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ON RINGS WHOSE CYCLIC FAITHFUL MODULES ARE GENERATORS

HIROSHI YOSHIMURA

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0. Introduction

For each positive integer \( n \), we temporarily say that a ring \( R \) is \( n \)-PF (\( n \)-pseudo-Frobenius) if every faithful right \( R \)-module generated by at most \( n \) elements is a generator for the category of all right \( R \)-modules. As well known, the ring which is \( n \)-PF for all positive integers \( n \) is called a right FPF (finitely pseudo-Frobenius) ring, and every FPF ring splits in a ring with essential singular ideal and a nonsingular ring. Nonsingular FPF rings were investigated in S. Kobayashi [9] and S. Page [11], [12], [13], etc.; in particular, S. Page [11] characterized (von Neumann) regular right FPF rings as self-injective regular rings having bounded index, and S. Kobayashi [9] gave a characterization of nonsingular right FPF rings. The aim of this paper is to study nonsingular 1-PF rings, which were to some extent investigated in G.F. Birkenmeier [2], [3] and S. Kobayashi [10].

Modifying the proof of [10, Proposition 1] and observing that the converse of the proposition is also true, we see, as will be noted in §3, that for a fixed integer \( n \geq 2 \), a ring \( R \) is right nonsingular and \( n \)-PF if and only if \( R \) satisfies the condition (C\(_n\)) that:

(i) \( R \) is right bounded, i.e., every essential right ideal of \( R \) contains a two-sided ideal which is essential in \( R \) as a right ideal,

(ii) For every right ideal \( A \) generated by at most \( n \) elements, \( R = Tr_R(A) \oplus r_R(A) \), where \( Tr_R(A) \) (respectively, \( r_R(A) \)) is the trace (resp. the right annihilator) ideal of \( A \), and

(iii) Every nonsingular right \( R \)-module generated by at most \( n \) elements can be embedded in a free right \( R \)-module.

However, such the result as above is, in general, false in the case \( n = 1 \). Moreover, for regular or commutative semiprime rings, the FPF condition is, as noted in [10], equivalent to the \( n \)-PF condition for each \( n \geq 2 \), although it is not
to the 1-PF condition. Thus it seems that the 1-PF condition is not so much related to the \((C_n)\) and to the FPF as the \(n\)-PF \((n \geq 2)\); actually, the class of 1-PF rings is much larger than that of \(n\)-PF \((n \geq 2)\) rings.

In this paper we shall refer 1-PF rings to as right GFC rings as in [2], [3], and be concerned with nonsingular right GFC rings. We shall show that the quasi-Baer right GFC rings are precisely the rings satisfying the condition \((C_1)\), so that they may be regarded as a natural generalization of nonsingular \(n\)-PF \((n \geq 2)\) rings, and hence, of nonsingular right FPF rings. On the other hand, the structure of right GFC regular rings \(R\), under the assumption that every nonzero two-sided ideal of \(R\) contains a nonzero central idempotent, was determined, in [10, Theorem 1], as finite direct products of abelian regular rings and full matrix rings over self-injective abelian regular rings. We shall generally show that even without the assumption, any right GFC regular ring has the same structure and is characterized as a regular ring having bounded index such that every cyclic faithful nonsingular right module is projective.

In Section 1 of this paper is assembled a summary of notation and terminology. Section 2 contains preliminary results on right GFC rings, which will be used afterward. There we shall show that if \(R\) is a right nonsingular right GFC ring, then the maximal right quotient ring of \(R\) is FPF (Theorem 2.8), which is the key to our study in the following sections. Section 3 is concerned with quasi-Baer or right p.p. right GFC rings. We shall characterize quasi-Baer right GFC rings as rings satisfying the condition \((C_1)\) (Theorem 3.3), and in addition give characterizations of right p.p. (and quasi-Baer) right GFC rings (Theorem 3.7, Theorem 3.8). The last section is devoted to (von Neumann) regular rings. We shall determine the structure of regular right GFC rings, and present other characterizations of those rings (Theorem 4.3). As a consequence, we see that the GFC condition is left-right symmetric for regular rings (Corollary 4.4).

1. Notation and terminology

Throughout this paper all rings are associative with identity, and all modules are unitary.

Let \(R\) be a ring, \(M\) an \(R\)-module, and \(X\) a subset of \(M\). We denote by \(r_R(X)\) (respectively, \(l_R(X)\)) the right (resp. left) annihilator of \(X\) in \(R\), by \(Tr_R(M)\) the trace ideal of \(M\), i.e., \(Tr_R(M) = \sum\{\text{Im } \varphi | \varphi \in \text{Hom}_R(M, R)\}\), and by \(Z(M)\) the (right) singular submodule of \(M\), i.e., \(Z(M) = \{x \in M | r_R(x) \text{ is essential in } R\}\). Given a positive integer \(n\), we denote by \(M^{(n)}\) the direct sum of \(n\) copies of \(M\). By ideals we usually mean two-sided ideals of \(R\). The notation \(N \subseteq M\) (resp. \(N \subseteq_e M\)) means that \(N\) is a submodule (resp. an essential submodule) of \(M\). In particular, the notation \(A \subseteq R\) signifies that \(A\) is a right ideal of \(R\). We use \(B(R)\) to denote the set of all central idempotents in \(R\). A complement for \(N\) in \(M\) is any submodule \(L\) of \(M\) which is maximal with respect to the property \(N \cap L = 0\).

We call a ring \(R\) a right GFC (resp. right FPF) ring if every cyclic (resp.
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finitely generated) faithful right $R$-module is a generator for Mod-$R$, the category of all right $R$-modules. A left GFC, or left FPF ring is defined similarly.

2. Properties of GFC rings

In this section, we shall provide preliminary results on right GFC rings, which will be used repeatedly throughout the sequel.

**Lemma 2.1.** (1) Let $I$ be an ideal of a ring $R$, and let $A$ be a right ideal of $R$ such that $I + A$ is essential in $R$. If $R/A$ is a generator for Mod-$R$, then $I$ is essential in $R$ as a right ideal.

(2) Let $R$ be a right GFC ring, and let $I$ be an ideal of $R$. Then $R/I$ is nonsingular as a right $R$-module if and only if $I$ is a semiprime ideal which has no proper essential extensions in $R$ as a right ideal.

In particular, $R$ is right nonsingular if and only if it is a semiprime ring.

Proof. See [15, Lemma 2 and Corollaries 5 and 6].

**Lemma 2.2.** Let $R$ be a right GFC ring. Let $I$ be an ideal of $R$ such that $R/I$ is nonsingular as a right $R$-module, let $A$ be a complement for $I$ in $R_\mathfrak{p}$, and set $J = r_R(R/A)$. Then

(1) $I = r_R(J) = l_R(J)$.

(2) If $R$ is right nonsingular, then $A = J = r_R(I) = l_R(I)$.

Proof. (1). If $B$ is a complement for $J$ in $A_\mathfrak{p}$, then $R/B$ is faithful, whence Lemma 2.1(1) implies that $I \oplus J \leq eR_\mathfrak{p}$. Thus $(J \oplus I)/I$ becomes an essential right ideal of the right nonsingular ring $R/I$, because $(R/I)_\mathfrak{p}$ is nonsingular. Consequently, we have $I = l_R(J)$. Moreover, noting by Lemma 2.1(2) that $I$ is a semiprime ideal, we obtain $l_R(J) = r_R(J)$.

(2). Since $R/A$ is an essential extension of $I_\mathfrak{p}$ and since $R$ is right nonsingular, the $R$-module $R/A$, and hence $(R/J)_\mathfrak{p}$, is nonsingular. Thus, it follows from the same argument as in (1) that $J = r_R(I) = l_R(I) = A$ as well.

**Lemma 2.3.** Let $R$ be a right GFC ring, and let $I$ be an ideal of $R$ such that $R/I$ is nonsingular as a right $R$-module. Then $R/I$ is a right nonsingular right GFC ring.

Proof. Since $R/I$ is a nonsingular right $R$-module, it is a right nonsingular ring. Let $A$ be a complement for $I$ in $R_\mathfrak{p}$, and set $J = r_R(R/A)$. If $B$ is a right ideal of $R$ such that $r_R(R/B) = I$, then $r_R(R/B) \leq I \cap J = 0$, so that $Tr_R(R/B) = l_R(BJ)R = R$. Setting $X = \{x \in R \mid rB \leq I\}$, by Lemma 2.2 we have $l_R(BJ) \leq X$, and hence $XR = R$. This means that $R/B$ generates $R/I$. Thus $R/I$ is a right GFC ring.
Lemma 2.4. Let $R$ be a right GFC ring, and let $I$ be an ideal of $R$. If $R/I$ is nonsingular as a right $R$-module, then it is nonsingular also as a left $R$-module.

