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BEHAVIOR OF SOLUTIONS AT THE INITIAL TIME IN NONLINEAR PARABOLIC DIFFERENTIAL EQUATION

Dedicated to Professor Hiroki Tanabe on his 60th birthday

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1. Introduction and Results

This note is concerned with the nonlinear parabolic differential equation

$$(\mathrm{E};\varphi)\qquad \frac{d}{dt}u(t)+\partial\varphi(u(t))\ni 0\,,\quad t>0\,,$$

where φ is a proper lower semi-continuous (l.s.c.) convex functional defined on a real Hilbert space H and $\partial\varphi$ denotes the subdifferential of φ. We call an H-valued function u on $(0, \infty)$ a solution of $(\mathrm{E};\varphi)$ if $u\in W^{1,2}_{\mathrm{loc}}((0,\infty)\colon H)$ and the relations $u(t)\in\mathcal{D}(\partial\varphi)$ and $-(d/dt)u(t)\in\partial\varphi(u(t))$ hold for a.e. $t>0$.

As is well known, the subdifferential $\partial\varphi$ of a proper l.s.c. convex functional φ on a real Hilbert space H is a maximal monotone operator in H. Hence $-\partial\varphi$ generates a possibly nonlinear semigroup $\{\exp(-t\partial\varphi)\colon t\geq 0\}$ on $\overline{\mathcal{D}(\partial\varphi)}$. In other words, for each $x\in\overline{\mathcal{D}(\partial\varphi)}$ the function $\exp(-(\cdot)\partial\varphi)x$ on $[0,\infty)$ is the unique solution of the initial value problem of $(\mathrm{E};\varphi)$ and $\text{s-lim}_{t\downarrow 0}u(t)=u(0)=x$.

In this note, our starting position is being given a solution $u\in W^{1,2}_{\mathrm{loc}}((0,\infty)\colon H)$ of $(\mathrm{E};\varphi)$, not being given an initial value of $\overline{\mathcal{D}(\partial\varphi)}$, and our purpose is to study the behavior of $u(t)$ as $t\downarrow 0$. Our results are the following.

Theorem 1.1. *Suppose that* $\dim H=\infty$. *Then there is a proper l.s.c. convex functional φ on H and a solution u of* $(\mathrm{E};\varphi)$ *such that $u(t)$ converges weakly, but not strongly, to a point of $\mathcal{D}(\partial\varphi)$ as $t\downarrow 0$.*

REMARK 1.1. Let $v(\cdot)$ be the solution of $(\mathrm{E};\varphi)$ in Theorem 1.1. Put $x=\text{w-lim}_{t\downarrow 0}v(t)\in\mathcal{D}(\partial\varphi)$. If we consider an initial value problem of $(\mathrm{E};\varphi)$ with a generalized initial condition

$$\text{w-lim}_{t\downarrow 0}u(t)=x\,,$$

then we have at least two solutions $v(\cdot)$ and $\exp\{-(\cdot)\partial\varphi\}x$, where $\{\exp(-t\partial\varphi)\colon t\geq 0\}$ denotes the nonlinear semigroup generated by $-\partial\varphi$.

REMARK 1.2. In the case where $\partial\varphi$ is linear, hence $\partial\varphi$ is a nonnegative self-adjoint operator in H by definition, then for each $-\tau<0$ there is a Hilbert space $X_{-\tau}$ satisfying the dense imbedding $H\subset X$ and a generator $A_{-\tau}$ such that every solution $u\in W_{\mathrm{loc}}^{1,2}((0, \infty): H)$ of (E; φ) can be extended uniquely on $(-\tau, \infty)$ as a solution of $(d/dt)u+\mathcal{A}_{-\tau}u\ni 0$, $t>-\tau$, in $X_{-\tau}$ (Arisawa [1]). However Theorem 1.1 shows that in nonlinear cases this extension may be impossible. In fact, if the solution v of Theorem 1.1 is extended on $[0, \infty)$ to X continuously in X-norm's topology for some space X satisfying the dense imbedding $H\subset X$, then the inclusion $X^*\subset H^*$ implies that $X\text{-s-lim}_{t\downarrow 0}v(t)=H\text{-w-lim}_{t\downarrow 0}v(t)\in\mathcal{D}(\partial\varphi)$. Hence there is no family $\{S(t): t\geq 0\}$ of single valued mappings in X such that $S(t)\supset\exp(-t\partial\varphi)$ for $t\geq 0$ and $X\text{-s-lim}_{t\downarrow 0}S(t)x=x$ for $x\in\mathcal{D}(\partial\varphi)$.

