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0. Introduction

Let $G$ be a finite group. A finite dimensional $G$-$CW$-complex $X$ is called a homotopy representation of $G$ if the $H$-fixed point set $X^H$ is homotopy equivalent to a $(\dim X^H)$-dimensional sphere or the empty set for each subgroup $H$ of $G$. Moreover if $X$ is $G$-homotopy equivalent to a finite $G$-$CW$-complex, then $X$ is called a finite homotopy representation of $G$ and if $X$ is $G$-homotopy equivalent to a unit sphere of a real representation of $G$, then $X$ is called a linear homotopy representation of $G$. T. tom Dieck and T. Petrie defined homotopy representation groups in order to study homotopy representations. Let $V^+(G, h^\infty)$ be the set of $G$-homotopy types of homotopy representations. We define the addition on $V^+(G, h^\infty)$ by the join and so $V^+(G, h^\infty)$ becomes a semigroup. The Grothendieck group of $V^+(G, h^\infty)$ is denoted by $V(G, h^\infty)$ and called the homotopy representation group. A similar group $V(G, h)$ [resp. $V(G, l)$] can be defined for finite [resp. linear] homotopy representations.

Let $\phi(G)$ denote the set of conjugacy classes of subgroups of $G$ and $C(G)$ the ring of functions from $\phi(G)$ to integers. For a homotopy representation $X$, the dimension function $\dim X$ in $C(G)$ is defined by $(\dim X)(H)=\dim X^H + 1$. (If $X^H$ is empty, then we set $\dim X^H = -1$.) Then

$$\dim X \star Y = \dim X + \dim Y$$

for any two homotopy representations. ("\star" means the join.) Hence one can define the homomorphism

$$\dim: V(G, \lambda) \to C(G) \quad (\lambda = h^\infty, h \text{ or } l)$$

by the natural way. The kernel of $\dim$ is denoted by $\vartheta(G, \lambda)$. tom Dieck and Petrie proved that $\vartheta(G, \lambda)$ is the torsion group of $V(G, \lambda)$ and

$$\vartheta(G, h^\infty) \approx \text{Pic} \Omega(G),$$

where $\text{Pic} \Omega(G)$ is the Picard group of the Burnside ring $\Omega(G)$.

There are the natural homomorphisms.
(0.2) \quad j_G: v(G, l) \rightarrow v(G, h) \\
\quad k_G: v(G, h) \rightarrow v(G, h^*).

The homomorphisms $j_G$ and $k_G$ are injective in general and hence we often regard $v(G, l)$ and $v(G, h)$ as the subgroups of $v(G, h^*)$ via these injective homomorphisms. We prove the following result in Section 2.

**Theorem A.** The homomorphism $k_G$ is an isomorphism if and only if the Swan subgroup $T(G)$ vanishes.

The definition and properties of the Swan subgroup are mentioned in Section 1. The finite groups with $T(G)=0$ are studied by Miyata and Endo [7]. The Swan subgroups play an important role in the computation of $v(G, h)$. In fact, the computation of $v(G, h)$ for an abelian group $G$ is completely reduced to the computation of the Swan subgroups. By computing the Swan subgroups of some groups, we prove the following result in Section 3.

**Theorem B.** Suppose that $G$ is an abelian group. Then $j_G$ is an isomorphism if and only if $G$ is isomorphic to one of the following groups.

(i) $C$ a cyclic group  
(ii) $G_2$ an abelian 2-group  
(iii) $G_3$ an abelian 3-group  
(iv) $\mathbb{Z}/2 \times G_3$  
(v) $G_2 \times \mathbb{Z}/3$  
(vi) $(\mathbb{Z}/2)^n \times (\mathbb{Z}/3)^m$

In Section 4 we determine the finite groups with $v(G, h^*)=0$ by using the results in Section 2.

**Theorem C.** The group $v(G, h^*)$ vanishes if and only if $G$ is isomorphic to one of the groups:

$\mathbb{Z}/n$ ($n=1, 2, 3, 4$ or $6$), $D(2n)$ ($n=2, 3, 4$ or $6$), $A_4$, $S_4$.

