Title	Global existence of solutions for nonlinear fourth-order Schrödinger equations
Author(s)	青木, 和貴
Citation	大阪大学, 2020, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/76358
rights	© Springer Nature Switzerland AG
Note	

Global existence of solutions for nonlinear fourth-order Schrödinger equations

(4 階非線形シュレディンガー方程式の時間大域解の存在)

Kazuki Aoki

Department of Mathematics, Graduate School of Science,
Osaka University

Doctoral thesis

2020

Contents

1 Introduction **3**

2 Global existence of small solutions for the fourth-order nonlinear Schrödinger equation **6**

2.1 Introduction 6
2.2 Preliminary estimates for $n = 1$ 9
2.3 Proof of Theorem 2.1.1 for $n = 1$ 13
2.4 Preliminary estimates for $n = 2$ 14
2.5 Proof of Theorem 2.1.1 for $n = 2$ 20

3 Global existence of small solutions for a quadratic nonlinear fourth-order Schrödinger equation in six space dimensions **23**

3.1 Introduction 23
3.2 Preliminary estimate 29
3.3 Normal form method 54
3.4 Proof of Theorem 3.1.2 55
3.5 Proof of Corollary 3.1.3 76

Acknowledgments **78**

List of the author's papers cited in this thesis **81**

Chapter 1

Introduction

In this thesis, we consider the Cauchy problem for nonlinear fourth-order Schrödinger equations:

$$\begin{cases} i\partial_t u - \frac{1}{4}\Delta^2 u = f(u, \overline{u}), & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.0.1}$$

where u_0 is a $\mathbf{C}$-valued known function which belongs to suitable weighted Sobolev space and small. $u = u(t, x)$ is a $\mathbf{C}$-valued unknown function and $\overline{u}$ is the complex conjugate of u. $f(u, \overline{u})$ is a nonlinearity. The class of the fourth-order nonlinear Schrödinger equations (1.0.1) describes deep water wave dynamics (see [6]). We consider a global existence of small solutions under the growth condition of

$$\left|\partial_u^a \partial_{\overline{u}}^b f(u, \overline{u})\right| \leq C|u|^{p-a-b}, \tag{1.0.2}$$

with $0 \leq a + b \leq p$. Since the pointwise time decay estimates of solutions to the free fourth-order Schrödinger equation is $O\left(t^{-\frac{n}{4}}\right)$ and the linear problem has the $\mathbf{L}^2$ conservation law, $\mathbf{L}^2$ norm of the nonlinearity $f(u, \overline{u})$ decays like $O\left(t^{-\frac{n}{4}(p-1)}\right)$. We find $\int_1^\infty t^{-\frac{n}{4}(p-1)} dt < \infty$ if $p > 1 + \frac{4}{n}$, therefore we expect a global existence of small solutions to (1.0.1) holds if $p > 1 + \frac{4}{n}$.

In [21], the $\mathbf{L}^{p+1}$-$\mathbf{L}^{1+\frac{1}{p}}$ time decay estimate of evolution operator $e^{\frac{1}{2}it\Delta}$ was applied to the nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = f(u, \overline{u}), & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.0.3}$$

to obtain a global existence of small solutions, when the initial data are small in $\mathbf{H}^1 \cap \mathbf{L}^{1+\frac{1}{p}}$ and the order of nonlinearity p satisfies $p_{2,s}(n) < p < p_{2,*}(n)$,

where

$$p_{2,s}(n) = \frac{1}{2}\left(1 + \frac{2}{n} + \sqrt{\left(1 + \frac{2}{n}\right)^2 + 4\left(\frac{2}{n}\right)}\right),$$

$$p_{2,*}(n) = \begin{cases} \infty & (n = 1, 2) \\ \frac{n+2}{n-2} & (n \geq 3). \end{cases}$$

More precisely,

Theorem 1.0.1. *We assume that* $p_{2,s}(n) < p < p_{2,*}(n)$*,* $u_0 \in \mathbf{H}^1 \cap \mathbf{L}^{1+\frac{1}{p}}$ *and* $\|u_0\|_{\mathbf{H}^1 \cap \mathbf{L}^{1+\frac{1}{p}}} < \varepsilon$*, then there exists an* $\varepsilon > 0$ *such that* (1.0.3) *has a unique global solution* $u \in \mathbf{C}\left([0, \infty) : \mathbf{L}^2 \cap \mathbf{L}^{p+1}\right)$*. Moreover, the following estimate*

$$\|u(t)\|_{\mathbf{L}^q} \leq C \langle t \rangle^{-\frac{n}{2}\left(1 - \frac{2}{q}\right)} \varepsilon,$$

is true for any $2 \leq q \leq p + 1$*.*

We remark that a global well-posedness and the $\mathbf{L}^\infty$ time decay estimate of solutions are unknown.

From the time decay of free solutions, in the case of the fourth-order nonlinear Schrödinger equation, $p_{2,s}(n)$ and $p_{2,*}(n)$ are replaced by $p_{2,s}\left(\frac{n}{2}\right)$ and $p_{2,*}\left(\frac{n}{2}\right)$ respectively. We write $p_{2,s}\left(\frac{n}{2}\right)$, $p_{2,*}\left(\frac{n}{2}\right)$ by $p_{4,s}(n)$, $p_{4,*}(n)$, respectively. Then,

$$p_{4,s}(n) = \frac{1}{2}\left(1 + \frac{4}{n} + \sqrt{\left(1 + \frac{4}{n}\right)^2 + 4\left(\frac{4}{n}\right)}\right),$$

$$p_{4,*}(n) = \begin{cases} \infty & (n = 1, 2, 3, 4) \\ \frac{n+4}{n-4} & (n \geq 5) \end{cases}.$$

Applying the method of Strauss, global in time of small solutions for (1.0.1) will be obtained if $p_{4,s}(n) < p < p_{4,*}(n)$. However there are no global result for the case $1 + \frac{4}{n} < p \leq p_{4,s}(n)$ as far as we know. Therefore, the purpose of this thesis is to consider the case when

$$1 + \frac{4}{n} < p \leq p_{4,s}(n).$$

In Chapter 2, we deal with the Cauchy problem (1.0.1) with the space dimension $n = 1, 2$. A typical example of the nonlinearity $f(u, \overline{u})$ is given by $|u|^{p-1} u$ and $|u|^p$. By Hayashi, Mendez-Navarro and Naumkin [12], a global existence of small solutions and $\mathbf{L}^q$ $(3 < q \leq \infty)$ time decay estimates of solutions to (1.0.1) in the case of $f(u, \overline{u}) = |u|^{p-1} u$ and one dimensional case $n = 1$ are obtained. We discuss the Cauchy problem (1.0.1) with more general nonlinearities and dimensions.

In Chapter 3, we study the Cauchy problem (1.0.1) with $n = 6$, $f(u, \overline{u}) = \lambda \overline{u}^2$ and $\lambda \in \mathbf{C}$. We remark that $p_{4,s}(6) = 2$. We show if the initial data u_0 is sufficiently small, regular and decay rapidly at infinity, then (1.0.1) has a unique global solution.

This thesis is a rearrangement of the author's papers [1] and [2]. Chapter 2 is a joint work with Professors Nakao Hayahi and Pavel I. Naumkin.

Before closing this Chapter, we introduce notation and function spaces. Let

$$\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \phi(x)\, dx$$

denote the Fourier transform of ϕ and

$$\mathcal{F}^{-1}\phi = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \phi(\xi)\, d\xi$$

denote the inverse Fourier transform of ϕ. The free evolution group is given by

$$\mathcal{U}(t) = \mathcal{F}^{-1} e^{-\frac{it}{4}|\xi|^4} \mathcal{F}.$$

The Lebesgue space is defined by

$$\mathbf{L}^p = \{\phi : \mathbf{R}^n \to \mathbf{C}\ ;\ \phi \text{ is measurable function and } \|\phi\|_{\mathbf{L}^p} < \infty\},$$

where the norm

$$\|\phi\|_{\mathbf{L}^p} = \begin{cases} \left(\int_{\mathbf{R}^n} |\phi(x)|^p\, dx\right)^{\frac{1}{p}}, & (1 \le p < \infty) \\ \text{ess.sup}_{x\in\mathbf{R}^n} |\phi(x)|, & (p = \infty) \end{cases}.$$

The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \left\{\phi \in \mathbf{L}^p; \|\phi\|_{\mathbf{H}_p^{m,s}} = \|\langle x\rangle^s \langle i\nabla\rangle^m \phi\|_{\mathbf{L}^p} < \infty\right\},$$

where $m, s \in \mathbf{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1 + |x|^2}$ and $\langle i\nabla\rangle = \mathcal{F}^{-1}\langle\xi\rangle\mathcal{F}$. Also define the homogeneous Sobolev space

$$\dot{\mathbf{H}}_p^{\alpha} = \left\{\phi \in \mathbf{S}'; \|\phi\|_{\dot{\mathbf{H}}_p^{\alpha}} = \||\partial_x|^{\alpha}\phi\|_{\mathbf{L}^p} < \infty\right\},$$

where $|\partial_x|^{\alpha} = \mathcal{F}^{-1}|\xi|^{\alpha}\mathcal{F}$. We denote $\partial_x^{\alpha} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, where $\alpha \in (\mathbf{N}\cup\{0\})^n$. We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$, $\dot{\mathbf{H}}^m = \dot{\mathbf{H}}_2^m$ shortly, if it dose not cause any confusion. Let $\mathbf{C}(\mathbf{I}, \mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C.

Chapter 2

Global existence of small solutions for the fourth-order nonlinear Schrödinger equation

2.1 Introduction

This Chapter is based on the joint work [2] with Nakao Hayashi and Pavel I. Naumkin. We consider the Cauchy problem for the fourth-order nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \frac{1}{4}\Delta^2 u = f(u, \overline{u}), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0, x) = u_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{2.1.1}$$

where the space dimension $n = 1$, 2 and the nonlinearity $f(u, \overline{u})$ satisfies the estimate

$$\left| \partial_u^a \partial_{\overline{u}}^b f(u, \overline{u}) \right| \le C |u|^{p-a-b}, \tag{2.1.2}$$

with $p > 1 + \frac{4}{n}$ and $0 \le a + b \le p$. A typical example of the nonlinearity $f(u, \overline{u})$ is given by $|u|^{p-1} u$ and $|u|^p$. Local and global existence of solutions for more general equations including (2.1.1) have been studied by many authors (see, e.g., [8], [16], [17], [19], [23]). In the case of the gauge invariant nonlinearity $|u|^{p-1} u$ with $p > 1 + \frac{4}{n}$, the small data global existence (SDGE) to (2.1.1) was shown in [10] in the one dimensional case $n = 1$. We are interested in (SDGE) to (2.1.1) with more general nonlinearities and dimensions. As far as we know, the critical power of the nonlinearity for (SDGE) is

$p=1+\frac{4}{n}$. Previously in [10] and [11], we have applied the operator

$$\begin{aligned}
\mathcal{J} &= \mathcal{U}(t)\, x\mathcal{U}(-t) \\
&= \left(e^{-\frac{it}{4}\Delta^2} x_1 e^{\frac{it}{4}\Delta^2}, \cdots, e^{-\frac{it}{4}\Delta^2} x_n e^{\frac{it}{4}\Delta^2}\right) \\
&= \left(x_1 - it\Delta\partial_{x_1}, \cdots, x_n - it\Delta\partial_{x_n}\right) \\
&= x - it\Delta\nabla
\end{aligned}$$

to show (SDGE) of (2.1.1) with the gauge invariant nonlinearity for $n=1$. A similar operator

$$e^{\frac{it}{2}\Delta} x e^{-\frac{it}{2}\Delta} = \left(x_1 + it\partial_{x_1}, \cdots, x_n + it\partial_{x_n}\right) = x + it\nabla$$

was widely used for the study of the large time asymptotic behavior of solutions to the nonlinear Schrödinger equations with a gauge invariant nonlinearity such that

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = |u|^{p-1} u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1.3}$$

where it works well. However it is known from [15] that the operator $e^{\frac{it}{2}\Delta} x e^{-\frac{it}{2}\Delta}$ does not work well for the nonlinearity of the form $|u|^p$. The operator $\mathcal{J} = e^{-\frac{it}{4}\Delta^2} x e^{\frac{it}{4}\Delta^2}$ is useful for obtaining the time decay estimates of solutions $e^{\frac{it}{4}\Delta^2} u_0$ to the linear problem. However, it is not clear if $\mathcal{J}$ acts well on the nonlinear term. In this Chapter, our aim is to show that the operator $\mathcal{J}$ works well not only on the gauge invariant nonlinearity $|u|^{p-1} u$, but also for $|u|^p$.

To state our results in this Chapter precisely, we introduce notation. In order to obtain the result for the two dimensional case, we use the dilation operator

$$\begin{aligned}
\mathcal{P} &= x \cdot \nabla + 4t\partial_t = \sum_{j=1}^{n} x_j \partial_{x_j} + 4t\partial_t \\
\left(\Omega_{j,k}\right)_{j,k=1,\cdots,n} &= \left(x_j \partial_{x_k} - x_k \partial_{x_j}\right)_{j,k=1,\cdots,n}.
\end{aligned}$$

which is related to the operator $\mathcal{J}$ by the identity

$$\begin{aligned}
\mathcal{J} \cdot \nabla &= \sum_{j=1}^{n} \mathcal{J}_j \partial_{x_j} = \sum_{j=1}^{n} \left(x_j \partial_{x_j} - it\Delta\partial_{x_j}^2\right) \\
&= x \cdot \nabla - it\Delta^2 = x \cdot \nabla + 4t\partial_t + 4it\left(i\partial_t - \frac{1}{4}\Delta^2\right) \\
&= \mathcal{P} + 4it\mathcal{L},
\end{aligned}$$

where $\mathcal{L} = i\partial_t - \frac{1}{4}\Delta^2$ is the linear part of equation (2.1.1). We have the commutation relations $[\mathcal{J}, \mathcal{L}] = 0$, and $[\mathcal{L}, \mathcal{P}] = 4\mathcal{L}$ and $[\mathcal{L}, \Omega_{j,k}] = 0$. We introduce the function space

$$\mathbf{Z}_\infty = \left\{v;\ \mathcal{U}(-t)\, v \in \mathbf{C}([0,\infty);\mathbf{Z}) \text{ and } \|v\|_{\mathbf{Z}_\infty} < \infty\right\},$$

where

$$\|v\|_{\mathbf{Z}_\infty} = \sup_{t\in[0,\infty)} \|\mathcal{U}(-t)\, v(t)\|_{\mathbf{Z}}$$

$$\mathbf{Z} = \mathbf{H}^{\frac{1}{2}+\delta} \cap \mathbf{H}^{0,1},$$

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{H}^{\frac{1}{2}+\delta}} + \|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\delta}} + \langle t\rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}}$$

for $n = 1$ and

$$\|v\|_{\mathbf{Z}_\infty} = \sup_{t\in[0,\infty)} \|\mathcal{U}(-t)\, v(t)\|_{\mathbf{Z}} + \sup_{t\in[0,\infty)} \|\mathcal{P}v(t)\|_{\mathbf{L}^2},$$

$$\mathbf{Z} = \mathbf{H}^{1+\delta} \cap \mathbf{H}^{1,1},$$

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{H}^{1+\delta}} + \|\phi\|_{\mathbf{H}^{1,1}}$$

for $n = 2$, where $\delta > 0$ is small enough.

Our main result is the following.

Theorem 2.1.1. *Assume that $p > 1 + \frac{4}{n}$, and the initial data are such that $\|u_0\|_{\mathbf{Z}} < \infty$. Then there exists an $\varepsilon_1 > 0$ such that* (2.1.1) *has a unique global solution $u \in \mathbf{Z}_\infty$ satisfying $\|u\|_{\mathbf{Z}_\infty} \le 2\widetilde{\varepsilon}$ for any $\widetilde{\varepsilon}$ such that $\|u_0\|_{\mathbf{Z}} \le \widetilde{\varepsilon} \le \varepsilon_1$.*

In view of the proof of the above theorem we have

Corollary 2.1.2. *Let u be the solution constructed in Theorem 2.1.1. Then the time decay estimates are fulfilled*

$$\|u(t)\|_{\mathbf{L}^r} \le Ct^{-\frac{n}{4}\left(1-\frac{1}{r}\right)},\ 3 < r < \infty,$$

$$\|u(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{4}},\ \ \|\partial_x u(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}},\ n = 1,$$

$$\|u(t)\|_{\dot{\mathbf{H}}^1_r} \le Ct^{-\frac{1}{4}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)},\ 6 < r < \infty,\ 1 > \frac{1}{q} - \frac{1}{r} > \frac{5}{6},\ n = 2.$$

Our proof is based on the estimate of the operator $\mathcal{J} = \mathcal{U}(t)\, x\mathcal{U}(-t) = x - it\Delta\nabla$ in the $\mathbf{L}^2$- norm. The main ingredient of the proof is that, when acting by $\mathcal{J}$ on the nonlinearity $f(u, \overline{u})$, we obtain the worst term $it\Delta\nabla$, which can be expressed again in terms of $\mathcal{J}$. Then the derivatives of the first and the second order give us gain of the time decay of solutions of order $t^{-\frac{1}{4}}$ and $t^{-\frac{1}{2}}$, respectively according to the properties of the linear evolutions group $\mathcal{U}(t)$. Also the multiplication of the nonlinearity $f(u, \overline{u})$ by x can be decomposed as follows $|xf(u, \overline{u})| = O\left((|x|^{\alpha}|u|)^{\frac{1}{\alpha}} |u|^{p-\frac{1}{\alpha}}\right)$, so we need to estimate the loss of the time decay, when multiplying the solution u by $|x|^{\frac{1}{\alpha}}$, where $\frac{1}{p} < \alpha \leq \frac{n}{3}$.

We organize the rest of our Chapter as follows. Section 2.2 and Section 2.4 are devoted to the estimates for the solution of the linear problem in $\mathbf{L}^p$ and weighted $\mathbf{L}^{\infty}$ (Lemmas 2.2.1-2.4.3) and the estimates of $\mathcal{J}f(u, \overline{u})$ (Lemmas 2.2.3, 2.4.4) in the one dimensional and two dimensional cases, respectively. We prove the main result in Section 2.3 and Section 2.5 in the one dimensional and two dimensional cases, respectively.

2.2 Preliminary estimates for $n = 1$

We first estimate the solutions of linear problem in the following lemma.

Lemma 2.2.1. *Let $\mu > 0$ be small. Then the following estimates*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^q} \leq C\langle t\rangle^{-\frac{1}{4}\left(1-\frac{1}{q}\right)} \left(\|\phi\|_{\mathbf{H}^{\frac{1}{2}+\mu}} + \|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}}\right)$$

if $3 < q \leq \infty$,

$$\|\partial_x \mathcal{U}(t)\phi\|_{\mathbf{L}^{\infty}} \leq Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^{\frac{1}{2}+\mu}}$$

and

$$\left\|\left\langle xt^{-\frac{1}{4}}\right\rangle^{-\frac{1}{3}} \partial_x^2 \mathcal{U}(t)\phi\right\|_{\mathbf{L}^{\infty}} \leq Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t\rangle^{\frac{1}{8}} \left(\|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}} + \langle t\rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}}\right)$$

are true for all $t > 0$.

Proof. We have

$$\begin{aligned} \mathcal{U}(t)\phi &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + \frac{it}{4}|\xi|^4} \widehat{\phi}(\xi)\, d\xi \\ &= \int_{\mathbb{R}} A(x-y, t)\,\phi(y)\, dy, \end{aligned}$$

where the kernel

$$A(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + \frac{it}{4}|\xi|^4} d\xi.$$

By the estimate of the kernel $|A(x,t)| \le Ct^{-\frac{1+j}{4}} \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1-j}{3}}$ given in paper [3], we obtain

$$\left|\partial_x^j \mathcal{U}(t)\phi\right| \le Ct^{-\frac{1+j}{4}} \int_{\mathbb{R}} \left\langle (x-y)\,t^{-\frac{1}{4}} \right\rangle^{-\frac{1-j}{3}} |\phi(y)|\,dy. \tag{2.2.1}$$

Therefore using the Young's inequality

$$\|F * G\|_{\mathbf{L}^s} \le C\,\|F\|_{\mathbf{L}^r}\,\|G\|_{\mathbf{L}^q}\,,$$

where $\frac{1}{s} = \frac{1}{r} + \frac{1}{q} - 1$ and the convolution $(F*G)(x) = \int F(x-y)\,G(y)\,dy$, we get

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^q} \le Ct^{-\frac{1}{4}\left(1-\frac{1}{q}\right)}\,\|\phi\|_{\mathbf{L}^1} \le Ct^{-\frac{1}{4}\left(1-\frac{1}{q}\right)}\,\|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}}$$

if $3 < q \le \infty$, and

$$\|\partial_x \mathcal{U}(t)\phi\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}}\,\|\phi\|_{\mathbf{L}^1} \le Ct^{-\frac{1}{2}}\,\|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}}\,.$$

Then the first estimate of the lemma follows by the Sobolev embedding theorem. We consider the third estimate to find that

$$\begin{aligned}
&\left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \left|\partial_x^2 \mathcal{U}(t)\phi\right| \\
&\le Ct^{-\frac{3}{4}} \int_{\mathbb{R}} \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \left\langle (x-y)\,t^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} \left\langle yt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \left\langle yt^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} |\phi(y)|\,dy \\
&\le Ct^{-\frac{3}{4}} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^1} \le Ct^{-\frac{3}{4}} \left\| x^{\frac{1}{2}+\mu} \left\langle xt^{-\frac{1}{4}} \right\rangle^{\frac{1}{2}-\mu} \phi \right\|_{\mathbf{L}^1} \\
&\le Ct^{-\frac{3}{4}} \left(\|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}} + t^{\frac{\mu}{4}-\frac{1}{8}}\,\|\phi\|_{\mathbf{H}^{0,1}} \right) \\
&\le Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t\rangle^{\frac{1}{8}} \left(\|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}} + \langle t\rangle^{-\frac{1}{8}}\,\|\phi\|_{\mathbf{H}^{0,1}} \right).
\end{aligned}$$

□

In order to estimate the action of $\mathcal{J}$ on the nonlinearity we need the following lemmas.

Lemma 2.2.2. *Let $p > \frac{9}{2}$. Then there exists small $\mu > 0$ such that*

$$\left\| |x|^{\frac{1}{3}}\,\mathcal{U}(t)\phi \right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{6}-\frac{\mu}{12}} \langle t\rangle^{\frac{\mu}{12}}\,\|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}}$$

is true for all $t > 0$.

Proof. By (2.2.1) and the Hölder's inequality with $\frac{1}{r}+\frac{1}{s}=1$

$$\begin{aligned}\left| |x|^{\frac{1}{3}} \mathcal{U}(t)\phi \right| &\leq Ct^{-\frac{1}{4}} \int |x-y|^{\frac{1}{3}} \left\langle (x-y)\, t^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} |\phi(y)|\, dy \\ &+ Ct^{-\frac{1}{4}} \int \left\langle (x-y)\, t^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} |y|^{\frac{1}{3}} |\phi(y)|\, dy \\ &\leq Ct^{-\frac{1}{4}+\frac{1}{12}} \|\phi\|_{\mathbf{L}^1} + Ct^{-\frac{1}{4}} \left\| \left\langle x t^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \right\|_{\mathbf{L}^s} \left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^r} \\ &\leq Ct^{-\frac{1}{6}} \|\phi\|_{\mathbf{L}^1} + Ct^{-\frac{1}{4r}} \left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^r}\end{aligned}$$

if $s>3$, i.e. $1 \leq r < \frac{3}{2}$. We choose $r=\frac{3}{2+\mu}$, where $\mu>0$ is small enough. Then we get

$$\left\| |x|^{\frac{1}{3}} \mathcal{U}(t)\phi \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{6}} \|\phi\|_{\mathbf{L}^1} + Ct^{-\frac{1}{6}-\frac{\mu}{12}} \left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^{\frac{3}{2+\mu}}}.$$

Since $\|\phi\|_{\mathbf{L}^1} \leq C \|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}}$ and

$$\left\| |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^{\frac{3}{2+\mu}}} \leq \left\| \langle x\rangle^{\frac{1}{6}+\mu} |x|^{\frac{1}{3}} \phi \right\|_{\mathbf{L}^2} \left\| \langle x\rangle^{-\frac{1}{6}-\mu} \right\|_{\mathbf{L}^{\frac{6}{1+2\mu}}} \leq C \|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}},$$

we obtain

$$\left\| |x|^{\frac{1}{3}} \mathcal{U}(t)\phi \right\|_{\mathbf{L}^\infty} \leq C \left(t^{-\frac{1}{6}} + t^{-\frac{1}{6}-\frac{\mu}{12}} \right) \|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}}.$$

□

Next we estimate the action of the operator $\mathcal{J}=\mathcal{U}(t)\, x\mathcal{U}(-t) = x - it\partial_x^3$ on the nonlinearity $f(u,\overline{u})$. Define the norms

$$\|u\|_{\mathbf{V}} = \|u\|_{\mathbf{H}^{\frac{1}{2}+\mu}} + \|\mathcal{U}(-t)\, u\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}},$$

$$\|u\|_{\mathbf{W}} = \|\mathcal{U}(-t)\, u\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}} + \langle t\rangle^{-\frac{1}{8}} \|\mathcal{U}(-t)\, u\|_{\mathbf{H}^{0,1}}.$$

Lemma 2.2.3. *Let $p>\frac{9}{2}$. Then there exists a small $\mu>0$ such that*

$$\|\mathcal{J} f(u,\overline{u})\|_{\mathbf{L}^2} \leq Ct^{-\frac{5}{8}} \langle t\rangle^{\frac{\mu}{4}-\frac{1}{4}(p-4)} \|u\|_{\mathbf{V}}^{p-1} \|u\|_{\mathbf{W}}$$

is valid for all $t>0$.

Proof. By a direct computation, we have

$$\begin{aligned}|\mathcal{J} f(u,\overline{u})| &\leq Ct\, |u|^{p-3} |\partial_x u|^3 + Ct\, |u|^{p-2} |\partial_x u| \left|\partial_x^2 u\right| \\ &+ C\,|x|\, |u|^p + C\,|u|^{p-1} |\mathcal{J} u|. \qquad (2.2.2)\end{aligned}$$

By the second and third estimates of Lemma 2.2.1 with $\phi = \mathcal{U}(-t)\, u$, we get

$$\|\partial_x u\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{2}} \|u\|_{\mathbf{V}}$$

and

$$\begin{aligned}\left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \partial_x^2 u \right\|_{\mathbf{L}^\infty} &\le Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t\rangle^{\frac{1}{8}} \left(\|\mathcal{U}(-t)\, u\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}} + \langle t\rangle^{-\frac{1}{8}} \|\mathcal{U}(-t)\, u\|_{\mathbf{H}^{0,1}} \right) \\ &\le Ct^{\frac{\mu}{4}-\frac{7}{8}} \langle t\rangle^{\frac{1}{8}} \|u\|_{\mathbf{W}} .\end{aligned}$$

Hence the first term of the right-hand side of (2.2.2) is estimated as

$$t \left\| |u|^{p-3} (\partial_x u)^3 \right\|_{\mathbf{L}^2} \le Ct^{-\frac{1}{2}} \|u\|_{\mathbf{V}}^3 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3}$$

and for the second term by Lemma 2.2.2 we obtain

$$\begin{aligned}& t \left\| |u|^{p-2} (\partial_x u)\, \partial_x^2 u \right\|_{\mathbf{L}^2} \\ &\le Ct \|\partial_x u\|_{\mathbf{L}^\infty} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{1}{3}} \partial_x^2 u \right\|_{\mathbf{L}^\infty} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{\frac{1}{3}} u\, |u|^{p-3} \right\|_{\mathbf{L}^2} \\ &\le Ct^{\frac{\mu}{4}-\frac{3}{8}} \langle t\rangle^{\frac{1}{8}} \|u\|_{\mathbf{V}} \|u\|_{\mathbf{W}} \|u\|_{\mathbf{L}^{2(p-2)}}^{p-2} \\ &+ Ct^{\frac{\mu}{4}-\frac{3}{8}-\frac{1}{12}} \langle t\rangle^{\frac{1}{8}} \|u\|_{\mathbf{V}} \|u\|_{\mathbf{W}} \left\| |x|^{\frac{1}{3}} u \right\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \\ &\le Ct^{\frac{\mu}{4}-\frac{3}{8}} \langle t\rangle^{\frac{1}{8}} \|u\|_{\mathbf{V}} \|u\|_{\mathbf{W}} \|u\|_{\mathbf{L}^{2(p-2)}}^{p-2} \\ &+ Ct^{\frac{\mu}{6}-\frac{5}{8}} \langle t\rangle^{\frac{1}{8}+\frac{\mu}{12}} \|u\|_{\mathbf{W}} \|u\|_{\mathbf{V}}^2 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} .\end{aligned}$$

By the first estimate of Lemma 2.2.1 we get

$$\|u\|_{\mathbf{L}^{2(p-2)}}^{p-2} \le C \langle t\rangle^{-\frac{1}{4}\left(p-\frac{5}{2}\right)} \|u\|_{\mathbf{V}}^{p-2}$$

and

$$\|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \le C \langle t\rangle^{-\frac{1}{4}\left(p-\frac{7}{2}\right)} \|u\|_{\mathbf{V}}^{p-3}$$

if $2(p-3) > 3$. Therefore

$$\begin{aligned}& t \left\| |u|^{p-3} (\partial_x u)^3 \right\|_{\mathbf{L}^2} + t \left\| |u|^{p-2} (\partial_x u)\, \partial_x^2 u \right\|_{\mathbf{L}^2} \\ &\le Ct^{-\frac{5}{8}} \langle t\rangle^{\frac{\mu}{4}-\frac{1}{4}(p-4)} \|u\|_{\mathbf{V}}^{p-1} \|u\|_{\mathbf{W}} .\end{aligned}$$

By Lemma 2.2.2, we also have for the third summand of the right-hand side of (2.2.2)

$$\begin{aligned}\|x\, |u|^p\|_{\mathbf{L}^2} &\le \left\| |x|^{\frac{1}{3}} u \right\|_{\mathbf{L}^\infty}^3 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \\ &\le Ct^{-\frac{1}{2}-\frac{\mu}{4}} \langle t\rangle^{\frac{\mu}{4}} \|\mathcal{U}(-t)\, u\|_{\mathbf{H}^{0,\frac{1}{2}+\mu}}^3 \|u\|_{\mathbf{L}^{2(p-3)}}^{p-3} \\ &\le Ct^{-\frac{1}{2}-\frac{\mu}{4}} \langle t\rangle^{-\frac{1}{4}\left(p-\frac{7}{2}-\mu\right)} \|u\|_{\mathbf{V}}^p .\end{aligned}$$

The last term of the right-hand side of (2.2.2) is estimated by Lemma 2.2.1 as

$$\left\| |u|^{p-1} \mathcal{J}u \right\|_{\mathbf{L}^2} \leq \|u\|_{\mathbf{L}^\infty}^{p-1} \|\mathcal{J}u\|_{\mathbf{L}^2} \leq C \langle t \rangle^{-\frac{1}{4}\left(p-\frac{3}{2}\right)} \|u\|_{\mathbf{V}}^{p-1} \|u\|_{\mathbf{W}}.$$

Therefore we get the desired estimate. □

2.3 Proof of Theorem 2.1.1 for $n = 1$

We consider the linearized equation

$$i\partial_t u - \frac{1}{4}\Delta^2 u = f(v, \overline{v}), u(0) = u_0, \tag{2.3.1}$$

where

$$v \in \mathbf{Z}_{\infty,\rho} = \left\{ v \in \mathbf{Z}_\infty ; \|v\|_{\mathbf{Z}_\infty} \leq \rho \right\},$$

$$\|v\|_{\mathbf{Z}_\infty} = \sup_{t \in [0,\infty)} \|\mathcal{U}(-t) v(t)\|_{\mathbf{Z}},$$

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{H}^{\frac{1}{2}+\delta}} + \|\phi\|_{\mathbf{H}^{0,\frac{1}{2}+\delta}} + \langle t \rangle^{-\frac{1}{8}} \|\phi\|_{\mathbf{H}^{0,1}}.$$

We have the integral equation associated with (2.3.1)

$$u(t) = Sv(t) = \mathcal{U}(t) u_0 - i \int_0^t \mathcal{U}(t-s) f(v, \overline{v}) \, ds. \tag{2.3.2}$$

By Lemma 4 in [14], we obtain

$$\begin{aligned} \sup_{t \in [0,T]} \|u(t)\|_{\mathbf{H}^{\frac{1}{2}+\delta}} &\leq \|u_0\|_{\mathbf{H}^{\frac{1}{2}+\delta}} + C \int_0^t \|v(s)\|_{\mathbf{L}^\infty}^{p-1} \|v(s)\|_{\mathbf{H}^{\frac{1}{2}+\delta}} \, ds \\ &\leq \|u_0\|_{\mathbf{H}^{\frac{1}{2}+\delta}} + C\rho^p. \end{aligned} \tag{2.3.3}$$

By Lemma 2.2.3 and (2.3.2) we get

$$\begin{aligned} \|\mathcal{J}u(t)\|_{\mathbf{L}^2} &\leq \|xu_0\|_{\mathbf{L}^2} + C\rho^p \int_0^t s^{-\frac{5}{8}} \langle s \rangle^{\frac{\mu}{4}-\frac{1}{4}(p-4)} \, ds \\ &\leq \|xu_0\|_{\mathbf{L}^2} + C\rho^p t^{\frac{1}{8}} \end{aligned} \tag{2.3.4}$$

and also

$$\begin{aligned} &\left\| |x|^{\frac{1}{2}+\delta} \mathcal{U}(-t) u(t) \right\|_{\mathbf{L}^2} \\ &\leq \left\| |x|^{\frac{1}{2}+\delta} u_0 \right\|_{\mathbf{L}^2} + C \int_0^\infty \left\| |x|^{\frac{1}{2}+\delta} \mathcal{U}(-s) f(v, \overline{v}) \right\|_{\mathbf{L}^2} ds. \end{aligned} \tag{2.3.5}$$

By Hölder's inequality, Lemma 2.2.3 and the first estimate of Lemma 2.2.1 we have

$$\begin{aligned} \left\| |x|^{\frac{1}{2}+\delta} \mathcal{U}(-s) f(v,\overline{v}) \right\|_{\mathbf{L}^2} &\le \| x\mathcal{U}(-s) f(v,\overline{v})\|_{\mathbf{L}^2}^{\frac{1}{2}+\delta} \| f(v,\overline{v})\|_{\mathbf{L}^2}^{\frac{1}{2}-\delta} \\ &\le C\rho^p s^{-\frac{5}{16}-\frac{5}{8}\delta} \langle s\rangle^{-\frac{1}{4}\left(p-\frac{9}{4}\right)+\frac{7}{8}\delta+\frac{1}{4}\mu\delta+\frac{1}{8}\mu}. \qquad (2.3.6) \end{aligned}$$

Therefore applying (2.3.6) to (2.3.5) we get

$$\begin{aligned} &\left\| |x|^{\frac{1}{2}+\delta} \mathcal{U}(-t) u(t) \right\|_{\mathbf{L}^2} \\ &\le \left\| |x|^{\frac{1}{2}+\delta} u_0 \right\|_{\mathbf{L}^2} + C\rho^p + C\rho^p \int_1^\infty s^{-\frac{1}{4}(p-1)+\frac{1}{4}(\mu+\delta+\mu\delta)} ds \\ &\le \left\| |x|^{\frac{1}{2}+\delta} u_0 \right\|_{\mathbf{L}^2} + C\rho^p \qquad (2.3.7) \end{aligned}$$

if $p > 5+\mu+\delta+\mu\delta$. Thus (2.3.3), (2.3.4) and (2.3.7) show that the mapping S defined by $u = Sv$ transforms $\mathbf{Z}_{\infty,\rho}$ into itself if the data are sufficiently small in $\mathbf{Z}_\infty$. We also find

$$\sup_{t\in[0,\infty)} \| Sv_1(t) - Sv_2(t)\|_{\mathbf{L}^2} \le C\rho^{p-1} \sup_{t\in[0,\infty)} \| v_1(t) - v_2(t)\|_{\mathbf{L}^2}. \qquad (2.3.8)$$

Therefore the contraction mapping shows that there exists a unique global solution u such that

$$\mathcal{U}(-t) u \in \mathbf{L}^\infty \left(0,\infty; \mathbf{H}^{\frac{1}{2}+\delta} \cap \mathbf{H}^{0,\frac{1}{2}+\delta}\right), \langle t\rangle^{\frac{1}{8}} \mathcal{U}(-t) u \in \mathbf{L}^\infty \left(0,\infty; \mathbf{H}^{0,1}\right)$$

and

$$u \in \mathbf{C}\left([0,\infty); \mathbf{L}^2\right).$$

The continuity in time of solutions $\mathcal{U}(-t) u$ in $\mathbf{H}^{\frac{1}{2}+\delta} \cap \mathbf{H}^{0,1}$ follows from (2.3.3) and (2.3.4).