Theorem 1.2. *Suppose that φ satisfies a generalized evenness condition*

$$\varphi(-cx)\leq\varphi(x)\,, \quad x\in\mathcal{D}(\varphi) \tag{1.1}$$

for some positive constant c. Let u be an arbitrary solution of (E; φ) *such that the orbit* $\{u(t): t\in(0, 1]\}$ *is bounded. Then u converges strongly as* $t\downarrow 0$. *In particular, if a solution u of* (E; φ) *converges weakly as* $t\downarrow 0$, *then the strong convergence* $\text{s-lim}_{t\downarrow 0}u(t)\in H$ *holds.*

REMARK 1.3. In Theorem 1.2, the assumption of the boundedness of the orbit $\{u(t): t\in(0, 1]\}$ is essential to get the strong convergence of $u(t)$ in H as $t\downarrow 0$. In fact, there is a functional φ such that (i) the generalized evenness condition (1.1) holds; and (ii) there is a solution u of (E; φ) with the orbit $\{u(t): t\in(0, 1]\}$ unbounded (hence, $u(t)$ does not converge strongly as $t\downarrow 0$). To see this, we put, for example, $H=\boldsymbol{R}$ and $\varphi(x)=3^{-1}|x|^3$, $x\in\boldsymbol{R}$. Let $u\in W_{\mathrm{loc}}^{1,2}((0,1]; \boldsymbol{R})$ be the solution of (E; φ) satisfying $u(1)=1$. Then, one has $u(t)\uparrow +\infty$ as $t\downarrow 0$.

REMARK 1.4. The generalized evenness condition (1.1) is known to be sufficient for that all solutions of (E; φ) converge strongly as $t\to\infty$ (eg. [6]).

2. Proof of Theorem 1.1

Given an infinite dimentional Hilbert space H with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, let $H=l^2\oplus H_1$. To define the aimed functional $\varphi: H\to(-\infty, \infty]$, we first define a function $f_\lambda: \boldsymbol{R}^2\to[0, \infty]$, $\lambda>1$, by

$$f_\lambda(\xi, \eta)=\begin{cases}(\xi^2+\eta^2)^{1/2}\,\{\mathrm{Tan}^{-1}(\eta/\xi)\}^\lambda\,, & \text{if } \xi>0\,,\ \eta\geq 0\,,\\ \eta(2^{-1}\pi)^\lambda\,, & \text{if } \xi=0\,,\ \eta\geq 0\,,\\ +\infty\,, & \text{otherwise.}\end{cases} \tag{2.1}$$

Then, for each $\lambda>1$, f_λ is l.s.c. and convex on $\boldsymbol{R}^2$ (see Baillon [2; Lemma 1]). Fix a number $b>1$ and put

$$\lambda_i = \frac{\pi^2}{8}\frac{b}{b-1}b^i\,, \quad i=1, 2, \cdots . \tag{2.2}$$

For each sequence $\alpha=\{\alpha_i\}$ of positive number, we define a proper l.s.c. convex functional $\varphi_\alpha: H\to[0, +\infty]$ by

$$\mathcal{D}(\varphi_\alpha) = \{(x_i)_{i=1}^\infty+0\in l^2\oplus H_1: \sum_{i=1}^\infty \alpha_i f_{\lambda_i}(x_i, x_{i+1})<\infty\}$$

$$\varphi_\alpha(x) = \begin{cases} \sum_{i=1}^\infty \alpha_i f_{\lambda_i}(x_i, x_{i+1})\,, & x = (x_i)_{i=1}^\infty+0\in\mathcal{D}(\varphi_\alpha)\,, \\ +\infty\,, & \text{otherwise.}\end{cases}$$

Next, let $\{a_n\}$ be a sequence in l^2 defined by

$$\begin{aligned} a_1 &= (1, 0, 0, \cdots)\,, \\ a_2 &= (0, \exp(\frac{\pi^2}{8}\frac{1}{\lambda_1}), 0, 0, \cdots)\,, \\ a_n &= (0, \cdots, 0, \exp[\frac{\pi^2}{8}(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots+\frac{1}{\lambda_{n-1}})], 0, \cdots)\,. \end{aligned} \tag{2.3}$$

Then $\{a_n+0\}$ converges to $0\in H$ weakly as $n\to\infty$, but does not converge strongly, since $\lim_{n\to\infty}\|a_n+0\|=\exp(1/b)<+\infty$ by (2.2).