Proof. Set $\bar{R} = R/I$. If $\bar{R}$ is nonsingular, then by Lemma 2.2 there exists an ideal $J$ of $R$ such that $I = r_{\bar{R}}(J)$ and $I \cap J = 0$. Given any essential left ideal $L$ of $R$, we see that $(L + I)/I \leq e \bar{R}$. Set $A/I = r_{\bar{R}}((L + I)/I)$ where $A \leq R$, and let $B/I$ be a complement for $A/I$ in $\bar{R}$ where $B \leq R$. Then the nonsingularity of $\bar{R}$ implies that $(R/B)_{\bar{R}}$ is nonsingular and that $l_{\bar{R}}(A/I) \cap l_{\bar{R}}(B/I) = 0$; hence $l_{\bar{R}}(B/I) = 0$, because $(L + I)/I \leq e l_{\bar{R}}(A/I) \leq e \bar{R}$. On the other hand, by Lemma 2.3 the $R$-module $R/B$ generates $R/r_{\bar{R}}(R/B)$; hence $XR = R$, where $X = \{r \in R \mid rB \leq r_{\bar{R}}(R/B)\}$. Since $AXB \leq A \cap r_{\bar{R}}(R/B) = I$ and $l_{\bar{R}}(B/I) = 0$, it follows that $A = AXR = I$. Thus we obtain $r_{\bar{R}}((L + I)/I) = 0$, which shows that $\bar{R}$ is nonsingular.

As an easy consequence of the lemmas above, we obtain the following results on left and right GFC rings.

Corollary (c.f. [12, Proposition 4]). If $R$ is a right nonsingular ring which is right and left GFC, then the maximal right quotient ring of $R$ is also the maximal left quotient ring of $R$.

Proof. Note by Lemma 2.4 that $R$ is also left nonsingular. Then, by virtue of Utumi's Theorem (c.f. [5, Theorem 2.38]), it suffices to show that if $A$ is a right ideal of $R$ such that $R/A$ is nonsingular, then $A$ is a right annihilator ideal.

We shall show that $A$ is essential in $r_{\bar{R}}(A)_{\bar{R}}$, which will obviously imply that $A = r_{\bar{R}}l_{\bar{R}}(A)$, as desired. So, let $B$ be a right ideal of $R$ such that $B \leq r_{\bar{R}}l_{\bar{R}}(A)$ and $A \cap B = 0$, and let $C$ be a complement for $B$ in $\bar{R}$ such that $A \leq C$. It then follows from Lemma 2.3 that $XR = R$, where $X = \{r \in R \mid rC \leq r_{\bar{R}}(R/C)\}$. Since $BXC = 0$ and $l_{\bar{R}}(C) \leq l_{\bar{R}}(C) \cap l_{\bar{R}}(B) = 0$, we have $BX = 0$; hence $B = 0$. Thus $A$ is essential in $r_{\bar{R}}l_{\bar{R}}(A)_{\bar{R}}$.

We give a proof for the following easy fact, which will be well known.

Lemma 2.5. Let $R$ be a right nonsingular and semiprime ring, and let $Q$ be the maximal right quotient ring of $R$. For every ideal $I$ of $R$, there exists $e \in B(Q)$ such that $I \leq e \in Q_{e}$.

Proof. An idempotent $e$ in $Q$ can be taken to satisfy $I \leq e \in Q_{e}$. We claim that $e$ is central in $Q$. To see this, let $A$ be an essential right ideal of $R$ such that $eA \leq I$. Then $(1 - e)ReA \leq (1 - e)I = 0$. Since $Q_{e}$ is nonsingular, it follows that $(1 - e)Re = 0$. Also, setting $B = eR(1 - e)R \cap R$, we see by the semiprimeness of $R$
that $B = 0$, and hence $eR(1 - e) = 0$. Thus $e$ commutes with all elements of $R$. Now, given any $x \in Q$, we take $C \leq eR$ such that $xC \leq R$. It follows that $(ex - xe)C = 0$, so that $ex = xe$. Therefore $e$ is indeed central in $Q$.

If every essential right ideal of $R$ contains an ideal which is essential in $R$ as a right ideal, then $R$ is said to be right essentially bounded. Note that in [9] such rings are referred to as right bounded rings.

**Lemma 2.6** ([9, Lemma 2]). Let $R$ be a right nonsingular ring which is right essentially bounded. If $M$ is a finitely generated faithful right $R$-module, then $M/Z(M)$ is also faithful.

**Lemma 2.7.** Let $R$ be a right GFC ring. Then

1. $R$ is right essentially bounded.
2. If $N$ is an essential submodule of a finitely generated nonsingular right $R$-module $M$, then $r_R(N) = r_R(M)$.

**Proof.** (1). See [15, Proposition 4].

(2). Let $x_1, \ldots, x_n$ generate $M_R$, and for each $i = 1, \ldots, n$, set $A_i = \{a \in R | x_i a \in N\}$. Then, by the essentiality of $N$, each $A_i$ is essential in $R$; hence, according to (1) we see, by noting $\prod_{i=1}^n r_R(R/A_i) \leq r_R(M/N)$, that $r_R(M/N) \leq eR_R$. As a result, $r_R(M/N)/r_R(M)$ is an essential right ideal of the ring $R/r_R(M)$, because $(R/r_R(M))_R$ is nonsingular. Moreover, since $r_R(M)$ is a semiprime ideal by Lemma 2.1(2) and since $r_R(M/N)r_R(N) \leq r_R(M)$, it follows that $r_R(N)r_R(M/N) \leq r_R(M)$. Thus the nonsingularity of $R/r_R(M)$ implies that $r_R(N) = r_R(M)$.

S. Page proved in [12, Theorem 2] that if $R$ is a right nonsingular right FPF ring, then the maximal right quotient ring of $R$ is also FPF, while G.F. Birkenmeier [3, Corollary 3.6] obtained the same result for right nonsingular quasi-Baer right GFC rings. The next theorem more generally shows that for any right nonsingular ring, the right GFC condition has the same effect on the maximal right quotient ring. This is useful to our study in the following sections.

**Theorem 2.8.** Let $R$ be a right nonsingular right GFC ring. Then the maximal right quotient ring of $R$ is a left and right FPF ring.

**Proof.** Let $Q$ denote the maximal right quotient ring of $R$. By [2, Theorem 2] and [11, Corollary 9.2] it suffices to prove that $Q$ is right GFC. Thus, given any right ideal $X$ of $Q$ such that $Q/X$ is faithful, we must show that $Q/X$ is a generator for $\text{Mod}-Q$. To this end, set $Y/X = Z((Q/X)_Q)$, the singular submodule of $Q/X$ as a $Q$-module, and note that $Y/X$ is the singular submodule of $Q/X$.
also as an \( R \)-module; hence \( Q/Y \) is nonsingular both as a \( Q \)-module and as an \( R \)-module.

First we claim that \( Q/Y \) is faithful. Since \( (Q/r_Q(Q/Y))Q \) is a nonsingular \( Q \)-module, there exists \( e \in B(Q) \) such that \( r_Q(Q/Y) = eQ \). Observing that the cyclic \( R \)-module \( (eR+X)/X \) is singular because it is contained in the singular module \( Y/X \), we see by Lemma 2.7(1) that \( r_R((eR+X)/X) \) is essential in \( R \). Now, let \( a \) be an arbitrary element of \( eR \cap r_R((eR+X)/X) \). Set \( I = l_Rr_R(aR) \), and let \( x \) be an idempotent in \( Q \) such that \( aQ = xQ \), and so \( x = ay \) for some \( y \in Q \). Also, by Lemma 2.1(2) and Lemma 2.5, let \( f \in B(Q) \) such that \( I \leq efR \). Then, \( ef = 0 \). To see this, note that \( aR \) is essential in \( xR \), and then by Lemma 2.7(2) that \( r_R(xR) = r_R(aR) = r_R(I) = r_R(fR) \). Thus, according to Lemma 2.3, the \( R \)-module \( xR \) generates \( fR \), whence there exist \( \varphi_1, \ldots, \varphi_n \in Hom_R(xR, fR) \) such that \( f \in \sum_{i=1}^n \varphi_i(xR) \). Each \( R \)-homomorphism \( \varphi_i \) may be extended to a \( Q \)-homomorphism from \( xQ \) to \( fQ \), so that \( ef \in \sum_{i=1}^n \varphi_i(xR) = \sum_{i=1}^n \varphi_i(x)xR \leq efRayR \leq eRayfR \leq X \) (the last inclusion of which is obtained from \( a \in r_R((eR+X)/X) \), i.e., \( ef \in X \). Since \( ef \) is central and \( Q/X \) is faithful, we obtain \( ef = 0 \), as desired. This implies that \( a = 0 \), because \( a \in eR \cap fR \). Thus \( eR \cap r_R((eR+X)/X) = 0 \), whence the essentiality of \( r_R((eR+X)/X) \) shows that \( e = 0 \). Therefore, \( r_Q(Q/Y) \) must be zero, as claimed.