2.4 Preliminary estimates for $n = 2$

We obtain

$$\begin{aligned} |\partial_x|^\alpha \mathcal{U}(t) \phi &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi+\frac{it}{4}|\xi|^4} |\xi|^\alpha \widehat{\phi}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} A_\alpha(x-y,t) \phi(y) dy, \end{aligned}$$

where $0 \le \alpha < 3$ and the kernel

$$A_\alpha(x.t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi+\frac{it}{4}|\xi|^4} |\xi|^\alpha d\xi.$$

The estimate of the kernel $A_\alpha(x,t)$ with an integer α was obtained in paper [3]. We are interested here in fractional α. By the scalling $xt^{-\frac{1}{4}} = y$, $\xi t^{\frac{1}{4}} = \eta$ and then changing the variable of integration $\eta = z\,|y|^{\frac{1}{3}}$, we get

$$\begin{aligned}
A_\alpha(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi + \frac{it}{4}|\xi|^4} |\xi|^\alpha \, d\xi \\
&= t^{-\frac{2+\alpha}{4}} \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{iy\cdot\eta + \frac{i}{4}|\eta|^4} |\eta|^\alpha \, d\eta \\
&= t^{-\frac{2+\alpha}{4}} |y|^{\frac{n+\alpha}{3}} \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{i|y|^{\frac{4}{3}}\left(\frac{y}{|y|}\cdot z + \frac{1}{4}|z|^4\right)} |z|^\alpha \, dz \\
&= t^{-\frac{2+\alpha}{4}} |y|^{\frac{n+\alpha}{3}} \widetilde{A_\alpha}(y),
\end{aligned}$$

where

$$\widetilde{A_\alpha}(y) = \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{i|y|^{\frac{4}{3}}\left(\frac{y}{|y|}\cdot z + \frac{1}{4}|z|^4\right)} |z|^\alpha \, dz.$$

It is easy to see that $\left|\widetilde{A_\alpha}(y)\right| < \infty$ for $|y| \le 1$ since $0 \le \alpha < 3$. To estimate $\widetilde{A_\alpha}(y)$ for large y we apply the stationary phase method (see [7], p. 110)

$$\begin{aligned}
\int_{\mathbb{R}^n} F(x)\, e^{i\lambda S(x)} dx = {}& e^{i\lambda S(x_0)} \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} \left|\det S''(x_0)\right|^{-\frac{1}{2}} e^{i\frac{\pi}{4}\mathrm{sgn} S''(x_0)} F(x_0) \\
& + O\left(\lambda^{-\frac{n}{2}-1}\right),
\end{aligned}$$

where $\lambda = |y|^{\frac{4}{3}}$, $S(x) = \frac{y}{|y|}\cdot x + \frac{1}{4}|x|^4$ and the stationary point $x_0 = (x_{0,1}, \cdots, x_{0,n})$ is defined by the equation

$$\begin{aligned}
\nabla S(x_0) &= (\partial_1 S(x_0), \cdots, \partial_n S(x_0)) \\
&= \left(\frac{y_1}{|y|} + |x_0|^2 x_{0,1}, \cdots, \frac{y_n}{|y|} + |x_0|^2 x_{0,n}\right) \\
&= (0, \cdots, 0).
\end{aligned}$$

We have $x_0 = -\frac{y}{|y|}$ and for $n = 2$

$$\det S''(x_0) = \begin{vmatrix} |x_0|^2 + 2x_{0,1}^2 & 2x_{0,1}x_{0,2} \\ 2x_{0,1}x_{0,2} & |x_0|^2 + 2x_{0,2}^2 \end{vmatrix} = 3\,|x_0|^4 = 3,$$

$\mathrm{sgn} S''(x_0)$ is the difference between the number of positive and negative eigenvalues of $S''(x_0)$ and $\mathrm{sgn} S''(x_0) = 2$, $f(x_0) = |x_0|^\alpha = 1$. Then

$$\widetilde{A_\alpha}(y) = \frac{1}{\sqrt{3}} i\,|y|^{-\frac{4}{3}} e^{-\frac{3}{4}i|y|^{\frac{4}{3}}} + O\left(|y|^{-\frac{8}{3}}\right)$$

for $|y| \to \infty$. Thus we obtain the estimate

$$|A_\alpha(x,t)| \le Ct^{-\frac{2+\alpha}{4}} \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{2-\alpha}{3}}.$$

In the two dimensional case we would like to avoid the use of the norm $\|\phi\|_{\mathbf{L}^1}$ since we restrict to the weighted Sobolev space $\|\phi\|_{\mathbf{H}^{1,1}}$.

Lemma 2.4.1. *Let $\mu > 0$ be small. Then the following estimates are true for all $t > 0$*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^q} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{q}-\frac{\mu}{6}\right)} \left(\|\phi\|_{\mathbf{H}^{1+\mu}} + \|\phi\|_{\mathbf{H}^{0,1}}\right)$$

if $\frac{6}{2-\mu} < q \le \infty$,

$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^1_q} \le Ct^{-\frac{3}{8}}\langle t\rangle^{-\frac{1}{2}\left(\frac{3}{4}-\frac{1}{q}-\frac{\mu}{6}\right)} \|\phi\|_{\mathbf{H}^{1,1}}$$

if $\frac{6}{1-\mu} < q \le \infty$, and

$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^2_q} \le Ct^{-\frac{1}{2}\left(2-\frac{1}{q}-\frac{7\mu}{12}\right)} \|\phi\|_{\mathbf{H}^{1,1}}$$

if $\frac{12}{\mu} < q \le \infty$.

Proof. As in the proof of Lemma 2.2.1, applying the Young's inequality with $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{s}$, we get

$$\begin{aligned}
\left\| |\partial_x|^j \mathcal{U}(t)\phi \right\|_{\mathbf{L}^q} &\le Ct^{-\frac{2+j}{4}} \left\| \int_{\mathbb{R}^2} \left\langle (x-y)t^{-\frac{1}{4}} \right\rangle^{-\frac{2-j}{3}} |\phi(y)|\, dy \right\|_{\mathbf{L}^q} \\
&\le Ct^{-\frac{2+j}{4}} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{2-j}{3}} \right\|_{\mathbf{L}^r} \|\phi\|_{\mathbf{L}^s} \\
&\le Ct^{-\frac{2+j}{4}-\frac{1}{2r}} \|\phi\|_{\mathbf{L}^s},
\end{aligned}$$

since

$$\left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{2-j}{3}} \right\|_{\mathbf{L}^r}^r = t^{\frac{1}{2}} \int_{\mathbb{R}^2} \langle x\rangle^{-\frac{2-j}{3}r}\, dx \le Ct^{\frac{1}{2}}$$

if $0 \le 1 + \frac{1}{q} - \frac{1}{s} = \frac{1}{r} < \frac{2-j}{6}$, $j = 0, 1$. Hence taking $\frac{1}{s} = 1 - \frac{\mu}{6}$, we have

$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^j_q} \le Ct^{-\frac{1}{2}\left(\frac{2+j}{2}-\frac{1}{q}-\frac{\mu}{6}\right)} \|\phi\|_{\mathbf{H}^{0,1}},$$

if $\frac{6}{2-j-\mu} < q \le \infty$, $j = 0, 1$. For the case of $0 < t < 1$, we apply the Sobolev embedding theorem $\|\mathcal{U}(t)\phi\|_{\mathbf{L}^q} \le C\|\mathcal{U}(t)\phi\|_{\mathbf{H}^{1+\mu}} = C\|\phi\|_{\mathbf{H}^{1+\mu}}$ for

$1 \leq q \leq \infty$, to obtain the first estimate of the lemma. For the second estimate in the case of $0 < t < 1$, we estimate

$$\|\nabla \mathcal{U}(t)\phi\|_{\mathbf{L}^q} \leq Ct^{-\frac{1}{2}} \left\| \int_{\mathbb{R}^2} \left\langle (x-y)\, t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} |\nabla \phi(y)|\, dy \right\|_{\mathbf{L}^q}$$
$$\leq Ct^{-\frac{1}{2}} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} \right\|_{\mathbf{L}^r} \|\nabla \phi\|_{\mathbf{L}^s} \leq Ct^{-\frac{1}{2}+\frac{1}{2r}} \|\nabla \phi\|_{\mathbf{L}^s} \leq Ct^{-\frac{3}{8}} \|\phi\|_{\mathbf{H}^{1,1}},$$

if $0 \leq 1 + \frac{1}{q} - \frac{1}{s} = \frac{1}{r} < \frac{1}{3}$, taking $r = 4$, $\frac{1}{s} = \frac{3}{4} + \frac{1}{q}$ we get the second estimate of the lemma. To prove the third estimate of the lemma, we use the Young's inequality with $\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{s}$, $\alpha = 2 - \mu$, to obtain

$$\left\| |\partial_x|^2 \mathcal{U}(t)\phi \right\|_{\mathbf{L}^q} = \left\| |\partial_x|^{2-\mu} \mathcal{U}(t) |\partial_x|^{\mu} \phi \right\|_{\mathbf{L}^q}$$
$$\leq Ct^{\frac{\mu}{4}-1} \left\| \int_{\mathbb{R}^2} \left\langle (x-y)\, t^{-\frac{1}{4}} \right\rangle^{-\frac{\mu}{3}} \left| |\partial_y|^{\mu} \phi(y) \right| dy \right\|_{\mathbf{L}^q}$$
$$\leq Ct^{\frac{\mu}{4}-1} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{\mu}{3}} \right\|_{\mathbf{L}^r} \left\| |\partial_x|^{\mu} \phi \right\|_{\mathbf{L}^s}.$$

Then using $\left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{\mu}{3}} \right\|_{\mathbf{L}^r} \leq Ct^{\frac{1}{2r}}$ for $r > \frac{6}{\mu}$, and choosing $\frac{1}{s} = 1 - \frac{\mu}{12}$, we get

$$\left\| |\partial_x|^2 \mathcal{U}(t)\phi \right\|_{\mathbf{L}^q} \leq Ct^{-1+\frac{1}{2q}+\frac{7\mu}{24}} \left\| |\partial_x|^{\mu} \phi \right\|_{\mathbf{L}^s} \leq Ct^{-\frac{1}{2}\left(2-\frac{1}{q}-\frac{7\mu}{12}\right)} \|\phi\|_{\mathbf{H}^{1,1}}$$

if $\frac{12}{\mu} < q \leq \infty$. □

In order to estimate the action of $\mathcal{J}$ on the nonlinearity we need the following lemmas.

Lemma 2.4.2. *Let $\mu > 0$ be small. Then the estimate*

$$\left\| |x|^{\frac{2}{3}-\mu} \mathcal{U}(t)\phi \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{3}-\frac{\mu}{12}} \|\phi\|_{\mathbf{H}^{0,1}}$$

is true for all $t > 0$.

Proof. Using the Young's inequality with $1 = \frac{1}{r} + \frac{1}{r'} = \frac{1}{s} + \frac{1}{s'}$, we get

$$\left\| |x|^a \mathcal{U}(t)\phi \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{2}} \left\| \int_{\mathbb{R}^2} |x-y|^a \left\langle (x-y)\, t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} |\phi(y)|\, dy \right\|_{\mathbf{L}^\infty}$$
$$+ Ct^{-\frac{1}{2}} \left\| \int_{\mathbb{R}^2} \left\langle (x-y)\, t^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} |y|^a |\phi(y)|\, dy \right\|_{\mathbf{L}^\infty}$$
$$\leq Ct^{-\frac{1}{2}+\frac{a}{4}} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{a-\frac{2}{3}} \right\|_{\mathbf{L}^{r'}} \|\phi\|_{\mathbf{L}^r}$$
$$+ Ct^{-\frac{1}{2}} \left\| \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{2}{3}} \right\|_{\mathbf{L}^{s'}} \left\| |x|^a \phi \right\|_{\mathbf{L}^s}$$
$$\leq Ct^{-\frac{1}{2}+\frac{a}{4}+\frac{1}{2r'}} \|\phi\|_{\mathbf{L}^r} + Ct^{-\frac{1}{2}+\frac{1}{2s'}} \left\| |x|^a \phi \right\|_{\mathbf{L}^s}$$

if $\left(\frac{2}{3}-a\right)r'>2$, $\frac{2}{3}s'>2$, i.e. $1\le r<\frac{6}{4+3a}$, $1\le s<\frac{3}{2}$. Therefore we have

$$\left\||x|^{a}\mathcal{U}(t)\phi\right\|_{\mathbf{L}^{\infty}}\le Ct^{-\frac{1}{2r}+\frac{a}{4}}\|\phi\|_{\mathbf{L}^{r}}+Ct^{-\frac{1}{2s}}\left\||x|^{a}\phi\right\|_{\mathbf{L}^{s}}$$

for $0\le a<\frac{2}{3}$, $1\le r<\frac{6}{4+3a}$, $1\le s<\frac{3}{2}$. Taking $a=\frac{2}{3}-\mu$, $\frac{1}{r}=1-\frac{\mu}{3}$, $\frac{1}{s}=\frac{2}{3}+\frac{\mu}{6}$ we find

$$\left\||x|^{\frac{2}{3}-\mu}\mathcal{U}(t)\phi\right\|_{\mathbf{L}^{\infty}}\le Ct^{-\frac{1}{3}-\frac{\mu}{12}}\|\phi\|_{\mathbf{L}^{\frac{3}{3-\mu}}}+Ct^{-\frac{1}{3}-\frac{\mu}{12}}\left\||x|^{\frac{2}{3}-\mu}\phi\right\|_{\mathbf{L}^{\frac{6}{4+\mu}}}.$$

Since

$$\|\phi\|_{\mathbf{L}^{\frac{3}{3-\mu}}}\le\|\langle x\rangle\phi\|_{\mathbf{L}^{2}}\left\|\langle x\rangle^{-1}\right\|_{\mathbf{L}^{\frac{6}{3-2\mu}}}\le C\|\phi\|_{\mathbf{H}^{0,1}}$$

and

$$\left\||x|^{\frac{2}{3}-\mu}\phi\right\|_{\mathbf{L}^{\frac{6}{4+\mu}}}\le\left\|\langle x\rangle^{\frac{1}{3}+\mu}|x|^{\frac{2}{3}-\mu}\phi\right\|_{\mathbf{L}^{2}}\left\|\langle x\rangle^{-\frac{1}{3}-\mu}\right\|_{\mathbf{L}^{\frac{6}{1+\mu}}}\le C\|\phi\|_{\mathbf{H}^{0,1}},$$

we obtain the estimate of the lemma

$$\left\||x|^{\frac{2}{3}-\mu}\mathcal{U}(t)\phi\right\|_{\mathbf{L}^{\infty}}\le Ct^{-\frac{1}{3}-\frac{\mu}{12}}\|\phi\|_{\mathbf{H}^{0,1}}.$$

□

Define the norm

$$\|\phi\|_{\mathbf{Z}}=\|\phi\|_{\mathbf{H}^{1+\mu}}+\|\phi\|_{\mathbf{H}^{1,1}}.$$

Lemma 2.4.3. *Let $p>3$. Then there exists a small $\mu>0$ such that the estimate*

$$\|x\,|\mathcal{U}(t)\phi|^{p}\|_{\mathbf{L}^{2}}\le Ct^{-\frac{3}{4}}\langle t\rangle^{-\frac{1}{4}-\mu}\|\phi\|_{\mathbf{Z}}^{p}$$

holds for all $t>0$.

Proof. We use Lemma 2.4.2 to find

$$\begin{aligned}\left\||x|\,|\mathcal{U}(t)\phi|^{p}\right\|_{\mathbf{L}^{2}}&=\left\|\left||x|^{\frac{2}{3}-\mu}|\mathcal{U}(t)\phi|\right|^{\frac{3}{2-3\mu}}|\mathcal{U}(t)\phi|^{p-\frac{3}{2-3\mu}}\right\|_{\mathbf{L}^{2}}\\&\le C\left\||x|^{\frac{2}{3}-\mu}|\mathcal{U}(t)\phi|\right\|_{\mathbf{L}^{\infty}}^{\frac{3}{2-3\mu}}\|\mathcal{U}(t)\phi\|_{\mathbf{L}^{2\left(p-\frac{3}{2-3\mu}\right)}}^{p-\frac{3}{2-3\mu}}\\&\le Ct^{-\left(1+\frac{\mu}{4}\right)\frac{1}{2-3\mu}}\|\phi\|_{\mathbf{H}^{0,1}}^{\frac{3}{2-3\mu}}\|\mathcal{U}(t)\phi\|_{\mathbf{L}^{2\left(p-\frac{3}{2-3\mu}\right)}}^{p-\frac{3}{2-3\mu}}.\end{aligned}$$

For a small $\mu>0$, we find that $2\left(p-\frac{3}{2-3\mu}\right)>\frac{6}{2-\mu}$, hence the first estimate of Lemma 2.4.1 says

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^{2\left(p-\frac{3}{2-3\mu}\right)}}^{p-\frac{3}{2-3\mu}}\le C\langle t\rangle^{-\frac{1}{2}\left(p-\frac{3}{2-3\mu}-\frac{1}{2}-\frac{\mu}{6}\left(p-\frac{3}{2-3\mu}\right)\right)}\|\phi\|_{\mathbf{Z}}^{p-\frac{3}{2-3\mu}}.$$

Therefore we obtain

$$
\begin{aligned}
\left\| |x| \left| \mathcal{U}(t)\phi \right|^p \right\|_{\mathbf{L}^2} &\le Ct^{-\left(1+\frac{\mu}{4}\right)\frac{1}{2-3\mu}} \langle t\rangle^{-\frac{1}{2}\left(p-\frac{3}{2-3\mu}-\frac{1}{2}-\frac{\mu}{6}\left(p-\frac{3}{2-3\mu}\right)\right)} \|\phi\|_{\mathbf{Z}}^p . \\
&\le Ct^{-\frac{3}{4}} \langle t\rangle^{-\frac{1}{4}-\mu} \|\phi\|_{\mathbf{Z}}^p
\end{aligned}
$$

since $p > 3$ and $\mu > 0$ is small enough. □

Next we estimate the action of the operator $\mathcal{J} = \mathcal{U}(t)\, x\mathcal{U}(-t) = x - it\Delta\nabla$ on the nonlinearity $f(u, \overline{u})$.

Lemma 2.4.4. *Let $p > 3$. Then there exists a small $\mu > 0$ such that the estimate*

$$
\|\mathcal{J} f(u,\overline{u})\|_{\mathbf{L}^2} \le Ct^{-\frac{3}{4}} \langle t\rangle^{-\frac{1}{4}-\mu} \|\mathcal{U}(-t)\, u\|_{\mathbf{Z}}^p
$$

is true for all $t > 0$.

Proof. As in (2.2.2) we start with

$$
\begin{aligned}
\|\mathcal{J} f(u,\overline{u})\|_{\mathbf{L}^2} \le Ct \sum_{|\beta|=1} I_\beta &+ Ct \sum_{|\alpha|=2,|\beta|=1} I_{\alpha,\beta} \\
&+ C\left\| x\, |u|^p \right\|_{\mathbf{L}^2} + C\left\| |u|^{p-1} \mathcal{J} u \right\|_{\mathbf{L}^2}, \qquad (2.4.1)
\end{aligned}
$$

where

$$
\begin{aligned}
I_\beta &= t\left\| |u|^{p-3} \left|\partial_x^\beta u\right|^3 \right\|_{\mathbf{L}^2}, \\
I_{\alpha,\beta} &= t\left\| |u|^{p-2} \left(\partial_x^\beta u\right) \partial_x^\alpha u \right\|_{\mathbf{L}^2} .
\end{aligned}
$$

By the Hölder's inequality with $\sum_{j=1}^{2} \frac{1}{r_j} = \frac{1}{2}$ we find

$$
I_\beta \le t \left\| \partial_x^\beta u \right\|_{\mathbf{L}^{3r_1}}^3 \|u\|_{\mathbf{L}^{r_2(p-3)}}^{p-3} .
$$

Then taking $\frac{1}{r_1} = \frac{1}{2} - \mu$, $\frac{1}{r_2} = \mu$, and using the first and the second estimates of Lemma 2.4.1 with $\phi = \mathcal{U}(-t)\, u$ we get

$$
\|u\|_{\mathbf{L}^{r_2(p-3)}}^{p-3} \le C \langle t\rangle^{-\frac{1}{2}\left(p-3-\frac{\mu}{6}(p+3)\right)} \|\mathcal{U}(-t)\, u\|_{\mathbf{Z}}^{p-3}
$$

and

$$
\left\| \partial_x^\beta u \right\|_{\mathbf{L}^{3r_1}}^3 \le Ct^{-\frac{9}{8}} \langle t\rangle^{-\frac{1}{4}\left(\frac{7}{2}+\mu\right)} \|\mathcal{U}(-t)\, u\|_{\mathbf{Z}}^3 .
$$

Hence

$$
\begin{aligned}
I_\beta &\le Ct^{-\frac{1}{8}} \langle t\rangle^{-\frac{1}{2}\left(p-\frac{5}{4}\right)+\frac{p}{12}\mu} \|\mathcal{U}(-t)\, u\|_{\mathbf{Z}}^p \\
&\le Ct^{-\frac{3}{4}} \langle t\rangle^{-\frac{1}{4}-\mu} \|\mathcal{U}(-t)\, u\|_{\mathbf{Z}}^p \qquad (2.4.2)
\end{aligned}
$$

if $p > 3$ and $\mu > 0$ is small. Similarly, by the Hölder's inequality with $\sum_{j=1}^{3} \frac{1}{q_j} = \frac{1}{2}$ we find

$$I_{\alpha,\beta} \leq Ct \left\| \partial_x^{\alpha} u \right\|_{\mathbf{L}^{q_1}} \left\| \partial_x^{\beta} u \right\|_{\mathbf{L}^{q_2}} \left\| u \right\|_{\mathbf{L}^{q_3(p-2)}}^{p-2}.$$

Then taking $\frac{1}{q_1} = \mu$, $\frac{1}{q_2} = \frac{1}{6} - \mu$, $\frac{1}{q_3} = \frac{1}{3}$, and applying Lemma 2.4.1 with $\phi = \mathcal{U}(-t)\, u$ we obtain

$$\left\| u \right\|_{\mathbf{L}^{q_3(p-2)}}^{p-2} \leq C \left\langle t \right\rangle^{-\frac{1}{2}\left(p-2-\frac{1}{3}-\frac{\mu}{6}(p-2)\right)} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}^{p-2},$$

$$\left\| \partial_x^{\beta} u \right\|_{\mathbf{L}^{q_2}} \leq Ct^{-\frac{3}{8}} \left\langle t \right\rangle^{-\frac{1}{2}\left(\frac{3}{4}-\frac{1}{6}+\frac{5\mu}{6}\right)} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}$$

and

$$\left\| \partial_x^{\alpha} u \right\|_{\mathbf{L}^{q_1}} \leq Ct^{-1+\frac{19}{24}\mu} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}.$$

Hence

$$\begin{aligned} I_{\alpha,\beta} &\leq Ct^{-\frac{3}{8}+\frac{19}{24}\mu} \left\langle t \right\rangle^{-\frac{1}{2}\left(p-\frac{7}{4}\right)+\frac{1}{12}\mu(p-7)} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}^{p} \\ &\leq Ct^{-\frac{3}{4}} \left\langle t \right\rangle^{-\frac{1}{4}-\mu} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}^{p}. \end{aligned} \tag{2.4.3}$$

By Lemma 2.4.4 we have

$$\left\| |x| \, |u|^p \right\|_{\mathbf{L}^2} \leq Ct^{-\frac{3}{4}} \left\langle t \right\rangle^{-\frac{1}{4}-\mu} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}^{p}. \tag{2.4.4}$$

Also the last term of the right-hand side of (2.4.1) is estimated by Lemma 2.4.1 as

$$\begin{aligned} \left\| |u|^{p-1} \mathcal{J} u \right\|_{\mathbf{L}^2} &\leq \left\| u \right\|_{\mathbf{L}^{\infty}}^{p-1} \left\| \mathcal{J} u \right\|_{\mathbf{L}^2} \\ &\leq C \left\langle t \right\rangle^{-\frac{p-1}{2}\left(1-\frac{\mu}{6}\right)} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}^{p} \\ &\leq Ct^{-\frac{3}{4}} \left\langle t \right\rangle^{-\frac{1}{4}-\mu} \left\| \mathcal{U}(-t)\, u \right\|_{\mathbf{Z}}^{p}. \end{aligned} \tag{2.4.5}$$

Therefore we get the desired estimate from (2.4.2) - (2.4.5). □

2.5 Proof of Theorem 2.1.1 for $n = 2$

We consider the linearized equation (2.3.1), where

$$v \in \mathbf{Z}_{\infty,\rho} = \left\{ v \in \mathbf{Z}_{\infty}; \left\| v \right\|_{\mathbf{Z}_{\infty}} \leq \rho \right\},$$

with the norm

$$\left\| v \right\|_{\mathbf{Z}_{\infty}} = \sup_{t\in[0,\infty)} \left\| \mathcal{U}(-t)\, v(t) \right\|_{\mathbf{Z}} + \sup_{t\in[0,\infty)} \left\| \mathcal{P} v(t) \right\|_{\mathbf{L}^2},$$

$$\|\phi\|_{\mathbf{Z}} = \|\phi\|_{\mathbf{H}^{1+\delta}} + \|\phi\|_{\mathbf{H}^{1,1}}.$$

We consider the integral equation (2.3.2) associated with (2.3.1). We apply Lemma 2.4.4 in (2.3.2) to get

$$\|\mathcal{J}u(t)\|_{\mathbf{L}^2} \le \|xu_0\|_{\mathbf{L}^2} + C\rho^p. \tag{2.5.1}$$

By Lemma 4 in [14], we obtain

$$\begin{aligned}\|u(t)\|_{\mathbf{H}^{1+\delta}} &\le \|u_0\|_{\mathbf{H}^{1+\delta}} + C\int_0^t \left\| f\left(v(s), \overline{v(s)}\right)\right\|_{\mathbf{H}^{1+\delta}} ds \\ &\le \|u_0\|_{\mathbf{H}^{1+\delta}} + C\int_0^t \|v(s)\|_{\mathbf{L}^\infty}^{p-1} \|v(s)\|_{\mathbf{H}^{1+\delta}} ds. \end{aligned}\tag{2.5.2}$$

We also get by the first estimate of Lemma 2.4.1

$$\|v(s)\|_{\mathbf{L}^\infty} \le C\langle s\rangle^{-\frac{1}{2}\left(1-\frac{\mu}{6}\right)}\rho.$$

Therefore, we find

$$\|u(t)\|_{\mathbf{H}^{1+\delta}} \le \|u_0\|_{\mathbf{H}^{1+\delta}} + C\rho^p. \tag{2.5.3}$$

By the identity

$$\mathcal{J}\cdot\nabla u = \mathcal{P}u + 4it\mathcal{L}u = \mathcal{P}u + 4itf(v,\overline{v}),$$

we have

$$\|\mathcal{J}\cdot\nabla u\|_{\mathbf{L}^2} \le \|\mathcal{P}u\|_{\mathbf{L}^2} + Ct\|f(v,\overline{v})\|_{\mathbf{L}^2} \le \|\mathcal{P}u\|_{\mathbf{L}^2} + C\rho^p. \tag{2.5.4}$$

Applying $\mathcal{P}$ to both sides of (2.3.2), we obtain

$$\mathcal{L}\mathcal{P}u = (4+\mathcal{P})f(v,\overline{v})$$

and by the energy estimate

$$\begin{aligned}\|\mathcal{P}u(t)\|_{\mathbf{L}^2} &\le \|x\cdot\nabla u_0\|_{\mathbf{L}^2} + C\int_0^t \|v(s)\|_{\mathbf{L}^\infty}^{p-1}\left(\|\mathcal{P}v(s)\|_{\mathbf{L}^2} + \|v(s)\|_{\mathbf{L}^2}\right) ds \\ &\le \|x\cdot\nabla u_0\|_{\mathbf{L}^2} + C\rho^p. \end{aligned}\tag{2.5.5}$$

We apply (2.5.5) to (2.5.4) to obtain

$$\|\mathcal{J}\cdot\nabla u(t)\|_{\mathbf{L}^2} \le \|x\cdot\nabla u_0\|_{\mathbf{L}^2} + C\rho^p. \tag{2.5.6}$$

Applying $\Omega_{j,k} = x_j\partial_k - x_k\partial_{x_j}$ to both sides of (2.3.2), we get

$$\mathcal{L}\Omega_{j,k}u = \Omega_{j,k}f(v,\overline{v})$$

and by the energy method

$$
\begin{aligned}
\|\Omega_{j,k}u(t)\|_{\mathbf{L}^2} &\le \|\Omega_{j,k}u_0\|_{\mathbf{L}^2} + C\int_0^t \|v\|_{\mathbf{L}^\infty}^{p-1}\|\Omega_{j,k}v\|_{\mathbf{L}^2}\,ds \\
&\le \|\Omega_{j,k}u_0\|_{\mathbf{L}^2} + C\rho^p.
\end{aligned} \tag{2.5.7}
$$

As in [9], we obtain by integration by parts

$$
\begin{aligned}
&\sum_{j,k=1}^{2}\|\mathcal{J}_j\partial_{x_k}u\|_{\mathbf{L}^2}^2 = \sum_{j,k=1}^{2}\left(\mathcal{J}_j\partial_{x_k}u, \mathcal{J}_j\partial_{x_k}u\right) \\
&= \sum_{j,k=1}^{2}\left(x_j\partial_{x_k}u, x_j\partial_{x_k}u\right) + \sum_{j,k=1}^{2}\left(x_j\partial_{x_k}u, -it\Delta\partial_{x_j}\partial_{x_k}u\right) \\
&+ \sum_{j,k=1}^{2}\left(-it\Delta\partial_{x_j}\partial_{x_k}u, x_j\partial_{x_k}u\right) + \sum_{j,k=1}^{2}\left(it\Delta\partial_{x_j}\partial_{x_k}u, it\Delta\partial_{x_j}\partial_{x_k}u\right) \\
&= \frac{1}{2}\sum_{j,k=1}^{2}\left(\Omega_{j,k}u, \Omega_{j,k}u\right) + \left((x\cdot\nabla)u, (x\cdot\nabla)u\right) + \left(-it\Delta\nabla\cdot\nabla u, (x\cdot\nabla)u\right) \\
&+ \left((x\cdot\nabla)u, -it\Delta\nabla\cdot\nabla u\right) + \left(it\Delta\nabla\cdot\nabla u, it\Delta\nabla\cdot\nabla u\right) \\
&= \frac{1}{2}\sum_{j,k=1}^{2}\|\Omega_{j,k}u\|_{\mathbf{L}^2} + \|\mathcal{J}\cdot\nabla u\|_{\mathbf{L}^2}^2
\end{aligned}
$$

where we have denoted the inner product by $(\cdot,\cdot)$. By (2.5.1), (2.5.3) and (2.5.6) we get

$$
\|u\|_{\mathbf{Z}_\infty} \le \varepsilon + C\rho^p. \tag{2.5.8}
$$

Hence the mapping S defined by $u = Sv$ transforms $\mathbf{Z}_{\infty,\rho}$ into itself. In the same way as in the proof of (2.3.8), we obtain (2.3.8) for $n = 2$. Therefore the contraction mapping shows that there exists a unique global solution u such that

$$
\mathcal{U}(-t)\,u \in \mathbf{L}^\infty\left(0,\infty;\mathbf{H}^{1+\delta}\cap\mathbf{H}^{1,1}\right),
$$

and

$$
u \in \mathbf{C}\left([0,\infty);\mathbf{L}^2\right).
$$

The continuity in time of solutions $\mathcal{U}(-t)\,u$ in $\mathbf{H}^{1+\delta}\cap\mathbf{H}^{1,1}$ follows from (2.5.2), (2.5.4) and (2.5.5).