Let $\varepsilon\in(0, 1)$. To prove Theorem 1.1, we have only to see that there is a sequence $\alpha=\{\alpha_i\}$ and a solution $u\in W^{1,2}_{\text{loc}}(0, \infty); H)$ of $(\text{E}; \varphi_\alpha)$ such that the estimate

$$\|u(\tau_n)-a_n\|<\varepsilon^n\,, \quad n=1, 2, \cdots, \tag{2.4}$$

holds for some sequence $\{\tau_n\}$ with $\tau_n\downarrow 0$ as $n\to\infty$. We verify this in a number of lemmas below.

The first lemma is a direct result of the definition (2.1).

Lemma 2.1.

(i)
$$\frac{\partial f_\lambda}{\partial\xi}(\xi, \eta) = \theta^{\lambda-1}(-\lambda\sin\theta+\theta\cos\theta)\,,$$
$$\frac{\partial f_\lambda}{\partial\eta}(\xi, \eta) = \theta^{\lambda-1}(\lambda\cos\theta+\theta\sin\theta)\,, \quad \xi, \eta>0\,,$$

where $\theta=\text{Tan}^{-1}(\eta/\xi)$.

(ii)
$$\partial f_\lambda(\xi, 0)\ni 0\,, \quad \xi\geq 0\,.$$

We define a family $\{F_n: n=1, 2, \cdots\}$ of functionals on H by

$$\mathcal{D}(F_n) = \{(x_i)_{i=1}^\infty+0\in l^2\oplus H_1: f_{\lambda_n}(x_n, x_{n+1})<\infty\}\,,$$

$$F_n(x) = \begin{cases} f_{\lambda_n}(x_n, x_{n+1}), & x = (x_i)_{i=1}^{\infty} + 0 \in \mathcal{D}(F_n), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then each F_n is l.s.c. and convex. Let $\{\exp(-t\partial\varphi_\alpha): t \geq 0\}$ and $\{\exp(-t\alpha_n \partial F_n): t \geq 0\}$, $\alpha_n > 0$, be the semigroups generated by $-\partial\varphi_\alpha$ and $-\alpha_n \partial F_n$, respectively. We note the following lemma.

Lemma 2.2. (Baillon [2; Lemma 2]) *For* $\alpha_n > 0$,

$$\text{s-lim}_{t\to\infty} \exp(-t\alpha_n \partial F_n) a_{n+1} = a_n .$$

Now, for each n, we put

$$|\partial F_n| \equiv \sup\{\|\partial F_n x\| : x = (x_i) \in \mathcal{D}(\partial F_n), \quad x_n, x_{n+1} > 0\}.$$

Then, by Lemma 2.1,

$$|\partial F_n| = (\pi/2)^{\lambda_n - 1}\{\lambda_n^2 + (\pi/2)^2\}^{1/2} . \tag{2.5}$$

Lemma 2.3. *Let* $\alpha = \{\alpha_i\}$ *be an arbitrary sequence. Then, for each* n,

$$\|\exp(-t\partial\varphi_\alpha) a_{n+1} - \exp(-t\alpha_n \partial F_n) a_{n+1}\| \leq t \sum_{i=1}^{n-1} \alpha_i |\partial F_i|, \quad t \geq 0 .$$

Proof. Fix an arbitrary integer n. Put

$$u_\alpha(t) = \exp(-t\partial\varphi_\alpha) a_{n+1}, \quad u_n(t) = \exp(-t\alpha_n \partial F_n) a_{n+1}, \quad t \geq 0 .$$

Since $(\partial F_i) a_{n+1} \ni 0$ for $i \geq n+1$ by Lemma 2.1 (ii), the well-known equation $\|(d/dt) u_\alpha(t)\| = \min\{\|x\| : x \in \partial\varphi_\alpha(u_\alpha(t))\}$, $t > 0$, implies that

$$u_\alpha(t) = (u_{\alpha,1}(t), \cdots, u_{\alpha,n+1}(t), 0, \cdots) + 0 \in l^2 \oplus H_1 ,$$
$$u_n(t) = (0, \cdots, 0, u_{n,n}(t), u_{n,n+1}(t), 0, \cdots) + 0 \in l^2 \oplus H_1 .$$

Hence, one has the estimate

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|u_\alpha(t) - u_n(t)\|^2 \\ &= \alpha_n(-\partial F_n(u_\alpha(t)) + \partial F_n(u_n(t)), u_\alpha(t) - u_n(t)) + \sum_{i=1}^{n-1} (\alpha_i \partial F_i(u_\alpha(t)), u_\alpha(t) - u_n(t)) \\ &\leq 0 + \sum_{i=1}^{n-1} \alpha_i |\partial F_i| \, \|u_\alpha(t) - u_n(t)\| , \qquad t > 0 , \end{aligned}$$

or

$$\frac{d}{dt} \|u_\alpha(t) - u_n(t)\| \leq \sum_{i=1}^{n-1} \alpha_i |\partial F_i| , \qquad t > 0 .$$

Therefore Lemma 2.3 was proved.