Now, set \( N = (R+Y)/Y \). According to Lemma 2.5, there exists \( g \in B(Q) \) such that \( r_R(N) \leq egR \), and then \( NgB = 0 \) for some \( B \leq eR \). Noting that \( Q/Y \) is nonsingular, we have \( Ng = 0 \), and hence \( g \in Y \). Since the \( Q \)-module \( Q/Y \), as seen above, is faithful, the central idempotent \( g \) is zero, that is, \( N_R \) is faithful. It then follows from the hypothesis of \( R \) that the cyclic faithful \( R \)-module \( N \) is a generator for \( Mod-R \); hence \( l_R(R \cap Y)R = R \). On the other hand, the essentiality of \( R \cap Y \) in \( Y_R \) implies that \( l_R(R \cap Y) \leq l_0(Y) \). Therefore we obtain \( l_0(Y)Q = Q \), which means that \( Q/Y \) is a generator for \( Mod-Q \). Obviously, \( Q/X \) generates \( Q/Y \), so that \( Q/X \) is indeed a generator for \( Mod-Q \), which completes the proof of the theorem.

A ring \( R \) has \textit{bounded index} if there exists a positive integer \( n \) such that \( x^n = 0 \) for all nilpotent elements \( x \) of \( R \). The least such positive integer is called the \textit{(bounded) index} of \( R \).

The theorem above implies the following.

\textbf{Corollary 2.9.} Let \( R \) be a right nonsingular right GFC ring. Then there exists a positive integer \( k \), and \( R \) is a subdirect product of prime rings, each of which is contained in a simple artinian ring of length at most \( k \).

Proof. Let \( Q \) denote the maximal right quotient ring of \( R \). It then follows
from [11, Theorem 9] and [6, Theorem 6.2 and Corollary 7.10] that $Q$ has index $k$ for some positive integer $k$, and that for every prime ideal $P$ of $Q$, the ring $Q/P$ is a simple artinian ring of length at most $k$. Thus it suffices to show that for every prime ideal $P$ of $Q$, the ring $R/(R \cap P)$ is a prime ring. So, let $a \in R$, and let $I$ be an ideal of $R$ such that $a \subseteq P$ and $aI \subseteq R \cap P$. Also, set $J = l_{R}r_{R}(aR)$. By Lemma 2.1(2) and Lemma 2.5, there exists $e \in B(Q)$ such that $J \subseteq eR_{R}$ and then by Lemma 2.7(2), $r_{R}(aR) = r_{R}(J) = r_{R}(eR)$. It then follows from Lemma 2.3 that $aR$ generates $eR$, whence there exists $\varphi_{1}, \ldots, \varphi_{n} \in \text{Hom}_{R}(aR, eR)$ such that $e \in \sum_{i=1}^{n} \varphi_{i}(aR)$. Extending each $\varphi_{i}$ to a $Q$-homomorphism $Q \to Q$, we have $eI \subseteq \sum_{i=1}^{n} \varphi_{i}(aR)I \subseteq \sum_{i=1}^{n} \varphi_{i}(1)I \subseteq P$. On the other hand, $e \in P$, because $a \in P$ and $a \in eR$. Thus the primeness of $P$ implies that $I \subseteq P \cap R$. Therefore, $R/(R \cap P)$ is a prime ring, as desired.

3. quasi-Baer or p.p. GFC rings

In this section we shall study quasi-Baer or right p.p. right GFC rings, and give characterizations of those rings.

Following [4], we call a ring $R$ a quasi-Baer ring if the right annihilator of every ideal in $R$ is generated by an idempotent in $R$. A right p.p. ring is one in which every principal right ideal is projective.

First we note the following on right nonsingular and semiprime rings, which will be well known.

**Lemma 3.1.** Let $R$ be a right nonsingular and semiprime ring, and let $Q$ be the maximal right quotient ring of $R$. Then the following conditions are equivalent:

(a) $R$ is quasi-Baer.

(b) For every ideal $I$ of $R$ such that $R/I$ is nonsingular as a right $R$-module, $I$ is generated by an idempotent in $R$.

(c) $B(R) = B(Q)$.

**Proof.** (a)$\Rightarrow$(b). It is only to note that if $I$ is an ideal of a semiprime ring $R$ such that $(R/I)_{R}$ is nonsingular, then we have $I = r_{R}l_{R}(I)$.

(b)$\Rightarrow$(c). Since any element of $Q$ commuting with all elements of $R$ is central in $Q$, we have only to show that $B(Q) \subseteq B(R)$. Thus, given any $e \in B(Q)$, we see by (b) that there exists $f \in B(R)$ such that $eQ \cap R = fR$. It follows that $eQ = fQ$, whence $e = f \in B(R)$.

(c)$\Rightarrow$(a). This follows immediately from Lemma 2.5.

As a result of the lemma above, we note by [13, Proposition 1] that right nonsingular right FPF rings are quasi-Baer right GFC rings.
Recall that a regular ring $R$ is abelian if all idempotents in $R$ are central, or equivalently, $R$ has bounded index 1. Also, an idempotent $e$ in a regular ring $R$ is said to be abelian whenever the ring $eRe$ is abelian.

We need the following lemma as in [11, Lemma A].

**Lemma 3.2.** Let $R$ be a right nonsingular right GFC ring, and let $Q$ be the maximal right quotient ring of $R$. If $M$ is a cyclic nonsingular right $R$-module, then there exist finitely many $e_1, \ldots, e_n \in B(Q)$ such that $M$ can be embedded in $e_1R \oplus \cdots \oplus e_nR$.

**Proof.** Let $M$ be a cyclic nonsingular right $R$-module. Then there exists an idempotent $f$ in $Q$ such that $M \cong fR$. Note from [6, Theorem 7.20] that any idempotent in a right self-injective regular ring $S$ having bounded index is a finite sum of orthogonal abelian idempotents in $S$. Since $Q$ has bounded index by Theorem 2.8 and [11, Theorem 9], the idempotent $f$ can be actually expressed as $f = \sum_{i=1}^k f_i$, where $f_1, \ldots, f_k$ are orthogonal abelian idempotents in $Q$. Thus $M$ is contained in $f_1R \oplus \cdots \oplus f_kR$, so that it suffices to embed each $f_iR$ in $(e_jR)^{(n_j)}$ for some integer $n_j$ and for some $e_j \in B(Q)$. Therefore we may furthermore assume that $f$ itself is abelian in $Q$.

Let $A$ be a complement for $r_k(M)$ in $R_k$. It then follows from Lemma 2.2 that $A = l_kr_k(M)$ and $r_k(A) = r_k(M)$, and from Lemma 2.5 that there exists $e \in B(Q)$ for which $A \leq eR_k$, and then by Lemma 2.7(2), $r_k(M) = r_k(eR)$. Consequently, Lemma 2.3 shows that $M$ generates $eR$, whence there exist $\varphi_1, \ldots, \varphi_n \in \text{Hom}_R(M, eR)$ such that $e \in \sum_{i=1}^n \varphi_i(M)$. Extending each $\varphi_i$, we may assume that $\varphi_i \in \text{Hom}_Q(fQ, eQ)$. Now, we consider a homomorphism $\varphi : fQ \rightarrow (eQ)^{(n)}$ defined by $\varphi(x) = (\varphi_i(x))_{i=1}^n$ for $x \in fQ$, and claim that $\varphi$ is monic. To this end, set $K = \ker \varphi$. Then, $fQ = K \oplus N$ for some $N \leq fQ$. Since $f$ is an abelian idempotent in $Q$, it follows from [6, Theorem 3.4] that $NK = 0$. Thus, $K \leq r_k(fQ/K)$, that is, $\varphi(fQK) = 0$ for all $i = 1, \ldots, n$, from which we obtain $e(K \cap R) \leq \sum_{i=1}^n \varphi_i(fR)(K \cap R) \leq \sum_{i=1}^n \varphi_i(fR(K \cap R)) = 0$. On the other hand, $K \cap R \leq M \cap R \leq l_kr_k(M) \leq eR$, so that $K \cap R$, and hence $K$, must be zero. Therefore, $\varphi$ is indeed monic, whence the restriction of $\varphi$ to $M$ obviously embeds $M$ into $(eR)^{(n)}$, which completes the proof of the lemma.

**Remark.** The proof of the lemma above shows that if $f$ is an abelian idempotent in $Q$, then $fR_k$ can be embedded in $(eR)^{(n)}$ for some integer $n$ and for some $e \in B(Q)$. 
As a corollary, we obtain the following.

