x *for* c *is convex.*

(vi) G_0 *is an upper-semicontinuous non-empty convex valued correspondence.*

3 Balancedness Condition for GSCE Framework

To show the existence of GSCE, we extend the notion of the *balanced game*. Given the set of all coalitions, $\mathbf{N} = \{A \subset N | \, A \neq \emptyset\}$, we say that a finite family, $\{B_s\}_{s=1}^m$, of elements of $\mathbf{N}$ is *balanced* if there are non-negative real numbers, $\alpha_1, \alpha_2, \ldots, \alpha_m$, such that for each $i \in N$, $\sum_{B_s \ni i} \alpha_s = 1$.[1] In the literature of cooperative game theory, it is said that a coalitional-form game without side payments, $V : \mathbf{N} \to R^N$, where $R^N = R^{\sharp N} = R^n$, is *balanced* if any utility allocation $(c_i)_{i\in N} \in R^N$ with a balanced

[1] In other words, by using $\sharp N - 1$ dimensional standard simplex $\Delta = \mathrm{co}\{e^i | \, i \in N\}$, if we identify each $B_s \subset N$ with barycenter b_s of its $\sharp B_s - 1$ dimensional face $\mathrm{co}\{e^i | \, i \in B_s\}$, then the balancedness condition is equivalent to saying that there is a convex combination among points b_s, $s = 1, 2, \ldots, m$, such that $\sum_{s=1}^m a_s b_s$ is the barycenter of Δ.

family $\{B_s\}_{s=1}^m$ such that $(c_i)_{i\in B_s} \in V(B_s)$ for each $s = 1, \ldots, m$, satisfies $(c_i)_{i\in N} \in V(N)$. (A utility allocation attainable for all coalitions in a certain balanced subfamily is also attainable in the society.) Ichiishi (1981) generalizes such condition to the SCE framework. In the following, we further extend the notion of balancedness to the GSCE structure.

As we formalized in section 2, for each $\mathfrak{T} \in \mathfrak{T}$, $t \in \Lambda$, parameter $x_0 \in X_0$, and an arbitrary strategy profile $x \in X_N$, coalition $S \in \mathbf{N}$ defines feasible strategy allocations and utility allocations for deviation as $\hat{K}^{\mathfrak{T},t,S}(x_0, x) \subset X_S$ and $\{(u_i^{\mathfrak{T},t,S}(x, y_S))_{i\in S} | \, y_S \in \hat{K}^{\mathfrak{T},t,S}(x_0, x)\}$, respectively. Therefore, for each $t \in \Lambda$ and $(x_0, x) \in X_0 \times X_N$, we can define a *generalized coalitional-form game without side payments*, $V^t_{x_0,x} : \mathbf{N} \to R^N$ as

$$V^t_{x_0,x}(S) = \{(c_i)_{i\in N} | \, \exists \mathfrak{T} \in \mathfrak{T}, \exists y_S \in \hat{K}^{\mathfrak{T},t,S}(x_0, x), \forall i \in S, c_i \leqq u_i^{\mathfrak{T},t,S}(x, y_S)\} \subset R^N. \tag{3}$$

We say that a generalized SCE game parametrized by elements of X_0 is said to be balanced if the following condition is satisfied.

(Balanced GSCE) Given $(x_0, x) \in X_0 \times X_N$, if for each $t \in \Lambda$, a utility allocation, $(c_i^t)_{i\in N} \in R^N$, is such that we have a balanced family, $\{B_s^t\}_{s=1}^{m(t)}$, satisfying that $c^t_{B^t_s} \in V^t_{x_0,x}(B^t_s)$ for all $s = 1, \ldots, m(t)$, then there exist a strategy profile $y \in X_N$ and a coalition structure $\mathfrak{T}^* = (\mathfrak{T}^*_1, \ldots, \mathfrak{T}^*_\lambda) \in \mathfrak{T}$ such that $y_S \in \bar{K}^{\mathfrak{T}^*,t,S}(x_0, x)$ for each $S \in \mathfrak{T}^*_t$ and $t \in \Lambda$ (y is feasible at (x_0, x) under $\mathfrak{T}^*$) and $c_i^t \leqq u_i^{\mathfrak{T}^*,t,S}(x, y_S)$ for all $i \in S$, $S \in \mathfrak{T}^*_t$ and $t \in \Lambda$ (utility allocation $(c_i^t)_{i\in N}$ attainable for balanced family $\{B_s^t\}_{s=1}^{m(t)}$ is also attainable under y for all $t \in \Lambda$).