Lemma 2.4. *For each $\varepsilon\in(0, 1)$, there is a sequence $\alpha=\{\alpha_i\}$ and positive numbers t_n, $n=1, 2, \cdots$ such that*

$$\|\exp(-t_n\partial\varphi_\alpha)a_{n+1}-a_n\|\leq\varepsilon^n, \qquad n=1, 2, 3, \cdots, \tag{2.6}$$

$$t_n\leq\varepsilon^n, \qquad n=1, 2, 3, \cdots. \tag{2.7}$$

Proof. We show the existence of the aimed sequences α and $\{t_n\}$ inductively.

First, by Lemma 2.2, there is $T_1>0$ such that

$$\|\exp(-T_1\partial F_1)a_2-a_1\|\leq\varepsilon .$$

Put $t_1=\varepsilon$. Let α be an arbitrary sequence satisfying $\alpha_1=t_1{}^{-1}T_1$. Then both (2.6) and (2.7) hold for $n=1$, since $\exp(-t\partial\varphi_\alpha)a_2=\exp(-t\alpha_1\partial F_1)a_2$, $t>0$.

Next, let k be an arbitrary integer. Assume that there are positive numbers $\alpha_1, \cdots, \alpha_k$ and $t_1, \cdots, t_k$ such that, for any sequence α with the first k numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$, estimates (2.6) and (2.7) hold for $n\leq k$. By Lemma 2.2, let T_{k+1} be a number such that

$$\|\exp(-T_{k+1}\partial F_{k+1})a_{k+2}-a_{k+1}\|\leq 2^{-1}\varepsilon^{k+1}. \tag{2.8}$$

Put

$$\alpha_{k+1}=\max\{\varepsilon^{-k+1}T_{k+1},\ 2T_{k+1}\varepsilon^{-k-1}\sum_{i=1}^{k}\alpha_i|\partial F_i|\}, \quad t_{k+1}=\alpha_{k+1}{}^{-1}T_{k+1}.$$

Then, estimate (2.7) holds for $n=k+1$. To verify (2.6) for $n=k+1$, let α be an arbitrary sequence whose first $k+1$ numbers are $\alpha_1, \alpha_2, \cdots, \alpha_{k+1}$, respectively. Lemma 2.3 implies

$$\|\exp(-t_{k+1}\partial\varphi_\alpha)a_{k+2}-\exp(-t_{k+1}\alpha_{k+1}\partial F_{k+1})a_{k+1}\|$$
$$\leq t_{k+1}\sum_{i=1}^{k}\alpha_i|\partial F_i|\leq 2^{-1}\varepsilon^{k+1}.$$

Noting that $T_{k+1}=t_{k+1}\alpha_{k+1}$ in (2.8), we get (2.6) for $n=k+1$.

Consequently, there are sequences α and $\{t_n\}$ satisfying (2.6) and (2.7) for every n.

Lemma 2.5. *Fix $\varepsilon\in(0, 1)$. Let $\alpha=\{\alpha_i\}$ and $\{t_n\}$ be as mentioned in Lemma 2.4. Put*

$$\tau_n=\sum_{i=n+1}^{\infty}t_i, \qquad n=1, 2, \cdots. \tag{2.9}$$

Then there is a solution $u\in W^{1,2}_{\mathrm{loc}}((0, \infty); H)$ of (E; φ_α) *such that estimate* (2.4) *holds.*

Proof. Define functions $v_n\in W^{1,2}_{\mathrm{loc}}([\tau_n, \infty); H)$, $n=1, 2, \cdots$, by

$$v_n(t)=\exp(-(t-\tau_n)\partial\varphi_\alpha)a_{n+1}, \quad t\geq\tau_n, \quad n=1, 2, \cdots. \tag{2.10}$$

Then by (2.6) and the nonexpansivity of the semigroup $\{\exp(-t\partial\varphi_\alpha)\}$, one has

$$(2.11) \qquad \|v_n(t)-v_m(t)\|\leq\sum_{i=m+1}^{n} \varepsilon^i, \qquad m<n, \quad t\geq\tau_m.$$

In particular,

$$(2.12) \qquad \|v_n(\tau_m)-a_m\|\leq\varepsilon^m, \qquad m<n.$$

Since $\tau_n\to 0$ as $n\to\infty$ by (2.7) and (2.9), there is a function u: $(0, \infty)\to H$ such that for each $\delta>0$

$$\text{s-lim}_{\substack{n\geq N(\delta)\\ n\to\infty}} v_n(t) = u(t) \qquad \text{uniformly on } [\delta, \infty).$$

By (2.12), u satisfies (2.4).