**Corollary** (c.f. [15, Corollary 9]). Let \( R \) be a right GFC ring. Let \( M \) be a cyclic nonsingular right \( R \)-module which has finite Goldie dimension. Then the endomorphism ring \( \text{End}_R(M) \) of \( M_R \) is a right order in a semisimple artinian ring.

Furthermore, if \( R \) is also left GFC, then \( \text{End}_R(M) \) is a two-sided order in a semisimple artinian ring.

Proof. According to Lemma 2.3, we may assume, by passing from \( R \) to \( R/rM \), that \( R \) is right nonsingular and \( M \) is faithful. Thus \( M \) generates \( R \), whence \( R \) can be embedded in a finite direct sum of copies of \( M \). As a result, \( R_M \) has finite Goldie dimension, because so does \( M_R \). It then follows from Lemma 2.1(2) and [5, Corollary 3.32] that \( R \) is a semiprime right Goldie ring, i.e., \( R \) is a right order in a semisimple artinian ring \( Q \). On the other hand, Lemma 3.2 shows that \( M \) can be embedded in \( e_1R \oplus \cdots \oplus e_nR \) for some central idempotents \( e_1, \ldots, e_n \) in \( Q \). Since each \( e_iR (\cong R/rR(e_iQ \cap R)) \) can be embedded in a direct product of copies of \( R_r \), so is \( M \). Therefore, [8, Theorem 2.2.14 and Theorem 2.2.17] implies that \( \text{End}_R(M) \) is a right order in a semisimple artinian ring.

If \( R \) is also left GFC, then by Lemma 2.4 we may also assume, as in the proof above, that \( R \) is right and left nonsingular and \( M \) is faithful. Thus it follows from Corollary following Lemma 2.4 and [5, Theorems 2.38 and 3.14] that the semi-prime ring \( R \) is right and left Goldie, so that [8, Theorem 2.2.17] again implies that \( \text{End}_R(M) \) is a two-sided order in a semisimple artinian ring.

Recall that a ring \( R \) is \( \text{n-PF} \) for some positive integer \( n \) if every faithful right \( R \)-module generated by at most \( n \) elements is a generator for \( \text{Mod}-R \), and also that a ring \( R \) satisfies the condition \( (C_n) \) for some positive integer \( n \) if \( R \) satisfies the following three conditions:

(i) \( R \) is right essentially bounded,
(ii) For every right ideal \( A \) generated by at most \( n \) elements, \( R = Tr_R(A) \oplus r_R(A) \),
(iii) Every nonsingular right \( R \)-module generated by at most \( n \) elements can be embedded in a free right \( R \)-module.

It is obvious that for each positive integer \( n \), the following implications hold:

\( \text{FPF} \Rightarrow (n+1-) \text{PF} \Rightarrow n \text{-PF} \Rightarrow 1 \text{-PF} \Leftrightarrow \text{GFC} \);

\( (C_{n+1}) \Rightarrow (C_n) \).

Here we note, as mentioned in Section 0, that for each \( n \geq 2 \), the conditions \( n \text{-PF} \) and \( (C_n) \) on right nonsingular rings are equivalent.
**Proposition A** (c.f. [10, Proposition 1]). Let \( n (\geq 2) \) be an integer. Then a ring \( R \) is right nonsingular and \( n \)-PF if and only if \( R \) satisfies \((C_n)\).

Proof. The "only if" part is obtained by modifying the proof of [10, Proposition 1] and noting [5, Theorem 5.17].

Conversely, assume that \( R \) satisfies \((C_n)\), and then note by the conditions (i) and (ii) of \((C_n)\) that \( R \) is right nonsingular. Let \( M \) be a faithful right \( R \)-module generated by at most \( n \) elements.

To prove that \( M \) is a generator, according to Lemma 2.6 we may assume, by replacing \( M \) with \( M/Z(M) \), that \( M \) is nonsingular; hence by (iii) there exists a positive integer \( k \), and a monomorphism \( \varphi : M \to R^k \). For each \( i = 1, \ldots, k \), letting \( p_i : R^k \to R \) be the \( i \)-th projection, we see by (ii) that \( R = \sum_{i=1}^k Tr_R(p_i \varphi(M)) = r_R(M) = 0 \), it follows that \( R = \prod_{i=1}^k Tr_R(p_i \varphi(M)) = Tr_R(M) \), as desired.

In the case \( n = 1 \), the proposition above can not hold in general. However, in the next theorem, we shall show that the right GFC (i.e., 1-PF) rings with quasi-Baer condition are precisely the rings satisfying the condition \((C_1)\). Thus it seems that the right GFC rings, under the quasi-Baer condition, fairly behave as well as nonsingular \( n \)-PF \((n \geq 2)\) rings, and hence, as FPF rings.

**Theorem 3.3.** For a ring \( R \), the following conditions are equivalent:

(a) \( R \) is a quasi-Baer right GFC ring,

(b) (i) \( R \) is right essentially bounded,

(ii) For every \( a \in R \),

\[
Tr_R(aR) \oplus r_R(aR),
\]

(iii) Every cyclic nonsingular right \( R \)-module can be embedded in a free right \( R \)-module.

(c) (i) \( R \) is right nonsingular and right essentially bounded,

(ii) For every cyclic nonsingular right \( R \)-module \( M \),

\[
R = Tr_R(M) \oplus r_R(M).
\]

Proof. (a)\(\Rightarrow\)(b). The condition (b)(i) follows from Lemma 2.7(1). Given any \( x \in R \) such that \( r_R(x) \leq eR \), we see by the right essentially boundedness of \( R \) that \( r_R(xR) \leq eR \). Since \( R \) is quasi-Baer, the ideal \( r_R(xR) \) is generated by an idempotent; hence \( x = 0 \). Thus \( R \) is right nonsingular, whence (b)(iii) is obtained from Lemma 2.1(2), Lemma 3.1 and Lemma 3.2. To prove (b)(ii), let \( a \in R \). Since \( R \) is quasi-Baer, there exists an ideal \( I \) of \( R \) such that \( R = r_R(aR) \oplus I \). Obviously, \( Tr_R(aR) = I \cap Tr_R(aR) \leq I \), while conversely, it follows from Lemma 2.3 that \( I \leq \)
Therefore we obtain $R = \text{Tr}_R(aR) \oplus r_R(aR)$.

(b)$\Rightarrow$(c). The conditions (b)(i)(ii) immediately shows that $R$ is right nonsingular. For (c)(ii), let $M$ be a cyclic nonsingular right $R$-module generated by $x$. According to (b)(iii), there exists a monomorphism $\varphi : M \rightarrow R^{(n)}$ for some positive integer $n$. Setting $\varphi(x) = (a_1, \cdots, a_n) \in R^{(n)}$, we have $r_R(M) = \bigoplus_{i=1}^n r_R(a_iR)$. Also, according to (b)(ii), for each $i = 1, \cdots, n$, there exists $e_i \in B(R)$ such that $\text{Tr}_R(a_iR) = e_iR$ and $r_R(a_iR) = (1 - e_i)R$. The element $\prod_{i=1}^n (1 - e_i)$ belongs to $r_R(M)$, so that we have $r_R(M) + \sum_{i=1}^n e_iR = R$. It is easy to see that $\text{Tr}_R(a_iR) \subseteq \text{Tr}_R(M)$ for all $i$; hence $R = \text{Tr}_R(M) + r_R(M)$. Moreover, since $(\text{Tr}_R(M) \cap r_R(M))^2 = 0$, the condition (b)(ii) implies that $\text{Tr}_R(M) \cap r_R(M) = 0$. Therefore we obtain $R = \text{Tr}_R(M) \oplus r_R(M)$.

(c)$\Rightarrow$(a). Let $M$ be a cyclic faithful right $R$-module. It then follows from (c)(i) and Lemma 2.6 that $M/Z(M)$ is also faithful, and then from (c)(ii) that $R = \text{Tr}_R(M/Z(M))$. Thus $M/Z(M)$, and hence $M$, is a generator for Mod-$R$. Therefore $R$ is right GFC. Next, to prove that $R$ is quasi-Baer, let $I$ be an ideal of $R$, and set $J/I = Z((R/I)R)$, where $J \subseteq R$. Noting that $J$ becomes an ideal such that $(R/J)_R$ is nonsingular, by (c)(ii) we have $R = J \oplus K$, where $K = \text{Tr}_R((R/J)_R)$. Since $R$ is right nonsingular, $I$ is essential in $J_R$, whence Lemma 2.7(2) shows that $r_R(I) = r_R(J) = K$. Thus $R$ is quasi-Baer, which completes the proof of the theorem.

**Remark.** As can be seen above, the following implications on rings hold:

(C$_1$)$\Leftrightarrow$quasi-Baer and right GFC (i.e., 1-PF);
(C$_n$)$\Leftrightarrow$right nonsingular and $n$-PF for each $n \geq 2$.