4 Existence of Equilibrium

For a parameter $x_0 \in X_0$, a strategy profile $(x) = ((x_i)_{i\in N}) \in X_N$, and a utility profile $c = (c_i)_{i\in N} \in R^N$, let us consider the set of strategy profiles that are feasible and seem as good as level $c = (c_i)_{i\in N}$ at (x_0, x) for all members of each coalition in a certain admissible social coalition structure $\mathfrak{T} \in \mathfrak{T}$. We call set $U(x_0, x, c) = \{(y_i)_{i\in N} \in X_N | \, \exists \mathfrak{T} = (\mathfrak{T}_1, \ldots, \mathfrak{T}_\lambda) \in \mathfrak{T}, \forall s \in \Lambda, \forall S \in \mathfrak{T}_s, (y_i)_{i\in S} \in \bar{K}^{\mathfrak{T},k,S}(x_0, x)$ and $\forall i \in S,\ u_i^{\mathfrak{T},k,S}(x, y_S) \geqq c_i\}$, the *socially feasible upper-contour set* at (x_0, x) for c. We also denote by $\bar{K}(x_0, x)$ the set, $\{(y_i)_{i\in N} \in X_N | \, \exists \mathfrak{T} = (\mathfrak{T}_1, \ldots, \mathfrak{T}_\lambda) \in \mathfrak{T}, \forall t \in \Lambda, \forall S \in \mathfrak{T}_t, (y_i)_{i\in S} \in \bar{K}^{\mathfrak{T},t,S}(x_0, x)\}$, the *socially feasible set* at (x_0, x). Now we have the rigorous descripion of condition (v).

(v') For each $x \in X_N$ and $c \in R^N$, $U(x_0, x, c)$ is convex.

Theorem 1. *For society* $\big(X_0, (X_i)_{i\in N}, G_0, \big(\bar{K}^{\mathfrak{T},t,S}, \hat{K}^{\mathfrak{T},t,S}, (u_i^{\mathfrak{T},t,S})_{i\in S}, \mathfrak{T}\big)_{(\mathfrak{T},t,S)\in\mathfrak{T}\times\Lambda\times\mathbf{N}}\big)$, *social coalitional equilibrium* $(x_0^*, x^*, \mathfrak{T}^*) \in X_0 \times X_N \times \mathfrak{T}$ *exists if the conditions (i), (ii), (iii), (iv), (v') and (vi) are satisfied.*

Proof:

Let M be a positive real number greater than $u_i^{\mathfrak{T},t,S}(x, y_S)$ for all $i \in N$, $x \in X_N$, $y_S \in X_S$, $S \in \mathbf{N}$, $t \in \Lambda$, and $\mathfrak{T} \in \mathfrak{T}$. Such number M exists since N, $\mathbf{N}$, Λ, and $\mathfrak{T}$ are finite, all strategy sets are compact, and all utility functions are continuous. Given the base of $R^N = R^{\sharp N} = R^n$, $(e^i)_{i\in N}$, let D^N be simplex $\overline{(-(Mn)e^i)_{i\in N}}$ in non-positive orthant $-R^N_+$. Then, for each $(x_0, x) \in X_0 \times X_N$ and $t \in \Lambda$, we obtain a continuous function, $r^t_{x_0,x} : D^N \to R_+$, such that for each $a \in D^N$,