Chapter 3

Global existence of small solutions for a quadratic nonlinear fourth-order Schrödinger equation in six space dimensions

3.1 Introduction

This Chapter is based on the author's work [1]. We consider the Cauchy problem for a quadratic nonlinear fourth-order Schrödinger equation in six space dimensions

$$\begin{cases} i\partial_t u - \frac{1}{4}\Delta^2 u = \lambda \overline{u}^2, & (t,x) \in (0,\infty) \times \mathbf{R}^6, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^6, \end{cases} \tag{3.1.1}$$

where, $\overline{u}$ is the complex conjugate of u and $\lambda \in \mathbf{C}$. The aim of this Chapter is to prove a global existence of small solutions and $\mathbf{L}^q\,(3 < q \leq \infty)$ time decay estimates of solutions to (3.1.1). We recall known results of global existence of small solutions for

$$\begin{cases} i\partial_t u - \frac{1}{4}\Delta^2 u = f(u,\overline{u}), & (t,x) \in (0,\infty) \times \mathbf{R}^n \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{3.1.2}$$

where the nonlinearity $f(u,\overline{u})$ satisfies the growth condition $\left|\partial_u^a \partial_{\overline{u}}^b f(u,\overline{u})\right| \leq C|u|^{p-a-b}$ with $0 \leq a+b \leq p$. Since the pointwise time decay estimates of solutions to the free fourth-order Schrödinger equation is $O\left(t^{-\frac{n}{4}}\right)$ and the linear problem has the $\mathbf{L}^2$ conservation law, $\mathbf{L}^2$ norm of the nonlinearity $f(u,\overline{u})$ decays like $O\left(t^{-\frac{n}{4}(p-1)}\right)$. We find $\int_1^\infty t^{-\frac{n}{4}(p-1)}dt < \infty$ if $p > 1 + \frac{4}{n}$, therefore we expect a global existence of small solutions to (3.1.2) holds if $p > 1 + \frac{4}{n}$. In the previous paper [2], we have applied the operator

$$\begin{aligned} \mathcal{J} &= e^{-\frac{it}{4}\Delta^2} x e^{\frac{it}{4}\Delta^2} = \left(e^{-\frac{it}{4}\Delta^2} x_1 e^{\frac{it}{4}\Delta^2}, \cdots, e^{-\frac{it}{4}\Delta^2} x_n e^{\frac{it}{4}\Delta^2} \right) \\ &= (x_1 - it\Delta\partial_1, \cdots, x_n - it\Delta\partial_n) = x - it\Delta\nabla \end{aligned}$$

(3.1.2) to show a global existence of small solutions to (3.1.2) with the space dimensions $n = 1$ or 2 when the order of nonlinearity $p > 1 + \frac{4}{n}$. We have shown that the operator $\mathcal{J}$ works well for the power nonlinearities $f(u, \overline{u})$ in lower space dimensions as we have seen in Chapter 2. We are interested in a global existence of small solutions to (3.1.2) when $n \geq 3$ and $p > 1 + \frac{4}{n}$. In [21], the $\mathbf{L}^{p+1}$-$\mathbf{L}^{1+\frac{1}{p}}$ time decay estimate of evolution operator $e^{\frac{1}{2}it\Delta}$ was applied to the nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = f(u, \overline{u}), & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{3.1.3}$$

to obtain a global existence of small solutions, when the initial data are small in $\mathbf{H}^{1,0} \cap \mathbf{L}^{1+\frac{1}{p}}$ and the order of nonlinearity p satisfies $p_{2,s}(n) < p < p_{2,*}(n)$, where

$$\begin{aligned} p_{2,s}(n) &= \frac{1}{2}\left(1 + \frac{2}{n} + \sqrt{\left(1 + \frac{2}{n}\right)^2 + 4\left(\frac{2}{n}\right)} \right), \\ p_{2,*}(n) &= \begin{cases} \infty & (n = 1, 2) \\ \frac{n+2}{n-2} & (n \geq 3). \end{cases} \end{aligned}$$

More precisely,

Theorem 3.1.1. *We assume that* $p_{2,s}(n) < p < p_{2,*}(n)$, $u_0 \in \mathbf{H}^{1,0} \cap \mathbf{L}^{1+\frac{1}{p}}$ *and* $\|u_0\|_{\mathbf{H}^{1,0} \cap \mathbf{L}^{1+\frac{1}{p}}} < \varepsilon$, *then there exists an* $\varepsilon > 0$ *such that* (3.1.3) *has a unique global solution* $u \in \mathbf{C}\left([0, \infty) : \mathbf{L}^2 \cap \mathbf{L}^{p+1}\right)$. *Moreover, the following estimate*

$$\|u(t)\|_{\mathbf{L}^q} \leq C \langle t \rangle^{-\frac{n}{2}\left(1 - \frac{2}{q}\right)} \varepsilon,$$

is true for any $2 \leq q \leq p + 1$.

We remark that a global existence of small solutions to (3.1.3) with $n = 4$, 5 and $f(u, \overline{u}) = u^2 + \overline{u}^2 + |u|^2$ was solved in the sense of Theorem 3.1.1 since $p_{2,s}(n) < 2 < p_{2,*}(n)$ for $n = 4$, 5. However global well-posedness and the $\mathbf{L}^\infty$ time decay estimate of solutions are unknown. From the time decay of free solutions, in the case of the fourth-order nonlinear Schrödinger equation,

$p_{2,s}(n)$ and $p_{2,*}(n)$ are replaced by $p_{2,s}\left(\frac{n}{2}\right)$ and $p_{2,*}\left(\frac{n}{2}\right)$ respectively. We write $p_{2,s}\left(\frac{n}{2}\right)$, $p_{2,*}\left(\frac{n}{2}\right)$ by $p_{4,s}(n)$, $p_{4,*}(n)$, respectively. Then,

$$p_{4,s}(n) = \frac{1}{2}\left(1+\frac{4}{n}+\sqrt{\left(1+\frac{4}{n}\right)^2+4\left(\frac{4}{n}\right)}\right),$$

$$p_{4,*}(n) = \begin{cases} \infty & (n=1,2,3,4) \\ \frac{n+4}{n-4} & (n \geq 5) \end{cases}.$$

Applying the method of Strauss, global in time of small solutions for (3.1.2) will be obtained if $p_{4,s}(n) < p < p_{4,*}(n)$. However there are no global result for the case $1+\frac{4}{n} < p \leq p_{4,s}(n)$ as far as we know. Note that $p_{4,s}(6) = p_{2,s}(3) = 2$. In [12], [13] and [18], global results in time and $\mathbf{L}^\infty$ time decay estimate of solutions to (3.1.3)) were shown when $n = 3$ and $f(u,\overline{u}) = u^2+\overline{u}^2$. In these papers the factorization technique of the free evolution group was used. If we apply the same method as in [12], [13] and [18] to (3.1.3) with $n \geq 4$ and $f(u,\overline{u}) = u^2+\overline{u}^2$, we use the operator $\widetilde{\mathcal{J}}$ to the equation $\left[\frac{n}{2}\right]+1$ times, where

$$\begin{aligned}\widetilde{\mathcal{J}} &= e^{-\frac{it}{2}\Delta} x e^{\frac{it}{2}\Delta} = \left(e^{-\frac{it}{2}\Delta} x_1 e^{\frac{it}{2}\Delta}, \cdots, e^{-\frac{it}{2}\Delta} x_n e^{\frac{it}{2}\Delta}\right) \\ &= (x_1 + it\partial_{x_1}, \cdots, x_n + it\partial_{x_n}) = x + it\nabla.\end{aligned}$$

It seems that the iterative use of $\widetilde{\mathcal{J}}$ makes the problem difficult, therefore as far as we know, global results for higher space dimensions are still open problem. We have the same difficulty in our problem of this Chapter.

We define the following operators

$$\begin{aligned}&\mathcal{P} = x\cdot\nabla_x + 4t\partial_t, \\ &\widetilde{\mathcal{P}} = -\xi\cdot\nabla_\xi + 4t\partial_t, \\ &(\Omega_{j,k})_{j,k=1,\cdots,6} = \left(x_j\partial_{x_k} - x_k\partial_{x_j}\right)_{j,k=1,\cdots,6}.\end{aligned}$$

The operator $\mathcal{P}$ is related to the operator $\mathcal{J}$ through the identity

$$\begin{aligned}\mathcal{J}\cdot\nabla &= \sum_{j=1}^{6}\mathcal{J}_j\partial_{x_j} = \sum_{j=1}^{6}\left(x_j\partial_{x_j} - it\Delta\partial_{x_j}^2\right) = x\cdot\nabla - it\Delta^2 \\ &= x\cdot\nabla + 4t\partial_t + 4it\left(i\partial_t - \frac{1}{4}\Delta^2\right) = \mathcal{P} + 4it\mathcal{L}\end{aligned}$$

where $\mathcal{L} = i\partial_t - \frac{1}{4}\Delta^2$ is the linear part of equation (3.1.1). We have the commutation relations $[\mathcal{J},\mathcal{L}] = 0$, $[\mathcal{L},\mathcal{P}] = 4\mathcal{L}$ and $[\mathcal{L},\Omega_{j,k}] = 0$. To state our results in this Chapter precisely, we introduce the notation.

$$\mathbf{X} = \bigcap_{j=0}^{3}\mathbf{H}^{12-3j,j}, \|v\|_{\mathbf{X}} = \sum_{j=0}^{3}\|v\|_{\mathbf{H}^{12-3j,j}}.$$

Our main result is the following.

Theorem 3.1.2. *Let $u_0 \in \mathbf{X}$, then there exists an $\varepsilon > 0$ such that* (3.1.1) *has a unique global solution u satisfying $\mathcal{U}(-t)\,u \in \mathbf{C}([0,\infty) : \mathbf{X})$ for any u_0 satisfying $\|u_0\|_{\mathbf{X}} < \varepsilon$. Moreover, the time decay estimates*

$$\|u\|_{\dot{\mathbf{H}}^{\alpha}_{q}} \le Ct^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{\alpha}{4}+\gamma}$$

are fulfilled, where $0 \le \alpha < 3$, $\frac{18}{6-\alpha} < r < q \le \infty$ and $\gamma > \frac{3}{2}\left(\frac{1}{r}-\frac{1}{q}\right)$.

We note here that we have $\mathbf{L}^{\infty}$ time decay of solutions such that

$$\|u\|_{\mathbf{L}^{\infty}} \le Ct^{-\frac{3}{2}+\gamma},$$

where $\gamma > \frac{3}{2r}$, $r > 3$. Therefore order of time decay is worse compared to the one of solutions to the linear problem $O\left(t^{-\frac{3}{2}}\right)$. This fact comes from the reason why the use of the operator $\sum_{|\alpha|=4} \mathcal{J}^{\alpha}$ yields the strong time growth $O\left(t^{4}\right)$ in the nonlinearity which is difficult to treat, see our strategy below.

From the proof of the above theorem, we have the scattering result.

Corollary 3.1.3. *Let u be the solution constructed in Theorem 3.1.2. Then for any $u_0 \in \mathbf{X}$ satisfying $\|u_0\|_{\mathbf{X}} < \varepsilon$ there exists a unique scattering state $u_+ \in \mathbf{X}_+ = \bigcap_{j=0}^{2} \mathbf{H}^{12-3j,j} \supset \mathbf{X}$, such that $\|u_+\|_{\mathbf{X}_+} < 2\varepsilon$ and*

$$\lim_{t\to\infty} \|\mathcal{U}(-t)\,u - u_+\|_{\mathbf{X}_+} = 0.$$

We state our strategy of the proof. The operator $\mathcal{J} = (\mathcal{J}_j)_{j=1,2,3,4,5,6}$, $\mathcal{J}_j = x_j - it\Delta\partial x_j$ is useful to get time decay estimates of solutions to fourth-order nonlinear Schrödinger equations. However, when we apply the operator $\mathcal{J}$ to the nonlinearity $\overline{u}^2$ iteratively, we encounter a difficulty of the explicit time growth. Indeed we have from equation (3.1.1)

$$\mathcal{L}\sum_{|\alpha|=3} \mathcal{J}^{\alpha} u = \lambda \sum_{|\alpha|=3} \mathcal{J}^{\alpha}\overline{u}^2 = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{R}_{\mathcal{B}}, \tag{3.1.4}$$

where

$$\mathcal{B}_0 = \sum_{|\alpha|+|\beta|=3, 0\le|\beta|\le 1} C_{\alpha,\beta}\overline{\mathcal{J}^{\alpha}u}\,\overline{\mathcal{J}^{\beta}u},$$

$$\mathcal{B}_1 = t \sum_{|\alpha|+|\beta|=2, 0\le|\beta|\le 1, |\gamma|+|\delta|=3, 0\le|\delta|\le 3} C_{\alpha,\beta}^{\gamma,\delta}\overline{\partial_x^{\gamma}\mathcal{J}^{\alpha}u\partial_x^{\delta}\mathcal{J}^{\beta}u}$$

$$\mathcal{B}_2 = t^2 \sum_{|\alpha|=1, |\gamma|+|\delta|=6, 0\le|\delta|\le 6} C_{\alpha}^{\gamma,\delta}\overline{\partial_x^{\gamma}\mathcal{J}^{\alpha}u\partial_x^{\delta}u} + t^2 \sum_{|\gamma|+|\delta|=5, 0\le|\delta|\le 2} C^{\gamma,\delta}\overline{\partial_x^{\gamma}u\partial_x^{\delta}u},$$

$$\mathcal{B}_3 = t^3 \sum_{|\gamma|+|\delta|=9, 0\le|\delta|\le 4} C^{\gamma,\delta}\overline{\partial_x^{\gamma}u\partial_x^{\delta}u},$$

$$\mathcal{R}_{\mathcal{B}} = t\sum_{|\gamma|=1} C^{\gamma}\overline{\partial_x^{\gamma}u}\overline{u} + t \sum_{|\alpha|=1, |\gamma|+|\delta|=2, 0\le|\delta|\le 2} C_{\alpha}^{\gamma,\delta}\overline{\partial_x^{\gamma}\mathcal{J}^{\alpha}u\partial_x^{\delta}u}.$$

It seems to be difficult to get the desired estimates for $\mathcal{B}_2$ and $\mathcal{B}_3$ which are needed to show the theorem. In this Chapter, we overcome the difficulty to use the normal form method. Our approach to this problem is based on [20], where quadratic nonlinear terms were transformed to cubic nonlinearities with faster time decay properties in nonlinear Klein-Gordon equations with quadratic nonlinearities. The normal form method was applied to nonlinear Schrödinger equations successfully by [4] in two space dimensions, where a global existence of small solutions to the Cauchy problem (3.1.3) with $n=2$ and $f = f(u,\overline{u}) = \sum_{j,k=1}^{2} \lambda_{jk} \left(\partial_{x_j}\overline{u}\right)\left(\partial_{x_k}\overline{u}\right)$, $\lambda_{jk} \in \mathbf{C}$ was shown. In [4], (3.1.3) was shown that the identity

$$f = \mathcal{L}_2 \mathcal{G}_1 \left(\overline{u}, \overline{u}\right) + 2\mathcal{G}_1 \left(\overline{\mathcal{L}_2 u}, \overline{u}\right) \tag{3.1.5}$$

holds for any smooth function u, where $\mathcal{L}_2 = i\partial_t + \frac{1}{2}\Delta$ and the symmetric bilinear operator $\mathcal{G}_1$ is defined by the convolution

$$\mathcal{G}_1 (\phi, \psi) = \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} g_1 (y,z)\, \phi (x-y)\, \psi (x-z)\, dydz$$

and the kernel $g_1(y,z)$is given by the inverse Fourier transform of

$$\hat{\hat{g}}_1 (\xi, \eta) = \frac{2}{(2\pi)^2} \sum_{j,k=1}^{2} \lambda_{jk} \frac{\xi_j \eta_k}{|\xi|^2 + \xi \cdot \eta + |\eta|^2}.$$

with respect to y and z, namely

$$g_1 (y,z) = \frac{2}{(2\pi)^4} \sum_{j,k=1}^{2} \lambda_{jk} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{\xi_j \eta_k}{|\xi|^2 + \xi \cdot \eta + |\eta|^2} e^{iy\cdot\xi + iz\cdot\eta} d\xi d\eta$$

We remark that $\mathcal{G}_1(\phi,\psi)$ can be rewritten as

$$\mathcal{G}_1 (\phi, \psi) = C \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \hat{\hat{g}}_1 (\xi, \eta)\, \hat{\phi}(\xi)\, \hat{\psi}(\eta)\, e^{ix\cdot(\xi+\eta)} d\xi d\eta.$$

If u is the solution of (3.1.3), then

$$\mathcal{L}_2 u = f = \mathcal{L}_2 \mathcal{G}_1 \left(\overline{u}, \overline{u}\right) + 2\mathcal{G}_1 \left(\overline{f}, \overline{u}\right)$$

which yields the nonlinear Schrödinger equation

$$\mathcal{L}_2 \left(u - \mathcal{G}_1 \left(\overline{u}, \overline{u}\right)\right) = 2\mathcal{G}_1 \left(\overline{f}, \overline{u}\right),$$

If one tries to apply this method to the nonlinear terms

$$f = \sum_{j,k=1}^{2} \mu_{jk} \left(\partial_{x_j} u\right)\left(\partial_{x_k} u\right) \quad \text{or} \quad f = \sum_{j,k=1}^{2} \nu_{jk} \left(\partial_{x_j} \overline{u}\right)\left(\partial_{x_k} u\right), \quad \mu_{jk}, \nu_{jk} \in \mathbf{C}$$

then the functions $\hat{\hat{g}}_2$and $\hat{\hat{g}}_3$ will have a form

$$\hat{\hat{g}}_2(\xi,\eta)=\frac{2}{(2\pi)^2}\sum_{j,k=1}^{2}\mu_{jk}\frac{\xi_j\eta_k}{\xi\cdot\eta}\quad\text{or}\quad\hat{\hat{g}}_3(\xi,\eta)=\frac{2}{(2\pi)^2}\sum_{j,k=1}^{2}\nu_{jk}\frac{\xi_j\eta_k}{|\xi|^2+\xi\cdot\eta},$$

respectively. Note the first function $\hat{\hat{g}}_1$ dose not have singularity at the origin and so $\mathcal{G}_1\left(\overline{f},\overline{u}\right)$ can be considered as a cubic nonlinearity through the result of Coifman-Mayer [5]. Their result was improved by [22] as

Proposition 3.1.4. *Let*

$$\Lambda(\phi,\psi)(x)=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}e^{ix\cdot(\xi+\eta)}m(\xi,\eta)\,\hat{\phi}(\xi)\,\hat{\psi}(\eta)\,d\xi d\eta,$$

where $1<p,q,r<\infty$ *and* $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. *If* $m\in\mathbf{C}^{n+1}\left(\mathbf{R}^{2n}\setminus\{0\}\right)$ *satisfies*

$$\left|\partial_\eta^\alpha\partial_\zeta^\beta m(\xi,\eta)\right|\le C(|\xi|+|\eta|)^{-|\alpha|-|\beta|}$$

for all $|\alpha|+|\beta|\le n+1$ *and* $(\xi,\eta)\neq(0,0)$*, then*

$$\|\Lambda(\phi,\psi)\|_{\mathbf{L}^r}\le C\|\phi\|_{\mathbf{L}^p}\|\psi\|_{\mathbf{L}^q}$$

is true.

If we put $\hat{\hat{g}}_1(\xi,\eta)=m(\xi,\eta)$ in Proposition 3.1.4, then $\hat{\hat{g}}_1(\xi,\eta)$ satisfies the Coifman-Mayer condition of the proposition. Hence we have the estimate

$$\left\|\mathcal{G}_1\left(\overline{f(u,\overline{u})},\overline{u}\right)\right\|_{\mathbf{L}^r}\le C\left(\sum_{|\alpha|=1}\|\partial_x^\alpha u\|_{\mathbf{L}^{p_1}}\right)\left(\sum_{|\alpha|=1}\|\partial_x^\alpha u\|_{\mathbf{L}^{p_2}}\right)\|u\|_{\mathbf{L}^{p_3}},$$

where

$$\frac{1}{r}=\sum_{1\le j\le 3}\frac{1}{p_j},\quad 1<p_j,\quad r<\infty.$$

However, other two functions $\hat{\hat{g}}_2$ and $\hat{\hat{g}}_3$ have singularities at the origin, hence, $\mathcal{G}_2(f,u)$ or $\mathcal{G}_3\left(\overline{f},u\right)$ does not satisfies the Coifman-Mayer condition except $\hat{\hat{g}}_2(\xi,\eta)=\frac{1}{2\pi^2}(\mu_{11}+\mu_{22})$, $\mu_{11}=\mu_{22}=1$. In this case $\mathcal{G}_2(u,u)=u^2$ and $f=\sum_{j=1,2}\left(\partial_{x_j}u\right)\left(\partial_{x_j}u\right)$. Therefore we have

$$f=\mathcal{L}_2u^2-2u\mathcal{L}_2u$$

which implies if u is the solution of (3.1.3), then we have

$$\mathcal{L}_2\left(u-u^2\right)=-2u\mathcal{L}_2u=-2uf.$$

We now turn to our problem. We consider the problem

$$\begin{cases} \mathcal{L}u = f_j, & (t,x) \in (0,\infty) \times \mathbf{R}^6, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^6, \end{cases} \tag{3.1.6}$$

where

$$f_1 = \sum_{|\alpha|+|\beta|=4} (\partial_x^\alpha \overline{u}) \left(\partial_x^\beta \overline{u}\right), f_2 = \sum_{|\alpha|+|\beta|=4} (\partial_x^\alpha u) \left(\partial_x^\beta u\right), f_3 = \sum_{|\alpha|+|\beta|=4} (\partial_x^\alpha u) \left(\partial_x^\beta \overline{u}\right)$$

to explain our situation. In the same way as in the proof of (3.1.5), we have

$$\begin{aligned} f_1 &= -\mathcal{L}\mathcal{G}_1(\overline{u},\overline{u}) - 2\mathcal{G}_1\left(\overline{\mathcal{L}u},\overline{u}\right), \\ f_2 &= -\mathcal{L}\mathcal{G}_2(u,u) - 2\mathcal{G}_2(\mathcal{L}u,u) \end{aligned} \tag{3.1.7}$$

and

$$f_3 = -\mathcal{L}\mathcal{G}_3(u,\overline{u}) - 2\mathcal{G}_3(\mathcal{L}u,\overline{u})$$

where

$$\begin{aligned} \hat{\hat{g}}_1(\xi,\eta) &= \frac{4}{(2\pi)^6} \sum_{|\alpha|+|\beta|=4} \frac{\xi^\alpha \eta^\beta}{|\xi|^4 + |\eta|^4 + |\xi+\eta|^4}, \\ \hat{\hat{g}}_2(\xi,\eta) &= -\frac{4}{(2\pi)^6} \sum_{|\alpha|+|\beta|=4} \frac{\xi^\alpha \eta^\beta}{|\xi|^4 + |\eta|^4 - |\xi+\eta|^4}, \end{aligned}$$

and

$$\hat{\hat{g}}_3(\xi,\eta) = \frac{4}{(2\pi)^6} \sum_{|\alpha|+|\beta|=4} \frac{\xi^\alpha \eta^\beta}{|\xi|^4 - |\eta|^4 + |\xi+\eta|^4}.$$

Nonlinear terms $\mathcal{B}_2$ and $\mathcal{B}_3$ which are in the right hand sides of (3.1.4) will be represented as the similar form as the right hand side of (3.1.7).

In the case of the fourth order Schödinger equation, $\hat{\hat{g}}_2$ and $\hat{\hat{g}}_3$ have singularities at the origin. This is the reason why the nonlinearity of (3.1.1) does not include the nonlinearity u^2 or $|u|^2$.

We organize the Chapter as follows. In Sections 3.2, we prove preliminary estimates. In Section 3.3, We state the normal form method in the case of the fourth-order nonlinear Schrödinger equation. We prove the main result in Section 3.4.

3.2 Preliminary estimate

We summarize some estimates of the solutions of linear problem.

Lemma 3.2.1. *We define*

$$\mathcal{A}(t,x) = \mathcal{F}^{-1}E = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{-\frac{1}{4}it|\xi|^4 + ix\cdot\xi} d\xi.$$

If $0 \leq \alpha < 3$, *then the following estimate*

$$\left| |\partial_x|^{\alpha} \mathcal{A}(t,x) \right| \leq Ct^{-\frac{n+|\alpha|}{4}} \left\langle xt^{-\frac{1}{4}} \right\rangle^{-\frac{n-|\alpha|}{3}}$$

is true for any $t > 0$, *and* $x \in \mathbf{R}^n$.

Lemma 3.2.2. *Let* $n = 6, 2 \leq q \leq \infty$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$. *Then*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^q} \leq Ct^{-\frac{3}{2}\left(1-\frac{2}{q}\right)} \|\phi\|_{\mathbf{L}^{q'}}$$

for any $t > 0$.

Proof. By Lemma 3.2.1, we obtain

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^\infty} \leq C\|A * \phi\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{n}{4}} \|\phi\|_{\mathbf{L}^1},$$

where $(F * G)(x) = \int_{\mathbf{R}^6} F(x-y)G(y)\,dy$. Also $\mathcal{U}(t)$ is unitary operator in $\mathbf{L}^2$. Therefore, we have the desire estimate by the Riesz-Thorin interpolation theorem. □

Since by the Sobolev embedding $\|\phi\|_{\mathbf{L}^{q_1}} \leq C\|\phi\|_{\mathbf{H}^1}$, $\|\phi\|_{\mathbf{L}^{q_2}} \leq C\|\phi\|_{\mathbf{H}^2}$, where $2 \leq q_1 < 3, 2 \leq q_2 < 6$ and by the Hölder inequality $\|\phi\|_{\mathbf{L}^{q_1'}} \leq \|\phi\|_{\mathbf{H}^{0,1}}$, $\|\phi\|_{\mathbf{L}^{q_2'}} \leq \|\phi\|_{\mathbf{H}^{0,2}}$, where $\frac{1}{q_1} + \frac{1}{q_1'} = \frac{1}{q_2} + \frac{1}{q_2'} = 1$, we get from Lemma 3.2.2

Corollary 3.2.3. *Let* $n = 6$, $2 \leq q_1 < 3$ *and* $2 \leq q_2 < 6$. *Then the following estimates*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^{q_1}} \leq C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{q_1}\right)} \left(\|\phi\|_{\mathbf{H}^1} + \|\phi\|_{\mathbf{H}^{0,1}}\right),$$
$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^{q_2}} \leq C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{q_2}\right)} \left(\|\phi\|_{\mathbf{H}^2} + \|\phi\|_{\mathbf{H}^{0,2}}\right)$$

are true for any $t \geq 0$.

Next two lemmas say that time decay of solutions to linear problem is similar to that of fourth order parabolic equation if we restrict $\mathbf{L}^q$ time decay of solutions with $q > 3$.

Lemma 3.2.4. *Let* $n = 6$, $0 \leq a < 3$, $\frac{18}{6-a} < q \leq \infty$, $\frac{18}{6-a} < r_1 \leq q$ *and* $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{r_1} + 1$. *Then the following estimate*

$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}_q^a} \leq Ct^{-\frac{3}{2}\left(1-\frac{1}{r_1}\right)-\frac{a}{4}} \|\phi\|_{\mathbf{L}^{r_2}}$$

is valid for any $t > 0$.

Proof. We obtain

$$|\partial_x|^a \mathcal{U}(t)\phi = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^6} e^{-\frac{1}{4}it|\xi|^4+ix\cdot\xi}|\xi|^a\hat{\phi}(\xi)\,d\xi = \int_{\mathbf{R}^6}\mathcal{A}_a(t,x-y)\phi(y)\,dy,$$

where the kernel

$$\mathcal{A}_a(t,x) = |\partial_x|^a\mathcal{A}(t,x) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^6} e^{-\frac{1}{4}it|\xi|^4+ix\cdot\xi}|\xi|^a\,d\xi.$$

By Lemma 3.2.1, we get

$$\begin{aligned}||\partial_x|^a\mathcal{U}(t)\phi| &\le C\int_{\mathbf{R}^6}|\mathcal{A}_a(t,x-y)||\phi(y)|\,dy \\ &\le Ct^{-\frac{3}{2}-\frac{a}{4}}\int_{\mathbf{R}^6}\left\langle (x-y)t^{-\frac{1}{4}}\right\rangle^{-\frac{6-a}{3}}|\phi(y)|\,dy.\end{aligned}$$

Therefore applying the Young's inequality

$$\|F*G\|_{\mathbf{L}^q} \le \|F\|_{\mathbf{L}^{r_1}}\|G\|_{\mathbf{L}^{r_2}}$$

where $\frac{1}{q}=\frac{1}{r_1}+\frac{1}{r_2}-1$, we have

$$\begin{aligned}\||\partial_x|^a\mathcal{U}(t)\phi\|_{\mathbf{L}^q} &\le Ct^{-\frac{3}{2}-\frac{a}{4}}\left\|\left\langle xt^{-\frac{1}{4}}\right\rangle^{-\frac{6-a}{3}}\right\|_{\mathbf{L}^{r_1}}\|\phi\|_{\mathbf{L}^{r_2}} \\ &\le Ct^{-\frac{3}{2}\left(1-\frac{1}{r_1}\right)-\frac{a}{4}}\|\phi\|_{\mathbf{L}^{r_2}}\end{aligned}$$

if $\frac{18}{6-a}<q\le\infty$, $\frac{18}{6-a}<r_1\le q$, $\frac{1}{r_2}=\frac{1}{q}-\frac{1}{r_1}+1$, which implies the lemma. $\square$

Remark 3.2.1. *In Lemma 3.2.4, we put $r_1=q$, $r_2=1$, then*

$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^a_q} \le Ct^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{a}{4}}\|\phi\|_{\mathbf{L}^1},$$

where $\frac{18}{6-a}<q\le\infty, 0\le a<3$. Time decay estimate of solutions for the fourth-order Schrödinger equations is the same as that for the fourth-order heat equations if we assume the restriction on q from the below. This is the different point from the usual second order Schrödinger equations.

As a corollary of Lemma 3.2.4, we get

Corollary 3.2.5. *Let $0\le a<3$, $b>3+a$, $\frac{18}{6-a}<q_1\le\infty$, $\frac{18}{6-a}<r_1<q_1$ and $\frac{18}{6-a}<r_2<6$, $\frac{6r_2}{6-r_2}<q_2\le\infty$. Then the following estimates*

$$\begin{aligned}\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^a_{q_1}} &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r_1}\right)-\frac{a}{4}}\left(\|\phi\|_{\mathbf{H}^b}+\|\phi\|_{\mathbf{H}^{0,3}}\right) \\ \|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^a_{q_2}} &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r_2}\right)-\frac{a}{4}}\left(\|\phi\|_{\mathbf{H}^b}+\|\phi\|_{\mathbf{H}^{0,2}}\right)\end{aligned}$$

are true for any $t\ge 0$.

Proof. By lemma 3.2.4, we find

$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^a_{q_1}} \leq Ct^{-\frac{3}{2}\left(1-\frac{1}{r_1}\right)-\frac{a}{4}} \|\phi\|_{\mathbf{L}^{s_1}} \leq Ct^{-\frac{3}{2}\left(1-\frac{1}{r_1}\right)-\frac{a}{4}} \|\phi\|_{\mathbf{H}^{0,3}},$$
$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^a_{q_2}} \leq Ct^{-\frac{3}{2}\left(1-\frac{1}{r_2}\right)-\frac{a}{4}} \|\phi\|_{\mathbf{L}^{s_2}} \leq Ct^{-\frac{3}{2}\left(1-\frac{1}{r_2}\right)-\frac{a}{4}} \|\phi\|_{\mathbf{H}^{0,2}},$$

where $\frac{1}{s_j} = \frac{1}{q_j} - \frac{1}{r_j} + 1$ $(j = 1, 2)$. Applying the Sobolev inequality, we get

$$\|\mathcal{U}(t)\phi\|_{\dot{\mathbf{H}}^a_{q_j}} \leq C\|\mathcal{U}(t)\phi\|_{\mathbf{H}^b} \leq C\|\phi\|_{\mathbf{H}^b},$$

where $j = 1, 2$ and $b > a + 3$. Therefore, Corollary 3.2.5 is proved. □

In order to estimate the action of $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t)$ on the nonlinearity we need the following lemma. We use the lemma by putting $\phi = \mathcal{U}(-t)u$.