Concerning rings satisfying the condition (C$_n$) for $n \geq 1$, we may improve Corollary 2.9 on nonsingular GFC rings as follows.

**Proposition B.** Let $R$ be a ring satisfying the condition (C$_n$) for some positive integer $n$. Then there exists a positive integer $k$, and $R$ is a subdirect product of prime ring $R_i$'s, where each $R_i$ is contained in a simple artinian ring of length at most $k$ such that every nonzero right ideal of $R_i$ generated by at most $n$ elements is a generator for Mod-$R_i$.

In particular, if $R$ is a right nonsingular right FPF ring, then each the ring $R_i$ above may be taken to satisfy the condition that every nonzero finitely generated right ideal of $R_i$ is a generator for Mod-$R_i$.

Proof. Let $Q$ denote the maximal right quotient ring of $R$. Since the intersection of all minimal prime ideals of $Q$ is zero, it suffices, as in the proof of
Corollary 2.9, to show that for every minimal prime ideal \( P \) of \( Q \), every nonzero right ideal of the ring \( R/(R \cap P) \) generated by at most \( n \) elements is a generator for \( \text{Mod}(R/(R \cap P)) \). So, let \( a_1, \cdots, a_n \in R \) such that \( a_i \notin P \), and set \( A = \sum_{i=1}^{n} a_i R \).

Then we must show that \( (A+ (R \cap P))/(R \cap P) \approx A/(A \cap P) \) generates \( R/(R \cap P) \). By hypothesis, there exists \( e \in B(R) (= B(Q) \) by Lemma 3.1) such that \( r_\delta(A) = eR \) and \( Tr_\delta(A) = (1-e)R \). Since \( (1-e)a_i = a_i \notin P \), it follows that \( 1-e \notin P \); hence \( e \notin P \). Thus we have \( r_\delta(A) \subseteq R \cap P \), whence \( A \) generates \( R/(R \cap P) \), that is, there exists an epimorphism \( \varphi : A^{(m)} \rightarrow R/(R \cap P) \) for some positive integer \( m \). Now, observing by [6, Theorem 8.26 and Corollary 9.15] that \( P = \{ \exists x | e \in P \cap B(R) ; x \in Q \} \) and hence \( R \cap P = \{ er | e \in P \cap B(R) ; r \in R \} \), we see that \( \varphi \) induces an epimorphism from \( (A/(A \cap P))^{(m)} \) onto \( R/(R \cap P) \). Therefore, \( A/(A \cap P) \) generates \( R/(R \cap P) \), as desired.

The second assertion is now obvious.

Here we shall present the following examples to illustrate the conditions of Theorem 3.3.

**Example 1.** (1) There exists a ring \( R \) which satisfies the conditions (b)(i) (ii) and (c)(i) of Theorem 3.3, but \( R \) is not quasi-Baer right GFC.

Choose a commutative domain \( D \), set \( D_n = D \) for all \( n = 1, 2, \cdots \), and set \( T = \prod_{n=1}^{\infty} D_n \) and \( R = D \cdot 1 + \bigoplus_{n=1}^{\infty} D_n \subseteq T \). Then, we see that \( R \) satisfies the conditions (b) (i) (ii) and (c)(i) of Theorem 3.3.

Set \( x = (x_n) \in T \) such that \( x_n = 0 \) if \( n \) is odd; \( x_n = 1 \) if \( n \) is even. Then, it is easy to see that \( xR \) can not be embedded in a free \( R \)-module. Therefore \( R \) is not a quasi-Baer right GFC ring.

(2) There exists a ring \( R \) which satisfies the conditions (b)(ii) (iii) and (c)(ii) of Theorem 3.3, but \( R \) is not quasi-Baer right GFC.

Let \( R \) be a simple noetherian ring which is not artinian (e.g. the Weyl algebra over a field of characteristic 0). Obviously, \( R \) satisfies the condition (b)(ii) of Theorem 3.3, while by [8, Theorem 2.2.15] it does also the condition (b)(iii) and (c)(ii) of Theorem 3.3.

But, \( R \) is not right essentially bounded, because \( R \) is a simple non-artinian ring. Therefore \( R \) is not a quasi-Baer right GFC ring.

(3) There exists a ring \( R \) which satisfies the conditions (b)(i) (iii) of Theorem 3.3, but \( R \) is not quasi-Baer right GFC.

Let \( R \) be a right artinian ring such that \( Z(R) \leq eR \) (e.g. let \( p \) be a prime number and \( k \) \( (\geq 2) \), \( n \) positive integers, and let \( R \) be the ring of all lower triangular \( n \times n \) matrices over \( \mathbb{Z}/p^k\mathbb{Z} \), where \( \mathbb{Z} \) is the ring of integers). Then \( R \)}
obviously satisfies the conditions (b)(i) (iii) of Theorem 3.3.

But, choosing \(0 \neq a \in Z(R_T)\), we see that \(r_T(aR) \cap T_r(aR) \neq 0\). Therefore \(R\) is not a quasi-Baer right GFC ring.

From now on, we shall be concerned with right p.p. right GFC rings.
We need the following lemma on right p.p ring.

**Lemma 3.4** (c.f. [14, Proposition I, 6.9]). Let \(R\) be a right p.p. ring. Then every cyclic submodule of a free right \(R\)-module is isomorphic to a direct sum of principal right ideals of \(R\); in particular, it is projective.

**Proof.** Let \(M\) be a cyclic submodule of a free right \(R\)-module \(F = \bigoplus_{i=1}^{n} R_{i}\), where each \(R_{i} = R\). The proof is by induction on \(n\).

The case \(n = 1\) is clear. Now let \(n > 1\), and let \(p\) be the \(n\)-th projection \(F \to R_n\). Since \(R\) is right p.p., the epimorphism \(p : M \to p(M)\) splits, so that \(M \cong p(M) \oplus (\operatorname{Ker} p \cap M)\). Noting that \(\operatorname{Ker} p \cap M\) is a cyclic submodule of \(\bigoplus_{i=1}^{n-1} R_{i}\), we see by the induction hypothesis that \(\operatorname{Ker} p \cap M\), and hence \(M\), is isomorphic to a direct sum of principal right ideals of \(R\).

We call an idempotent \(e\) in \(R\) a **faithful** idempotent if the \(R\)-module \(eR\) is faithful.

The following is a categorical result on right p.p. right GFC rings (c.f. [13, Corollary 1B]).

** Proposition 3.5.** Let \(R\) be a right p.p. right GFC ring, and let \(Q\) be the maximal right quotient ring of \(R\). Then there exists a faithful idempotent \(e\) in \(R\) such that \(eQE\) is a self-injective abelian regular ring which is the maximal right quotient ring of \(eRE\).

In particular, \(R\) is Morita equivalent to a right nonsingular ring whose the maximal right quotient ring is a self-injective abelian regular ring.

**Proof.** By virtue of Theorem 2.8 and [11, Theorem 9], there exists an idempotent \(f\) in \(Q\) such that \(f\) is faithful and abelian, which means that the \(Q\)-module \(fQ\) is faithful and the regular ring \(fQf\) is abelian. According to Lemma 2.5, there exists \(g \in B(Q)\) for which \(r_T(fR) \leq gQ\), and then \(gA \leq r_T(fR)\) for some \(A \leq sR_e\). Noting that \(R\) is right nonsingular and that \(fgA = 0\), we have \(fg = 0\). Since \(fQ\) is faithful, it follows that \(g = 0\), that is, the \(R\)-module \(fR\) is faithful. Now, by Remark following Lemma 3.2, and Lemma 3.4, the faithful module \(fR \cong R/(1-f)Q \cap R\) is projective, whence there exists an idempotent \(e\) in \(R\) such that \((1-f)Q \cap R = (1-e)R\), and then \(eR \cong fR\) is faithful. Moreover, we have \(eQE \cong\)
End\(\Phi(eQ) \cong \text{End}_\Phi(fQ) \cong fQf\), whence \(eQe\) is a self-injective abelian regular ring. Thus, to show the first assertion, it suffices to prove that \(eRe\) is essential in \((eQe)_{eRe}\). To this end, given any nonzero element \(exe\) in \(eQe\), where \(x \in Q\), we take an essential right ideal \(B\) of \(R\) such that \(0 \neq exeB \leq R\). Since \(R\) is semiprime by Lemma 2.1(2), it follows that \((exeB)^2\), and hence \(exe(eRe) \cap eRe\), is nonzero. Consequently, \(eRe\) is essential in \((eQe)_{eRe}\) as desired.

The assumption of \(R\) also implies that \(eR\) is a generator for \(\text{Mod}-R\). Therefore, \(R\) is Morita equivalent to the right nonsingular ring \(eRe\) whose the maximal right quotient ring is a self-injective abelian regular ring, thereby completing the proof of the proposition.