Here $D(2n)$ denotes the dihedral group of order $2n$ and $S_n[A_n]$ denotes the symmetric [alternating] group on $n$ symbols.

1. **The Swan subgroup**

In this Section, we shall summarize the well-known results on the Swan subgroup. Let $\Sigma_G$ be the sum of elements of $G$ in the integral group ring $\mathbb{Z}G$ and $[r, \Sigma_G]$ be the left ideal generated by $r$ and $\Sigma_G$, where $r \in \mathbb{Z}$ is prime to the order $|G|$ of $G$. The ideal $[r, \Sigma_G]$ is projective as a $\mathbb{Z}G$-module. Hence $[r, \Sigma_G]$ decides the element of the reduced projective class group $\mathcal{K}_G(\mathbb{Z}G)$. From [9], a homomorphism
is defined by \( S_G(r) = ([r, \Sigma_G]) \), where \( \mathbb{Z}/|G|^* \) is the unit group of \( \mathbb{Z}/|G| \). We put \( u(G) = (\mathbb{Z}/|G|^*)/\pm 1 \). Since \( S_G(\pm 1) = 0 \), \( S_G \) induces

\[
S_G: u(G) \rightarrow \tilde{K}_0(\mathbb{Z}G),
\]

which is called the Swan homomorphism. The image of \( S_G \) is called the Swan subgroup of \( G \) and denoted by \( T(G) \).

The following results are well-known.

**Theorem 1.1** ([9]). If \( G \) is a cyclic group, then \( T(G) = 0 \).

**Theorem 1.2** ([11]).

(i) \( T(G) \) is a quotient group of \( u(G) \).

(ii) If \( f: G \rightarrow G' \) is a surjective homomorphism, then the natural map \( \tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Z}G') \) sends \( ([r, \Sigma_G]) \) to \( ([r, \Sigma_{G'}]) \), hence \( T(G) \) onto \( T(G') \).

(iii) The restriction map \( \tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Z}H) \) sends \( ([r, \Sigma_G]) \) to \( ([r, \Sigma_H]) \), hence \( T(G) \) onto \( T(H) \).

(iv) The exponent of \( T(G) \) divides the Artin exponent \( A(G) \). (For the Artin exponent, see [5].)

(v) \( T(D(2^n)) = 0 (n \geq 2), T(Q(2^n)) = \mathbb{Z}/2 (n \geq 3) \) and \( T(SD(2^n)) = \mathbb{Z}/2 (n \geq 4) \), where \( D(2^n) \) [resp. \( Q(2^n), SD(2^n) \)] is the dihedral [resp. quaternion, semi-dihedral] group of order \( 2^n \). These groups are called the exceptional groups.

**Theorem 1.3** ([10]).

(i) If \( G \) is a non-cyclic \( p \)-group (\( p: \) an odd prime), then \( T(G) \) is the cyclic group of order \( |G|/p \).

(ii) If \( G \) is a non-cyclic and non-exceptional 2-group, then \( T(G) \) is the cyclic group of order \( |G|/4 \).

Let \( G_p \) denote a \( p \)-Sylow subgroup of \( G \).

**Corollary 1.4.** If \( T(G) \) vanishes, then \( G_p \) is cyclic when \( p \) is odd and \( G_2 \) is cyclic or dihedral.

2. **The inclusion \( k_G \)**

tom Dieck and Petrie defined the finiteness obstruction map

\[
\rho: v(G, h^o) \rightarrow \bigoplus_{\mathbb{Z}^G} \tilde{K}_0(\mathbb{Z}WH),
\]

where \( WH = NH/H \) and \( NH \) is the normalizer of \( H \) in \( G \). They proved that the following sequence is exact.