Lemma 3.2.6. *Let* $0 \leq a < 3$, $b > 3 + a$, $\frac{18}{3-a} < q \leq \infty$, $\frac{18}{3-a} < r_1 < q$, $\frac{18}{6-a} < r_3 < 6$ *and* $\frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2} - 1 = \frac{1}{r_3} + \frac{1}{r_4} - 1$. *Then*

$$\begin{aligned}&\||x||\partial_x|^a\mathcal{U}(t)\phi\|_{\mathbf{L}^q}\\ &\leq C\left(\langle t\rangle^{-\frac{5+a}{4}+\frac{3}{2r_1}} + \langle t\rangle^{-\frac{6+a}{4}+\frac{3}{2r_3}}\right)\left(\|\phi\|_{\mathbf{H}^{b+3}} + \|\phi\|_{\mathbf{H}^{b,1}} + \|\phi\|_{\mathbf{H}^{0,3}}\right)\end{aligned}$$

for any $t \geq 0$.

Proof. By Lemma 3.2.1, we find

$$\begin{aligned}||x||\partial_x|^a\mathcal{U}(t)\phi| &\leq Ct^{-\frac{6+a}{4}}\int_{\mathbf{R}^6}|x-y|\left\langle(x-y)t^{-\frac{1}{4}}\right\rangle^{-\frac{6-a}{3}}|\phi(y)|\,dy\\ &\quad+Ct^{-\frac{6+a}{4}}\int_{\mathbf{R}^6}\left\langle(x-y)t^{-\frac{1}{4}}\right\rangle^{-\frac{6-a}{3}}|y||\phi(y)|\,dy\\ &\leq Ct^{-\frac{5+a}{4}}\int_{\mathbf{R}^6}\left\langle(x-y)t^{-\frac{1}{4}}\right\rangle^{-\frac{3-a}{3}}|\phi(y)|\,dy\\ &\quad+Ct^{-\frac{6+a}{4}}\int_{\mathbf{R}^6}\left\langle(x-y)t^{-\frac{1}{4}}\right\rangle^{-\frac{6-a}{3}}|y||\phi(y)|\,dy.\end{aligned}$$

Applying the Young inequality, we get

$$\begin{aligned}&\||x||\partial_x|^a\mathcal{U}(t)\phi\|_{\mathbf{L}^q}\\ &\leq Ct^{-\frac{5+a}{4}}\left\|\left\langle xt^{-\frac{1}{4}}\right\rangle^{-\frac{3-a}{3}}\right\|_{\mathbf{L}^{r_1}}\|\phi\|_{\mathbf{L}^{r_2}} + Ct^{-\frac{6+a}{4}}\left\|\left\langle xt^{-\frac{1}{4}}\right\rangle^{-\frac{6-a}{3}}\right\|_{\mathbf{L}^{r_3}}\||x|\phi\|_{\mathbf{L}^{r_4}}\\ &\leq C\left(t^{-\frac{5+a}{4}+\frac{3}{2r_1}} + t^{-\frac{6+a}{4}+\frac{3}{2r_3}}\right)\|\phi\|_{\mathbf{H}^{0,3}}.\end{aligned}$$

Let $t \leq 1$. By $x_j = \mathcal{J}_j - it\Delta\partial_{x_j}$, we obtain

$$\begin{aligned}\left\| |x| \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{L}^q} &\leq C \sum_{j=1}^{6} \left\| x_j \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{L}^q} \\ &\leq C \sum_{j=1}^{6} \left(\left\| \mathcal{J}_j \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{L}^q} + t \left\| \Delta\partial_{x_j} \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{L}^q} \right).\end{aligned}$$

Applying the Sobolev inequality and the identity $\mathcal{J}_j = \mathcal{U}(t)\,x_j\mathcal{U}(-t)$, we have

$$\begin{aligned}&\left\| \mathcal{J}_j \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{L}^q} + t \left\| \Delta\partial_{x_j} \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{L}^q} \\ &\leq C \left(\left\| \mathcal{J}_j \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{H}^{b-a}} + t \left\| \Delta\partial_{x_j} \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{H}^{b-a}} \right) \\ &\leq C \left(\left\| x_j \left|\partial_x\right|^a \phi \right\|_{\mathbf{H}^{b-a}} + \|\phi\|_{\mathbf{H}^{b+3}} \right) \\ &\leq C \left(\|\phi\|_{\mathbf{H}^{b,1}} + \|\phi\|_{\mathbf{H}^{b+3}} \right)\end{aligned}$$

with $j = 1, \cdots, 6$. Thus, we get

$$\left\| |x| \left|\partial_x\right|^a \mathcal{U}(t)\,\phi \right\|_{\mathbf{L}^q} \leq C \left(\|\phi\|_{\mathbf{H}^{b,1}} + \|\phi\|_{\mathbf{H}^{b+3}} \right).$$

Therefore we obtain the desire estimate. $\square$

The next lemma is used for obtaining estimates of $\sum_{|\alpha|=3} \|\mathcal{J}^\alpha u\|_{\dot{\mathbf{H}}^1}$ and $\left\| \langle\cdot\rangle^3 \mathcal{F}\mathcal{U}(-t)\,u \right\|_{\mathbf{L}^q}$.

Lemma 3.2.7. *Let $q > 6$. Then we have the following estimates*

$$\left\| \langle\cdot\rangle^4 \mathcal{F}\mathcal{U}(-t)\,\overline{u}^2 \right\|_{\mathbf{L}^\infty} \leq C \max\left(t^{-\frac{3(q-2)}{2q}}, t^{-\frac{3(q-1)}{2q}+\gamma_0}, t^{-\frac{3}{2}+2\gamma_0} \right) \|u\|_{\mathbf{Y}}^2,$$
$$\left\| \nabla\mathcal{F}\mathcal{U}(-t)\,\overline{u}^2 \right\|_{\mathbf{L}^\infty} \leq C \max\left(t^{-\frac{3(3q-4)}{4q}}, t^{-\frac{3(3q-2)}{4q}+\gamma_0}, t^{-\frac{9}{4}+2\gamma_0} \right) \|u\|_{\mathbf{Y}}^2$$

for any $t \geq 1$, where

$$\|v\|_{\mathbf{Y}} = \|v\|_{\mathbf{L}^2} + \langle t\rangle^{-\gamma_0} \sum_{|\alpha|=3} \|\mathcal{J}^\alpha v\|_{\mathbf{L}^2} + \|\mathcal{F}\mathcal{U}(-t)\,v\|_{\mathbf{L}^q},$$

and $0 < \gamma_0$.

Proof. Applying the operator $\mathcal{FU}(-t)$ to $\overline{u}^2$, we find

$$\begin{aligned}
&\mathcal{FU}(-t)\,\overline{u}^2 = (2\pi)^3 e^{\frac{1}{4}it|\xi|^4}\int_{\mathbf{R}^6}\hat{\overline{u}}(\xi-\eta)\,\hat{\overline{u}}(\eta)\,d\eta\\
&= (2\pi)^3 e^{\frac{1}{4}it|\xi|^4}\int_{\mathbf{R}^6}\overline{\hat{u}(\eta-\xi)\,\hat{u}(-\eta)}d\eta\\
&= (2\pi)^3 e^{\frac{1}{4}it|\xi|^4}\int_{\mathbf{R}^6}\overline{\hat{u}\left(\eta-\frac{\xi}{2}\right)\hat{u}\left(-\left(\eta+\frac{\xi}{2}\right)\right)}d\eta\\
&= (2\pi)^3\int_{\mathbf{R}^6} e^{\frac{1}{4}it\left(|\xi|^4+\left|\eta-\frac{\xi}{2}\right|^4+\left|-\left(\eta+\frac{\xi}{2}\right)\right|^4\right)}\\
&\times\overline{e^{\frac{1}{4}it\left|\eta-\frac{\xi}{2}\right|^4}\hat{u}\left(\eta-\frac{\xi}{2}\right)e^{\frac{1}{4}it\left|-\left(\eta+\frac{\xi}{2}\right)\right|^4}\hat{u}\left(-\left(\eta+\frac{\xi}{2}\right)\right)}d\eta\\
&= (2\pi)^3\int_{\mathbf{R}^6} e^{\frac{1}{4}it\left(|\xi|^4+\left|\eta-\frac{\xi}{2}\right|^4+\left|-\left(\eta+\frac{\xi}{2}\right)\right|^4\right)}\overline{\hat{\varphi}\left(\eta-\frac{\xi}{2}\right)\hat{\varphi}\left(-\left(\eta+\frac{\xi}{2}\right)\right)}d\eta\\
&= (2\pi)^3\int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)}\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}d\eta,
\end{aligned}$$

where

$$\begin{aligned}
S(\eta,\xi) &= |\xi|^4+\left|\eta-\frac{\xi}{2}\right|^4+\left|-\left(\eta+\frac{\xi}{2}\right)\right|^4\\
&= 2|\eta|^4+2(\eta\cdot\xi)^2+|\eta|^2|\xi|^2+\frac{9}{8}|\xi|^4,
\end{aligned}$$

$$\hat{\varphi} = \mathcal{FU}(-t)\,u, z=\eta-\frac{\xi}{2}, w=-\left(\eta+\frac{\xi}{2}\right).$$

By the identity

$$e^{\frac{1}{4}itS(\eta,\xi)} = H\nabla_\eta\cdot\left(\eta e^{\frac{1}{4}itS(\eta,\xi)}\right)$$

with

$$H=\left(6+\frac{1}{4}it\eta\cdot\nabla_\eta S(\eta,\xi)\right)^{-1}=\left(6+it\left(2|\eta|^4+(\eta\cdot\xi)^2+\frac{1}{2}|\xi|^2|\eta|^2\right)\right)^{-1}$$

and integration by parts, we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)}\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}d\eta = \int_{\mathbf{R}^6} H\nabla_\eta\cdot\left(\eta e^{\frac{1}{4}itS(\eta,\xi)}\right)\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}d\eta\\
&= -\int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)}\left(\eta\cdot\nabla_\eta H\right)\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}d\eta\\
&- \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H\eta\cdot\nabla_\eta\left(\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}\right)d\eta\\
&= \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H\left(\left(\eta\cdot\nabla_\eta\right)^2 H\right)\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}d\eta\\
&+ \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)}\left(\eta\cdot\nabla_\eta H\right)^2\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}d\eta\\
&+ 2\int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H\left(\eta\cdot\nabla_\eta H\right)\eta\cdot\nabla_\eta\left(\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}\right)d\eta\\
&+ \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H^2\left(\left(\eta\cdot\nabla_\eta\right)^2\left(\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}\right)\right)d\eta. \qquad (3.2.1)
\end{aligned}
$$

By applying the estimates

$$
\begin{aligned}
&\left|\eta\cdot\nabla_\eta H\right| \le CW(\eta,\xi)\\
&\left|H\left(\left(\eta\cdot\nabla_\eta\right)^2 H\right)\right| + \left|H\left(\eta\cdot\nabla_\eta\right)H\right| + \left|H^2\right| \le C\left(W(\eta,\xi)\right)^2
\end{aligned}
$$

with

$$
W(\eta,\xi) = \left(1 + t\,|\eta|^2\left(|\eta|^2 + |\xi|^2\right)\right)^{-1}
$$

and the estimate

$$
\begin{aligned}
&\left|\left(\eta\cdot\nabla_\eta\right)^2\left(\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}\right)\right|\\
&\le C\left|\left(\eta\cdot\nabla_\eta\right)\left(\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}\right)\right| + \left|\eta\cdot\left(\left(\eta\cdot\nabla_\eta\right)\nabla_\eta\left(\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}\right)\right)\right|\\
&\le C\,|\eta|\sum_{|\alpha|+|\beta|=1}\left|\partial^\alpha\hat{\varphi}(z)\right|\left|\partial^\beta\hat{\varphi}(w)\right| + C\,|\eta|^2\sum_{|\alpha|+|\beta|=2}\left|\partial^\alpha\hat{\varphi}(z)\right|\left|\partial^\beta\hat{\varphi}(w)\right|,
\end{aligned}
$$

we find

$$
\left|\int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)}\overline{\hat{\varphi}(z)\,\hat{\varphi}(w)}d\eta\right| \le C\sum_{j=0}^{2} I_{1,j},
$$

where

$$
I_{1,j} = \int_{\mathbf{R}^6} |\eta|^j\left(W(\eta,\xi)\right)^2\sum_{|\alpha|+|\beta|=j}\left|\partial^\alpha\hat{\varphi}(z)\right|\left|\partial^\beta\hat{\varphi}(w)\right|d\eta
$$

with $j = 0, 1, 2$. We make a change of variable $\eta = t^{-\frac{1}{4}}\widetilde{\eta}$ to obtain

$$\begin{aligned}\left\| |\eta|^{\beta} W(\eta,\xi)^2 \right\|_{\mathbf{L}^{\alpha}_{\eta}}^{\alpha} &\le C \int \left(|\eta|^{\beta} \left(1 + t|\eta|^4\right)^{-2} \right)^{\alpha} d\eta \\ &\le C t^{-\frac{3}{2}-\frac{\alpha\beta}{4}} \int \left(|\widetilde{\eta}|^{\beta} \left(1 + |\widetilde{\eta}|^4\right)^{-2} \right)^{\alpha} d\widetilde{\eta} \\ &\le C t^{-\frac{3}{2}-\frac{\alpha\beta}{4}} \end{aligned} \tag{3.2.2}$$

if $\alpha > 1$ and $(8-\beta)\alpha > 6$. By (3.2.2), the Hölder and the Sobolev inequalities, we estimate

$$\begin{aligned} I_{1,0} &\le C \int_{\mathbf{R}^6} W(\eta,\xi)^2 |\hat{\varphi}(z)| |\hat{\varphi}(w)| \, d\eta \\ &\le C \left\| W(\eta,\xi)^2 \right\|_{\mathbf{L}_{\eta}^{\frac{q}{q-2}}} \|\hat{\varphi}\|_{\mathbf{L}^q}^2 \\ &\le C t^{-\frac{3(q-2)}{2q}} \|\hat{\varphi}\|_{\mathbf{L}^q}^2 , \end{aligned}$$

$$\begin{aligned} I_{1,1} &\le C \int_{\mathbf{R}^6} |\eta| W(\eta,\xi)^2 \sum_{|\alpha|+|\beta|=1} |\partial^{\alpha}\hat{\varphi}(z)| \left|\partial^{\beta}\hat{\varphi}(w)\right| d\eta \\ &\le C \left\| |\eta| W(\eta,\xi)^2 \right\|_{\mathbf{L}_{\eta}^{\frac{6q}{5q-6}}} \sum_{|\alpha|=1} \|\partial^{\alpha}\hat{\varphi}\|_{\mathbf{L}^6} \|\hat{\varphi}\|_{\mathbf{L}^q} \\ &\le C t^{-\frac{3(q-1)}{2q}} \|\hat{\varphi}\|_{\dot{\mathbf{H}}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} \end{aligned}$$

and

$$\begin{aligned} I_{1,2} &\le C \int_{\mathbf{R}^6} |\eta|^2 W(\eta,\xi)^2 \sum_{|\alpha|+|\beta|=2} |\partial^{\alpha}\hat{\varphi}(z)| \left|\partial^{\beta}\hat{\varphi}(w)\right| d\eta \\ &\le C \left\| |\eta|^2 W(\eta,\xi)^2 \right\|_{\mathbf{L}^{\frac{3q}{2q-3}}} \sum_{|\alpha|=2} \|\partial^{\alpha}\hat{\varphi}\|_{\mathbf{L}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} \\ &+ C \left\| |\eta|^2 W(\eta,\xi)^2 \right\|_{\mathbf{L}^{\frac{3}{2}}} \left(\sum_{|\alpha|=1} \|\partial^{\alpha}\hat{\varphi}\|_{\mathbf{L}^6} \right)^2 \\ &\le C t^{-\frac{3(q-1)}{2q}} \|\hat{\varphi}\|_{\dot{\mathbf{H}}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} + C t^{-\frac{3}{2}} \|\hat{\varphi}\|_{\dot{\mathbf{H}}^3}^2 , \end{aligned}$$

where $q > 2$.

By the Hölder inequality, we obtain

$$\begin{aligned} |\xi|^4 I_{1,0} &\le \int_{\mathbf{R}^6} |\xi|^4 \left(1 + t|\eta|^2 \left(|\eta|^2 + |\xi|^2\right)\right)^{-2} |\hat{\varphi}(z)| |\hat{\varphi}(w)| \, d\eta \\ &\le t^{-2} \int_{\mathbf{R}^6} |\eta|^{-4} |\hat{\varphi}(z)| |\hat{\varphi}(w)| \, d\eta \\ &\le t^{-2} \left(\|\hat{\varphi}\|_{\mathbf{L}^2}^2 + \|\hat{\varphi}\|_{\mathbf{L}^q}^2 \right) \end{aligned}$$

if $q > 6$. In the same way

$$
\begin{aligned}
|\xi|^4 I_{1,1} &\le \int_{\mathbf{R}^6} |\xi|^4 |\eta| \left(1 + t|\eta|^2 \left(|\eta|^2 + |\xi|^2\right)\right)^{-2} \sum_{|\alpha|+|\beta|=1} |\partial^\alpha \hat{\varphi}(z)| \left|\partial^\beta \hat{\varphi}(w)\right| d\eta \\
&\le t^{-2} \int |\eta|^{-3} \sum_{|\alpha|+|\beta|=1} |\partial^\alpha \hat{\varphi}(z)| \left|\partial^\beta \hat{\varphi}(w)\right| d\eta \\
&\le Ct^{-2} \left(\|\hat{\varphi}\|_{\mathbf{H}^1} \|\hat{\varphi}\|_{\mathbf{L}^2} + \sum_{|\alpha|=1} \|\partial^\alpha \hat{\varphi}\|_{\mathbf{L}^6} \|\hat{\varphi}\|_{\mathbf{L}^q} \right) \\
&\le Ct^{-2} \left(\|\hat{\varphi}\|_{\mathbf{H}^1} \|\hat{\varphi}\|_{\mathbf{L}^2} + \|\hat{\varphi}\|_{\dot{\mathbf{H}}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} \right)
\end{aligned}
$$

if $q > 3$ and

$$
\begin{aligned}
|\xi|^4 I_{1,2} &\le \int_{\mathbf{R}^6} |\xi|^4 |\eta|^2 \left(1 + t|\eta|^2 \left(|\eta|^2 + |\xi|^2\right)\right)^{-2} \sum_{|\alpha|+|\beta|=2} |\partial^\alpha \hat{\varphi}(z)| \left|\partial^\beta \hat{\varphi}(w)\right| d\eta \\
&\le t^{-2} \int |\eta|^{-2} \sum_{|\alpha|+|\beta|=2} |\partial^\alpha \hat{\varphi}(z)| \left|\partial^\beta \hat{\varphi}(w)\right| d\eta \\
&\le Ct^{-2} \left(\|\hat{\varphi}\|_{\mathbf{H}^2} \|\hat{\varphi}\|_{\mathbf{H}^1} + \sum_{|\alpha|=2} \|\partial^\alpha \hat{\varphi}\|_{\mathbf{L}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} + \sum_{|\alpha|=1} \|\partial^\alpha \hat{\varphi}\|_{\mathbf{L}^6}^2 \right) \\
&\le Ct^{-2} \left(\|\hat{\varphi}\|_{\mathbf{H}^2} \|\hat{\varphi}\|_{\mathbf{H}^1} + \|\hat{\varphi}\|_{\dot{\mathbf{H}}^3} \left(\|\hat{\varphi}\|_{\dot{\mathbf{H}}^3} + \|\hat{\varphi}\|_{\mathbf{L}^q} \right) \right),
\end{aligned}
$$

if $q > 6$. Thus we find

$$
\left\| \langle \cdot \rangle^4 I_{1,j} \right\|_{\mathbf{L}^\infty} \le C \max \left(t^{-\frac{3(q-2)}{2q}}, t^{-\frac{3(q-1)}{2q} + \gamma_0}, t^{-\frac{3}{2} + 2\gamma_0} \right) \|u\|_{\mathbf{Y}}^2
$$

for any $t \ge 1$. Therefore, the first estimate of the lemma follows.

To obtain the second estimate, we compute

$$
\begin{aligned}
\nabla_\xi \mathcal{F} \mathcal{U}(-t) \overline{u}^2 &= (2\pi)^{-3} \nabla_\xi \int_{\mathbf{R}^6} e^{\frac{1}{4} itS(\eta,\xi)} \overline{\hat{\varphi}(z) \hat{\varphi}(w)} d\eta \\
&= \frac{(2\pi)^{-3} it}{4} \int_{\mathbf{R}^6} e^{\frac{1}{4} itS(\eta,\xi)} N(\eta, \xi) \overline{\hat{\varphi}(z) \hat{\varphi}(w)} d\eta \\
&+ (2\pi)^{-3} \int_{\mathbf{R}^6} e^{\frac{1}{4} itS(\eta,\xi)} \nabla_\xi \left(\overline{\hat{\varphi}(z) \hat{\varphi}(w)} \right) d\eta \\
&= J_1 + J_2,
\end{aligned}
$$

where $N(\eta, \xi) = 4(\eta \cdot \xi)\eta + \frac{9}{2} |\xi|^2 \xi + 2|\eta|^2 \xi$. In the same way as in the proof of (3.2.1), we have

$$
|J_1| \le Ct \sum_{j=0}^{3} J_{1,j}, \quad |J_2| \le C \sum_{j=1}^{3} J_{2,j}
$$

where

$$J_{1,j}=\int_{\mathbf{R}^6}\left(\left(|\xi|^3+|\xi|\,|\eta|^2\right)|\eta|^j\right)(W(\eta,\xi))^3\sum_{|\alpha|+|\beta|=j}|\partial^\alpha\hat{\varphi}(z)|\left|\partial^\beta\hat{\varphi}(w)\right|d\eta,$$

$j=0,\,1,\,2,\,3$ and

$$J_{2,j}=\int_{\mathbf{R}^6}|\eta|^{j-1}(W(\eta,\xi))^2\sum_{|\alpha|+|\beta|=j}|\partial^\alpha\hat{\varphi}(z)|\left|\partial^\beta\hat{\varphi}(w)\right|d\eta.$$

In order to estimate $J_{1,j}\,(j=1,2,3)$, we apply the Hölder and the Sobolev inequalities to obtain for $0\le a,b\le 2$

$$\begin{aligned}
J_{1,0}&=\int_{\mathbf{R}^6}\left(|\xi|^3+|\xi|\,|\eta|^2\right)\left(1+t\,|\eta|^2\left(|\eta|^2+|\xi|^2\right)\right)^{-3}|\hat{\varphi}(z)|\,|\hat{\varphi}(w)|\,d\eta\\
&\le C\,|\xi|^3\int_{\mathbf{R}^6}\left(1+t\,|\eta|^{4-a}\,|\xi|^a\right)^{-3}|\hat{\varphi}(z)|\,|\hat{\varphi}(w)|\,d\eta\\
&+C\,|\xi|\int_{\mathbf{R}^6}|\eta|^2\left(1+t\,|\eta|^{4-b}\,|\xi|^b\right)^{-3}|\hat{\varphi}(z)|\,|\hat{\varphi}(w)|\,d\eta\\
&\le C\,|\xi|^3\left(\int_{\mathbf{R}^6}\left(1+t\,|\eta|^{4-a}\,|\xi|^a\right)^{-3\left(\frac{q}{q-2}\right)}d\eta\right)^{\frac{q-2}{q}}\|\hat{\varphi}\|^2_{\mathbf{L}^q}\\
&+C\,|\xi|\left(\int_{\mathbf{R}^6}|\eta|^{2\left(\frac{q}{q-2}\right)}\left(1+t\,|\eta|^{4-b}\,|\xi|^b\right)^{-3\left(\frac{q}{q-2}\right)}d\eta\right)^{\frac{q-2}{q}}\|\hat{\varphi}\|^2_{\mathbf{L}^q}\,.
\end{aligned}$$

By changing of variable $t^{\frac{1}{4-a}}\,|\xi|^{\frac{a}{4-a}}\,\eta=\widetilde{\eta}$, we get

$$\begin{aligned}
&\int_{\mathbf{R}^6}\left(1+t\,|\eta|^{4-a}\,|\xi|^a\right)^{-3\left(\frac{q}{q-2}\right)}d\eta\\
&=t^{-\frac{6}{4-a}}\,|\xi|^{-\frac{6a}{4-a}}\int_{\mathbf{R}^6}\left(1+|\widetilde{\eta}|^{4-a}\right)^{-3\left(\frac{q}{q-2}\right)}d\widetilde{\eta}\\
&\le Ct^{-\frac{6}{4-a}}\,|\xi|^{-\frac{6a}{4-a}}
\end{aligned}$$

if $3\left(\frac{q}{q-2}\right)(4-a)>6$. In the same way,

$$\begin{aligned}
&\int_{\mathbf{R}^6}|\eta|^{2\left(\frac{q}{q-2}\right)}\left(1+t\,|\eta|^{4-b}\,|\xi|^b\right)^{-3\left(\frac{q}{q-2}\right)}d\eta\\
&=t^{-\frac{6}{4-b}}\,|\xi|^{-\frac{6a}{4-b}}\left(t^{-\frac{1}{4-b}}\,|\xi|^{-\frac{b}{4-b}}\right)^{2\left(\frac{q}{q-2}\right)}\int_{\mathbf{R}^6}|\widetilde{\eta}|^{2\left(\frac{q}{q-2}\right)}\left(1+|\widetilde{\eta}|^{4-b}\right)^{-3\left(\frac{q}{q-2}\right)}d\widetilde{\eta}\\
&\le Ct^{-\frac{6}{4-b}}\,|\xi|^{-\frac{6b}{4-b}}\left(t^{-\frac{1}{4-b}}\,|\xi|^{-\frac{b}{4-b}}\right)^{2\left(\frac{q}{q-2}\right)}
\end{aligned}$$

if $-2\left(\frac{q}{q-2}\right)+3\left(\frac{q}{q-2}\right)(4-b)>6$. Therefore

$$\begin{aligned} J_{1,0} &\leq Ct^{-\frac{6}{4-a}\left(\frac{q-2}{q}\right)}\left|\xi\right|^{3-\frac{6a}{4-a}\left(\frac{q-2}{q}\right)}\left\|\hat{\varphi}\right\|_{\mathbf{L}^q}^2 \\ &\quad + Ct^{-\frac{6}{4-b}\left(\frac{q-2}{q}\right)-\frac{2}{4-b}}\left|\xi\right|^{1-\frac{6b}{4-b}\left(\frac{q-2}{q}\right)-\frac{2b}{4-b}}\left\|\hat{\varphi}\right\|_{\mathbf{L}^q}^2. \end{aligned}$$

We put $a=\frac{4q}{3q-4}$, $b=\frac{4q}{3(3q-4)}=\frac{1}{3}a$, then $3-\frac{6a}{4-a}\left(\frac{q-2}{q}\right)=0$, $1-\frac{6b}{4-b}\left(\frac{q-2}{q}\right)-\frac{2b}{4-b}=0$. We also find that $3\left(\frac{q}{q-2}\right)(4-a)>6$ if $q>4$ and $-2\left(\frac{q}{q-2}\right)+3\left(\frac{q}{q-2}\right)(4-b)>6$ if $q>4$. Hence

$$\begin{aligned} J_{1,0} &\leq Ct^{-\frac{6}{4-a}\left(\frac{q-2}{q}\right)}\left\|\hat{\varphi}\right\|_{\mathbf{L}^q}^2+Ct^{-\frac{6}{4-b}\left(\frac{q-2}{q}\right)-\frac{2}{4-b}}\left\|\hat{\varphi}\right\|_{\mathbf{L}^q}^2 \\ &\leq Ct^{-\frac{3(3q-4)}{4q}}\left\|\hat{\varphi}\right\|_{\mathbf{L}^q}^2 \end{aligned} \tag{3.2.3}$$

for $q>4$.

We also obtain

$$\begin{aligned} J_{1,j} &\leq C\left|\xi\right|^3\int_{\mathbf{R}^6}\left|\eta\right|^j\left(1+t\left|\eta\right|^{4-a}\left|\xi\right|^a\right)^{-3}\sum_{|\alpha|+|\beta|=j}\left|\partial^\alpha\hat{\varphi}(z)\right|\left|\partial^\beta\hat{\varphi}(w)\right|d\eta \\ &\quad + C\left|\xi\right|\int_{\mathbf{R}^6}\left|\eta\right|^{2+j}\left(1+t\left|\eta\right|^{4-b}\left|\xi\right|^b\right)^{-3}\sum_{|\alpha|+|\beta|=j}\left|\partial^\alpha\hat{\varphi}(z)\right|\left|\partial^\beta\hat{\varphi}(w)\right|d\eta. \end{aligned}$$

We take $a=\frac{4q}{3q-2}$, $b=\frac{4q}{3(3q-2)}$, then in the same way as in the proof of (3.2.3), we find

$$J_{1,1}\leq Ct^{-\frac{3(3q-2)}{4q}}\sum_{|\alpha|=1}\left\|\partial^\alpha\hat{\varphi}\right\|_{\mathbf{L}^6}\left\|\hat{\varphi}\right\|_{\mathbf{L}^q}\leq Ct^{-\frac{3(3q-2)}{4q}}\left\|\hat{\varphi}\right\|_{\dot{\mathbf{H}}^3}\left\|\hat{\varphi}\right\|_{\mathbf{L}^q}$$

for $q>2$. We obtain

$$\begin{aligned} J_{1,2} &\leq C\left|\xi\right|^3\int_{\mathbf{R}^6}\left|\eta\right|^2\left(1+t\left|\eta\right|^{4-a}\left|\xi\right|^a\right)^{-3}\sum_{|\alpha|=2}\left|\partial^\alpha\hat{\varphi}(z)\right|\left|\hat{\varphi}(w)\right|d\eta \\ &\quad + C\left|\xi\right|\int_{\mathbf{R}^6}\left|\eta\right|^4\left(1+t\left|\eta\right|^{4-b}\left|\xi\right|^b\right)^{-3}\sum_{|\alpha|=2}\left|\partial^\alpha\hat{\varphi}(z)\right|\left|\hat{\varphi}(w)\right|d\eta \\ &\quad + C\left|\xi\right|^3\int_{\mathbf{R}^6}\left|\eta\right|^2\left(1+t\left|\eta\right|^{4-c}\left|\xi\right|^c\right)^{-3}\sum_{|\alpha|=1}\left|\partial^\alpha\hat{\varphi}(z)\right|\sum_{|\beta|=1}\left|\partial^\beta\hat{\varphi}(w)\right|d\eta \\ &\quad + C\left|\xi\right|\int_{\mathbf{R}^6}\left|\eta\right|^4\left(1+t\left|\eta\right|^{4-d}\left|\xi\right|^d\right)^{-3}\sum_{|\alpha|=1}\left|\partial^\alpha\hat{\varphi}(z)\right|\sum_{|\beta|=1}\left|\partial^\beta\hat{\varphi}(w)\right|d\eta. \end{aligned}$$

We put $a = \frac{4q}{3q-2}$, $b = \frac{a}{3}$, $c = \frac{4}{3}$, $d = \frac{c}{3}$. Then in the same way as in the proof of (3.2.3), we find

$$J_{1,2} \le Ct^{-\frac{3(3q-2)}{4q}} \sum_{|\alpha|=2} \|\partial^\alpha \hat{\varphi}\|_{\mathbf{L}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} + Ct^{-\frac{9}{4}} \sum_{|\alpha|=1} \|\partial^\alpha \hat{\varphi}\|^2_{\mathbf{L}^6}$$
$$\le Ct^{-\frac{3(3q-2)}{4q}} \|\hat{\varphi}\|_{\dot{\mathbf{H}}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} + Ct^{-\frac{9}{4}} \|\hat{\varphi}\|^2_{\dot{\mathbf{H}}^3} \tag{3.2.4}$$

for $q > 2$. Similarly,

$$J_{1,3} \le Ct^{-\frac{3(3q-2)}{4q}} \sum_{|\alpha|=3} \|\partial^\alpha \hat{\varphi}\|_{\mathbf{L}^2} \|\hat{\varphi}\|_{\mathbf{L}^q} + Ct^{-\frac{9}{4}} \sum_{|\alpha|=2,|\beta|=1} \|\partial^\alpha \hat{\varphi}\|_{\mathbf{L}^3} \left\|\partial^\beta \hat{\varphi}\right\|_{\mathbf{L}^6}$$
$$\le Ct^{-\frac{3(3q-2)}{4q}} \|\hat{\varphi}\|_{\dot{\mathbf{H}}^3} \|\hat{\varphi}\|_{\mathbf{L}^q} + Ct^{-\frac{9}{4}} \|\hat{\varphi}\|^2_{\dot{\mathbf{H}}^3} \tag{3.2.5}$$

for $q > 2$. By (3.2.3)-(3.2.5)

$$\|J_1\|_{\mathbf{L}^\infty} \le C \max\left(t^{-\frac{3(3q-4)}{4q}}, t^{-\frac{3(3q-2)}{4q}+\gamma_0}, t^{-\frac{9}{4}+2\gamma_0}\right) \|u\|^2_{\mathbf{Y}} \tag{3.2.6}$$

for $t \ge 1$. In the same way as in the proof of (3.2.6), we obtain

$$\|J_2\|_{\mathbf{L}^\infty} \le C \max\left(t^{-\frac{3(3q-4)}{4q}}, t^{-\frac{3(3q-2)}{4q}+\gamma_0}, t^{-\frac{9}{4}+2\gamma_0}\right) \|u\|^2_{\mathbf{Y}}. \tag{3.2.7}$$

Therefore, by (3.2.6) and (3.2.7), Lemma 3.2.7 is proved. □

In order to estimate $\sum_{|\alpha|=3} \|\mathcal{J}^\alpha u\|_{\dot{\mathbf{H}}^1}$ we need Lemma 3.2.8 - Lemma 3.2.10 below.