**Lemma 3.6.** Let \(R\) be a right nonsingular right GFC ring, and let \(f\) be an idempotent in \(R\). Then \(R = (RfR) \oplus r_p(fR)\).

**Proof.** According to Lemma 2.3, the \(R\)-module \(fR\) generates \(R / r_p(fR)\), whence there exist \(a_1, \ldots, a_n, b_1, \ldots, b_n \in R\) such that \(\sum a_i b_i = 1\) and \(a_i(1-f) \in r_p(fR)\) for \(i = 1, \ldots, n\). Thus we have \(1 = \sum a_i (1-f) b_i + \sum a_i f b_i\), which implies that \(R = r_p(fR) + RfR\). Moreover, since \(R\) is a semiprime ring by Lemma 2.1(2), it follows that \(r_p(fR) \cap RfR = 0\).

Concerning right p.p. (and quasi-Baer) rings, we obtain the following results.

**Theorem 3.7.** For a right p.p. ring \(R\), the following conditions are equivalent:

(a) \(R\) is a right GFC ring.

(b) (i) \(R\) is right essentially bounded,

(ii) For every idempotent \(f\) in \(R\), the ideal \(RfR\) is generated by a central idempotent in \(R\),

(iii) Every cyclic faithful nonsingular right \(R\)-module has a direct summand which is faithful and projective.

**Proof.** (a)\(\Rightarrow\)(b). The conditions (b)(i) (ii) follow from Lemma 2.7(1) and Lemma 3.6.

For (b)(iii), let \(C\) be a cyclic faithful nonsingular right \(R\)-module. Then, \(C \cong fR_R\) for some idempotent \(f\) in the maximal right quotient ring \(Q\) of \(R\). As in the proof of Lemma 3.2, the idempotent \(f\) can be expressed as \(f = \sum f_i\), where \(f_i\), \(\ldots, f_k\) are orthogonal abelian idempotents in \(Q\). Since \(fQ_Q\) is faithful, i.e., \(f\) is a faithful idempotent in \(Q\), we may take \(f_i\) to be a faithful idempotent in \(Q\), so that by the first half of the proof of Proposition 3.5, the \(R\)-module \(f_i R_R\) is faithful and
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projective. Thus we have a split epimorphism \( C \cong (f_1 + \cdots + f_n)R \to f_iR \), which
implies (b)(iii).

(b) \implies (a). To prove that \( R \) is right GFC, it suffices by (b)(i) (iii) and Lemma 2.6 to show that every cyclic faithful projective right \( R \)-module \( M \) is a generator for \( \text{Mod-}R \). Since \( M \) is cyclic projective, \( M \cong fR \) for some idempotent \( f \) in \( R \). By (b)(ii), there exists \( e \in B(R) \) such that \( RfR = eR \), and then \( 1 - e \in r_\theta(fR) = r_\theta(M) = 0 \). Thus we obtain \( Tr_\theta(M) = RfR = R \), as desired.

**Theorem 3.8.** For a ring \( R \), the following conditions are equivalent:

(a) \( R \) is a quasi-Baer right p.p. right GFC ring.

(b) (i) \( R \) is right nonsingular and right essentially bounded,

(ii) For every idempotent \( f \) in \( R \), the ideal \( RfR \) is generated by a central idempotent in \( R \),

(iii) Every cyclic nonsingular right \( R \)-module is projective.

Proof. (a) \implies (b). This follows immediately from Lemma 3.1, Lemma 3.2, Lemma 3.4 and Theorem 3.7.

(b) \implies (a). It follows from (b)(i) (iii) that \( R \) is a quasi-Baer right p.p. ring, while Theorem 3.7 implies that \( R \) is right GFC.

Let \( S \) be a ring, and let \( n \) be a positive integer. We denote by \( M_n(S) \) the ring of all \( n \times n \) matrices over \( S \), and by \( e_{ij} \) (\( 1 \leq i, j \leq n \)) the matrix units in \( M_n(S) \), i.e., \( e_{ij} \) has a \( 1 \) in the \((i, j)\) position as its only nonzero entry.

We shall illustrate the conditions of Theorem 3.7 by the following examples, in which all rings considered are regular rings.

**Example 2.** (1) There exists a regular ring \( R \) which satisfies the conditions (b) (i) (ii) of Theorem 3.7, but \( R \) is not right GFC.

Choose an abelian regular ring \( S \) which is not self-injective (e.g., let \( S = D \cdot 1 + \bigoplus_{n=1}^\infty D_n \) be as in Example 1(1), where each \( D_n = D \) is a division ring). Let \( n \) be an integer \( \geq 2 \), and set \( R = M_n(S) \) and \( Q = M_n(Q(S)) \), where \( Q(S) \) is the maximal quotient ring of \( S \). It then follows from [6, Lemma 6.20] that \( R \) satisfies the condition (b)(i) of Theorem 3.7, while it is easy to see that \( R \) satisfies the condition (b)(ii) of Theorem 3.7.

But, there exists a cyclic faithful nonsingular right \( R \)-module which has no faithful and projective direct summands. Indeed, choose \( x \in Q(S) - S \) and set \( e = e_{11} + xe_{1n} \), and \( C = eR = \begin{pmatrix} S + xS & \cdots & S + xS \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \). Then we see that \( e \) is a faithful
abelian idempotent in \( Q \), and \( C_R \) is a cyclic faithful nonsingular \( R \)-module. Now, suppose that \( C_R = C_1 \oplus C_2 \), where \( C_1 \) is faithful and projective. Then, \( eQ = E(C_1) \oplus E(C_2) \), where \( E(C_i) \) is the \( R \)-injective hull of \( C_i \). Since \( E(C_1) \) is a faithful principal right ideal of \( Q \), and hence, a generator for \( \text{Mod-}Q \), and since by [6, Theorem 3.4], \( \text{Hom}_Q(E(C_1), E(C_2)) = 0 \), it follows that \( E(C_2) = 0 \), so that \( C_R = C_1 \) is projective. Consequently, the \( S \)-module \( S + xS \) is projective, whence \( x \in S \), a contradiction. Thus, \( C_R \) must have no faithful and projective direct summands. Therefore \( R \) is not a right GFC ring.

(2) There exists a regular ring \( R \) which satisfies the conditions (b)(ii) (iii) of Theorem 3.7, but \( R \) is not right GFC.

Choose an infinite dimensional vector space \( V \) over a field \( F \), and set \( S = \text{End}_F(V) \) and \( K = \{ x \in S | \text{dim}_F(xV < \text{dim}_F(V)) \} \). Let \( R \) be the maximal right quotient ring of \( S/K \) (See [6, Example 10.11]). Since \( R \) is a simple right self-injective regular ring, it obviously satisfies the conditions (b)(ii) (iii) of Theorem 3.7.

But, \( R \) is a simple non-artinian ring, whence it is not right essentially bounded. Therefore \( R \) is not a right GFC ring.

(3) There exists a regular ring \( R \) which satisfies the conditions (b)(i) (iii) of Theorem 3.7, but \( R \) is not right GFC.

For each \( n = 1, 2, \ldots \), choose a regular ring \( R_n \) having bounded index \( i_n \) such that the supremum of all \( i_n \)'s is infinite (e.g. as a simple such \( R_n \), we may take a simple artinian ring of length \( n \)), and set \( R = \prod_{n=1}^{\infty} R_n \). First we shall show that \( R \) does not satisfy the condition (b)(ii) of Theorem 3.7. By [6, Theorem 7.2], for each \( n \), there exist nonzero orthogonal idempotents \( f_{n,1}, f_{n,2}, \ldots, f_{n,i_n} \) in \( R_n \) such that \( f_{n,j}R_n \cong f_{n,j}R_n \) for all \( j \in \{1, 2, \ldots, i_n \} \). Now, set \( f = (f_{1,1}, f_{2,1}, \ldots) \in R \), and claim that the ideal \( RfR \) can not be generated by any central idempotents in \( R \). Suppose, to the contrary, that \( RfR = gR \) for some \( g \in B(R) \). Obviously, \( g = (g_1, g_2, \ldots) \in R \), where \( g_n \in B(R_n) \) for all \( n \). For each \( n \), we have \( \bigoplus_{j=1}^{i_n} f_{n,j}R_n \leq Tr_{R_n}(f_{n,1}R_n) = R_nf_{n,1}R_n = g_nR_n \); hence \( \bigoplus_{j=1}^{i_n} f_{n,j}R_n \oplus X_n = g_nR_n \) for some \( R_n \)-submodule \( X_n \) of \( g_nR_n \). On the other hand, since \( g \in RfR \), there exists a positive integer \( s \), and for each \( n \) there exist \( x_{n,1}, x_{n,2}, \ldots, x_{n,s}, y_{n,1}, y_{n,2}, \ldots, y_{n,s} \in R_n \) such that \( g_n = \sum_{k=1}^{s} x_{n,k}f_{n,1}y_{n,k} \). By the assumption of \( R_n \), an integer \( m \) can be taken to satisfy \( i_m \geq s + 1 \). Defining a map \( \varphi : (f_{m,1}R_m)^{(s)} \rightarrow R_m \) by \( (f_{m,1}z_k)_{k=1}^{s} \mapsto \sum_{k=1}^{s} x_{m,k}f_{m,1}z_k \) for \( z_k \in R_m \), and noting that \( g_mR_m \leq \text{Im} \varphi \), we obtain \( (f_{m,1}R_m)^{(s)} \cong g_mR_m \oplus Y \) for some right
Rings whose cyclic faithful modules are generators