$$r^t_{x_0,x}(a) = \max\{r \in R | \, a + re \in V^t_{x_0,x}(S), S \in \mathbf{N}\}, \tag{4}$$

where $e = \sum_{i \in N} e^i = (1, 1, \ldots, 1) \in R^N$. One can assure the continuity of $r^t_{x_0,x}$ by the routine method through Berge's maximum theorem. Let us define a function, $f^t_{x_0,x} : D^N \to R^N$, for each $t \in \Lambda$ as

$$f^t_{x_0,x}(a) = a + r^t_{x_0,x}(a)e, \tag{5}$$

for each $a \in D^N$. Function $f^t_{x_0,x}$ is also continuous.

For each $t \in \Lambda$, $(x_0, x) \in X_0 \times X_N$, and $S \in \mathbf{N}$, define $C^t_S(x_0, x) \subset D^N$ as

$$C^t_S(x_0, x) = \{b \in D^N | \, f^t_{x_0,x}(b) \in V^t_{x_0,x}(S)\}. \tag{6}$$

Note that for each t and $S \in \mathbf{N}$, the graph of correspondence $C^t_S : X_0 \times X_N \to D^N$ is closed since the graph of correspondence $V^t_{(\cdot)}(S) : X_0 \times X_N \ni (x_0, x) \mapsto V^t_{x_0,x}(S) \subset R^N$ is closed under the finiteness of $\mathfrak{T}$. Moreover, for each $t \in \Lambda$ and $(x_0, x) \in X_0 \times X_N$, we can verify that class $\{C^t_S(x_0, x) | \, S \in \mathbf{N}\}$ satisfies the following KKMS-condition:

$$\forall T \in \mathbf{N},\ (\sharp T - 1)\text{-dimensional face } D^T = \overline{(-(Mn)e^i)_{i \in T}} \text{ of } D^N \text{ is a subset of } \textstyle\bigcup_{S \subset T} C^t_S(x_0, x). \tag{7}$$

Indeed, class $\{C^t_S(x_0, x) | \, S \in \mathbf{N}\}$ cearly covers D^N. So if $b = (b_i)_{i \in N} \in D^T$ exists such that $b \notin C^t_S(x_0, x)$ for all $S \subset T$, then since $T \neq N$, we can take S' and $j \in S'$ such that $b \in C^t_{S'}(x_0, x)$ and $j \in S' \setminus T$. Since $b_j = 0$, and since at b, $b + r^t_{x_0,x}(b)e$ must be an element of $R^{S'}_+ = \{(c_i)_{i \in I} | \, \forall i \in S', c_i \geqq 0\}$, j-th coordinate of $b + r^t_{x_0,x}(b)e$ must be greater than the distance between D^T and R^N_+. Hence, j-th coordinate of $b + r^t_{x_0,x}(b)e = f^t_{x_0,x}(b) \in V^t_{x_0,x}(S')$ must be greater than M, a contradiction. Therefore, by KKMS-Theorem (Shapley 1973, Theorem 3.1.2), for each $t \in \Lambda$ and $(x_0, x) \in X_0 \times X_N$, balanced family $\mathfrak{B}^t_{x_0,x} \subset \mathbf{N}$ exists such that $\bigcap_{B \in \mathfrak{B}^t_{x_0,x}} C^t_B(x_0, x) \neq \emptyset$.

Under the balancedness condition for the society, for λ types of elements $a^t \in \bigcap_{B \in B^t_x} C^t_B(x_0, x)$, $t \in \Lambda$, there exist a feasible stragegy profile, $y = (y_i)_{i \in N} \in X_N$ for an admissible social coalition structure, $\mathfrak{T} = (\mathfrak{T}_1, \ldots, \mathfrak{T}_\lambda) \in \mathfrak{T}$, (i.e., $(y_i)_{i \in T} \in \bar{K}^{\mathfrak{T},s,T}(x_0, x)$ for each T in $\mathfrak{T}_s$ for each $s \in \Lambda$,) such that for each $t \in \Lambda$, $(c^t_j)_{j \in N} = f^t_{x_0,x}(a^t)$ satisfies $\forall T \in \mathfrak{T}_t$, $(c^t_j)_{j \in T} \leqq (u^{\mathfrak{T},t,T}_j(x, (y_i)_{i \in T}))_{j \in T}$. It follows that