Lemma 3.2.8. *Let*

$$\kappa_{1,j} = \int_{\mathbf{R}^6} |\xi|^6 (W(\eta,\xi))^3 \sum_{a+b=j} |(\eta\cdot\nabla)^a \hat{\varphi}(z)| \left|(\eta\cdot\nabla)^b \hat{\varphi}(w)\right| d\eta,$$

where $j = 0,\ 1,\ 2,\ 3,$

$$W(\eta,\xi) = \left(1 + t|\eta|^2\left(|\eta|^2 + |\xi|^2\right)\right)^{-1},$$

$\hat{\varphi} = \mathcal{F}\mathcal{U}(-t)u$, $z = \eta - \frac{\xi}{2}$ *and* $w = -\left(\eta + \frac{\xi}{2}\right)$. *Then we have*

$$\sum_{j=0}^{3}\left(\sup_{|\xi|\le 1}\left||\xi|^2 \kappa_{1,j}\right| + \sup_{1\le|\xi|}\left||\xi|^{3+\delta}\kappa_{1,j}\right|\right) \le Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}} + Ct^{-2}\|u\|_{\widetilde{\mathbf{Y}}}(D+E)$$

for any $t \ge 1$ *where* $q > 6$, $\gamma_1 = \frac{6}{q}$, $0 < \delta \le 1 - \gamma_1$, $0 < \gamma_0 < \frac{1}{4}$,

$$\|v\|_{\widetilde{\mathbf{Y}}} = \sum_{0\le|\alpha|\le 2} \|\mathcal{J}^\alpha v\|_{\mathbf{H}^{12-3|\alpha|}} + \sum_{j=0}^{1} \langle t\rangle^{-\gamma_j} \sum_{|\alpha|=3} \|\mathcal{J}^\alpha v\|_{\dot{\mathbf{H}}^j} + \sum_{1\le|\alpha|\le 3} \|Q^\alpha v\|_{\mathbf{H}^{12-4|\alpha|}}$$
$$+ \sum_{1\le|\alpha|\le 2,|\beta|=1} \left\|\mathcal{J}^\alpha Q^\beta v\right\|_{\mathbf{H}^2} + \left\|\langle\cdot\rangle^3 \mathcal{F}\mathcal{U}(-t)v\right\|_{\mathbf{L}^q},$$

$$Q = (Q_1, \cdots, Q_{17}) = \left(1, \mathcal{P}, (\Omega_{j,k})_{j,k=1,\cdots,6,\ j\geq k}\right),$$

$$D = \sup_{|\xi|\leq 1} \left| |\xi|^2 \Delta \mathcal{F} \mathcal{U}(-t) \overline{u}^2 \right|$$

and

$$E = \sup_{1\leq |\xi|} \left| |\xi|^{3+\delta} \Delta \mathcal{F} \mathcal{U}(-t) \overline{u}^2 \right|.$$

Proof. We make a change of variable $\eta = t^{-\frac{1}{2}} |\xi|^{-1} \widetilde{\eta}$ to obtain

$$\begin{aligned}
\left\| W(\cdot,\xi)^3 \right\|_{\mathbf{L}^\alpha_\eta}^\alpha &\leq C \int \left(\left(1 + t|\xi|^2 |\eta|^2\right)^{-3} \right)^\alpha d\eta \\
&\leq Ct^{-3} |\xi|^{-6} \int \left(1 + |\widetilde{\eta}|^2\right)^{-3\alpha} d\widetilde{\eta} \\
&\leq Ct^{-3} |\xi|^{-6} \qquad (3.2.8)
\end{aligned}$$

if $\alpha > 1$.

First we estimate $\kappa_{1,0}$. Let $|\xi| \leq 1$. Applying the Hölder inequality and (3.2.8), we have

$$\begin{aligned}
\kappa_{1,0} &\leq |\xi|^6 \left\| (W(\cdot,\xi))^3 \right\|_{\mathbf{L}^{\frac{q}{q-2}}} \|\hat{\varphi}\|_{\mathbf{L}^q}^2 \\
&\leq Ct^{-3+\frac{6}{q}} |\xi|^{\frac{12}{q}} \|\hat{\varphi}\|_{\mathbf{L}^q}^2, \qquad (3.2.9)
\end{aligned}$$

where $q > 2$. Let $|\xi| \geq 1$. By

$$|\xi|^6 \leq C\left(|\eta|^3 + |z|^3\right)\left(|\eta|^3 + |w|^3\right) \leq C\langle\eta\rangle^6 \langle z\rangle^3 \langle w\rangle^3$$

we find

$$\begin{aligned}
|\xi|^{3+\delta} \kappa_{1,0} &\leq C|\xi|^{3+\delta} \int_{\mathbf{R}^6} \langle\eta\rangle^6 W(\eta,\xi)^3 \left|\langle z\rangle^3 \hat{\varphi}(z)\right| \left|\langle w\rangle^3 \hat{\varphi}(w)\right| d\eta \\
&\leq C|\xi|^{3+\delta} \int_{\mathbf{R}^6} W(\eta,\xi)^3 \left|\langle z\rangle^3 \hat{\varphi}(z)\right| \left|\langle w\rangle^3 \hat{\varphi}(w)\right| d\eta \\
&+ Ct^{-3} |\xi|^{-3+\delta} \int_{\mathbf{R}^6} \left|\langle z\rangle^3 \hat{\varphi}(z)\right| \left|\langle w\rangle^3 \hat{\varphi}(w)\right| d\eta \\
&\leq C|\xi|^{3+\delta} \left\| W(\cdot,\xi)^3 \right\|_{\mathbf{L}^{\frac{q}{q-2}}} \left\| \langle\cdot\rangle^3 \hat{\varphi} \right\|_{\mathbf{L}^q}^2 + Ct^{-3} |\xi|^{-3+\delta} \|\hat{\varphi}\|_{\mathbf{H}^{0,3}}^2,
\end{aligned}$$

where $q > 2$. Therefore we get

$$|\xi|^{3+\delta} \kappa_{1,0} \leq Ct^{-3+\frac{6}{q}} |\xi|^{-3+\delta+\frac{12}{q}} \left\| \langle\cdot\rangle^3 \hat{\varphi} \right\|_{\mathbf{L}^q}^2 + Ct^{-3} |\xi|^{-3+\delta} \|\hat{\varphi}\|_{\mathbf{H}^{0,3}}^2. \qquad (3.2.10)$$

By (3.2.9) and (3.2.10), we obtain

$$\sup_{|\xi|\leq 1}\left||\xi|^2\kappa_{1,0}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,0}\right|\leq Ct^{-3+\gamma_1}\|u\|_{\widetilde{\mathbf{Y}}}^2. \tag{3.2.11}$$

We next consider the term $\kappa_{1,1}$. We use the estimate

$$|\eta\cdot\nabla\hat{\varphi}(z)|\leq C\frac{|\eta|}{|z|}\left(\left|\widetilde{\mathcal{P}}\hat{\varphi}(z)\right|+\sum_{|\alpha|=1}|\Omega^\alpha\hat{\varphi}(z)|+t|\partial_t\hat{\varphi}(z)|\right)\equiv C\frac{|\eta|}{|z|}\rho(z)$$

(see [13]) to find that

$$\begin{aligned}
\kappa_{1,1}&\leq\int_{\mathbf{R}^6}|\xi|^6W(\eta,\xi)^3\sum_{a+b=1}|(\eta\cdot\nabla)^a\hat{\varphi}(z)|\left|(\eta\cdot\nabla)^b\hat{\varphi}(w)\right|d\eta\\
&\leq|\xi|^6\int_{\mathbf{R}^6}\frac{|\eta|}{|z|}W(\eta,\xi)^3\rho(z)|\hat{\varphi}(w)|d\eta\\
&+|\xi|^6\int_{\mathbf{R}^6}\frac{|\eta|}{|w|}W(\eta,\xi)^3|\hat{\varphi}(z)|\rho(w)d\eta\\
&=\kappa_{1,1,1}+\kappa_{1,1,2}.
\end{aligned}\tag{3.2.12}$$

Let $|\xi|\leq 1$. In order to remove a singularity or gain faster time decay, we apply $|\xi|\leq C(|\eta|+|z|)$. We get

$$\kappa_{1,1,1}\leq C|\xi|^5\left\|\left(\frac{|\eta|^2}{|z|}+|\eta|\right)W(\eta,\xi)^3\right\|_{\mathbf{L}_\eta^{\frac{6q}{5q-6}}}\|\rho\|_{\mathbf{L}^6}\|\hat{\varphi}\|_{\mathbf{L}^q}.$$

We make a change of variable $\eta=t^{-\frac{1}{2}}|\xi|^{-1}\widetilde{\eta}$ to obtain

$$\begin{aligned}
&\left\|\left(\frac{|\eta|^{\beta+1}}{|z|}+|\eta|^\beta\right)W(\eta,\xi)^3\right\|_{\mathbf{L}_\eta^\alpha}^\alpha\\
&\leq C\int\left(\left(\frac{|\eta|^{\beta+1}}{\left|\eta-\frac{\xi}{2}\right|}+|\eta|^\beta\right)\left(1+t|\eta|^2|\xi|^2\right)^{-3}\right)^\alpha d\eta\\
&\leq Ct^{-3-\frac{\alpha\beta}{2}}|\xi|^{-6-\alpha\beta}\int\left(\left(\frac{|\widetilde{\eta}|^{\beta+1}}{\left|\widetilde{\eta}-\frac{1}{2}\xi|\xi|t^{\frac{1}{2}}\right|}+|\widetilde{\eta}|^\beta\right)\left(1+|\widetilde{\eta}|^2\right)^{-3}\right)^\alpha d\widetilde{\eta}.
\end{aligned}$$

By the Hölder inequality,

$$\int\frac{1}{|\widetilde{\eta}-\gamma|^\alpha(1+|\widetilde{\eta}|)^{(5-\beta)\alpha}}d\widetilde{\eta}<\infty$$

if $\gamma \in \mathbf{R}^6$, $1 < \alpha < 6$ and $(6-\beta)\alpha > 6$. Therefore, we find

$$\left\| \left(\frac{|\eta|^{\beta+1}}{|z|} + |\eta|^{\beta} \right) W(\eta,\xi)^3 \right\|_{\mathbf{L}^{\alpha}_{\eta}}^{\alpha} \le Ct^{-3-\frac{\alpha\beta}{2}} |\xi|^{-6-\alpha\beta}. \tag{3.2.13}$$

Thus, we get

$$\kappa_{1,1,1} \le Ct^{-3+\frac{3}{q}} |\xi|^{-1+\frac{6}{q}} \left(\left\| \widetilde{\mathcal{P}}\hat{\varphi} \right\|_{\mathbf{H}^2} + \sum_{|\alpha|=1} \|\Omega^{\alpha}\hat{\varphi}\|_{\mathbf{H}^2} + t\|\partial_t\hat{\varphi}\|_{\mathbf{L}^6} \right) \|\hat{\varphi}\|_{\mathbf{L}^q}, \tag{3.2.14}$$

where $q > \frac{6}{5}$. Let $|\xi| \ge 1$. By applying the estimate $|\xi| \le C(|\eta| + |w|)$, we obtain

$$\begin{aligned} |\xi|^{3+\delta}\kappa_{1,1,1} &\le C|\xi|^{6+\delta} \int_{\mathbf{R}^6} \frac{|\eta|^4}{|z|} (W(\eta,\xi))^3 \rho(z) |\hat{\varphi}(w)| \, d\eta \\ &\quad + C|\xi|^{6+\delta} \int_{\mathbf{R}^6} \frac{|\eta|}{|z|} (W(\eta,\xi))^3 \rho(z) \left| |w|^3 \hat{\varphi}(w) \right| d\eta \\ &= \kappa_{1,1,1,1} + \kappa_{1,1,1,2}. \end{aligned} \tag{3.2.15}$$

By $|\xi|^2 |\eta|^2 W(\xi,\eta) \le Ct^{-1}$ and $|\xi| \le C(|\eta| + |z|)$, we get

$$\begin{aligned} \kappa_{1,1,1,1} &\le Ct^{-3} \int_{\mathbf{R}^6} \frac{|\xi|^{\delta}}{|\eta|^2 |z|} \rho(z) |\hat{\varphi}(w)| \, d\eta \\ &\le Ct^{-3} \int_{\mathbf{R}^6} \left(\frac{1}{|\eta|^{2-\delta}|z|} + \frac{1}{|\eta|^2 |z|^{1-\delta}} \right) \rho(z) |\hat{\varphi}(w)| \, d\eta. \end{aligned}$$

We use the Hölder inequality to find that

$$\kappa_{1,1,1,1} \le Ct^{-3} \left(\|\rho\|_{\mathbf{L}^2} \|\hat{\varphi}\|_{\mathbf{L}^2} + \|\rho\|_{\mathbf{L}^6} \|\hat{\varphi}\|_{\mathbf{L}^q} \right), \tag{3.2.16}$$

where $\frac{6}{2+\delta} < q$.

By $|\xi| \le C(|\eta| + |z|)$, we have

$$\kappa_{1,1,1,2} \le C|\xi|^{5+\delta} \int_{\mathbf{R}^6} \left(\frac{|\eta|^2}{|z|} + |\eta| \right) (W(\eta,\xi))^3 \rho(z) \left| |w|^3 \hat{\varphi}(w) \right| d\eta$$

In the same way as in the proof of (3.2.14), we find

$$\kappa_{1,1,1,2} \le Ct^{-3+\frac{3}{q}} |\xi|^{-1+\frac{6}{q}+\delta} \|\rho\|_{\mathbf{L}^6} \left\| |\cdot|^3 \hat{\varphi} \right\|_{\mathbf{L}^q}, \tag{3.2.17}$$

where $q > \frac{6}{5}$. Applying the Sobolev inequality, we obtain

$$\|\rho\|_{\mathbf{L}^2} + \|\rho\|_{\mathbf{L}^6} \le C \left(\left\| \widetilde{\mathcal{P}}\hat{\varphi} \right\|_{\mathbf{H}^2} + \sum_{|\alpha|=1} \|\Omega^{\alpha}\hat{\varphi}\|_{\mathbf{H}^2} + t\|\partial_t\hat{\varphi}\|_{\mathbf{L}^2} + t\|\partial_t\hat{\varphi}\|_{\mathbf{L}^6} \right).$$

We obtain

$$\left\|\widetilde{\mathcal{P}}\hat{\varphi}\right\|_{\mathbf{H}^2}+\sum_{|\alpha|=1}\left\|\Omega^{\alpha}\hat{\varphi}\right\|_{\mathbf{H}^2}\leq C\left(\left\|\mathcal{U}\left(-t\right)\mathcal{P}u\right\|_{\mathbf{H}^{0,2}}+\sum_{|\alpha|=1}\left\|\mathcal{U}\left(-t\right)\Omega^{\alpha}u\right\|_{\mathbf{H}^{0,2}}\right)$$
$$\leq C\left\|u\right\|_{\widetilde{\mathbf{Y}}}. \tag{3.2.18}$$

By the equation $i\partial_t\hat{\varphi}=\lambda\mathcal{F}\mathcal{U}\left(-t\right)\overline{u}^2$ and Lemma 3.2.7 with $q>6$ and $0<\gamma_0\leq\frac{1}{4}$, we find

$$t\left\|\partial_t\hat{\varphi}\right\|_{\mathbf{L}^2}+t\left\|\partial_t\hat{\varphi}\right\|_{\mathbf{L}^6}\leq Ct\left\|\langle\cdot\rangle^{3+a}\mathcal{F}\mathcal{U}\left(-t\right)\overline{u}^2\right\|_{\mathbf{L}^{\infty}}\leq C\left\|u\right\|_{\mathbf{Y}}^2,$$

where $0<a\leq1$. Combining the estimates (3.2.14)-(3.2.18), we get

$$\sup_{|\xi|\leq1}\left||\xi|^2\kappa_{1,1,1}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,1,1}\right|\leq Ct^{-3+\gamma_1}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^2,$$

where $\gamma_1=\frac{6}{q}$. Similarly

$$\sup_{|\xi|\leq1}\left||\xi|^2\kappa_{1,1,2}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,1,2}\right|\leq Ct^{-3+\gamma_1}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^2.$$

Therefore we find

$$\sup_{|\xi|\leq1}\left||\xi|^2\kappa_{1,1}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,1}\right|\leq Ct^{-3+\gamma_1}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^2. \tag{3.2.19}$$

Next we estimate $\kappa_{1,2}$. We obtain

$$\kappa_{1,2}\leq\int_{\mathbf{R}^6}|\xi|^6\left(W\left(\eta,\xi\right)\right)^3\left|\left(\eta\cdot\nabla\right)\hat{\varphi}\left(z\right)\right|\left|\left(\eta\cdot\nabla\right)\hat{\varphi}\left(w\right)\right|d\eta$$
$$+\int_{\mathbf{R}^6}|\xi|^6\left(W\left(\eta,\xi\right)\right)^3\left(\left|\left(\eta\cdot\nabla\right)^2\hat{\varphi}\left(z\right)\right|\left|\hat{\varphi}\left(w\right)\right|+\left|\hat{\varphi}\left(z\right)\right|\left|\left(\eta\cdot\nabla\right)^2\hat{\varphi}\left(w\right)\right|\right)d\eta$$
$$=\kappa_{1,2,1}+\kappa_{1,2,2}$$

By the estimates

$$\left|\eta\cdot\nabla\hat{\varphi}\left(z\right)\right|\leq C\frac{|\eta|}{|z|}\left(\left|\widetilde{\mathcal{P}}\hat{\varphi}\left(z\right)\right|+\sum_{|\alpha|=1}\left|\Omega^{\alpha}\hat{\varphi}\left(z\right)\right|+t\left|\partial_t\hat{\varphi}\left(z\right)\right|\right)=C\frac{|\eta|}{|z|}\rho\left(z\right)$$

and $|\xi|\leq C\left(|z|+|w|\right)$, we obtain

$$\kappa_{1,2,1}\leq C\int_{\mathbf{R}^6}\frac{|\xi|^6|\eta|^2}{|z||w|}\left(W\left(\eta,\xi\right)\right)^3\rho\left(z\right)\rho\left(w\right)d\eta$$
$$\leq C|\xi|^5\int_{\mathbf{R}^6}\frac{|\eta|^2}{|z|}\left(W\left(\eta,\xi\right)\right)^3\rho\left(z\right)\rho\left(w\right)d\eta$$
$$+C|\xi|^5\int_{\mathbf{R}^6}\frac{|\eta|^2}{|w|}\left(W\left(\eta,\xi\right)\right)^3\rho\left(z\right)\rho\left(w\right)d\eta$$
$$=\kappa_{1,2,1,1}+\kappa_{1,2,1,2}.$$

Let $|\xi| \le 1$. By $|\xi| \le C(|\eta| + |z|)$, the Hölder inequality and (3.2.13), we estimate

$$\begin{aligned}
\kappa_{1,2,1,1} &\le C|\xi|^4 \int_{\mathbf{R}^6} \left(\frac{|\eta|^3}{|z|} + |\eta|^2\right) (W(\eta,\xi))^3 \rho(z)\rho(w)\,d\eta \\
&\le C|\xi|^4 \left\| \left(\frac{|\eta|^3}{|z|} + |\eta|^2\right)(W(\eta,\xi))^3 \right\|_{\mathbf{L}_\eta^{\frac{3+\theta}{2}}} \|\rho\|^2_{\mathbf{L}^{\frac{6+2\theta}{1+\theta}}} \\
&\le Ct^{-3+\frac{2\theta}{3+\theta}} |\xi|^{-2+\frac{4\theta}{3+\theta}} \left(\left\|\widetilde{\mathcal{P}}\hat{\varphi}\right\|_{\mathbf{H}^2} + \sum_{|\alpha|=1} \|\Omega^\alpha \hat{\varphi}\|_{\mathbf{H}^2} + t\|\partial_t \hat{\varphi}\|_{\mathbf{L}^{\frac{6+2\theta}{1+\theta}}} \right)^2,
\end{aligned} \tag{3.2.20}$$

where $0 < \theta \le \frac{3\gamma_1}{2-\gamma_1}$. Let $|\xi| \ge 1$. Using $|\xi| \le \langle z\rangle\langle w\rangle$, we obtain

$$|\xi|^{3+\delta}\kappa_{1,2,1,1} \le C|\xi|^{6+\delta} \int_{\mathbf{R}^6} \frac{|\eta|^2}{|z|} (W(\eta,\xi))^3 \langle z\rangle^2 \rho(z) \langle w\rangle^2 \rho(w)\,d\eta.$$

In the same way as in the proof of (3.2.20), we find

$$\begin{aligned}
|\xi|^{3+\delta}\kappa_{1,2,1,1} &\le Ct^{-3+\frac{4\theta}{3+2\theta}} |\xi|^{-1+\delta+\frac{8\theta}{3+2\theta}} \\
&\quad \times \left(\left\|\widetilde{\mathcal{P}}\hat{\varphi}\right\|_{\mathbf{H}^{2,2}} + \sum_{|\alpha|=1} \|\Omega^\alpha \hat{\varphi}\|_{\mathbf{H}^{2,2}} + t\left\|\langle\cdot\rangle^2 \partial_t \hat{\varphi}\right\|_{\mathbf{L}^{\frac{6+2\theta}{1+\theta}}} \right)^2.
\end{aligned} \tag{3.2.21}$$

We obtain

$$\begin{aligned}
\left\|\widetilde{\mathcal{P}}\hat{\varphi}\right\|_{\mathbf{H}^{2,2}} + \sum_{|\alpha|=1} \|\Omega^\alpha \hat{\varphi}\|_{\mathbf{H}^{2,2}} &\le C\left(\|\mathcal{U}(-t)\mathcal{P}u\|_{\mathbf{H}^{2,2}} + \sum_{|\alpha|=1} \|\mathcal{U}(-t)\Omega^\alpha u\|_{\mathbf{H}^{2,2}} \right) \\
&\le C\|u\|_{\widetilde{\mathbf{Y}}}.
\end{aligned} \tag{3.2.22}$$

Due to Lemma 3.2.7 with $q > 6$ and $0 < \gamma_0 \le \frac{1}{4}$, we get

$$t\left\|\langle\cdot\rangle^2 \partial_t \hat{\varphi}\right\|_{\mathbf{L}^{\frac{6+2\theta}{1+\theta}}} \le Ct\left\|\langle\cdot\rangle^3 \mathcal{F}\mathcal{U}(-t)\overline{u}^2\right\|_{\mathbf{L}^\infty} \le C\|u\|^2_{\mathbf{Y}}. \tag{3.2.23}$$

By (3.2.20)-(3.2.23), we have

$$\sup_{|\xi|\le 1}\left||\xi|^2 \kappa_{1,2,1,1}\right| + \sup_{1\le|\xi|}\left||\xi|^{3+\delta}\kappa_{1,2,1,1}\right| \le Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}}.$$

Similarly, we get

$$\sup_{|\xi|\le 1}\left||\xi|^2 \kappa_{1,2,1,2}\right| + \sup_{1\le|\xi|}\left||\xi|^{3+\delta}\kappa_{1,2,1,2}\right| \le Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}}.$$

Therefore

$$\sup_{|\xi|\leq 1}\left||\xi|^2\kappa_{1,2,1}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,2,1}\right|\leq Ct^{-3+\gamma_1}\|u\|_{\widetilde{\mathbf{Y}}}^2. \tag{3.2.24}$$

Next we consider $\kappa_{1,2,2}$ to obtain

$$\begin{aligned}\kappa_{1,2,2}&\leq C\kappa_{1,1}+C\int_{\mathbf{R}^6}\frac{|\xi|^6|\eta|^2}{|z|}(W(\eta,\xi))^3\varsigma(z)|\hat{\varphi}(w)|\,d\eta\\&+C\int_{\mathbf{R}^6}\frac{|\xi|^6|\eta|^2}{|w|}(W(\eta,\xi))^3|\hat{\varphi}(z)|\varsigma(w)\,d\eta\\&=C\kappa_{1,1}+\kappa_{1,2,2,1}+\kappa_{1,2,2,2},\end{aligned}$$

where

$$\varsigma(z)=\left|\widetilde{\mathcal{P}}\nabla\hat{\varphi}(z)\right|+\sum_{|\alpha|=1}|\Omega^\alpha\nabla\hat{\varphi}(z)|+t|\partial_t\nabla\hat{\varphi}(z)|.$$

Let $|\xi|\leq 1$. By $|\xi|\leq C(|\eta|+|z|)$, the Hölder inequality and (3.2.13), we find

$$\begin{aligned}\kappa_{1,2,2,1}&\leq C|\xi|^5\int_{\mathbf{R}^6}\left(\frac{|\eta|^3}{|z|}+|\eta|^2\right)(W(\eta,\xi))^3\varsigma(z)|\hat{\varphi}(w)|\,d\eta\\&\leq C|\xi|^5\left\|\left(\frac{|\eta|^3}{|z|}+|\eta|^2\right)(W(\eta,\xi))^3\right\|_{\mathbf{L}_\eta^{\frac{3q}{2q-3}}}\|\varsigma\|_{\mathbf{L}^3}\|\hat{\varphi}\|_{\mathbf{L}^q}\\&\leq Ct^{-3+\frac{3}{q}}|\xi|^{-1+\frac{6}{q}}\|\varsigma\|_{\mathbf{L}^3}\|\hat{\varphi}\|_{\mathbf{L}^q},\end{aligned} \tag{3.2.25}$$

where $q>\frac{3}{2}$. Let $|\xi|\geq 1$. By $|\xi|\leq C(|\eta|+|w|)$, we have

$$\begin{aligned}|\xi|^{3+\delta}\kappa_{1,2,2,1}&\leq C|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^5}{|z|}(W(\eta,\xi))^3\varsigma(z)|\hat{\varphi}(w)|\,d\eta\\&+C|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^2}{|z|}(W(\eta,\xi))^3\varsigma(z)\left||w|^3\hat{\varphi}(w)\right|d\eta.\end{aligned}$$

In the same way as in the proofs of (3.2.16) and (3.2.25), we obtain

$$\begin{aligned}&|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^5}{|z|}(W(\eta,\xi))^3\varsigma(z)|\hat{\varphi}(w)|\,d\eta\\&\leq Ct^{-3}\|\varsigma\|_{\mathbf{L}^2}\|\hat{\varphi}\|_{\mathbf{L}^2}+Ct^{-3}\|\varsigma\|_{\mathbf{L}^3}\|\hat{\varphi}\|_{\mathbf{L}^q},\end{aligned}$$

where $\frac{6}{2+\delta}<q$ and

$$\begin{aligned}&|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^2}{|z|}(W(\eta,\xi))^3\varsigma(z)\left||w|^3\hat{\varphi}(w)\right|d\eta\\&\leq Ct^{-3+\frac{3}{q}}|\xi|^{-1+\delta+\frac{6}{q}}\|\varsigma\|_{\mathbf{L}^3}\left\||\cdot|^3\hat{\varphi}\right\|_{\mathbf{L}^q},\end{aligned}$$

where $q > \frac{3}{2}$ and $0 < \delta \le 1 - \gamma_1$. Therefore, we find

$$|\xi|^{3+\delta} \kappa_{1,2,2,1} \le Ct^{-3} \left(\left\| \widetilde{\mathcal{P}} \hat{\varphi} \right\|_{\mathbf{H}^1} + \sum_{|\alpha|=1} \|\Omega^\alpha \hat{\varphi}\|_{\mathbf{H}^1} + t \|\nabla \partial_t \hat{\varphi}\|_{\mathbf{L}^2} \right) \|\hat{\varphi}\|_{\mathbf{L}^2}$$
$$+ C \left(t^{-3} + t^{-3+\frac{3}{q}} |\xi|^{-1+\delta+\frac{6}{q}} \right)$$
$$\times \left(\left\| \widetilde{\mathcal{P}} \hat{\varphi} \right\|_{\mathbf{H}^2} + \sum_{|\alpha|=1} \|\Omega^\alpha \hat{\varphi}\|_{\mathbf{H}^2} + t \|\nabla \partial_t \hat{\varphi}\|_{\mathbf{L}^3} \right) \left\| \langle \cdot \rangle^3 \hat{\varphi} \right\|_{\mathbf{L}^q}. \tag{3.2.26}$$

Here we have used the Sobolev inequality. By the Hölder inequality, we get

$$\begin{aligned} t \|\nabla \partial_t \hat{\varphi}\|_{\mathbf{L}^2} &\le Ct \sum_{|\alpha|=1} \left\| \mathcal{J}^\alpha \overline{u}^2 \right\|_{\mathbf{L}^2} \\ &\le Ct \sum_{|\alpha|=1} \left(\left\| x^\alpha \overline{u}^2 \right\|_{\mathbf{L}^2} + \left\| it \Delta \partial_x^\alpha \overline{u}^2 \right\|_{\mathbf{L}^2} \right) \\ &\le Ct \sum_{|\alpha|=1} \left\| x^\alpha \overline{u}^2 \right\|_{\mathbf{L}^2} + Ct^2 \left\| \overline{u} \overline{(-\Delta)^{\frac{3}{2}} u} \right\|_{\mathbf{L}^2} \\ &\le Ct \left\| |x| u \right\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^2} + Ct^2 \left\| (-\Delta)^{\frac{3}{2}} u \right\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^2}. \end{aligned}$$

Applying Corollary 3.2.5, we obtain

$$\begin{aligned} t^2 \left\| (-\Delta)^{\frac{3}{2}} u \right\|_{\mathbf{L}^\infty} &\le Ct^2 \left\| (-\Delta)^{\theta} u \right\|_{\dot{\mathbf{H}}^{3-2\theta}_\infty} \\ &\le C \langle t \rangle^{-\frac{1}{4} + \frac{3}{2r} + \frac{\theta}{2} + (\gamma_0 + 4\theta(\gamma_1 - \gamma_0))} \|u\|_{\widetilde{\mathbf{Y}}} \\ &\le C \|u\|_{\widetilde{\mathbf{Y}}}, \end{aligned}$$

where $0 < \theta$ is small, $0 < \gamma_0 < \frac{1}{4}$ and $\frac{6}{1-4\gamma_0} < r < \infty$. By Lemma 3.2.6, we find

$$t \left\| |x| u \right\|_{\mathbf{L}^\infty} \le Ct \langle t \rangle^{-\frac{5}{4} + \frac{3}{2r} + \gamma_0} \|u\|_{\widetilde{\mathbf{Y}}} \le C \|u\|_{\widetilde{\mathbf{Y}}},$$

where $0 < \gamma_0 < \frac{1}{4}$ and $\frac{6}{1-4\gamma_0} \le r < \infty$. Therefore, we obtain

$$t \|\nabla \partial_t \hat{\varphi}\|_{\mathbf{L}^2} \le C \|u\|_{\widetilde{\mathbf{Y}}}^2.$$

Using the Hölder inequality and Lemma 3.2.7 with $q > \frac{12}{5}$ and $0 < \gamma_0 < \frac{5}{8}$, we find

$$\begin{aligned} t \|\nabla \partial_t \hat{\varphi}\|_{\mathbf{L}^3} &\le t \|\nabla \partial_t \hat{\varphi}\|_{\mathbf{L}^2}^{\frac{2}{3}} \|\nabla \partial_t \hat{\varphi}\|_{\mathbf{L}^\infty}^{\frac{1}{3}} \\ &\le Ct \left(\sum_{|\alpha|=1} \left\| \mathcal{J}^\alpha \overline{u}^2 \right\|_{\mathbf{L}^2} \right)^{\frac{2}{3}} \left\| \nabla \mathcal{F} \mathcal{U}(-t) \overline{u}^2 \right\|_{\mathbf{L}^\infty}^{\frac{1}{3}} \le C \|u\|_{\widetilde{\mathbf{Y}}}^2. \end{aligned} \tag{3.2.27}$$

By (3.2.25)-(3.2.27), we obtain

$$\sup_{|\xi|\leq 1}\left||\xi|^{2}\kappa_{1,2,2,1}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,2,2,1}\right|\leq Ct^{-3+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}.$$

Similarly, we have

$$\sup_{|\xi|\leq 1}\left||\xi|^{2}\kappa_{1,2,2,2}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,2,2,2}\right|\leq Ct^{-3+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}.$$

Therefore, we get

$$\sup_{|\xi|\leq 1}\left||\xi|^{2}\kappa_{1,2,2}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,2,2}\right|\leq Ct^{-3+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}. \tag{3.2.28}$$

Due to (3.2.24) and (3.2.28), we have

$$\sup_{|\xi|\leq 1}\left||\xi|^{2}\kappa_{1,2}\right|+\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,2}\right|\leq Ct^{-3+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}. \tag{3.2.29}$$