Thus, it follows that \( g_m R_m = (\bigoplus_{j=1}^{m} f_{m,j} R_m) \oplus X_m \cong (f_{m,1} R_m)^{(m)} \oplus X_m \cong g_m R_m \oplus Y \oplus (f_{m,1} R_m)^{(m-1)} \oplus X_m \). But, this contradicts the fact that every finitely generated projective \( R_m \)-module is directly finite (see [6, Corollary 7.11 and Proposition 5.2]). Therefore \( R \) does not satisfy the condition (b)(ii) of Theorem 3.7.

Now, assume, in addition, that each \( R_n \) is self-injective. Since \( R = \prod_{n=1}^{\infty} R_n \) is self-injective, it obviously satisfies the condition (b)(iii) of Theorem 3.7. Furthermore, it is easy to see that any direct product of right essentially bounded rings is also right essentially bounded, whence [6, Lemma 6.20 and Corollary 7.10] implies that \( R \) satisfies the condition (b)(i) of Theorem 3.7.

4. Regular GFC rings

In this section we shall characterize regular right GFC rings and determine the structure of those rings.

We use the following easy fact on matrix rings.

**Lemma 4.1.** Let \( S \) be a ring, and let \( n \geq 2 \) be an integer, and set \( T = M_n(S) \). For each \( a \in S \) and for \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \), set \( a^{(i,j)} = e_{ij} + ae_{ij} \in T \). Then

(1) Both \( a^{(i,j)} \) and \( 1 - a^{(i,j)} \) are faithful idempotents in \( T \).

(2) The ring \( T \) is generated by all the idempotents \( a^{(i,j)} \) \((a \in S ; 1 \leq i, j \leq n \text{ with } i \neq j)\) as a ring.

**Proof.**

(1). Since \( a^{(i,j)} T = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \) \(i\)-th and \( (1-a^{(i,j)}) T = \begin{pmatrix} S & \cdots & S \\ \vdots & \ddots & \vdots \\ S & \cdots & S \end{pmatrix} \) \(i\)-th, it follows that both the idempotents \( a^{(i,j)} \) and \( 1 - a^{(i,j)} \) are faithful.

(2). It suffices to show that for any \( x \in S \) and for \( i, j \in \{1, \ldots, n\} \), the element \( xe_{ij} \) is expressed as a product of such the elements \( a^{(i,j)} \). If \( i \neq j \), then \( xe_{ij} = x^{(i,j)} \cdot 0^{(j,i)} \). On the other hand, in case \( i = j \), choose \( k \neq i \) so that we obtain \( xe_{ii} = \)}
\[ x^{(k,i)} \cdot 0^{(k,i)} \cdot 1^{(k,i)} \cdot 0^{(k,i)}, \] thereby completing the proof of the lemma.

**Lemma 4.2.** Let \( R \) be a semiprime ring, and let \( Q \) be the maximal right quotient ring of \( R \). Then the following conditions are equivalent:

(a) (i) \( R \) contains all the faithful idempotents in \( Q \).
(ii) \( R \) has bounded index.

(b) (i) For right ideals \( A_1 \) and \( A_2 \) of \( R \) such that both \( R/A_1 \) and \( R/A_2 \) are faithful and nonsingular and \( A_1 \cap A_2 = 0 \), the sum \( A_1 \oplus A_2 \) is a direct summand of \( R_\infty \).
(ii) \( R \) has bounded index.

(c) \( R \) is isomorphic to a finite direct product of a ring whose maximal right quotient ring is a self-injective abelian regular ring, and full matrix rings over self-injective abelian regular rings.

Proof. Note from [7, Proposition 4] that any semiprime ring having bounded index is right (and left) nonsingular, and hence its maximal right quotient ring is a right self-injective regular ring.

(a) \( \Rightarrow \) (b). Let \( A_1, A_2 \) be right ideals of \( R \) such that both \( R/A_1 \) and \( R/A_2 \) are faithful and nonsingular. Taking idempotents \( e_1, e_2 \in Q \) to satisfy \( A_i \leq e_i Q \) for \( i = 1, 2 \), we have \( e_1 Q \cap e_2 Q = 0 \). Since \( Q \) is a regular ring, we may assume that \( e_1, e_2 \) are orthogonal. Also, \( A_i = e_i Q \cap R \), because \( R/A_i \) is nonsingular. Thus \( R/A_i \cong (1-e_i)R_\infty \) is faithful, whence so is \( (1-e_i)Q \). The condition (a)(i) now implies that each \( e_i \) belongs to \( R \), from which we obtain \( A_i = e_i R \) for \( i = 1, 2 \). Therefore \( A_1 \oplus A_2 = (e_1 + e_2) R \) is a direct summand of \( R \).

(b) \( \Rightarrow \) (c). By (b)(ii), \( R \) has index \( n \geq 1 \). First we claim that \( Q \) has index at most \( n \). Suppose not. Then, according to [6, Theorem 7.2], \( Q \) contains a direct sum of \( n+1 \) nonzero pairwise isomorphic right ideals; hence there exist nonzero orthogonal idempotents \( e_1, e_2, \ldots, e_{n+1} \) in \( Q \) such that \( e_i Q \cong e_j Q \) for all \( i, j \), because \( Q \) is a regular ring. Observe that for each \( i = 1, \ldots, n \), both \( (1-(e_1+\cdots+e_i))Q \) and \( (1-e_i)Q \) are faithful. It then follows, as in the first half of the proof of Proposition 3.5, that for each \( i \), both \( (1-(e_1+\cdots+e_i))R_\infty \cong R/(e_1+\cdots+e_i)Q \cap R \) and \( (1-e_i)R_\infty \cong R/(e_i+\cdots+e_n)Q \cap R \) are faithful. Noting that if \( (e_1 Q \cap R) \oplus \cdots \oplus (e_i Q \cap R) \) is a direct summand of \( R_\infty \), then \( (e_1 Q \cap R) \oplus \cdots \oplus (e_i Q \cap R) = (e_1+\cdots+e_i)Q \cap R \), and using (b)(i) \( n \) times in succession, we conclude that \( (e_1 Q \cap R) \oplus (e_2 Q \cap R) \oplus \cdots \oplus (e_{n+1} Q \cap R) \) is a direct summand of \( R_\infty \), so that there exist orthogonal idempotents \( e_1, \ldots, e_{n+1} \) in \( R \) such that \( e_i Q \cap R = e_i R \) for all \( i \). Since \( e_i Q = e_i Q \), we may take each \( e_i \) to be in \( R \). For each \( i = 1, \ldots, n \), let \( \varphi_i : e_i Q \to e_{i+1} Q \) be an isomorphism. Then, as in the proof of [6, Corollary 7.4], it is easy to see by the essentiality of \( e_i R \) in \( e_i Q \) that there exists a nonzero submodule \( A_1 \) of \( e_i R_\infty \) such that \( \varphi_i \varphi_{i-1} \cdots \varphi_1 (A_1) \) is a nonzero submodule of \( e_{i+1} R_\infty \) for all \( i \). For each \( i = 2, 3, \ldots, n+1 \), set \( A_i = \varphi_{i-1} \varphi_{i-2} \cdots \varphi_1 (A_1) \). Then, \( A_1, A_2, \ldots, A_{n+1} \) must be zero. Indeed, for each \( i = 1, \ldots, n \), let \( a_i \) be an arbitrary element of \( A_i \), and as
in the proof of (a)⇒(c) in [6, Theorem 7.2], set \( a = a_1e_2 + a_2e_3 + \cdots + a_ne_{n+1} \). Then, each \( a_i \) belongs to \( e_iR \), so that \( a^n = a_1a_2^{n-1}a_n + a_2a_3^{n-1}a_n + \cdots + a_na_1^{n-1}a_n \) and then \( a^{n+1} = 0 \); hence \( a^n = 0 \), because \( R \) has index \( n \). This shows that \( A_1A_2\cdots A_{n+1} = A_1A_2\cdots A_{n+1}R = 0 \). Thus there exists \( k \in \{2, 3, \cdots, n+1\} \) such that \( A_{k-1}A_k\cdots A_{n+1} = 0 \) and \( A_kA_{k+1}\cdots A_{n+1} = 0 \). But, \( A_kA_kA_k\cdots A_{n+1} = \varphi_{k-1}(A_{k-1})A_kA_k\cdots A_{n+1} = \varphi_{k-1}(A_{k-1}A_k\cdots A_{n+1}) = 0 \), whence \( (A_kA_kA_k\cdots A_{n+1})^2 = 0 \). Since \( R \) is a semiprime ring, it follows that \( A_kA_kA_k\cdots A_{n+1} = 0 \), which is a contradiction. Therefore \( Q \) has index at most \( n \), as claimed.