$$\forall t \in \Lambda, y \in U(x_0, x, f^t_{x_0,x}(a^t)) \subset \bar{K}(x_0, x), \tag{8}$$

i.e., feasible strategy profile y belongs to the socially feasible upper contour set at (x_0, x) for $f^t_{x_0,x}(a^t)$ for each $t \in \Lambda$. This, especially, means that for each (x_0, x) closed set $\bar{K}(x_0, x)$ is non-empty.

Denote by $(D^N)^\lambda$ the λ-times product of D^N. Now, we can define two mappings on $X_0 \times X_N \times (D^N)^\lambda$ to itself. Let b_T be the barycenter of D^T for each $T \in \mathbf{N}$ and consider mapping $F : X_0 \times X_N \times (D^N)^\lambda \to X_0 \times X_N \times (D^N)^\lambda$ as follows:

$$F(x_0, x, a^1, \ldots, a^\lambda) = \{(x_0, x)\} \times \mathrm{co}\{b_T | \, a^1 \in C^1_T(x_0, x)\} \times \cdots \times \mathrm{co}\{b_T | \, a^\lambda \in C^\lambda_T(x_0, x)\}, \tag{9}$$

where $\mathrm{co}A$ denotes the convex hull of set A. F is non-empty valued correspondence having closed graph (since every C^t_T has). Furthermore, for each $(x_0, x) \in X_0 \times X_N$ and $(a^t)_{t \in \Lambda} \in (D^N)^\lambda$, consider a distance between the set of socially attainable utility allocations and $f^1_{x_0,x}(a^1), \ldots, f^\lambda_{x_0,x}(a^\lambda)$ as follows:

$$V(x_0, x, (a^t)_{t\in\Lambda}) = \operatorname*{argmin}_v\{\|v\| \,|\, \exists y \in \bar{K}(x_0, x), \forall t \in \Lambda, f^t_{x_0,x}(a^t) - v \leqq (u_i^{\mathfrak{T}(y),t,T(i)}(x, y_{T(i)}))_{i\in N}\}, \quad (10)$$

where $\mathfrak{T}(y) = (\mathfrak{T}_1, \ldots, \mathfrak{T}_\lambda) \in \mathfrak{T}$ denotes a social coalition structure under which y is feasible, $T(i)$ denotes the unique coalition in $\mathfrak{T}_t$ that includes i, and $y_{T(i)} = (y_j)_{j\in T(i)}$ for $y = (y_i)_{i\in N}$. Mapping $V : (x_0, x, (a^t)_{t\in\Lambda}) \mapsto R$ has a closed graph since $\bar{K} : (x_0, x) \mapsto \bar{K}(x_0, x)$ has. Define mapping $G : X_0 \times X_N \times (D^N)^\lambda \to X_0 \times X_N \times (D^N)^\lambda$ as

$$G(x_0, x, a) = \mathrm{co}\left(G_0(x_0, x) \times \bigcup_{v\in V(x_0,x,a)} \left(\bigcap_{t\in\Lambda} U(x_0, x, f^t_{x_0,x}(a^t) - v)\right)\right) \times \{b_N\} \times \cdots \times \{b_N\}, \quad (11)$$