Next we estimate $\kappa_{1,3}$. We find that $\kappa_{1,3}$ is estimated from above as

$$\begin{aligned}
\kappa_{1,3} \leq & \int_{\mathbf{R}^{6}}|\xi|^{6}\left(W\left(\eta,\xi\right)\right)^{3} \\
& \times\left(\left|\eta\cdot\left(\left(\eta\cdot\nabla\right)\nabla\hat{\varphi}\left(z\right)\right)\right|\left|\left(\eta\cdot\nabla\right)\hat{\varphi}\left(w\right)\right|+\left|\left(\eta\cdot\nabla\right)\hat{\varphi}\left(z\right)\right|\left|\eta\cdot\left(\left(\eta\cdot\nabla\right)\nabla\hat{\varphi}\left(w\right)\right)\right|\right)d\eta \\
& +\int_{\mathbf{R}^{6}}|\xi|^{6}|\eta|^{2}\left(W\left(\eta,\xi\right)\right)^{3} \\
& \times\left(\left|\left(\eta\cdot\nabla\right)\Delta\hat{\varphi}\left(z\right)\right|\left|\hat{\varphi}\left(w\right)\right|+\left|\hat{\varphi}\left(z\right)\right|\left|\left(\eta\cdot\nabla\right)\Delta\hat{\varphi}\left(w\right)\right|\right)d\eta+C\kappa_{1,1}+C\kappa_{1,2} \\
& =\kappa_{1,3,1}+\kappa_{1,3,2}+\kappa_{1,3,3}+\kappa_{1,3,4}+C\kappa_{1,1}+C\kappa_{1,2}.
\end{aligned}$$

By $|\xi|\leq|z|+|w|$, we obtain

$$\begin{aligned}
\kappa_{1,3,1} & \leq C\int_{\mathbf{R}^{6}}\frac{|\xi|^{6}|\eta|^{3}}{|z||w|}\left(W\left(\eta,\xi\right)\right)^{3}\varsigma\left(z\right)\rho\left(w\right)d\eta \\
& \leq C\int_{\mathbf{R}^{6}}\frac{|\xi|^{5}|\eta|^{3}}{|z|}\left(W\left(\eta,\xi\right)\right)^{3}\varsigma\left(z\right)\rho\left(w\right)d\eta \\
& +C\int_{\mathbf{R}^{6}}\frac{|\xi|^{5}|\eta|^{3}}{|w|}\left(W\left(\eta,\xi\right)\right)^{3}\varsigma\left(z\right)\rho\left(w\right)d\eta \\
& =\kappa_{1,3,1,1}+\kappa_{1,3,1,2},
\end{aligned}$$

where we recall that

$$\rho\left(z\right)=\left|\widetilde{\mathcal{P}}\hat{\varphi}\left(z\right)\right|+\sum_{|\alpha|=1}\left|\Omega^{\alpha}\hat{\varphi}\left(z\right)\right|+t\left|\partial_{t}\hat{\varphi}\left(z\right)\right|$$

and

$$\varsigma(z)=\left|\widetilde{\mathcal{P}}\nabla\hat{\varphi}(z)\right|+\sum_{|\alpha|=1}|\Omega^{\alpha}\nabla\hat{\varphi}(z)|+t|\partial_t\nabla\hat{\varphi}(z)|.$$

Let $|\xi|\le 1$. By $|\xi|\le C(|\eta|+|z|)$ and (3.2.13), we get

$$\begin{aligned}\kappa_{1,3,1,1}&\le C|\xi|^4\int_{\mathbf{R}^6}\left(\frac{|\eta|^4}{|z|}+|\eta|^3\right)(W(\eta,\xi))^3\varsigma(z)\rho(w)\,d\eta\\&\le C|\xi|^4\left\|\left(\frac{|\eta|^4}{|z|}+|\eta|^3\right)(W(\eta,\xi))^3\right\|_{\mathbf{L}_{\eta}^{2+\theta}}\|\varsigma\|_{\mathbf{L}^3}\|\rho\|_{\mathbf{L}^{\frac{6+3\theta}{1+2\theta}}}\\&\le Ct^{-3+\frac{3\theta}{2(2+\theta)}}|\xi|^{-2+\frac{3\theta}{2+\theta}}\|\varsigma\|_{\mathbf{L}^3}\|\rho\|_{\mathbf{L}^{\frac{6+3\theta}{1+2\theta}}},\end{aligned}\tag{3.2.30}$$

where $0<\theta\le\frac{4\gamma_1}{3-2\gamma_1}$. Let $|\xi|\ge 1$. By using $|\xi|\le C(|\eta|+|w|)$, we have

$$\begin{aligned}|\xi|^{3+\delta}\kappa_{1,3,1,1}&\le C|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^5}{|z|}(W(\eta,\xi))^3\varsigma(z)\rho(w)\,d\eta\\&\quad+C|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^3}{|z|}(W(\eta,\xi))^3\varsigma(z)|w|^2\rho(w)\,d\eta.\end{aligned}$$

In the same way as in the proofs of (3.2.16) and (3.2.30), we obtain

$$\begin{aligned}&|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^5}{|z|}(W(\eta,\xi))^3\varsigma(z)\rho(w)\,d\eta\\&\le Ct^{-3}\|\varsigma\|_{\mathbf{L}^2}\|\rho\|_{\mathbf{L}^2}+Ct^{-3}\|\varsigma\|_{\mathbf{L}^3}\|\rho\|_{\mathbf{L}^{\frac{6+3\theta}{1+2\theta}}},\end{aligned}$$

$$\begin{aligned}&|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^3}{|z|}(W(\eta,\xi))^3\varsigma(z)|w|^2\rho(w)\,d\eta\\&\le Ct^{-3+\frac{3\theta}{2(2+\theta)}}|\xi|^{-1+\frac{3\theta}{2+\theta}+\delta}\|\varsigma\|_{\mathbf{L}^3}\left\||\cdot|^2\rho\right\|_{\mathbf{L}^{\frac{6+3\theta}{1+2\theta}}},\end{aligned}$$

where $0<\delta\le 1-\gamma_1$ and $0<\theta\le\frac{2\gamma_1}{3-\gamma_1}$. Therefore, we get

$$\begin{aligned}|\xi|^{3+\delta}\kappa_{1,3,1,1}&\le C\left(t^{-3}+t^{-3+\frac{3\theta}{2(2+\theta)}}|\xi|^{-1+\frac{3\theta}{2+\theta}+\delta}\right)\|\varsigma\|_{\mathbf{L}^3}\left\|\langle\cdot\rangle^2\rho\right\|_{\mathbf{L}^{\frac{6+3\theta}{1+2\theta}}}\\&\quad+Ct^{-3}\|\varsigma\|_{\mathbf{L}^2}\|\rho\|_{\mathbf{L}^2}.\end{aligned}\tag{3.2.31}$$

Due to (3.2.30) and (3.2.31), we have

$$\sup_{|\xi|\le 1}\left||\xi|^2\kappa_{1,3,1,1}\right|+\sup_{1\le|\xi|}\left||\xi|^{3+\delta}\kappa_{1,3,1,1}\right|\le Ct^{-3+\gamma_1}\|u\|_{\tilde{\mathbf{Y}}}^2.$$

Similarly, we find

$$\sup_{|\xi|\leq 1}\left||\xi|^2\kappa_{1,3,1,2}\right| + \sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,3,1,2}\right| \leq Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}}.$$

Therefore, we get

$$\sup_{|\xi|\leq 1}\left||\xi|^2\kappa_{1,3,1}\right| + \sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,3,1}\right| \leq Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}}. \tag{3.2.32}$$

In the same way as in the proof of (3.2.32), we have

$$\sup_{|\xi|\leq 1}\left||\xi|^2\kappa_{1,3,2}\right| + \sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,3,2}\right| \leq Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}}. \tag{3.2.33}$$

Next we estimate

$$\kappa_{1,3,3} \leq C\int_{\mathbf{R}^6}\frac{|\xi|^6|\eta|^3}{|z|}(W(\eta,\xi))^3\sigma_1(z)|\hat{\varphi}(w)|\,d\eta,$$

where

$$\sigma_1(z) = \left|\widetilde{\mathcal{P}}\Delta\hat{\varphi}(z)\right| + \sum_{|\alpha|=1}|\Omega^\alpha\Delta\hat{\varphi}(z)| + t|\partial_t\Delta\hat{\varphi}(z)|.$$

We divide $\sigma_1(z)$ into two parts

$$\begin{aligned}\kappa_{1,3,3} &\leq C\int_{\mathbf{R}^6}\frac{|\xi|^6|\eta|^3}{|z|}(W(\eta,\xi))^3\sigma_2(z)|\hat{\varphi}(w)|\,d\eta\\ &\quad + Ct\int_{\mathbf{R}^6}\frac{|\xi|^6|\eta|^3}{|z|}(W(\eta,\xi))^3|\partial_t\Delta\hat{\varphi}(z)||\hat{\varphi}(w)|\,d\eta\\ &= \kappa_{1,3,3,1} + \kappa_{1,3,3,2}\end{aligned}$$

where

$$\sigma_2(z) = \left|\widetilde{\mathcal{P}}\Delta\hat{\varphi}(z)\right| + \sum_{|\alpha|=1}|\Omega^\alpha\Delta\hat{\varphi}(z)|.$$

Let $|\xi|\leq 1$. By $|\xi|\leq C(|\eta|+|z|)$, the Hölder inequality and (3.2.13), we get

$$\begin{aligned}\kappa_{1,3,3,1} &\leq C|\xi|^5\int_{\mathbf{R}^6}\left(\frac{|\eta|^4}{|z|}+|\eta|^3\right)(W(\eta,\xi))^3\sigma_2(z)|\hat{\varphi}(w)|\,d\eta\\ &\leq C|\xi|^5\left\|\left(\frac{|\eta|^4}{|z|}+|\eta|^3\right)(W(\eta,\xi))^3\right\|_{\mathbf{L}_\eta^{\frac{2q}{q-2}}}\|\sigma_2\|_{\mathbf{L}^2}\|\hat{\varphi}\|_{\mathbf{L}^q}\\ &\leq Ct^{-3+\frac{3}{q}}|\xi|^{-1+\frac{6}{q}}\|\sigma_2\|_{\mathbf{L}^2}\|\hat{\varphi}\|_{\mathbf{L}^q},\end{aligned} \tag{3.2.34}$$

where $q > 2$. By $|\xi|^2 |\eta|^2 W(\xi, \eta) \leq Ct^{-1}$, we have

$$\begin{aligned} \kappa_{1,3,3,2} &\leq Ct|\xi|^5 \int_{\mathbf{R}^6} \left(\frac{|\eta|^4}{|z|} + |\eta|^3 \right) (W(\eta, \xi))^3 |\partial_t \Delta \hat{\varphi}(z)| |\hat{\varphi}(w)| d\eta \\ &\leq Ct^{-2} |\xi|^{-1} \int_{\mathbf{R}^6} \left(\frac{1}{|\eta|^2 |z|} + \frac{1}{|\eta|^3} \right) |\partial_t \Delta \hat{\varphi}(z)| |\hat{\varphi}(w)| d\eta. \end{aligned} \tag{3.2.35}$$

By (3.1.1), we find

$$i\partial_t \Delta \hat{\varphi} = \lambda \Delta \mathcal{F} \mathcal{U}(-t) \overline{u}^2.$$

We define $B = \left\{ \eta \in \mathbf{R}^6; 1 \leq |z| \right\}$. By definitions

$$D = \sup_{|\xi| \leq 1} \left| |\xi|^2 \Delta \mathcal{F} \mathcal{U}(-t) \overline{u}^2 \right|$$

and

$$E = \sup_{1 \leq |\xi|} \left| |\xi|^{3+\delta} \Delta \mathcal{F} \mathcal{U}(-t) \overline{u}^2 \right|,$$

we obtain

$$\begin{aligned} &\int_{\mathbf{R}^6} \frac{1}{|\eta|^2 |z|} |\partial_t \Delta \hat{\varphi}(z)| |\hat{\varphi}(w)| d\eta \\ &\leq E \int_B \frac{1}{|\eta|^2 |z|^{4+\delta}} |\hat{\varphi}(w)| d\eta + D \int_{B^c} \frac{1}{|\eta|^2 |z|^3} |\hat{\varphi}(w)| d\eta. \end{aligned}$$

The Hölder inequality gives us

$$\begin{aligned} &\int_{\mathbf{R}^6} \frac{1}{|\eta|^2 |z|} |\partial_t \Delta \hat{\varphi}(z)| |\hat{\varphi}(w)| d\eta \\ &\leq E \left\| \frac{1}{|\eta|^2 |z|^{4+\delta}} \right\|_{\mathbf{L}_B^{\frac{q}{q-1}}} \|\hat{\varphi}\|_{\mathbf{L}^q} + D \left\| \frac{1}{|\eta|^2 |z|^3} \right\|_{\mathbf{L}_{B^c}^{\frac{q}{q-1}}} \|\hat{\varphi}\|_{\mathbf{L}^q} \\ &\leq C(D+E) \|\hat{\varphi}\|_{\mathbf{L}^q} \end{aligned} \tag{3.2.36}$$

where $q > 6$.

Similarly, we get

$$\int_{\mathbf{R}^6} \frac{1}{|\eta|^3} |\partial_t \Delta \hat{\varphi}(z)| |\hat{\varphi}(w)| d\eta \leq C(D+E) \|\hat{\varphi}\|_{\mathbf{L}^q}. \tag{3.2.37}$$

Collecting (3.2.34)-(3.2.37), we have

$$\sup_{|\xi| \leq 1} \left| |\xi|^2 \kappa_{1,3,3} \right| \leq Ct^{-3+\gamma_1} \|u\|_{\widetilde{\mathbf{Y}}}^2 + Ct^{-2}(D+E) \|u\|_{\widetilde{\mathbf{Y}}}. \tag{3.2.38}$$

Let $|\xi| \geq 1$. By $|\xi| \leq C(|\eta| + |w|)$, we get

$$|\xi|^{3+\delta}\kappa_{1,3,3,1} \leq C|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^6}{|z|}(W(\eta,\xi))^3\sigma_2(z)|\hat{\varphi}(w)|\,d\eta$$
$$+ C|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^3}{|z|}(W(\eta,\xi))^3\sigma_2(z)\left||w|^3\hat{\varphi}(w)\right|d\eta.$$

In the same way of the proofs of (3.2.16) and (3.2.34), we find

$$|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^6}{|z|}(W(\eta,\xi))^3\sigma_2(z)|\hat{\varphi}(w)|\,d\eta$$
$$\leq Ct^{-3}\|\sigma_2\|_{\mathbf{L}^2}\left\|\langle\cdot\rangle^\delta\hat{\varphi}\right\|_{\mathbf{L}^2} + Ct^{-3}\|\sigma_2\|_{\mathbf{L}^2}\left\|\langle\cdot\rangle^\delta\hat{\varphi}\right\|_{\mathbf{L}^q},$$

where $q > 3$ and

$$|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^3}{|z|}(W(\eta,\xi))^3\sigma_2(z)\left||w|^3\hat{\varphi}(w)\right|d\eta$$
$$\leq Ct^{-3+\frac{3}{q}}|\xi|^{-1+\frac{6}{q}+\delta}\|\sigma_2\|_{\mathbf{L}^2}\left\||\cdot|^3\hat{\varphi}\right\|_{\mathbf{L}^q},$$

where $q > 2$. Therefore, we have

$$|\xi|^{3+\delta}\kappa_{1,3,3,1} \leq C\left(t^{-3} + t^{-3+\frac{3}{q}}|\xi|^{-1+\frac{6}{q}+\delta}\right)\|\sigma_2\|_{\mathbf{L}^2}\left(\left\|\langle\cdot\rangle^\delta\hat{\varphi}\right\|_{\mathbf{L}^2} + \left\|\langle\cdot\rangle^3\hat{\varphi}\right\|_{\mathbf{L}^q}\right).$$

We obtain

$$|\xi|^{3+\delta}\kappa_{1,3,3,2} \leq Ct|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^6}{|z|}(W(\eta,\xi))^3|\partial_t\Delta\hat{\varphi}(z)||\hat{\varphi}(w)|\,d\eta$$
$$+ Ct|\xi|^{6+\delta}\int_{\mathbf{R}^6}\frac{|\eta|^3}{|z|}(W(\eta,\xi))^3|\partial_t\Delta\hat{\varphi}(z)|\left||w|^3\hat{\varphi}(w)\right|d\eta.$$

Similarly, we get

$$|\xi|^{3+\delta}\kappa_{1,3,3,2} \leq Ct^{-2}(D+E)\left(\left\|\langle\cdot\rangle^\delta\hat{\varphi}\right\|_{\mathbf{L}^2} + \left\|\langle\cdot\rangle^\delta\hat{\varphi}\right\|_{\mathbf{L}^q}\right)$$
$$+ Ct^{-2}|\xi|^{-1+\delta}(D+E)\left\||\cdot|^3\hat{\varphi}\right\|_{\mathbf{L}^q}.$$

Thus, we find

$$\sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,3,3}\right| \leq Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}} + Ct^{-2}(D+E)\|u\|_{\widetilde{\mathbf{Y}}}. \tag{3.2.39}$$

Analogously, we get

$$\sup_{|\xi|\leq 1}\left||\xi|^2\kappa_{1,3,4}\right| + \sup_{1\leq|\xi|}\left||\xi|^{3+\delta}\kappa_{1,3,4}\right|$$
$$\leq Ct^{-3+\gamma_1}\|u\|^2_{\widetilde{\mathbf{Y}}} + Ct^{-2}(D+E)\|u\|_{\widetilde{\mathbf{Y}}}. \tag{3.2.40}$$

Therefore, we have

$$\sup_{|\xi|\leq 1}\left|\left|\xi\right|^{2}\kappa_{1,3}\right|+\sup_{1\leq|\xi|}\left|\left|\xi\right|^{3+\delta}\kappa_{1,3}\right|\leq Ct^{-3+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}+Ct^{-2}\left\|u\right\|_{\widetilde{\mathbf{Y}}}\left(D+E\right).$$

Lemma 3.2.8 is proved. □

In the same way of the proof of Lemma 3.2.8, the following lemmas are true.

Lemma 3.2.9. *Let*

$$\kappa_{2,1,j}=\int_{\mathbf{R}^{6}}\left(\left|\xi\right|^{2}+\left|\eta\right|^{2}\right)\left(W\left(\eta,\xi\right)\right)^{2}\sum_{a+b=j}\left|\left(\eta\cdot\nabla\right)^{a}\hat{\varphi}\left(z\right)\right|\left|\left(\eta\cdot\nabla\right)^{b}\hat{\varphi}\left(w\right)\right|d\eta,$$

with $j=0,\ 1,\ 2,\ 3,$ *and*

$$\kappa_{2,2,\widetilde{j}}=\int_{\mathbf{R}^{6}}\left(\left|\xi\right|^{2}+\left|\eta\right|^{2}\right)\left(W\left(\eta,\xi\right)\right)^{2}\sum_{a+b=\widetilde{j}}\left|\left(\eta\cdot\nabla\right)^{a}\left(z\cdot\nabla\right)\hat{\varphi}\left(z\right)\right|\left|\left(\eta\cdot\nabla\right)^{b}\hat{\varphi}\left(w\right)\right|d\eta,$$

$$\kappa_{2,3,\widetilde{j}}=\int_{\mathbf{R}^{6}}\left(\left|\xi\right|^{2}+\left|\eta\right|^{2}\right)\left(W\left(\eta,\xi\right)\right)^{2}\sum_{a+b=\widetilde{j}}\left|\left(\eta\cdot\nabla\right)^{a}\hat{\varphi}\left(z\right)\right|\left|\left(\eta\cdot\nabla\right)^{b}\left(w\cdot\nabla\right)\hat{\varphi}\left(w\right)\right|d\eta$$

with $\widetilde{j}=1,\ 2,$ *where*

$$W\left(\eta,\xi\right)=\left(1+t\left|\eta\right|^{2}\left(\left|\eta\right|^{2}+\left|\xi\right|^{2}\right)\right)^{-1}.$$

Then the following estimates

$$\begin{aligned}&\sum_{0\leq j\leq 3}\left(\sup_{|\xi|\leq 1}\left|\left|\xi\right|^{2}\kappa_{2,1,j}\right|+\sup_{1\leq|\xi|}\left|\left|\xi\right|^{3+\delta}\kappa_{2,1,j}\right|\right)\\&\leq Ct^{-2+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}+Ct^{-1}\left\|u\right\|_{\widetilde{\mathbf{Y}}}\left(D+E\right),\\&\sum_{1\leq\widetilde{j}\leq 2,2\leq k\leq 3}\left(\sup_{|\xi|\leq 1}\left|\left|\xi\right|^{2}\kappa_{2,k,\widetilde{j}}\right|+\sup_{1\leq|\xi|}\left|\left|\xi\right|^{3+\delta}\kappa_{2,k,\widetilde{j}}\right|\right)\\&\leq Ct^{-2+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}+Ct^{-1}\left\|u\right\|_{\widetilde{\mathbf{Y}}}\left(D+E\right)\end{aligned}$$

are true for any $t\geq 1.$

Lemma 3.2.10. *Let*

$$\kappa_{3,j}=\int_{\mathbf{R}^{6}}W\left(\eta,\xi\right)\sum_{|\alpha|+|\beta|=2,a+b=j}\left|\left(\eta\cdot\nabla\right)^{a}\partial^{\alpha}\hat{\varphi}\left(z\right)\right|\left|\left(\eta\cdot\nabla\right)^{b}\partial^{\beta}\hat{\varphi}\left(w\right)\right|d\eta,$$

where $j=0,\ 1.$ *Then we have*

$$\sum_{0\leq j\leq 1}\left(\sup_{|\xi|\leq 1}\left|\left|\xi\right|^{2}\kappa_{3,j}\right|+\sup_{1\leq|\xi|}\left|\left|\xi\right|^{3+\delta}\kappa_{3,j}\right|\right)\leq Ct^{-1+\gamma_{1}}\left\|u\right\|_{\widetilde{\mathbf{Y}}}^{2}+C\left\|u\right\|_{\widetilde{\mathbf{Y}}}\left(D+E\right)$$

for any $t\geq 1.$

3.3 Normal form method

In this section, we state the normal form method in the case of the fourth-order Schrödinger equation. We consider the transformation of

$$q_{\alpha,\beta} = \sum_{|\alpha'|+|\beta'|=4} c_{\alpha',\beta'} \left(\partial^{\alpha'} \overline{\mathcal{S}^{\alpha} u}\right) \left(\partial^{\beta'} \overline{\mathcal{S}^{\beta} u}\right),$$

where $\alpha, \beta \in (\mathbf{N} \cup \{0\})^{13}$ and $\mathcal{S} = \left(1, \nabla, (\mathcal{J}_j)_{j=1,\cdots,6}\right)$. For scalar functions ϕ, ψ and a single distribution Ω, we define

$$[\phi, \Omega, \psi] = \int_{\mathbf{R}^6} \int_{\mathbf{R}^6} \overline{\phi\,(x-y)} \Omega\,(y,z)\, \overline{\psi\,(x-z)} dydz.$$

Multiplying both side of equation (3.1.1) by $\mathcal{S}$, we obtain

$$i\partial_t \mathcal{S}^{\alpha} u - \frac{1}{4}\Delta^2 \mathcal{S}^{\alpha} u = \lambda \mathcal{S}^{\alpha} \overline{u}^2. \tag{3.3.1}$$

As in [4], by the equation (3.3.1), we represent

$$q_{\alpha,\beta} = -\mathcal{L}\left([U_\alpha, M, U_\beta]\right) - [F_\alpha, M, U_\beta] - [U_\alpha, M, F_\beta],$$

where $U_\alpha = \mathcal{S}^\alpha u$, $F_\alpha = \lambda \mathcal{S}^\alpha \overline{u}^2$, $\mathcal{L} = i\partial_t - \frac{1}{4}\Delta^2$ and

$$\hat{M}\,(\eta,\zeta) = \frac{4}{(2\pi)^6} \sum_{|\beta|+|\gamma|=4} c_{\beta,\gamma} \frac{\eta^\beta \zeta^\gamma}{|\zeta|^4 + |\eta|^4 + |\zeta+\eta|^4}.$$

In order to estimate $[\cdot, M, \cdot]$, we use the following result.

Theorem 3.3.1. *([5], [22]). Let*

$$\Lambda\,(\phi,\psi)\,(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\cdot(\eta+\zeta)} m\,(\eta,\zeta)\, \hat{\phi}\,(\eta)\, \hat{\psi}\,(\zeta)\, d\eta d\zeta,$$

and let $1 < p, q, r < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $m \in \mathbf{C}^{n+1}\left(\mathbf{R}^{2n} \setminus \{0\}\right)$ satisfies

$$\left|\partial_\eta^\alpha \partial_\zeta^\beta m\,(\eta,\zeta)\right| \le C\,(|\eta| + |\zeta|)^{-(|\alpha|+|\beta|)}$$

for all $|\alpha| + |\beta| \le n+1$ and $(\eta,\zeta) \ne (0,0)$, then

$$\|\Lambda\,(\phi,\psi)\|_{\mathbf{L}^r} \le C\,\|\phi\|_{\mathbf{L}^p}\,\|\psi\|_{\mathbf{L}^q}$$

is true.

Corollary 3.3.2. *If $1 < p, q, r < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $\hat{M}\,(\eta,\zeta)$ satisfies the hypothesis Theorem 3.3.1 on $m\,(\eta,\zeta)$, then*

$$\|[\phi, M, \psi]\|_{\mathbf{L}^r} \le C\,\|\phi\|_{\mathbf{L}^p}\,\|\psi\|_{\mathbf{L}^q}.$$

3.4 Proof of Theorem 3.1.2

We use the notation $\mathbf{X} = \bigcap_{j=0}^{3} \mathbf{H}^{12-3j,j}$, $\|v\|_{\mathbf{X}} = \sum_{j=0}^{3} \|v\|_{\mathbf{H}^{12-3j,j}}$ and

$$\|v\|_{\mathbf{X}'} = \sum_{0\le|\alpha|\le 2} \|\mathcal{J}^{\alpha} v\|_{\mathbf{H}^{12-3|\alpha|}} + \sum_{j=0}^{3} \langle t\rangle^{-\gamma_j} \sum_{|\alpha|=3} \|\mathcal{J}^{\alpha} v\|_{\dot{\mathbf{H}}^{j}} + \sum_{1\le|\alpha|\le 3} \|Q^{\alpha} v\|_{\mathbf{H}^{12-4|\alpha|}}$$
$$+ \sum_{1\le|\alpha|\le 2, |\beta|=1} \left\|\mathcal{J}^{\alpha} Q^{\beta} v\right\|_{\mathbf{H}^{2}} + \left\|\langle\cdot\rangle^{3} \mathcal{F}\mathcal{U}(-t) v\right\|_{\mathbf{L}^{q}},$$

where

$$Q = (Q_1, \cdots, Q_{17}) = \left(1, \mathcal{P}, (\Omega_{j,k})_{j,k=1,\cdots,6,\ j\ge k}\right),$$

$$0 < \gamma_0 < \frac{1}{12}, 0 < \gamma_1 < \frac{1}{2}\gamma_0, \gamma_2 = 3\gamma_1, \gamma_3 = 5\gamma_1$$

and $\frac{6}{\gamma_1} \le q < \infty$.

By the contraction mapping principle, we prove local existence of solutions.

Proposition 3.4.1. *Let $u_0 \in \mathbf{X}$, then there exists an $\varepsilon > 0$ such that* (3.1.1) *has a unique solution u satisfying $\mathcal{U}(-t) u \in \mathbf{C}([0,T]; \mathbf{X})$ with $T > 1$ and*

$$\sup_{t\in[0,T]} \|u(t)\|_{\mathbf{X}'} < \varepsilon^{\frac{1}{2}}$$

for any u_0 satisfying $\|u_0\|_{\mathbf{X}} < \varepsilon$.

Proof. We consider the mapping $u = \mathcal{M}v$ defined by the linearized Cauchy problem corresponding to (3.1.1):

$$\begin{cases} \mathcal{L}u = \lambda \overline{v}^2, & (t,x) \in (0,\infty) \times \mathbf{R}^6, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^6, \end{cases}$$

where $v \in \mathbf{Y}_{T,\rho} = \left\{v \in \mathbf{C}([0,T]; \mathbf{X}'); \sup_{t\in[0,T]} \|v\|_{\mathbf{X}'} \le \rho\right\}$. By virtue of the classical energy method we obtain the estimates

$$\sup_{t\in[0,T]} \sum_{|\alpha|=0}^{3} \|\mathcal{J}^{\alpha} u(t)\|_{\mathbf{H}^{12-3|\alpha|}}$$
$$\le \|u_0\|_{\mathbf{X}} + C\langle T\rangle^{4} \left(\sup_{t\in[0,T]} \sum_{|\alpha|=0}^{3} \|\mathcal{J}^{\alpha} v(t)\|_{\mathbf{H}^{12-3|\alpha|}}\right)^{2}$$
$$\le \varepsilon + C\rho^{2}\langle T\rangle^{4} \le \frac{\rho}{3},$$

$$
\begin{aligned}
&\left\| \langle \cdot \rangle^3 \mathcal{F}\mathcal{U}(-t)\, u \right\|_{\mathbf{L}^q} \\
&\leq C \|u_0\|_{\mathbf{H}^{3,3}} + C \langle T \rangle^4 \left(\sup_{t\in[0,T]} \sum_{0\leq|\alpha|\leq 3} \|\mathcal{J}^\alpha v(t)\|_{\mathbf{H}^{12-3|\alpha|}} \right)^2 \\
&\leq C\varepsilon + C\rho^2 \langle T \rangle^4 \leq \frac{\rho}{3}
\end{aligned}
$$

and

$$
\sup_{t\in[0,T]} \left(\sum_{1\leq|\alpha|\leq 3} \|Q^\alpha u\|_{\mathbf{H}^{12-4|\alpha|}} + \sum_{1\leq|\alpha|\leq 2, |\beta|=1} \left\| \mathcal{J}^\alpha Q^\beta u \right\|_{\mathbf{H}^2} \right) \leq 2\varepsilon + C\rho^2 \langle T \rangle^4 \leq \frac{\rho}{3}
$$

if $\max\{2\varepsilon, C\varepsilon\} = \frac{\rho}{6}$ and $C\rho \langle T \rangle^4 \leq \frac{1}{6}$. Thus, we get $\sup_{t\in[0,T]} \|\mathcal{M}v(t)\|_{\mathbf{X}'} \leq \rho$. Similarly, we have

$$
\sup_{t\in[0,T]} \|\mathcal{M}v_1 - \mathcal{M}v_2\|_{\mathbf{X}'} \leq C\rho \langle T \rangle^4 \sup_{t\in[0,T]} \|v_1 - v_2\|_{\mathbf{X}'} \leq \frac{1}{2} \sup_{t\in[0,T]} \|v_1 - v_2\|_{\mathbf{X}'}
$$

if we take ε satisfying $C\rho \langle T \rangle^4 \leq \frac{1}{2}$. Therefore, we get a unique fixed point $u = \mathcal{M}u$ such that $u \in \mathbf{C}([0,T]; \mathbf{X}')$ with $T > 1$. We find that $\mathcal{U}(-t)\, u \in \mathbf{C}([0,T]; \mathbf{X})$. Moreover, we have

$$
\sup_{t\in[0,T]} \|u(t)\|_{\mathbf{X}'} \leq \rho = 6\max\{2, C\}\varepsilon < \varepsilon^{\frac{1}{2}}
$$

if $6\max\{2, C\}\varepsilon^{\frac{1}{2}} < 1$. Proposition 3.4.1 is proved. □

Global existence of small solutions for Cauchy problem (3.1.1) is obtained in the following proposition.

Proposition 3.4.2. *Let* $u_0 \in \mathbf{X}$, *then there exists an* $\varepsilon > 0$ *such that* (3.1.1) *has a unique global solution* u *satisfying* $\mathcal{U}(-t)\, u \in \mathbf{C}([0,\infty); \mathbf{X})$ *and*

$$
\sup_{t\in[0,\infty)} \|u(t)\|_{\mathbf{X}'} < \varepsilon^{\frac{1}{2}}
$$

for any u_0 *satisfying* $\|u_0\|_{\mathbf{X}} < \varepsilon$.