\[
0 \rightarrow v(G, h^o) \xrightarrow{k_G} v(G, h^o) \xrightarrow{\rho} \bigoplus_{\mathbb{Z}^G} \tilde{K}_0(\mathbb{Z}WH). \]
We recall the map $\rho$. (For details, see [3].) For any element $x$ of $v(G, h^\omega)$, there exist homotopy representations $X$, $Y$ and a $G$-map $f: X \to Y$ such that $x=X-Y$ in $v(G, h^\omega)$ and $\deg f^H$ is prime to $|G|$ for each subgroup $H$ of $G$. A function $d \in C(G)$ is defined by $d(H)=\deg f^H$ for any $(H)$ and called the invertible degree function of $x$. Conversely, any $d \in C(G)$ with $(d(H), |G|)=1$ for any $(H)$ is the invertible degree function of some $x$ in $v(G, h^\omega)$. The finiteness obstruction map $\rho$ is described as follows. The $(H)$-component $\rho_H(x)$ of $\rho(x)$ is equal to

\begin{equation}
S_{WH}(d(H)) = \sum_{1 \leq K \leq W} a_{K,L} \ind_{L}^{W} \res_{K}^{L} S_{NK}(d(K)),
\end{equation}

where $\hat{K}$ is the subgroup of $G$ such that $\hat{K}/H=K$ and $a_{K,L}$ are certain integers and $d$ is the invertible degree function of $x$.

Proof of Theorem A. For any $r$ which is prime to $|G|$, we take the function $d(=C(G)$ such that $d(1)=r$ and $d(H)=1$ for $(H)\neq(1)$. By (2.3), we have $\rho_0(x)=S_0(r)$ and $\rho_H(x)=0$ for $(H)\neq(1)$, where $x$ denotes the element of $v(G, h^\omega)$ represented by $d$. Hence $T(G)=0$ if $\rho=0$. Conversely if $T(G)$ vanishes, then $S_K=0$ for any subquotient group $K$ of $G$ by Theorem 1.2. Hence $\rho=0$ and so $k_G$ is an isomorphism.

Corollary 2.4. Let $G$ be $D(2^n)$, $Q(2^n)$ or $SD(2^n)$. Then $v(G, h)=v(G, l)$.

Proof. In the case of $D(2^n)$, we have proved it in [8]. In the cases of $Q(2^n)$ and $SD(2^n)$, $v(G, l)$ is the subgroup of index 2 of $v(G, h^\omega)$ ([8]). On the other hand $v(G, h)$ is a proper subgroup of $v(G, h^\omega)$ since $T(G)=\mathbb{Z}/2$. Hence $v(G, h)=v(G, l)$.

Remark 2.5. If $G$ is nilpotent, then $\dim V(G, l)=\dim V(G, h^\omega)$ ([3]) and hence $V(G, h)=V(G, l)$ for the above groups.

Corollary 2.6. If $v(G, h^\omega)$ vanishes, then $T(G)$ also vanishes.

3. The inclusion $j_0$

Let $G$ be an abelian group. Then $v(G, l)$ and $v(G, h^\omega)$ were computed by Kawakubo [5] and tom Dieck-Petrie [3] respectively and it is known that the following diagram is commutative.

\[
\begin{array}{ccc}
v(G, l) & \longrightarrow & v(G, h^\omega) \\
\downarrow \cong & \ & \downarrow \cong \\
\prod_{H \in \mathbb{H}} u(G|H) & \subset & \prod_{H \in \mathbb{H}} u(G|H) \\
\end{array}
\]

\[\theta/H: \text{cyclic}\]
Here \( u(G/H) = (\mathbb{Z}/|G/H|^*)/\pm 1 \).

Furthermore, tom Dieck and Petrie showed the following commutative diagram.

\[
\begin{array}{c}
\psi(G, h^n) \\
\cong \alpha \\
\prod_h u(G/H) \xrightarrow{\rho} \bigoplus H S_{G/H} \oplus K_5(\mathbb{Z}[G/H]) \\
\end{array}
\]

Hence we obtain

**Proposition 3.3.** Let \( G \) be an abelian group. Then

(i) \( \psi(G, h) \cong \psi(G, I) \times N(G) \),

where \( N(G) = \prod H \text{Ker } S_{G/H} \). (If \( G \) is cyclic, then we put \( N(G) = 1 \)).

(ii) \( \psi(G, h^n)/\psi(G, h) \cong \bigoplus H T(G/H) \).

Proof. These are obtained from the exactness of the sequence (2.2) and the fact that \( T(G/H) = 0 \) if \( G/H \) is cyclic.