Now to complete the proof of Lemma 2.5, we have only to see that u belongs to $W^{1,2}_{\text{loc}}((0, \infty); H)$ and is a solution of (E; φ_α). To verify this, it is enough to see that for each $k=1, 2, \cdots$, the set $\{\partial\varphi(v_n(\tau_k)): n=k+1, k+2, \cdots\}$ is bounded in H, since $\partial\varphi$ is strongly-weakly continuous from H to H. Fix arbitrary k. From Lemma 2.1 (i) and (2.12), it follows that

$$\int_{\tau_{k+1}}^{\tau_k} \|(d/dt)v_n(t)\|\, dt<\int_0^{\tau_k-\tau_{k+1}} \|(d/dt)\{\exp(-t\alpha_k\partial F_k)a_{k+1})\}\|dt+\varepsilon\equiv c(k), \qquad n\geq k+1.$$

Since $\|(d/dt)v_n(\cdot)\|$ are decreasing,

$$\|(d/dt)v_n(\tau_k)\|<(\tau_k-\tau_{k+1})^{-1}c(k), \qquad n\geq k+1.$$

Hence the set $\{\partial\varphi_\alpha(v_n(\tau_k)): n\geq k+1\}$ is bounded, and Lemma 2.5 was proved.

REMARK 2.1. In the above example, the weak limit of the solution $u(t)$ as $t\downarrow 0$ happened to be a minimum point of φ_α. But we can revise the functional φ of Theorem 1.1 such that the set of minimum point of φ is empty. In fact, we can define the aimed functional φ as below. Put $H=\{re_0: r\in\boldsymbol{R}\}\oplus H_0$, where $e_0\in H\setminus\{0\}$. Let φ_α: $H_0\to[0, \infty]$ and u_0: $[0, \infty)\to H_0$ be the functional and the solution, respectively, obtained in the above proof of Theorem 1.1. Put

$$\mathcal{D}(\varphi) = \{re_0: t\in\boldsymbol{R}\}+\mathcal{D}(\varphi_\alpha)\subset\{re_0: r\in\boldsymbol{R}\}\oplus H_0,$$
$$\varphi(x) = (x, e_0)+\varphi_\alpha(\text{Proj}_{H_0} x), \quad \text{if } x\in\mathcal{D}(\varphi); \quad = +\infty, \text{ otherwise.}$$

Then φ does not attain the minimum in H. The H-valued function $u(t)=-te_0+u_0(t)$ on $t\in(0, \infty)$ is a solution of (E; φ) and converges weakly to $0\in H$ as $t\downarrow 0$, but does not converge strongly.

3. Proof of Theorem 1.2

Let $u\in W^{1,2}_{\text{loc}}((0, \infty); H)$ be a solution of (E; φ). Then, since

$$\frac{d}{dt}\varphi(u(t)) = -\|\frac{d}{dt}u(t)\|^2, \qquad \text{a.e. } t>0,$$

the value $\varphi(u(t))$ is decreasing on $(0, \infty)$. The definition of the subdifferential and condition (1.1) yield

$$(-cu(t)-u(\tau), -u'(\tau)) \leq \varphi(-cu(t))-\varphi(u(\tau)) \leq \varphi(u(t))-\varphi(u(\tau)) \leq 0, \qquad \text{a.e. } \tau\in(0, t), \quad t>0,$$

or

$$(-u(t), -u'(\tau)) \leq c^{-1}(u(\tau), -u'(\tau)), \qquad \text{a.e. } \tau\in(0, t), \quad t>0.$$

Hence

$$(3.1) \quad \|u(t)-u(s)\|^2 = \int_s^t\left\{-\frac{d}{d\tau}\|u(t)-u(\tau)\|^2\right\}d\tau = \int_s^t 2(u(t)-u(\tau), u'(\tau))\,d\tau \leq \int_s^t 2(1+c^{-1})(u(\tau), -u'(\tau))d\tau = (1+c^{-1})\{\|u(s)\|^2-\|u(t)\|^2\},$$

By (3.1) we first see that $\|u(\cdot)\|^2$ is decreasing on $(0, \infty)$. Hence, in the case where $\{\|u(t)\|: t\in(0, 1]\}$ is bounded, then $\|u(t)\|^2$ converges as $t\downarrow 0$. Therefore , using (3.1) again yields that $u(t)$ converges strongly as $t\downarrow 0$.

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