where $a = (a^1, \ldots, a^\lambda) \in (D^N)^\lambda$ and $\{b_N\} \times \cdots \times \{b_N\}$ denotes the λ times product of $\{b_N\}$. Since we define V so as to ensure the non-emptiness for the intersection among $U(x_0, x, f^t_{x_0,x}(a^t) - v)$'s, G is non-empty and convex valued. G has a closed graph since U and V have. (Correspondence U has a closed graph since $\bar{K}$ is continuous.) Remember that $X_0 \times X_N$ and D^N are subsets of vector spaces, $R^\ell \times (R^k)^n$ and R^n, respectively. Note that for each $(x_0, x, a) \in X_0 \times X_N \times (D^N)^\lambda$, $(x_0, x, a) + (G(x_0, x, a) - F(x_0, x, a))$ is a subset of $X_0 \times X_N \times (D^N)^\lambda$. Moreover, at each (x_0, x, a) such that $0 \notin G(x_0, x, a) - F(x_0, x, a)$, a closed hyperplane $H(x_0, x, a) \subset R^\ell \times (R^k)^n \times (R^n)^\lambda$ (a continuous linear form on $R^\ell \times (R^k)^n \times (R^n)^\lambda$) exists such that $F(x_0, x, a)$ and $G(x_0, x, a)$ are strictly separated by $H(x_0, x, a)$. Therefore, if we define mapping φ on $X_0 \times X_N \times (D^N)^\lambda$ to itself as

$$\varphi(x_0, x, a) = (x_0, x, a) + (G(x_0, x, a) - F(x_0, x, a)), \quad (12)$$

correspondence φ satisfies condition (K1) of fixed-point theorem in Urai (2000, Theorem 1) (see also Urai (2010, p.36, Theorem 2.1.10)). Hence, φ has a fixed point, (x_0^*, x^*, a^*), where $a^* = (a^{1*}, \ldots, a^{\lambda*})$, so F and G has a coincidence point, (x_0^*, x^*, a^*), in $F(x_0^*, x^*, a^*) \cap G(x_0^*, x^*, a^*)$.

By (9) and (11), family of $T \subset N$ satisfying $a^{t*} \in C^t_T(x_0^*, x^*)$ is balanced for all $t \in \Lambda$. It follows that as we see at (8), socially feasible strategy profile y and $\mathfrak{T} \in \mathfrak{T}$ exist such that $y \in U(x_0^*, x^*, f^t_{x_0^*,x^*}(a^{t*}))$ for each $t \in \Lambda$. This especially means, however, by definitions of V (see (10)), $V(x_0^*, x^*, a^*) = \{0\}$. Therefore, by (9) and (11), since each $U(x_0, x, c)$ is convex by (v), we have

$$x^* \in \bigcap_{t\in\Lambda} U(x_0^*, x^*, f^t_{x_0^*,x^*}(a^*)). \quad (13)$$

This also means under the balancedness condition that x^* is socially feasible under a certain $\mathfrak{T}^* = (\mathfrak{T}_1^*, \ldots, \mathfrak{T}_\lambda^*) \in \mathfrak{T}$ (GSCE1: Feasibility). Furthermore, condition that $\forall t \in \Lambda$, $\forall T \in \mathfrak{T}_t^*$, $\forall j \in T$, $u^{j,T,t,\mathfrak{T}^*}(x^*, (x_j^*)_{j\in T}) \geqq c_j^t$, where c_j^t is the j-th coordinate of $f^t_{x_0^*,x^*}(a^*)$, means (through definitions (4) and (5)) that no coalition of any type can improve the utility allocation under (x_0^*, x^*) (GSCE2: Stability). By the fixed point property, (GSCE3) is automatically satisfied. ■

5 Conclusion

This paper generalizes the social coalitional equilibrium (Ichiishi, 1981) and its framework in which the agents cooperate under the multiple coalition structures. For such social coalitional equilibrium settings, the message is treated like parameters in a cooperate game. The equilibrium outcomes depends on the multiple coalition structures and the generalization of the balancedness condition. Our result provides a useful framework to analyze the firm formation problem by incorporating it into the standard Arrow-Debreu private ownership economy, or a cooperative core theoretic nature in a non-cooperative market price mechanism. In such cases, the GSCE-parameter $x_0 \in X_0$ will be identified with (p, θ), the pair of a price vector and a vector of shareholding rates.

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