**Corollary 3.4.** Let \( G \) be an abelian group. Then

\[ V(G, h) \cong V(G, I) \times N(G) \, . \]

**Remark 3.5.** For any finite group, one can show that

\[ |\psi(G, h^n)/\psi(G, h)| \geq |\bigoplus H T(WH)| \, . \]

From now we shall prove Theorem B. Theorem B is proved by the following lemmas.

**Lemma 3.6.** If \( N(G) = 1 \) for a non-cyclic abelian group \( G \), then \( |G| = 2^n \cdot 3^m \) \((n, m \geq 0)\).

Proof. If a \( p \)-Sylow subgroup \( G_p \) \((p \geq 5)\) is non-cyclic, then there exists a subgroup \( L \) such that \( G/L \) is isomorphic to \( \mathbb{Z}/p \times \mathbb{Z}/p \). Since \( \text{Ker } S_{G/L} \) is non-trivial by Theorem 1.3, \( G_p \) must be cyclic. We may put \( G = G_2 \times G_3 \times C \), where \( C \) is a cyclic group with \((|C|, 6) = 1 \). We prove that \( C \) is trivial. Assume that \( C \) is non-trivial. Since \( G \) is non-cyclic, there exists a subgroup \( K \) such that \( G/K \) is isomorphic to \( \mathbb{Z}/q \times \mathbb{Z}/q \times \mathbb{Z}/p \) \((q = 2 \text{ or } 3, p \geq 5)\). The Artin exponent \( A(G/K) \) is equal to \( q \) and so \( T(G/K) \) is a \( q \)-group by Theorem 1.2. On the other hand, it is easily checked that the exponent of \( u(G/K) \) is not equal to \( q \). Hence \( \text{Ker } S_{G/K} \neq 1 \) and so \( N(G) \neq 1 \). This is a contradiction. Therefore \( C \) is trivial.
Lemma 3.7. Put \( G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3^n \) (\( m \geq 1 \)). Then \( \ker S_G \neq 1 \) if \( m \geq 2 \) and \( \ker S_G = 1 \) if \( m = 1 \).

Proof. Since the Artin exponent \( A(G) = 2 \) and \( |u(G)| = 2 \cdot 3^{m-1} \), the Swan subgroup \( T(G) \) is isomorphic to 1 or \( \mathbb{Z}/2 \). Moreover \( T(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3) = \mathbb{Z}/2 \) ([4], [7]). Hence \( T(G) = \mathbb{Z}/2 \). Thus the desired result holds.

Lemma 3.8. Put \( G = \mathbb{Z}/2^n \times \mathbb{Z}/3 \times \mathbb{Z}/3 \) (\( n \geq 1 \)). Then \( \ker S_G \neq 1 \) if \( n \geq 2 \) and \( \ker S_G = 1 \) if \( n = 1 \).

Proof. The proof is similar to the proof of Lemma 3.7. The details are omitted.

Lemma 3.9. Let \( G_2 \) be a non-cyclic abelian group of order \( 2^n \). We put \( G = G_2 \times \mathbb{Z}/3 \). Then \( \ker S_{G_2} = 1 \) and \( \ker S_G = 1 \).

Proof. By Theorem 1.3, it is clear that \( \ker S_{G_2} = 1 \). We consider the restriction map

\[ R = (\text{res}_{G_2}, \text{res}_K): T(G) \to T(G_2) \oplus T(K), \]

where \( K \) is a subgroup which is isomorphic to \( \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \). We show that \( R \) is surjective. Take any element \((a, b)\) in \( T(G_2) \oplus T(K) \). Then there exists \( r \in \mathbb{Z} \) with \( (r, |G|) = 1 \) such that \( \text{res}_{G_2} S_G(r) = S_{G_2}(r) = a \). Put \( c = \text{res}_K S_G(r) = S_K(r) \).

If \( c \neq b \), then take \((2^n - 1)r \) [resp. \((2^n + 1)r \)] instead of \( r \) when \( n \) is odd [resp. even]. Then

\[ \text{res}_{G_2} S_G(2^n \pm 1)r = S_{G_2}(r) = a \]

and

\[ \text{res}_K S_G(2^n \pm 1)r = \begin{cases} S_K(5) + S_K(r) & \text{if } n \text{ is even} \\ S_K(7) + S_K(r) & \text{if } n \text{ is odd} \end{cases} = b. \]

The last equality follows from the facts that \( T(K) = \mathbb{Z}/2 \), \( S_K(5) \neq 0 \) and \( S_K(7) \neq 0 \). Hence \( R \) is surjective.