According to [6, Theorem 7.20], \( Q = \prod_{h=1}^{m} M_{n(h)}(D_h) \), where \( n(1) = 1 \) and \( n(h) = 1 \) for \( h = 2, 3, \cdots, m \), and where \( D_1, D_2, \cdots, D_m \) are self-injective abelian regular rings. If \( Q \) is abelian, i.e., \( Q = D_1 \), then (c) obviously holds. Thus assume that \( Q \) is not abelian. Let \( f_1, f_2, \cdots, f_m \) denote the complete set of orthogonal central idempotents in \( Q \) such that \( f_1Q = D_1 \) and \( f_hQ = M_{n(h)}(D_h) \) for \( h = 2, 3, \cdots, m \). Then, to obtain (c), it suffices to prove that \( f_hQ \leq R \) for \( h = 2, 3, \cdots, m \). Let \( h \in \{2, 3, \cdots, m\} \) be fixed. In view of Lemma 4.1(2), it furthermore suffices to prove that \( a^{(i,j)} \in R \) for all \( a \in D_h \) and for all \( i, j \in \{1, 2, \cdots, n(h)\} \) with \( i \neq j \), where \( a^{(i,j)} = e_{ii} + ae_{ij} \in M_{n(h)}(D_h) \), and where \( e_{ij} \) (\( 1 \leq i, j \leq n(h) \)) are the \((i, j)\) matrix units in \( M_{n(h)}(D_h) \). To this end, set \( g_1 = a^{(i,j)} \) and \( g_2 = f_h - g_1 \), and set \( g_1^* = 1 - g_2 \) and \( g_2^* = 1 - g_1 \). Since by Lemma 4.1(1) both \( g_1 \) and \( g_2 \) are faithful idempotents in the ring \( f_hQ \), it follows that both \( g_1Q \) and \( g_2Q \) are faithful, whence \( g_1^*R \cong R/(g_1Q \cap R) \) is faithful for \( i \neq j \). Thus, the condition (b)(i) implies that \( (g_1Q \cap R) \oplus (g_2Q \cap R) \) is a direct summand of \( R_R \). Consequently, there exist orthogonal idempotents \( g_1, g_2 \) in \( R \) such that \( g_iQ \cap R = g_iR \) for \( i = 1, 2 \), from which we have \( g_1 = g_1g_1 = g_1(g_1 + g_2) = g_1f_h = g_1 \). Therefore we conclude that \( a^{(i,j)} = g_1 = g_1^* \in R \), as desired.

(c)⇒(a). If \( R = R_1 \times R_2 \), where the maximal right quotient ring of \( R_1 \) is a self-injective abelian regular ring, and where \( R_2 \) is a finite direct product of full matrix rings over self-injective abelian regular rings, then \( R_1 \) has bounded index at most 1 and \( R_2 \) also has bounded index by [6, Theorem 7.12]; hence \( R \) itself has bounded index. On the other hand, since both the rings \( R_1 \) and \( R_2 \) obviously satisfy the condition (a)(i), so does the ring \( R \), which completes the proof of the lemma.

Let \( R \) be a regular ring having bounded index with \( Q \) the maximal right quotient ring. Then by [6, Corollary 7.4 and Theorem 7.20], \( Q = \prod_{h=1}^{m} M_{n(h)}(D_h) \) = \( \prod_{h=1}^{m} f_hQ \), as in the proof of (b)⇒(c) in Lemma 4.2. For each \( h \), let \( e_{ij}^{(h)} \) denote the \((i, j)\) matrix unit in \( M_{n(h)}(D_h) \). Now, let \( g \) be an arbitrary faithful abelian idempotent in \( M_{n(h)}(D_h) \). Then it is easy to see that \( (f_1 + \cdots + f_{h-1} + g + f_{h+1} + \cdots + f_m)R_R \) can be embedded in \( \oplus \{e_{ij}^{(h)} + e_{ij}^{(h+1)} + e_{ij}^{(h+1)} + e_{ij}^{(m)} \} \) for \( i, j \leq n(j) \) for \( j \in \{1, \cdots, h-1, h+1, \cdots, m\} \), and each \( e_{ij}^{(h)} + e_{ij}^{(h+1)} + e_{ij}^{(h+1)} + e_{ij}^{(m)} \)
\[ + \cdots + e_{e_{m+1}}^{(m)} \text{ is a faithful abelian idempotent in } Q. \]

On the other hand, we see, by observing the proof of \((b) \Rightarrow (c)\) in Lemma 4.2, that if all such the \(R\)-modules \((e_i + \cdots + e_{(m-1)}^i + \cdots + e_{(n-1)}^i + g + e_{(n+1)}^i + \cdots + e_{(m+n)}^i)R_R\), and hence \((f_i + \cdots + f_{(n-1)} + g + f_{(n+1)} + \cdots + f_m)R_R\), are projective, then the \(R\)-modules \(g^*R_R \cong R/(g_iQ \cap R)\) (in the proof) are projective, that is, \((g_iQ \cap R) \oplus (g_iQ \cap R)\) is a direct summand of \(R_R\), whence \(R\) has the same structure as in \((c)\) of Lemma 4.2.

Thus we remark the following for the proof of the next theorem.

**REMARK.** For a regular ring \(R\) with \(Q\) the maximal right quotient ring, the equivalent conditions \((a),(b),(c)\) of Lemma 4.2 are also equivalent to the following condition:

\((b')\)
1. For every faithful abelian idempotent \(f\) in \(Q\), the \(R\)-module \(fR_R\) is projective,
2. \(R\) has bounded index.

At this point, applying the previous results to regular rings, we obtain the following theorem, in which the equivalence \((a) \Leftrightarrow (d)\), under the assumption that every nonzero ideal of \(R\) contains a nonzero central idempotent, is given in S. Kobayashi [10, Theorem 1].

**Theorem 4.3.** Let \(R\) be a regular ring, and let \(Q\) be the maximal right quotient ring of \(R\). Then the following conditions are equivalent:

\((a)\)
1. \(R\) is a right GFC ring.

\((b)\)
1. \(R\) contains all the faithful idempotents in \(Q\),
2. \(R\) has bounded index.

\((c)\)
1. Every cyclic faithful nonsingular right \(R\)-module is projective,
2. \(R\) has bounded index.

\((d)\)
1. \(R\) is isomorphic to a finite direct product of an abelian regular ring and full matrix rings over self-injective abelian regular rings.

**Proof.** \((a) \Rightarrow (b)\). It follows from Theorem 2.8 and [11, Theorem 9] that \(Q\), and hence \(R\), has bounded index, while the first half of the proof of Proposition 3.5 shows that for every faithful abelian idempotent \(f\) in \(Q\), the \(R\)-module \(fR\) is projective. Thus, the implication \((a) \Rightarrow (b)\) is obtained by the remark above.

\((b) \Leftrightarrow (c) \Leftrightarrow (d)\). This follows immediately from Lemma 4.2 and [6, Theorem 3.8].

\((d) \Rightarrow (a)\). Since every one-sided ideal of abelian regular rings is two-sided, those rings are obviously right GFC. Also, according to [11, Theorem 9] and [6, Theorem 7.12], full matrix rings over self-injective abelian regular rings are FPF rings. Thus it follows that \((d)\) implies \((a)\).

Using [11, Corollary 9.2], we obtain the following corollary.
Corollary 4.4. A regular ring $R$ is right GFC if and only if $R$ is left GFC.

The following corollaries are immediate.

Corollary 4.5. Let $R$ be a regular ring which contains no nonzero abelian central idempotents. Then, $R$ is right GFC if and only if $R$ is right FPF.

Corollary 4.6. Let $R$ be an indecomposable regular ring. If $R$ is right GFC, then it is a simple artinian ring.

Remark. Since the matrix ring $M_n(R)$ ($n \geq 2$) over any ring $R$ contains no nonzero abelian central idempotents, it follows immediately from Corollary 4.5 that a regular ring $R$ is right FPF if and only if the matrix ring $M_n(R)$ is right GFC for some $n \geq 2$. This fact is also noted in [10].

References