Proof. By Proposition 3.4.1, we find a $T > 1$ such that

$$
\sup_{t\in[0,T]} \|u\|_{\mathbf{X}'} < \varepsilon^{\frac{1}{2}}.
$$

We assume that there exists a time T such that

$$
\sup_{t\in[0,T]} \|u\|_{\mathbf{X}'} \leq \varepsilon^{\frac{1}{2}}.
$$

By applying a contradiction argument we prove that $T=\infty$. In order to prove that $T=\infty$, we will derive a priori estimates which do not depend on T. By Corollary 3.2.5, we obtain

$$\|u\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)}\left(\|u\|_{\mathbf{H}^4}+\sum_{|\alpha|\le 3}\|\mathcal{J}^\alpha u\|_{\mathbf{L}^2}\right)\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)+\gamma_0}\|u\|_{\mathbf{X}'},$$

where $3<r<\infty$. Due to the energy method we find

$$\frac{d}{dt}\|u\|_{\mathbf{H}^{12}} \le C\left\|u^2\right\|_{\mathbf{H}^{12}} \le C\|u\|_{\mathbf{H}^{12}}\|u\|_{\mathbf{L}^\infty}\le C\langle t\rangle^{-1-k}\varepsilon$$

where $0<k<\frac{1}{2}-\gamma_0$. Thus we obtain

$$\sup_{t\in[0,T]}\|u\|_{\mathbf{H}^{12}}\le C\varepsilon. \tag{3.4.1}$$

Next we estimate $\sum_{1\le|\alpha|\le 3}\|Q^\alpha v\|_{\mathbf{H}^{12-4|\alpha|}}$. Using Corollary 3.2.5, we get

$$\begin{aligned}\sum_{|\alpha|=1}\|Q^\alpha u\|_{\mathbf{L}^\infty} &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)}\left(\sum_{|\alpha|=1}\|Q^\alpha u\|_{\mathbf{H}^4}+\sum_{|\alpha|=1,|\beta|=2}\left\|\mathcal{J}^\beta Q^\alpha u\right\|_{\mathbf{L}^2}\right)\\ &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)}\|u\|_{\mathbf{X}'}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{J}\cdot\nabla u\|_{\mathbf{L}^\infty} &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)}\left(\sum_{|\alpha|\le 1}\|\mathcal{J}^\alpha u\|_{\mathbf{H}^5}+\sum_{|\alpha|\le 3}\|\mathcal{J}^\alpha u\|_{\dot{\mathbf{H}}^1}\right)\\ &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)+\gamma_1}\|u\|_{\mathbf{X}'}\end{aligned}$$

where $3<r<6$. Applying Lemma 3.2.6, we find

$$\begin{aligned}\|x\cdot\nabla u\|_{\mathbf{L}^\infty} &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)}\left(\|u\|_{\mathbf{H}^8}+\sum_{|\alpha|\le 1}\|\mathcal{J}^\alpha u\|_{\mathbf{H}^5}+\sum_{|\alpha|\le 3}\|\mathcal{J}^\alpha u\|_{\mathbf{L}^2}\right)\\ &\le C\langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)+\gamma_0}\|u\|_{\mathbf{X}'}\end{aligned}$$

where $9<r<\infty$. By the identities $[\mathcal{L},\mathcal{P}]=4\mathcal{L}$, $[\mathcal{L},\Omega_{j,k}]=0$ and $\Delta\nabla=\frac{1}{it}(x-\mathcal{J})$, Corollary 3.2.5, Lemma 3.2.6 and the energy method, we get

$$\begin{aligned}\frac{d}{dt}\sum_{|\alpha|=1}\|Q^\alpha u\|_{\mathbf{H}^8} &\le C\sum_{|\alpha|=1}\left\|Q^\alpha\overline{u}^2\right\|_{\mathbf{H}^8}\\ &\le C\sum_{|\alpha|=1}\|uQ^\alpha u\|_{\mathbf{H}^8}\\ &\le C\sum_{|\alpha|=1}\left(\|u\|_{\mathbf{H}^8}\|Q^\alpha u\|_{\mathbf{L}^\infty}+\|u\|_{\mathbf{L}^\infty}\|Q^\alpha u\|_{\mathbf{H}^8}\right)\\ &\le C\langle t\rangle^{-1-k}\varepsilon,\end{aligned}$$

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=2}\|Q^\alpha u\|_{\mathbf{H}^4}\le C\sum_{|\alpha|=2}\left\|Q^\alpha \overline{u}^2\right\|_{\mathbf{H}^4}\\
&\le C\sum_{|\alpha|=2}\|uQ^\alpha u\|_{\mathbf{H}^4}+C\sum_{|\alpha|,|\beta|=1}\left\|(Q^\alpha u)\left(Q^\beta u\right)\right\|_{\mathbf{H}^4}\\
&\le C\left(\|u\|_{\mathbf{L}^\infty}+\left\|\Delta^2 u\right\|_{\mathbf{L}^\infty}\right)\sum_{|\alpha|=2}\|Q^\alpha u\|_{\mathbf{H}^4}+\sum_{|\alpha|,|\beta|=1}\|Q^\alpha u\|_{\mathbf{L}^\infty}\left\|Q^\beta u\right\|_{\mathbf{H}^4}\\
&\le C\left(\|u\|_{\mathbf{L}^\infty}+\|\mathcal{J}\cdot\nabla u\|_{\mathbf{L}^\infty}+\|x\cdot\nabla u\|_{\mathbf{L}^\infty}\right)\varepsilon^{\frac12}+C\langle t\rangle^{-1-k}\varepsilon\\
&\le C\langle t\rangle^{-1-k}\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\sum_{|\alpha|=3}\|Q^\alpha u\|_{\mathbf{L}^2}&\le C\sum_{|\alpha|=3}\|Q^\alpha u\|_{\mathbf{L}^2}\|u\|_{\mathbf{L}^\infty}+C\sum_{|\alpha|=1,|\beta|=2}\|Q^\alpha u\|_{\mathbf{L}^\infty}\left\|Q^\beta u\right\|_{\mathbf{L}^2}\\
&\le C\langle t\rangle^{-1-k}\varepsilon,
\end{aligned}
$$

where $0<k<\frac14-\gamma_0$. Hence

$$
\sup_{t\in[0,T]}\sum_{1\le|\alpha|\le3}\|Q^\alpha v\|_{\mathbf{H}^{12-4|\alpha|}}\le C\varepsilon.
$$

Now we estimate $\sum_{|\alpha|=1}\|\mathcal{J}^\alpha u\|_{\mathbf{H}^9}$. By the Hölder inequality and (3.4.10), we estimate

$$
\sum_{|\alpha|=1}\|\mathcal{J}^\alpha u\|_{\mathbf{L}^2}\le\sum_{|\alpha|=2}\|\mathcal{J}^\alpha u\|_{\mathbf{L}^2}^{\frac12}\|u\|_{\mathbf{L}^2}^{\frac12}\le C\varepsilon. \tag{3.4.2}
$$

Using the identities $\mathcal{J}\cdot\nabla=\mathcal{P}+4it\mathcal{L}$ and

$$
\sum_{j,k=1}^{6}\|\mathcal{J}_j\partial_{x_k}\phi\|_{\mathbf{L}^2}^2=\frac12\sum_{j,k=1}^{6}\|\Omega_{j,k}\phi\|_{\mathbf{L}^2}^2+\|\mathcal{J}\cdot\nabla\phi\|_{\mathbf{L}^2}^2
$$

(see [2]), we get

$$
\begin{aligned}
\sum_{|\alpha|=1}\|\mathcal{J}^\alpha u\|_{\dot{\mathbf{H}}^9}&\le C\sum_{|\alpha|=1}\left\|\Delta^4 Q^\alpha u\right\|_{\mathbf{L}^2}+Ct\left\|\Delta^4\mathcal{L}u\right\|_{\mathbf{L}^2}\\
&\le C\varepsilon+Ct\left\|u\Delta^4 u\right\|_{\mathbf{L}^2}\\
&\le C\varepsilon+Ct\|u\|_{\mathbf{L}^\infty}\|u\|_{\mathbf{H}^8}\\
&\le C\varepsilon.
\end{aligned}
$$

Therefore, we find

$$
\sup_{t\in[0,T]}\sum_{|\alpha|=1}\|\mathcal{J}^\alpha u\|_{\mathbf{H}^9}\le C\varepsilon. \tag{3.4.3}
$$

Next we estimate $\sum_{|\alpha|=2} \|\mathcal{J}^\alpha u\|_{\mathbf{H}^6}$. Applying the operator $\mathcal{J}$ to equation (3.1.1) iteratively, we obtain

$$\mathcal{L} \sum_{|\alpha|=2} \mathcal{J}^\alpha u = \lambda \sum_{|\alpha|=2} \mathcal{J}^\alpha \overline{u}^2 = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{R}_{\mathcal{A}},$$

where

$$\mathcal{A}_0 = \sum_{\substack{|\alpha|+|\beta|=2 \\ 0\le|\beta|\le 1}} C_{\alpha,\beta} \overline{\mathcal{J}^\alpha u} \overline{\mathcal{J}^\beta u}, \quad \mathcal{A}_1 = t \sum_{\substack{|\alpha|=1 \\ |\gamma|+|\delta|=3 \\ 0\le|\delta|\le 3}} C_\alpha^{\gamma,\delta} \overline{\partial_x^\gamma \mathcal{J}^\alpha u \partial_x^\delta u},$$

$$\mathcal{A}_2 = t^2 \sum_{\substack{|\gamma|+|\delta|=6 \\ 0\le|\delta|\le 3}} C^{\gamma,\delta} \overline{\partial_x^\gamma u \partial_x^\delta u},$$

and

$$\mathcal{R}_{\mathcal{A}} = t \sum_{\substack{|\gamma|+|\delta|=2 \\ 0\le|\delta|\le 1}} C^{\gamma,\delta} \overline{\partial_x^\gamma u \partial_x^\delta u}.$$

Due to the following identity $\Delta\nabla = \frac{1}{it}(x - \mathcal{J})$, we have

$$C_\alpha^{\gamma,\delta} = 0 \quad \text{if} \quad (|\alpha|, |\gamma|, |\delta|) = (1,3,0).$$

By the normal form method, we obtain

$$\mathcal{L}\left(\sum_{|\alpha|=2} \mathcal{J}^\alpha u + t^2 \mathcal{A}_{2,1} \right) = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{R}_{\mathcal{A}} + 2it\mathcal{A}_{2,1} + t^2 \mathcal{A}_{2,2}, \qquad (3.4.4)$$

where

$$\mathcal{A}_{2,1} = \sum_{|\gamma|=2} [\partial_x^\gamma u, M_{1,\gamma}, u],$$
$$\mathcal{A}_{2,2} = \sum_{|\gamma|=2} \left(-[\partial_x^\gamma f, M_{1,\gamma}, u] - [\partial_x^\gamma u, M_{1,\gamma}, f] \right),$$

$f = \lambda \overline{u}^2$ and $M_{1,\gamma}$ is Coifman-Meyer kernel. Using the energy method to (3.4.4), we have

$$\frac{d}{dt} \left\| \sum_{|\alpha|=2} \mathcal{J}^\alpha u + t^2 \mathcal{A}_{2,1} \right\|_{\mathbf{L}^2}$$
$$\le \|\mathcal{A}_0\|_{\mathbf{L}^2} + \|\mathcal{A}_1\|_{\mathbf{L}^2} + \|\mathcal{R}_{\mathcal{A}}\|_{\mathbf{L}^2} + Ct \|\mathcal{A}_{2,1}\|_{\mathbf{L}^2} + t^2 \|\mathcal{A}_{2,2}\|_{\mathbf{L}^2}.$$

Applying Corollary 3.2.5, we obtain

$$\sum_{|\beta|=1}\left\|\mathcal{J}^{\beta}u\right\|_{\mathbf{L}^{\infty}}\leq C\left\langle t\right\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)}\left(\sum_{|\beta|=1}\left\|\mathcal{J}^{\beta}u\right\|_{\mathbf{H}^{4}}+\sum_{|\beta|\leq 3}\left\|\mathcal{J}^{\beta}u\right\|_{\mathbf{L}^{2}}\right)$$
$$\leq C\left\langle t\right\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)+\gamma_{0}}\left\|u\right\|_{\mathbf{X}'},$$
$$\left\|u\right\|_{\dot{\mathbf{H}}_{q}^{\alpha}}\leq C\left\langle t\right\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)-\frac{a}{4}}\left(\left\|u\right\|_{\mathbf{H}^{6}}+\sum_{|\alpha|\leq 3}\left\|\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{2}}\right)$$
$$\leq C\left\langle t\right\rangle^{-\frac{3}{2}\left(1-\frac{1}{r}\right)-\frac{a}{4}+\gamma_{0}}\left\|u\right\|_{\mathbf{X}'},$$

where $0\leq a<3$, $\frac{18}{6-a}<q\leq\infty$, $\frac{18}{6-a}<r<q$. By Corollary 3.2.3, we find

$$\sum_{|\alpha|,|\gamma|=1}\left\|\partial_{x}^{\gamma}\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{3}}\leq C\left\langle t\right\rangle^{-\frac{1}{2}}\left(\sum_{|\alpha|\leq 2}\left\|\mathcal{J}^{\alpha}u\right\|_{\mathbf{H}^{3}}+\sum_{|\alpha|\leq 3}\left\|\mathcal{J}^{\alpha}u\right\|_{\dot{\mathbf{H}}^{1}}\right)$$
$$\leq C\left\langle t\right\rangle^{-\frac{1}{2}+\gamma_{1}}\left\|u\right\|_{\mathbf{X}'},$$
$$\sum_{|\alpha|=1,|\gamma|=2}\left\|\partial_{x}^{\gamma}\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{4}}\leq C\left\langle t\right\rangle^{-\frac{3}{4}}\left(\sum_{|\alpha|\leq 2}\left\|\mathcal{J}^{\alpha}u\right\|_{\mathbf{H}^{4}}+\sum_{|\alpha|\leq 3}\left\|\mathcal{J}^{\alpha}u\right\|_{\dot{\mathbf{H}}^{2}}\right)$$
$$\leq C\left\langle t\right\rangle^{-\frac{3}{4}+\gamma_{2}}\left\|u\right\|_{\mathbf{X}'}.$$

Using the Hölder inequality and the identity $\Delta\nabla=\frac{1}{it}\left(x-\mathcal{J}\right)$, we estimate

$$\left\|\mathcal{A}_{0}\right\|_{\mathbf{L}^{2}}\leq C\sum_{\substack{|\alpha|+|\beta|=2\\0\leq|\beta|\leq 1}}\left\|\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{2}}\left\|\mathcal{J}^{\beta}u\right\|_{\mathbf{L}^{\infty}}\leq C\left\langle t\right\rangle^{-1-k}\varepsilon,$$

$$\left\|\mathcal{A}_{1}\right\|_{\mathbf{L}^{2}}\leq C\sum_{|\alpha|=1}\left\|\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{2}}\left(\sum_{|\alpha|=1}\left\|\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{\infty}}+\left\|\left|x\right|u\right\|_{\mathbf{L}^{\infty}}\right)$$
$$+Ct\sum_{\substack{|\alpha|,|\gamma|=1\\|\delta|=2}}\left\|\partial_{x}^{\gamma}\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{3}}\left\|\partial_{x}^{\delta}u\right\|_{\mathbf{L}^{6}}+Ct\sum_{\substack{|\alpha|,|\delta|=1\\|\gamma|=2}}\left\|\partial_{x}^{\gamma}\mathcal{J}^{\alpha}u\right\|_{\mathbf{L}^{4}}\left\|\partial_{x}^{\delta}u\right\|_{\mathbf{L}^{4}}$$
$$\leq C\left\langle t\right\rangle^{-1-k}\varepsilon,$$

$$\left\|\mathcal{R}_{A}\right\|_{\mathbf{L}^{2}}\leq Ct\left(\left(\sum_{|\gamma|=1}\left\|\partial_{x}^{\gamma}u\right\|_{\mathbf{L}^{4}}\right)^{2}+\sum_{|\gamma|=2}\left\|\partial_{x}^{\gamma}u\right\|_{\mathbf{L}^{5}}\left\|u\right\|_{\mathbf{L}^{\frac{10}{3}}}\right)\leq C\left\langle t\right\rangle^{-1-k}\varepsilon,$$

where $0<\gamma_{2}<\frac{1}{8}$ (i.e. $0<\gamma_{0}<\frac{1}{12}$) and $0<k<\frac{1}{8}-\gamma_{2}$. Applying the Coifman-Meyer inequality, we get

$$\|\mathcal{A}_{2,1}\|_{\mathbf{L}^2} \le C \sum_{|\gamma|=2} \|\partial_x^\gamma u\|_{\mathbf{L}^5} \|u\|_{\mathbf{L}^{\frac{10}{3}}} \le C \langle t \rangle^{-2-k} \varepsilon,$$

$$\|\mathcal{A}_{2,2}\|_{\mathbf{L}^2} \le C \left(\|u\|_{\mathbf{L}^6}^2 \sum_{|\gamma|=2} \|\partial_x^\gamma u\|_{\mathbf{L}^6} + \left(\sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^6} \right)^2 \|u\|_{\mathbf{L}^6} \right) \le C \langle t \rangle^{-3-k} \varepsilon^{\frac{3}{2}},$$

where $0 < k < \frac{3}{4} - 2\gamma_0$. Hence

$$\frac{d}{dt} \left\| \sum_{|\alpha|=2} \mathcal{J}^\alpha u + t^2 \mathcal{A}_{2,1} \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{-1-k} \varepsilon$$

from which it follows that

$$\left\| \sum_{|\alpha|=2} \mathcal{J}^\alpha u + t^2 \mathcal{A}_{2,1} \right\|_{\mathbf{L}^2} \le \|u_0\|_{\mathbf{H}^{0,2}} + C\varepsilon \le C\varepsilon.$$

By the triangle inequality, we find

$$\sum_{|\alpha|=2} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^2} \le C\varepsilon + t^2 \|\mathcal{A}_{2,1}\|_{\mathbf{L}^2} \le C\varepsilon. \tag{3.4.5}$$

Using the following identities $\mathcal{J} \cdot \nabla = \mathcal{P} + 4it\mathcal{L}$ and

$$\sum_{j,k=1}^{6} \|\mathcal{J}_j \partial_{x_k} \phi\|_{\mathbf{L}^2}^2 = \frac{1}{2} \sum_{j,k=1}^{6} \|\Omega_{j,k}\phi\|_{\mathbf{L}^2}^2 + \|\mathcal{J} \cdot \nabla \phi\|_{\mathbf{L}^2}^2,$$

we get

$$\begin{aligned} \sum_{|\alpha|=2} \|\mathcal{J}^\alpha u\|_{\dot{\mathbf{H}}^6} &\le C \sum_{|\alpha|=2} \left\|\Delta^2 Q^\alpha u\right\|_{\mathbf{L}^2} + Ct \sum_{|\alpha|=1} \left\|\Delta^2 Q^\alpha \mathcal{L} u\right\|_{\mathbf{L}^2} + Ct^2 \left\|\Delta^2 \mathcal{L}^2 u\right\|_{\mathbf{L}^2} \\ &\le C\varepsilon + Ct \sum_{|\alpha|=1} \left\|\Delta^2 Q^\alpha \overline{u}^2\right\|_{\mathbf{L}^2} + Ct^2 \left\|\Delta^2 \mathcal{L} \overline{u}^2\right\|_{\mathbf{L}^2} \\ &\le C\varepsilon + Ct \left(\|u\|_{\mathbf{L}^\infty} + \left\|\Delta^2 u\right\|_{\mathbf{L}^\infty} \right) \sum_{|\alpha|=1} \|Q^\alpha u\|_{\mathbf{H}^4} \\ &+ Ct^2 \left(\left\|u^2 \Delta^2 u\right\|_{\mathbf{L}^2} + \left\| \left(\Delta^2 u\right)^2 \right\|_{\mathbf{L}^2} + \left\|u \Delta^4 u\right\|_{\mathbf{L}^2} \right). \end{aligned} \tag{3.4.6}$$

By the following identity $\Delta \nabla = \frac{1}{it} (x - \mathcal{J})$, we find

$$t \left\|\Delta^2 u\right\|_{\mathbf{L}^\infty} \le \|\mathcal{J} \cdot \nabla u\|_{\mathbf{L}^\infty} + \|x \cdot \nabla u\|_{\mathbf{L}^\infty} \le \langle t \rangle^{-k} \varepsilon^{\frac{1}{2}},$$

where $0 < k < \frac{5}{4} - \gamma_0$. On account of Corollary 3.2.5, we obtain

$$t^2 \left\| u^2 \Delta^2 u \right\|_{\mathbf{L}^2} \le t^2 \left\| u \right\|^2_{\mathbf{L}^\infty} \left\| u \right\|_{\mathbf{H}^4} \le C \varepsilon^{\frac{3}{2}} \langle t \rangle^{-k}, \tag{3.4.7}$$

where $0 < k < 1 - 2\gamma_0$. By the following identity $\Delta \nabla = \frac{1}{it} (x - \mathcal{J})$, Corollary 3.2.5 and Lemma 3.2.6, we have

$$\begin{aligned} t^2 \left\| \left(\Delta^2 u\right)^2 \right\|_{\mathbf{L}^2} &\le t^2 \left\| \Delta^2 u \right\|_{\mathbf{L}^\infty} \left\| u \right\|_{\mathbf{H}^4} \\ &\le t \left(\left\| \mathcal{J} \cdot \nabla u \right\|_{\mathbf{L}^\infty} + \left\| x \cdot \nabla u \right\|_{\mathbf{L}^\infty} \right) \left\| u \right\|_{\mathbf{H}^4} \\ &\le C \varepsilon \langle t \rangle^{-k}, \end{aligned} \tag{3.4.8}$$

$$t^2 \left\| u \Delta^4 u \right\|_{\mathbf{L}^2} \le C t \left\| u \right\|_{\mathbf{L}^\infty} \sum_{|\alpha|=1} \left\| \mathcal{J}^\alpha u \right\|_{\mathbf{H}^5} + C t \left\| |x| \, u \right\|_{\mathbf{L}^\infty} \left\| u \right\|_{\mathbf{H}^5} \le C \varepsilon \langle t \rangle^{-k}, \tag{3.4.9}$$

where $0 < k < \frac{1}{4} - \gamma_0$. Thus, we obtain

$$\sum_{|\alpha|=2} \left\| \mathcal{J}^\alpha u \right\|_{\dot{\mathbf{H}}^6} \le C \varepsilon.$$

Therefore

$$\sup_{t \in [0,T]} \sum_{|\alpha|=2} \left\| \mathcal{J}^\alpha u \right\|_{\mathbf{H}^6} \le C \varepsilon. \tag{3.4.10}$$

Let us estimate $\sum_{1 \le |\alpha| \le 2, |\beta|=1} \left\| \mathcal{J}^\alpha Q^\beta v \right\|_{\mathbf{H}^2}$. Applying $\sum_{|\alpha|=2} \mathcal{J}^\alpha \mathcal{P}$ to both sides of (3.1.1), we get

$$\mathcal{L} \sum_{|\alpha|=2} \mathcal{J}^\alpha \mathcal{P} u = \lambda \sum_{|\alpha|=2} \mathcal{J}^\alpha \left(4 + \mathcal{P}\right) \overline{u}^2.$$

By the identity $\mathcal{P} = \mathcal{J} \cdot \nabla - 4it\mathcal{L}$, we have

$$\begin{aligned} \sum_{|\alpha|=2} \mathcal{J}^\alpha \mathcal{P} \overline{u}^2 &= 2 \sum_{|\alpha|=2} \mathcal{J}^\alpha \left(\overline{u \mathcal{P} u} \right) \\ &= \sum_{|\alpha|=2} \left(2 \mathcal{J}^\alpha \left(\overline{u \mathcal{J} \cdot \nabla u} \right) - 8 i \overline{\lambda} t \mathcal{J}^\alpha \left(\overline{u} u^2 \right) \right) = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{R}_{\mathcal{D}}, \end{aligned}$$

where

$$\mathcal{D}_0 = \sum_{\substack{|\alpha|+|\beta|=3 \\ 1\le|\beta|\le 3 \\ |\gamma|=1}} C^{\gamma}_{\alpha,\beta} \overline{\mathcal{J}^\alpha u} \overline{\partial_x^\gamma \mathcal{J}^\beta u}, \quad \mathcal{D}_1 = t \sum_{\substack{|\alpha|+|\beta|=2 \\ 1\le|\beta|\le 2 \\ |\gamma|+|\delta|=4 \\ 1\le|\delta|\le 4}} C^{\gamma,\delta}_{\alpha,\beta} \overline{\partial_x^\gamma \mathcal{J}^\alpha u \partial_x^\delta \mathcal{J}^\beta u},$$

$$\mathcal{D}_2 = t^2 \sum_{\substack{|\alpha|=1 \\ |\gamma|+|\delta|=7 \\ 1\le|\delta|\le 7}} C^{\gamma,\delta}_{\alpha} \overline{\partial_x^\gamma u \partial_x^\delta \mathcal{J}^\alpha u} + t^2 \sum_{\substack{|\gamma|+|\delta|=6 \\ 0\le|\delta|\le 3}} C^{\gamma,\delta} \overline{\partial_x^\gamma u \partial_x^\delta u},$$

$$\mathcal{R}_{\mathcal{D}} = t \sum_{\substack{|\alpha|=1 \\ |\gamma|+|\delta|=3 \\ 1\le|\delta|\le 3}} C^{\gamma,\delta}_{\alpha} \overline{\partial_x^\gamma u \partial_x^\delta \mathcal{J}^\alpha u} + t \sum_{\substack{|\gamma|+|\delta|=2 \\ 0\le|\delta|\le 1}} C^{\delta,\gamma} \overline{\partial_x^\gamma u \partial_x^\delta u} - 8i\overline{\lambda} t \sum_{|\alpha|=2} \mathcal{J}^\alpha \left(\overline{u} u^2\right).$$

Using the identity $\mathcal{J} = x - it\Delta\nabla$, we obtain

$$C^{\gamma,\delta}_{\alpha} = 0 \quad \text{if} \quad |\alpha| = 1 \quad \text{and} \quad (|\gamma|, |\delta|) = (0,7), (1,6), (2,5), (0,3).$$

By the normal form method, we get

$$\begin{aligned} \mathcal{D}_1 &= -\mathcal{L}\left(t\mathcal{D}_{1,1}\right) + i\mathcal{D}_{1,1} + t\mathcal{D}_{1,2}, \\ \mathcal{D}_2 &= -\mathcal{L}\left(t^2\mathcal{D}_{2,1}\right) + 2it\mathcal{D}_{2,1} + t^2\mathcal{D}_{2,2}, \end{aligned}$$

where

$$\mathcal{D}_{1,1} = \sum_{\substack{|\alpha|+|\beta|=2 \\ 1\le|\beta|\le 2}} \left[\mathcal{J}^\alpha u, M_{1,\alpha,\beta}, \mathcal{J}^\beta u\right],$$

$$\mathcal{D}_{1,2} = \sum_{\substack{|\alpha|+|\beta|=2 \\ 1\le|\beta|\le 2}} \left(-\left[\mathcal{J}^\alpha f, M_{1,\alpha,\beta}, \mathcal{J}^\beta u\right] - \left[\mathcal{J}^\alpha u, M_{1,\alpha,\beta}, \mathcal{J}^\beta f\right]\right),$$

$$\mathcal{D}_{2,1} = \sum_{\substack{|\alpha|=1 \\ |\delta|=3}} \left[\partial_x^\delta u, M_{2,\alpha,\delta}, \mathcal{J}^\alpha u\right] + \sum_{|\delta|=2} \left[u, M_{3,\delta}, \partial_x^\delta u\right],$$

$$\begin{aligned} \mathcal{D}_{2,2} = &\sum_{\substack{|\alpha|=1 \\ |\delta|=3}} \left(-\left[\partial_x^\delta f, M_{2,\alpha,\delta}, \mathcal{J}^\alpha u\right] - \left[\partial_x^\delta u, M_{2,\alpha,\delta}, \mathcal{J}^\alpha f\right]\right) \\ &+ \sum_{|\delta|=2} \left(-\left[f, M_{3,\delta}, \partial_x^\delta u\right] - \left[u, M_{3,\delta}, \partial_x^\delta f\right]\right), \end{aligned}$$

$f = \lambda \overline{u}^2$ and $M_{1,\alpha,\beta}$, $M_{2,\alpha,\gamma}$, $M_{3,\delta}$ are the Coifman-Meyer kernel. Applying Corollary 3.2.3, Corollary 3.2.5 and Lemma 3.2.6, we estimate

$$\begin{aligned}
\|\mathcal{D}_0\|_{\mathbf{L}^2} &\leq C \sum_{\substack{|\alpha|=2\\|\beta|=1,|\delta|=1}} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^2} \left\|\partial_x^\delta \mathcal{J}^\beta u\right\|_{\mathbf{L}^\infty} \\
&\quad + C \sum_{\substack{|\alpha|+|\beta|=3\\2\leq|\beta|\leq3,|\delta|=1}} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^\infty} \left\|\partial_x^\delta \mathcal{J}^\beta u\right\|_{\mathbf{L}^2} \\
&\leq C \langle t \rangle^{-1-k} \varepsilon,
\end{aligned}$$

$$\|\mathcal{D}_{1,1}\|_{\mathbf{L}^2} \leq C \sum_{\substack{|\alpha|+|\beta|=2\\1\leq|\beta|\leq2}} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{p_1}} \left\|\mathcal{J}^\beta u\right\|_{\mathbf{L}^{2+\theta}} \leq C \langle t \rangle^{-1-k} \varepsilon,$$

$$\begin{aligned}
&t\|\mathcal{D}_{1,2}\|_{\mathbf{L}^2} \\
&\leq Ct \sum_{|\alpha|=1} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{2+\theta}} \left(\left(\sum_{|\beta|=1} \left\|\mathcal{J}^\beta u\right\|_{\mathbf{L}^{p_2}} + \||x|\, u\|_{\mathbf{L}^{p_2}} \right) \|u\|_{\mathbf{L}^{p_2}} \right. \\
&\left. + t \sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^{p_2}} \sum_{|\delta|=2} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^{p_2}} \right) \\
&+ Ct\|u\|_{\mathbf{L}^{p_2}} \left(\|u\|_{\mathbf{L}^{p_2}} \sum_{|\alpha|=2} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{2+\theta}} + t \sum_{\substack{|\alpha|=1\\|\gamma|+|\delta|=3\\1\leq|\delta|\leq2}} \|\partial_x^\gamma \mathcal{J}^\alpha u\|_{\mathbf{L}^{2+\theta}} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^{p_2}} \right) \\
&\leq C \langle t \rangle^{-1-k} \varepsilon,
\end{aligned}$$

$$\begin{aligned}
&t\|\mathcal{D}_{2,1}\|_{\mathbf{L}^2} \\
&\leq C \sum_{|\alpha|=1} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{2+\theta}} \left(\sum_{|\beta|=1} \left\|\mathcal{J}^\beta u\right\|_{\mathbf{L}^{p_1}} + \||x|\, u\|_{\mathbf{L}^{p_1}} \right) + Ct\|u\|_{\mathbf{L}^4} \sum_{|\alpha|=1} \|\partial_x^\alpha u\|_{\mathbf{L}^4} \\
&\leq C \langle t \rangle^{-1-k} \varepsilon,
\end{aligned}$$

and

$$t^2 \|\mathcal{D}_{2,2}\|_{\mathbf{L}^2} \le Ct \sum_{|\alpha|=1} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{2+\theta}} \left(\left(\sum_{|\beta|=1} \left\|\mathcal{J}^\beta u\right\|_{\mathbf{L}^{p_2}} + \||x|\, u\|_{\mathbf{L}^{p_2}} \right) \|u\|_{\mathbf{L}^{p_2}} \right.$$

$$\left. +t \sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^{p_2}} \sum_{|\delta|=2} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^{p_2}} \right)$$

$$+ Ct^2 \left(\|u\|_{\mathbf{L}^6}^2 \sum_{|\gamma|=2} \|\partial_x^\gamma u\|_{\mathbf{L}^6} + \left(\sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^6} \right)^2 \|u\|_{\mathbf{L}^6} \right)$$

$$\le C \langle t \rangle^{-1-k} \varepsilon,$$

where $\theta > 0$ is small, $p_1 = \frac{2(2+\theta)}{\theta}$, $p_2 = \frac{4(2+\theta)}{\theta}$ and $0 < k < \frac{1}{4} - 2\gamma_0$. Due to the identity $\mathcal{J} = x - it\Delta\nabla$, we obtain

$$t \sum_{|\alpha|=2} \left\|\mathcal{J}^\alpha \left(\overline{u} u^2\right)\right\|_{\mathbf{L}^2}$$

$$\le Ct \sum_{\substack{|\alpha|+|\beta|=2 \\ 0\le|\beta|\le 1}} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^2} \left\|\mathcal{J}^\beta u\right\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^\infty} + Ct \sum_{|\alpha|=1} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^2} \||x|\, u\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^\infty}$$

$$+ Ct \||x|\, u\|_{\mathbf{L}^\infty}^2 \|u\|_{\mathbf{L}^2} + Ct^2 \sum_{\substack{|\alpha|=1 \\ |\beta|+|\gamma|+|\delta|=3 \\ |\beta|,|\gamma|,|\delta|\ne 3}} \left\|\partial_x^\beta \mathcal{J}^\alpha u\right\|_{\mathbf{L}^3} \|\partial_x^\gamma u\|_{\mathbf{L}^{12}} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^{12}}$$

$$+ Ct^3 \sum_{\substack{|\beta|=3 \\ |\gamma|=2 \\ |\delta|=1}} \left\|\partial_x^\beta u\right\|_{\mathbf{L}^{12}} \|\partial_x^\gamma u\|_{\mathbf{L}^6} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^4} + Ct^3 \left(\sum_{|\gamma|=2} \|\partial_x^\gamma u\|_{\mathbf{L}^6} \right)^3$$

$$\le C \langle t \rangle^{-1-k} \varepsilon.$$

Therefore, we find

$$\|\mathcal{R}_D\|_{\mathbf{L}^2}$$

$$\le Ct \left(\sum_{|\alpha|,|\gamma|=1} \|\partial_x^\gamma \mathcal{J}^\alpha u\|_{\mathbf{L}^3} \sum_{|\delta|=2} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^6} + Ct \sum_{\substack{|\alpha|=1 \\ |\gamma|=2}} \|\partial_x^\gamma \mathcal{J}^\alpha u\|_{\mathbf{L}^4} \sum_{|\delta|=1} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^4} \right)$$

$$+ Ct \left(\left(\sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^4} \right)^2 + \sum_{|\gamma|=2} \|\partial_x^\gamma u\|_{\mathbf{L}^5} \|u\|_{\mathbf{L}^{\frac{10}{3}}} \right) + C \langle t \rangle^{-1-k} \varepsilon$$

$$\le C \langle t \rangle^{-1-k} \varepsilon,$$

where $0 < k < \frac{1}{8} - \gamma_2$. Applying the energy method, we get

$$\sum_{|\alpha|=2} \|\mathcal{J}^\alpha \mathcal{P} u\|_{\mathbf{L}^2} \le C\varepsilon.$$

In the same way,

$$\sum_{|\alpha|=2} \|\Delta \mathcal{J}^\alpha \mathcal{P} u\|_{\mathbf{L}^2} \le C\varepsilon.$$

Using the identity $\Omega_{j,k} = x_j \partial_k - x_k \partial_{x_j} = \mathcal{J}_j \partial_k - \mathcal{J}_k \partial_{x_j}$, we analogously get

$$\sum_{\substack{|\alpha|=2\\|\beta|=1}} \left\| \mathcal{J}^\alpha \Omega^\beta u \right\|_{\mathbf{H}^2} \le C\varepsilon.$$