The orders of \( T(G_2) \) and \( T(K) \) are \( 2^{n-2} \) and 2 respectively. Since \( |u(G)| = 2^{n-2} \), \( |u(G)| = |T(G)| \). Hence \( \ker S_G = 1 \).

Lemma 3.10. Let \( G_3 \) be a non-cyclic abelian group of order \( 3^n \). We put \( G = \mathbb{Z}/2 \times G_3 \). Then \( \ker S_{G_3} = 1 \) and \( \ker S_G = 1 \).

Proof. This follows from the comparison between the orders of \( u(G) \) and \( T(G) \).

Lemma 3.11. We put \( G = G_2 \times G_3 \) for the above \( G_2 \) and \( G_3 \). Then \( \ker S_G = 1 \).
Proof. The restriction maps $T(G) \to T(G_2 \times \mathbb{Z}/3)$ and $T(G) \to T(G_3)$ are surjective. Since $|T(G_2 \times \mathbb{Z}/3)| = 2^{n-1}$ by Lemma 3.9 and $|T(G_3)| = 3^{m-1}$ by Theorem 1.3, we have $|T(G)| \geq 2^{n-1} \cdot 3^{m-1}$. Hence $\text{Ker} S_G = 1$.

Proof of Theorem B. Assume that $j_G$ is an isomorphism (i.e. $N(G)=1$). By Lemma 3.6, $G = G_2 \times G_3$. If both $G_2$ and $G_3$ are cyclic, then $G$ is cyclic. If $G_2$ is cyclic and $G_3$ is non-cyclic, then $G_3 = 1$ or $\mathbb{Z}/2$ by Lemma 3.8. If $G_2$ is non-cyclic and $G_3$ cyclic, then $G_3 = 1$ or $\mathbb{Z}/3$ by Lemma 3.7. If both $G_2$ and $G_3$ are non-cyclic, then $G = (\mathbb{Z}/2)^n \times (\mathbb{Z}/3)^m$ by Lemmas 3.7 and 3.8. Conversely, if $G$ is one of the groups (i)–(vi), then $N(G) = 1$ by Lemmas 3.7–3.11.

4. The finite groups $G$ with $v(G, h^\infty) = 0$

In this Section we determine the finite groups with $v(G, h^\infty) = 0$. We first show the following result.

**Proposition 4.1.** Let $C$ be a cyclic subgroup of $G$. Then the restriction map

$$\text{res}: v(G, h^\infty) \to v(C, h^\infty)$$

is surjective.

Proof. Let $d \in C(C)$ be an invertible degree function representing $x \in v(C, h^\infty)$. We can choose an integer $a_K$ such that $d(K) + a_K |C|$ is prime to $|G|$ for any subgroup $K$ of $C$. Then $d'(K) = d(K) + a_K |C|$ is also an invertible degree function representing $x$. (See [3].) We define $e \in C(G)$ by

$$e(H) = \begin{cases} d'(gHg^{-1}) & \text{if } (H) \in \phi(G) \text{ with } gHg^{-1} \subseteq C \\ 1 & \text{otherwise.} \end{cases}$$

This is well-defined since $C$ is cyclic. Let $y \in v(G, h^\infty)$ be the element represented by $e$. Then $\text{res} y = x$ since $d$ is an invertible degree function of $\text{res} y$.

In the abelian case, we have

**Lemma 4.2.** Let $G$ be an abelian group. Then $v(G, h^\infty) = 0$. If and only if $G$ is isomorphic to $1, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6$ or $D(4)$ (= $\mathbb{Z}/2 \times \mathbb{Z}/2$).

Proof. Using the isomorphism $v(G, h^\infty) \approx \prod_H u(G/H)$, one can easily see it.

By Lemmas 4.1 and 4.2, we have

**Lemma 4.3.** If $v(G, h^\infty)$ vanishes, then any cyclic subgroup $C$ of $G$ is isomorphic to $1, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4$ or $\mathbb{Z}/6$.

On the other hand, if $v(G, h^\infty)$ vanishes, then the Swan subgroup $T(G)$
also vanishes (Corollary 2.6) and hence we have the following conclusion by Lemma 4.3 and Corollary 1.4.

**Lemma 4.4.** If \( v(G, h^n) \) vanishes, then a 2-Sylow subgroup \( G_2 \) is isomorphic to 1, \( \mathbb{Z}/2, \mathbb{Z}/4, D(4) \) or \( D(8) \) and a 3-Sylow subgroup \( G_3 \) is isomorphic to 1, or \( \mathbb{Z}/3 \) and a \( p \)-Sylow subgroup \( G_p \) \((p \geq 5)\) is trivial.

We consider a non-abelian group \( G \). Suppose that \( v(G, h^n) \) vanishes. Then \(|G|=6, 8, 12\) or 24 by Lemma 4.4. If \(|G|=6\), then \( G \) is isomorphic to \( D(6) \). If \(|G|=8\), then \( G \) is isomorphic to \( D(8) \) by Lemma 4.4. If \(|G|=12\), then \( G \) is isomorphic to \( A_4, D(12) \) or \( Q(12) \). In the case \(|G|=24\), \( G \) is isomorphic to \( D(8) \) by Lemma 4.4. From Burnside's book ([1] Chap. 9, 126.), \( G \) is isomorphic to one of the groups: \( D(24), D(8) \times \mathbb{Z}/3, S_4 \) and \( K=(a \ y \ b, c\ a^{-1} \ b^{-1} \ ab=ca=c^{-1} \ b^2=c^2=1, bc=cb, b^{-1}ab=a^{-1}, a^{-1}ca=c^{-1}) \). However \( D(24) \) and \( D(8) \times \mathbb{Z}/3 \) are omitted by Lemma 4.3. Since \( K \) has a subgroup which is isomorphic to \( \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \), the Swan subgroup \( T(K) \) is non-trivial and \( K \) is also omitted by Corollary 2.6. Therefore, in the non-abelian case, if \( v(G, h^n) \) vanishes, then \( G \) is isomorphic to one of the groups: \( D(6), D(8), D(12), Q(12), A_4 \) and \( S_4 \).

We proved the following formula in [8]. (See also [2].)

**Proposition 4.4.** For any finite group,

\[
\left| v(G, h^n) \right| = 2^{-n} \left| \Omega(G)^* \right| \prod_{O(H)} \varphi(|WH|),
\]

where \( \varphi \) is the Euler function and \( n \) is the number of conjugacy classes of subgroups of \( G \).

By computing \( \left| v(G, h^n) \right| \) as in [8], one can see that \( \left| v(G, h^n) \right| = 1 \) for \( G = D(6), D(8), D(12), A_4 \) or \( S_4 \) and \( \left| v(G, h^n) \right| = 2 \) for \( G = Q(12) \). Therefore we have

**Theorem 4.6.** \( v(G, h^n) \) vanishes if and only if \( G \) is one of the following groups: \( \mathbb{Z}/n \) \((n=1, 2, 3, 4, 6), D(2n) \( (n=2, 3, 4, 6), A_4 \) and \( S_4 \).

As a remark, there exist infinitely many groups with \( v(G, h^n) = 0 \) \((\lambda = h \) or \( l) \). Indeed we have

**Proposition 4.7.** Let \( G \) be an abelian group. Then \( v(G, l) \) vanishes if and only if \( G = \mathbb{Z}/2^n \times (\mathbb{Z}/4)^m \) or \( (\mathbb{Z}/2)^n \times (\mathbb{Z}/3)^m \) \((n, m \geq 0)\).

Proof. One can see it by using the isomorphism \( v(G, l) \cong \prod_{H} \varphi(G/H) \).

By Proposition 4.7 and Theorem B, we have

**Corollary 4.8.** Let \( G \) be an abelian group. Then \( v(G, h) \) vanishes if and
only if \( v(G, l) \) vanishes.

References


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