Therefore,

$$\sup_{t\in[0,T]} \sum_{\substack{1\le|\alpha|\le 2\\|\beta|=1}} \left\| \mathcal{J}^\alpha Q^\beta v \right\|_{\mathbf{H}^2} \le C\varepsilon. \tag{3.4.11}$$

Next we estimate $\sum_{|\alpha|=3} \|\mathcal{J}^\alpha u\|_{\mathbf{H}^3}$. Applying the operator $\sum_{|\alpha|=3} \mathcal{J}^\alpha$ to equation (3.1.1), we obtain

$$\mathcal{L} \sum_{|\alpha|=3} \mathcal{J}^\alpha u = \lambda \sum_{|\alpha|=3} \mathcal{J}^\alpha \overline{u}^2 = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{R}_{\mathcal{B}}, \tag{3.4.12}$$

where

$$\mathcal{B}_0 = \sum_{\substack{|\alpha|+|\beta|=3\\0\le|\beta|\le 1}} C_{\alpha,\beta} \overline{\mathcal{J}^\alpha u}\,\overline{\mathcal{J}^\beta u}, \quad \mathcal{B}_1 = t \sum_{\substack{|\alpha|+|\beta|=2\\0\le|\beta|\le 1\\|\gamma|+|\delta|=3\\0\le|\delta|\le 3}} C^{\gamma,\delta}_{\alpha,\beta} \overline{\partial_x^\gamma \mathcal{J}^\alpha u \partial_x^\delta \mathcal{J}^\beta u},$$

$$\mathcal{B}_2 = t^2 \sum_{\substack{|\alpha|=1\\|\gamma|+|\delta|=6\\0\le|\delta|\le 6}} C^{\gamma,\delta}_{\alpha} \overline{\partial_x^\gamma \mathcal{J}^\alpha u \partial_x^\delta u} + t^2 \sum_{\substack{|\gamma|+|\delta|=5\\0\le|\delta|\le 2}} C^{\gamma,\delta} \overline{\partial_x^\gamma u \partial_x^\delta u},$$

$$\mathcal{B}_3 = t^3 \sum_{\substack{|\gamma|+|\delta|=9\\0\le|\delta|\le 4}} C^{\gamma,\delta} \overline{\partial_x^\gamma u \partial_x^\delta u},$$

$$\mathcal{R}_{\mathcal{B}} = t \sum_{|\gamma|=1} C^\gamma \overline{\partial_x^\gamma u}\,\overline{u} + t \sum_{\substack{|\alpha|=1\\|\gamma|+|\delta|=2\\0\le|\delta|\le 2}} C^{\gamma,\delta}_{\alpha} \overline{\partial_x^\gamma \mathcal{J}^\alpha u \partial_x^\delta u}.$$

By the identity $\mathcal{J} = x - it\Delta\nabla$, we have

$$C^{\gamma,\delta} = 0 \quad \text{if} \quad (|\gamma|, |\delta|) = (9,0), (8,1),$$

$$C_{\alpha}^{\gamma,\delta} = 0 \quad \text{if} \quad (|\alpha|, |\gamma|, |\delta|) = (1,6,0), (1,2,0)$$

and

$$C_{\alpha,\beta}^{\gamma,\delta} = 0 \quad \text{if} \quad (|\alpha|, |\beta|, |\gamma|, |\delta|) = (2,0,3,0), (2,0,2,1), (1,1,2,1), (1,1,3,0), (1,1,2,1).$$

Applying the normal form method, we get

$$\begin{aligned} &\mathcal{L}\left(\sum_{|\alpha|=3} \mathcal{J}^{\alpha} u + t^2 \mathcal{B}_{2,1} + t^3 \mathcal{B}_{3,1}\right) \\ &= \mathcal{B}_0 + \mathcal{B}_1 + 2it\mathcal{B}_{2,1} + t^2\mathcal{B}_{2,2} + 3it^2\mathcal{B}_{3,1} + t^3\mathcal{B}_{3,2} + \mathcal{R}_{\mathcal{B}}, \end{aligned}$$

where

$$\mathcal{B}_{2,1} = \sum_{\substack{|\alpha|=1 \\ |\gamma|+|\delta|=2 \\ 1\le|\delta|\le 2}} \left[\partial_x^{\gamma} \mathcal{J}^{\alpha} u, M_{\alpha}^{\gamma,\delta}, \partial_x^{\delta} u\right] + \sum_{|\gamma|=1} \left[\partial_x^{\gamma} u, M^{\gamma}, u\right],$$

$$\begin{aligned} \mathcal{B}_{2,2} = &\sum_{\substack{|\alpha|=1 \\ |\gamma|+|\delta|=2 \\ 1\le|\delta|\le 2}} \left(-\left[\partial_x^{\gamma} \mathcal{J}^{\alpha} f, M_{\alpha}^{\gamma,\delta}, \partial_x^{\delta} u\right] - \left[\partial_x^{\gamma} \mathcal{J}^{\alpha} U, M_{\alpha}^{\gamma,\delta}, \partial_x^{\delta} f\right]\right) \\ &+ \sum_{|\gamma|=1} \left(-\left[\partial_x^{\gamma} f, M^{\gamma}, u\right] - \left[\partial_x^{\gamma} u, M^{\gamma}, f\right]\right), \end{aligned}$$

$$\mathcal{B}_{3,1} = \sum_{\substack{|\gamma|=3 \\ |\delta|=2}} \left[\partial_x^{\gamma} u, M^{\gamma,\delta}, \partial_x^{\delta} u\right],$$

$$\mathcal{B}_{3,2} = \sum_{\substack{|\gamma|=3 \\ |\delta|=2}} \left(-\left[\partial_x^{\gamma} f, M^{\gamma,\delta}, \partial_x^{\delta} u\right] - \left[\partial_x^{\gamma} u, M^{\gamma,\delta}, \partial_x^{\delta} f\right]\right),$$

$f = \lambda \overline{u}^2$ and $M_{\alpha}^{\gamma,\delta}$, M^{γ}, $M^{\gamma,\delta}$ are Coifman-Meyer kernels. Applying the Hölder inequality, Corollary 3.2.3, Corollary 3.2.5 and Lemma 3.2.6, we estimate

$$\|\mathcal{B}_0\|_{\mathbf{L}^2} \le C \sum_{\substack{|\alpha|+|\beta|=3 \\ 0\le|\beta|\le 1}} \|\mathcal{J}^{\alpha} u\|_{\mathbf{L}^2} \left\|\mathcal{J}^{\beta} u\right\|_{\mathbf{L}^{\infty}} \le C \langle t \rangle^{-1-k} \varepsilon,$$

$$
\begin{aligned}
\|\mathcal{B}_1\|_{\mathbf{L}^2} &\le Ct \sum_{\substack{|\alpha|=2\\|\gamma|=1}} \|\partial_x^\gamma \mathcal{J}^\alpha u\|_{\mathbf{L}^{\frac{10}{3}}} \sum_{|\delta|=2} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^5} \\
&\quad + C\sum_{|\alpha|=2} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^2} \left(\sum_{|\beta|=1} \left\|\mathcal{J}^\beta u\right\|_{\mathbf{L}^\infty} + \||x|\, u\|_{\mathbf{L}^\infty}\right) \\
&\le C\langle t\rangle^{-1-k}\varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
\|\mathcal{R}_B\|_{\mathbf{L}^2} &\le Ct \left(\sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^4} \|u\|_{\mathbf{L}^4} + \sum_{\substack{|\alpha|=1\\|\delta|=2}} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{\frac{10}{3}}} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^5}\right. \\
&\quad \left. + \sum_{|\alpha|,|\gamma|,|\delta|=1} \|\partial_x^\gamma \mathcal{J}^\alpha u\|_{\mathbf{L}^4} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^4}\right) \\
&\le C\langle t\rangle^{-1-k}\varepsilon,
\end{aligned}
$$

where $0 < k < \frac{1}{8} - \gamma_0 - \gamma_1$. By the Coifman-Meyer inequality, we get

$$
\begin{aligned}
t\|\mathcal{B}_{2,1}\|_{\mathbf{L}^2} &\le Ct \left(\sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^4} \|u\|_{\mathbf{L}^4} + \sum_{|\alpha|=1,|\delta|=2} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{\frac{10}{3}}} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^5}\right. \\
&\quad \left. + \sum_{|\alpha|,|\gamma|,|\delta|=1} \|\partial_x^\gamma \mathcal{J}^\alpha u\|_{\mathbf{L}^4} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^4}\right) \\
&\le C\langle t\rangle^{-1-k}\varepsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
& t^2\|\mathcal{B}_{2,2}\|_{\mathbf{L}^2} \\
&\le Ct^2 \sum_{\substack{|\gamma|=1\\|\delta|=1}} \left(\sum_{|\alpha|=1} \|\partial_x^\gamma \mathcal{J}^\alpha u\|_{\mathbf{L}^{p_1}} + \||x|\cdot\partial_x^\gamma u\|_{\mathbf{L}^{p_1}}\right) \left\|\partial_x^\delta u\right\|_{\mathbf{L}^{4+\theta}} \|u\|_{\mathbf{L}^{4+\theta}} \\
&\quad + Ct^2 \left(\sum_{|\alpha|=1} \|\mathcal{J}^\alpha u\|_{\mathbf{L}^{p_1}} + \||x|\, u\|_{\mathbf{L}^{p_1}}\right) \left(\sum_{|\gamma|=1} \|\partial_x^\gamma u\|^2_{\mathbf{L}^{4+\theta}} + \sum_{|\gamma|=2} \|\partial_x^\gamma u\|_{\mathbf{L}^5} \|u\|_{\mathbf{L}^{p_2}}\right) \\
&\quad + Ct^3 \sum_{|\gamma|=2} \|\partial_x^\gamma u\|^2_{\mathbf{L}^6} \sum_{|\delta|=1} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^6} + Ct^2 \sum_{|\gamma|=1} \|\partial_x^\gamma u\|_{\mathbf{L}^6} \|u\|^2_{\mathbf{L}^6} \\
&\le C\langle t\rangle^{-1-k}\varepsilon,
\end{aligned}
$$

where $p_1 = \frac{2(4+\theta)}{\theta}$, $p_2 = \frac{5(4+\theta)}{6-\theta}$, $0 < \gamma_0 < \frac{1}{12}$ and $0 < k < \frac{1}{4} - 3\gamma_0$. By

Corollary 3.2.5, we obtain

$$
\begin{aligned}
t^2 \left\|\mathcal{B}_{3,1}\right\|_{\mathbf{L}^2} &\le Ct^2 \sum_{\substack{|\gamma|=3\\|\delta|=2}} \left\|\partial_x^\gamma u\right\|_{\mathbf{L}^{\frac{9}{2}}} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^{\frac{18}{5}}} \\
&\le Ct^2 \sum_{\substack{|\gamma|=3\\|\delta|=2}} \left\|\partial_x^\gamma u\right\|_{\mathbf{L}^{\frac{9}{2}+\Theta}}^{\frac{5(9+2\Theta)}{9(5+2\Theta)}} \left\|\partial_x^\gamma u\right\|_{\mathbf{L}^2}^{\frac{8\Theta}{9(5+2\Theta)}} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^{\frac{18}{5}+\Theta}}^{\frac{4(18+5\Theta)}{9(8+5\Theta)}} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^2}^{\frac{25\Theta}{9(8+5\Theta)}} \\
&\le C \langle t \rangle^{2-\frac{5(15+4\Theta)}{9(5+2\Theta)}-\frac{96+35\Theta}{9(8+5\Theta)}} \left(\|u\|_{\mathbf{H}^6} + \sum_{|\alpha|\le 3} \left\|\mathcal{J}^\alpha u\right\|_{\dot{\mathbf{H}}^1} \right)^{\frac{5(9+2\Theta)}{9(5+2\Theta)}+\frac{4(18+5\Theta)}{9(8+5\Theta)}} \\
&\times \varepsilon^{\frac{1}{2}\left(\frac{8\Theta}{9(5+2\Theta)}+\frac{25\Theta}{9(8+5\Theta)}\right)} \\
&\le C \langle t \rangle^{-1+\widetilde{\gamma}+2\gamma_1} \varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
&t^2 \left\|\mathcal{B}_{3,2}\right\|_{\mathbf{L}^2} \\
&\le Ct \left(\sum_{|\alpha|=1} \left\|\mathcal{J}^\alpha u\right\|_{\mathbf{L}^{p_1}} + \left\| |x|\, u \right\|_{\mathbf{L}^{p_1}} \right) \left(\sum_{|\gamma|=1} \left\|\partial_x^\gamma u\right\|_{\mathbf{L}^{4+\theta}}^2 + \sum_{|\gamma|=2} \left\|\partial_x^\gamma u\right\|_{\mathbf{L}^5} \|u\|_{\mathbf{L}^{p_2}} \right) \\
&+ Ct^2 \left(\sum_{|\gamma|=2} \left\|\partial_x^\gamma u\right\|_{\mathbf{L}^6} \right)^2 \sum_{|\delta|=1} \left\|\partial_x^\delta u\right\|_{\mathbf{L}^6} \\
&\le C \langle t \rangle^{-1-k} \varepsilon,
\end{aligned}
$$

where $\theta, \Theta > 0$ are small, $p_1 = \frac{2(4+\theta)}{\theta}$, $p_2 = \frac{5(4+\theta)}{6-\theta}$, $0 < \gamma_1 < \frac{1}{2}\gamma_0$, $0 < \widetilde{\gamma} \le \gamma_0 - 2\gamma_1$ and $0 < k < \frac{3}{4} - 3\gamma_0$. Applying the energy method, we get

$$
\sum_{|\alpha|=3} \left\|\mathcal{J}^\alpha u\right\|_{\mathbf{L}^2} \le C\varepsilon + t^{\widetilde{\gamma}+2\gamma_1}\varepsilon \le C \langle t \rangle^{\gamma_0} \varepsilon.
$$

Using the following identity $\mathcal{J} \cdot \nabla = \mathcal{P} + 4it\mathcal{L}$, we obtain

$$
\sum_{|\alpha|=2} \mathcal{J}^\alpha \mathcal{J} \cdot \nabla u = \sum_{|\alpha|=2} \left(\mathcal{J}^\alpha \mathcal{P} u + 4it \mathcal{J}^\alpha \mathcal{L} u\right) = \sum_{|\alpha|=2} \left(\mathcal{J}^\alpha \mathcal{P} u + 4i\lambda t \mathcal{J}^\alpha \overline{u}^2\right).
$$

Hence we find

$$
\begin{aligned}
\sum_{|\alpha|=2} \left\|\mathcal{J}^\alpha \mathcal{J} \cdot \nabla u\right\|_{\mathbf{L}^2} &\le \sum_{|\alpha|=2} \left(\left\|\mathcal{J}^\alpha \mathcal{P} u\right\|_{\mathbf{L}^2} + Ct \left\|\mathcal{J}^\alpha \overline{u}^2\right\|_{\mathbf{L}^2} \right) \\
&\le C\varepsilon + Ct \sum_{|\alpha|=2} \left\|\mathcal{J}^\alpha \overline{u}^2\right\|_{\mathbf{L}^2}. \qquad (3.4.13)
\end{aligned}
$$

Since

$$\sum_{|\alpha|=2} \left\| \mathcal{J}^{\alpha} \overline{u}^2 \right\|_{\mathbf{L}^2} \leq C \left\| \Delta \mathcal{F} \mathcal{U} (-t) \overline{u}^2 \right\|_{\mathbf{L}^2}$$

$$\leq C \left(\sup_{|\xi| \leq 1} \left| |\xi|^2 \Delta \mathcal{F} \mathcal{U} (-t) \overline{u}^2 \right| + \sup_{1 \leq |\xi|} \left| |\xi|^{3+\delta} \Delta \mathcal{F} \mathcal{U} (-t) \overline{u}^2 \right| \right), \tag{3.4.14}$$

we consider

$$\mathcal{F} \mathcal{U} (-t) \overline{u}^2 = (2\pi)^3 \int_{\mathbf{R}^6} \mathrm{e}^{\frac{1}{4} itS(\eta,\xi)} \overline{\hat{\varphi}(z) \hat{\varphi}(w)} d\eta$$

where

$$S(\eta, \xi) = 2 |\eta|^4 + 2 (\eta \cdot \xi)^2 + |\eta|^2 |\xi|^2 + \frac{9}{8} |\xi|^4,$$

$$\hat{\varphi} = \mathcal{F} \mathcal{U} (-t) u, \quad z = \eta - \frac{\xi}{2} \quad \text{and} \quad w = -\left(\eta + \frac{\xi}{2} \right).$$

We compute

$$\begin{aligned} \Delta_{\xi} \mathcal{F} \mathcal{U} (-t) \overline{u}^2 &= (2\pi)^3 \int_{\mathbf{R}^6} \left(\Delta_{\xi} e^{\frac{1}{4} itS(\eta,\xi)} \right) \overline{\hat{\varphi}(z) \hat{\varphi}(w)} d\eta \\ &+ 2 (2\pi)^3 \int_{\mathbf{R}^6} \left(\nabla_{\xi} e^{\frac{1}{4} itS(\eta,\xi)} \right) \cdot \nabla_{\xi} \left(\overline{\hat{\varphi}(z) \hat{\varphi}(w)} \right) d\eta \\ &+ (2\pi)^3 \int_{\mathbf{R}^6} e^{\frac{1}{4} itS(\eta,\xi)} \Delta_{\xi} \left(\overline{\hat{\varphi}(z) \hat{\varphi}(w)} \right) d\eta \\ &= K_1 + K_2 + K_3. \end{aligned} \tag{3.4.15}$$

Denoting

$$\begin{aligned} M(\eta, \xi) = &-\frac{t^2}{16} \left(8 (\eta \cdot \xi)^2 \left(4 |\eta|^2 + \frac{9}{2} |\xi|^2 \right) + \left(2 |\eta|^2 + \frac{9}{2} |\xi|^2 \right)^2 |\xi|^2 \right) \\ &+ it \left(4 |\eta|^2 + 9 |\xi|^2 \right), \end{aligned}$$

we write K_1 in the form

$$\begin{aligned} K_1 &= (2\pi)^3 \int_{\mathbf{R}^6} \left(\Delta_{\xi} e^{\frac{1}{4} itS(\eta,\xi)} \right) \overline{\hat{\varphi}(z) \hat{\varphi}(w)} d\eta \\ &= (2\pi)^3 \int_{\mathbf{R}^6} e^{\frac{1}{4} itS(\eta,\xi)} M(\eta, \xi) \overline{\hat{\varphi}(z) \hat{\varphi}(w)} d\eta. \end{aligned}$$

We integrate by parts via the identity

$$e^{\frac{1}{4} itS(\eta,\xi)} = H \nabla_{\eta} \cdot \left(\eta e^{\frac{1}{4} itS(\eta,\xi)} \right)$$

with

$$\begin{aligned} H &= \left(6 + \frac{1}{4} it\eta \cdot \nabla_\eta S(\eta, \xi)\right)^{-1} \\ &= \left(6 + it\left(2|\eta|^4 + (\eta \cdot \xi)^2 + \frac{1}{2}|\xi|^2|\eta|^2\right)\right)^{-1} \end{aligned}$$

to get

$$\begin{aligned} |K_1| \leq C &\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} \eta \cdot \nabla_\eta \left(H\eta \cdot \nabla_\eta \left(H\eta \cdot \nabla_\eta \left(HM(\eta,\xi)\right)\right)\right) \overline{\hat{\varphi}(z)\hat{\varphi}(w)} d\eta \right| \\ &+ C\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H\eta \cdot \nabla_\eta \left(H\eta \cdot \nabla_\eta \left(HM(\eta,\xi)\right)\right) \eta \cdot \nabla_\eta \left(\overline{\hat{\varphi}(z)\hat{\varphi}(w)}\right) d\eta \right| \\ &+ C\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} \eta \cdot \nabla_\eta \left(H^2\eta \cdot \nabla_\eta \left(HM(\eta,\xi)\right)\right) \eta \cdot \nabla_\eta \left(\overline{\hat{\varphi}(z)\hat{\varphi}(w)}\right) d\eta \right| \\ &+ C\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H^2\eta \cdot \nabla_\eta \left(HM(\eta,\xi)\right) (\eta \cdot \nabla_\eta)^2 \left(\overline{\hat{\varphi}(z)\hat{\varphi}(w)}\right) d\eta \right| \\ &+ C\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} \eta \cdot \nabla_\eta \left(H\eta \cdot \nabla_\eta \left(H^2M(\eta,\xi)\right)\right) \eta \cdot \nabla_\eta \left(\overline{\hat{\varphi}(z)\hat{\varphi}(w)}\right) d\eta \right| \\ &+ C\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H\eta \cdot \nabla_\eta \left(H^2M(\eta,\xi)\right) (\eta \cdot \nabla_\eta)^2 \left(\overline{\hat{\varphi}(z)\hat{\varphi}(w)}\right) d\eta \right| \\ &+ C\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} \eta \cdot \nabla_\eta \left(H^3M(\eta,\xi)\right) (\eta \cdot \nabla_\eta)^2 \left(\overline{\hat{\varphi}(z)\hat{\varphi}(w)}\right) d\eta \right| \\ &+ C\left| \int_{\mathbf{R}^6} e^{\frac{1}{4}itS(\eta,\xi)} H^3M(\eta,\xi) (\eta \cdot \nabla_\eta)^3 \left(\overline{\hat{\varphi}(z)\hat{\varphi}(w)}\right) d\eta \right|. \end{aligned}$$

Therefore we obtain

$$|K_1| \leq Ct^2 \sum_{j=0}^{3} \kappa_{1,j} + Ct \sum_{j=0}^{3} \kappa_{2,1,j} \tag{3.4.16}$$

where

$$\begin{aligned} \kappa_{1,j} &= \int_{\mathbf{R}^6} |\xi|^6 (W(\eta,\xi))^3 \sum_{a+b=j} |(\eta \cdot \nabla)^a \hat{\varphi}(z)| \left|(\eta \cdot \nabla)^b \hat{\varphi}(w)\right| d\eta, \\ \kappa_{2,1,j} &= \int_{\mathbf{R}^6} \left(|\xi|^2 + |\eta|^2\right) (W(\eta,\xi))^2 \sum_{a+b=j} |(\eta \cdot \nabla)^a \hat{\varphi}(z)| \left|(\eta \cdot \nabla)^b \hat{\varphi}(w)\right| d\eta \end{aligned}$$

and

$$W(\eta,\xi) = \left(1 + t|\eta|^2 \left(|\eta|^2 + |\xi|^2\right)\right)^{-1}.$$

In the same method, we find

$$|K_2| \le Ct\left(\sum_{j=1}^{3}\kappa_{2,1,j}+\sum_{j=1}^{2}\kappa_{2,2,j}+\sum_{j=1}^{2}\kappa_{2,3,j}\right),\quad |K_3| \le C\sum_{j=0}^{1}\kappa_{3,j} \tag{3.4.17}$$

where

$$\kappa_{2,2,j} = \int_{\mathbf{R}^6}\left(|\xi|^2+|\eta|^2\right)(W(\eta,\xi))^2\sum_{a+b=j}|(\eta\cdot\nabla)^a(z\cdot\nabla)\hat{\varphi}(z)|\left|(\eta\cdot\nabla)^b\hat{\varphi}(w)\right|d\eta,$$

$$\kappa_{2,3,j} = \int_{\mathbf{R}^6}\left(|\xi|^2+|\eta|^2\right)(W(\eta,\xi))^2\sum_{a+b=j}|(\eta\cdot\nabla)^a\hat{\varphi}(z)|\left|(\eta\cdot\nabla)^b(w\cdot\nabla)\hat{\varphi}(w)\right|d\eta,$$

$$\kappa_{3,j} = \int_{\mathbf{R}^6}W(\eta,\xi)\sum_{\substack{|\alpha|+|\beta|=2\\ a+b=j}}|(\eta\cdot\nabla)^a\partial^\alpha\hat{\varphi}(z)|\left|(\eta\cdot\nabla)^b\partial^\beta\hat{\varphi}(w)\right|d\eta.$$

By (3.4.16), (3.4.17), Lemma 3.2.8, Lemma 3.2.9 and Lemma 3.2.10, we find

$$\begin{aligned}&\sum_{j=1}^{3}\left(\sup_{|\xi|\le 1}\left||\xi|^2K_j\right|+\sup_{1\le|\xi|}\left||\xi|^{3+\delta}K_j\right|\right)\\ &\le Ct^{-1+\gamma_1}\varepsilon+C(D+E)\varepsilon^{\frac{1}{2}}.\end{aligned}$$

By (3.4.15), we have

$$\begin{aligned}D+E &= \sup_{|\xi|\le 1}\left||\xi|^2\Delta\mathcal{F}\mathcal{U}(-t)\overline{u}^2\right|+\sup_{1\le|\xi|}\left||\xi|^{3+\delta}\Delta\mathcal{F}\mathcal{U}(-t)\overline{u}^2\right|\\ &\le\sum_{j=1}^{3}\left(\sup_{|\xi|\le 1}\left||\xi|^2K_j\right|+\sup_{1\le|\xi|}\left||\xi|^{3+\delta}K_j\right|\right)\\ &\le Ct^{-1+\gamma_1}\varepsilon+C(D+E)\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Solving the inequality, we find

$$D+E \le Ct^{-1+\gamma_1}\frac{\varepsilon}{1-C\varepsilon^{\frac{1}{2}}} \le Ct^{-1+\gamma_1}\varepsilon.$$

Using (3.4.14), we obtain

$$\begin{aligned}\sum_{|\alpha|=2}\left\|\mathcal{J}^\alpha\overline{u}^2\right\|_{\mathbf{L}^2} &\le C\left(\sup_{|\xi|\le 1}\left||\xi|^2\Delta\mathcal{F}\mathcal{U}(-t)\overline{u}^2\right|+\sup_{1\le|\xi|}\left||\xi|^{3+\delta}\Delta\mathcal{F}\mathcal{U}(-t)\overline{u}^2\right|\right)\\ &\le C(D+E)\\ &\le Ct^{-1+\gamma_1}\varepsilon.\end{aligned}$$

By (3.4.13), we get

$$\sum_{|\alpha|=2} \left\| \mathcal{J}^{\alpha} \mathcal{J} \cdot \nabla u \right\|_{\mathbf{L}^2} \leq C\varepsilon + Ct \sum_{|\alpha|=2} \left\| \mathcal{J}^{\alpha} \overline{u}^2 \right\|_{\mathbf{L}^2}$$
$$\leq C t^{\gamma_1} \varepsilon.$$

Using the identity

$$\sum_{j,k=1}^{6} \left\| \mathcal{J}_j \partial_{x_k} \phi \right\|_{\mathbf{L}^2}^2 = \frac{1}{2} \sum_{j,k=1}^{6} \left\| \Omega_{j,k} \phi \right\|_{\mathbf{L}^2}^2 + \left\| \mathcal{J} \cdot \nabla \phi \right\|_{\mathbf{L}^2}^2,$$

we estimate

$$\sum_{|\alpha|=3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^1} \leq C \left(\sum_{1 \leq |\alpha| \leq 2, |\beta| \leq 1} \left\| \mathcal{J}^{\alpha} Q^{\beta} u \right\|_{\mathbf{L}^2} + \sum_{|\alpha|=2} \left\| \mathcal{J}^{\alpha} \mathcal{J} \cdot \nabla u \right\|_{\mathbf{L}^2} \right)$$
$$\leq C \langle t \rangle^{\gamma_1} \varepsilon.$$

Applying the Sobolev inequality, we obtain

$$\sum_{|\alpha|=3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^2} \leq \sum_{|\alpha|=3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^1}^{\frac{1}{2}} \sum_{|\alpha|=3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^3}^{\frac{1}{2}}$$
$$\leq C \langle t \rangle^{\frac{\gamma_1+\gamma_3}{2}} \varepsilon$$
$$\leq C \langle t \rangle^{\gamma_2} \varepsilon,$$

where $\gamma_2 = 3\gamma_1$ and $\gamma_3 = 5\gamma_1$. By the following identities $\mathcal{J} \cdot \nabla = \mathcal{P} + 4it\mathcal{L}$ and

$$\sum_{j,k=1}^{6} \left\| \mathcal{J}_j \partial_{x_k} \phi \right\|_{\mathbf{L}^2}^2 = \frac{1}{2} \sum_{j,k=1}^{6} \left\| \Omega_{j,k} \phi \right\|_{\mathbf{L}^2}^2 + \left\| \mathcal{J} \cdot \nabla \phi \right\|_{\mathbf{L}^2}^2,$$

we find

$$\sum_{|\alpha|=3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^3} \leq C\varepsilon + Ct^3 \left\| \mathcal{L}^3 u \right\|_{\mathbf{L}^2}.$$

By (3.1.1) and $\mathcal{L} = i\partial_t - \frac{1}{4}\Delta^2$, we obtain

$$\left\| \mathcal{L}^3 u \right\|_{\mathbf{L}^2} \leq C \left\| \mathcal{L}^2 \overline{u}^2 \right\|_{\mathbf{L}^2} \leq C \left(\left\| \partial_t^2 \overline{u}^2 \right\|_{\mathbf{L}^2} + \left\| \partial_t \Delta^2 \overline{u}^2 \right\|_{\mathbf{L}^2} + \left\| \Delta^4 \overline{u}^2 \right\|_{\mathbf{L}^2} \right)$$
$$\leq C \left(\left\| (\partial_t u)^2 \right\|_{\mathbf{L}^2} + \left\| \left(\partial_t^2 u \right) u \right\|_{\mathbf{L}^2} + \left\| \Delta^2 \left(u \partial_t u \right) \right\|_{\mathbf{L}^2} + \left\| u \Delta^4 u \right\|_{\mathbf{L}^2} \right)$$
$$\leq C \left(\left\| u^4 \right\|_{\mathbf{L}^2} + \left\| \left(\Delta^2 u^2 \right) u \right\|_{\mathbf{L}^2} + \left\| u^2 \Delta^2 u \right\|_{\mathbf{L}^2} + \left\| \left(\Delta^2 u \right)^2 \right\|_{\mathbf{L}^2} + \left\| u \Delta^4 u \right\|_{\mathbf{L}^2} \right).$$

Using Corollary 3.2.5, we have

$$\begin{aligned}
& t^3 \left(\left\|u^4\right\|_{\mathbf{L}^2} + \left\|\left(\Delta^2 u^2\right) u\right\|_{\mathbf{L}^2} + \left\|u^2 \Delta^2 u\right\|_{\mathbf{L}^2}\right) \\
& \leq t^3 \left(\|u\|_{\mathbf{L}^8}^4 + \left\|\Delta^2 u^2\right\|_{\mathbf{L}^2} \|u\|_{\mathbf{L}^\infty} + \left\|u^2 \Delta^2 u\right\|_{\mathbf{L}^2}\right) \\
& \leq t^3 \left(\|u\|_{\mathbf{L}^8}^4 + \left\|u \Delta^2 u\right\|_{\mathbf{L}^2} \|u\|_{\mathbf{L}^\infty} + \left\|u^2 \Delta^2 u\right\|_{\mathbf{L}^2}\right) \\
& \leq t^3 \left(\|u\|_{\mathbf{L}^8}^4 + \|u\|_{\mathbf{L}^2} \left\|\Delta \mathcal{U}(t) \mathcal{U}(-t) \Delta u\right\|_{\mathbf{L}^\infty} \|u\|_{\mathbf{L}^\infty}\right) \\
& \leq C \varepsilon^{\frac{3}{2}}.
\end{aligned}$$

Therefore, we obtain

$$\sum_{|\alpha|=3} \|\mathcal{J}^\alpha u\|_{\dot{\mathbf{H}}^3} \leq C\varepsilon + Ct^3 \left\|\left(\Delta^2 u\right)^2\right\|_{\mathbf{L}^2} + Ct^3 \left\|u \Delta^4 u\right\|_{\mathbf{L}^2}.$$

By the identity $\Delta \nabla = \frac{1}{it}(\mathcal{J} - x)$, we get

$$\left\|u \Delta^4 u\right\|_{\mathbf{L}^2} \leq Ct^{-1} \left(\left\|u \Delta^2 u\right\|_{\mathbf{L}^2} + \left\|u \Delta^2 \nabla \cdot \mathcal{J} u\right\|_{\mathbf{L}^2} + \left\|u \left(x \cdot \Delta^2 \nabla u\right)\right\|_{\mathbf{L}^2}\right).$$

Applying Corollary 3.2.5 and Lemma 3.2.6, we obtain

$$\begin{aligned}
t^2 \left\|u \Delta^2 u\right\|_{\mathbf{L}^2} & \leq Ct \left(\|u\|_{\mathbf{L}^\infty} \|\mathcal{J} \cdot \nabla u\|_{\mathbf{L}^2} + \||x| \, u\|_{\mathbf{L}^\infty} \|\nabla u\|_{\mathbf{L}^2}\right) \\
& \leq C\varepsilon,
\end{aligned}$$

$$\begin{aligned}
t^2 \left\|u \Delta^2 \nabla \cdot \mathcal{J} u\right\|_{\mathbf{L}^2} & \leq t \left\|u \Delta \left(x - \mathcal{J}\right) \cdot \mathcal{J} u\right\|_{\mathbf{L}^2} \\
& \leq Ct \left(\|u\|_{\mathbf{L}^\infty} + \||x| \, u\|_{\mathbf{L}^\infty}\right) \sum_{|\alpha| \leq 2} \|\mathcal{J}^\alpha u\|_{\mathbf{H}^2} \\
& \leq C\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
t^2 \left\|u \left(x \cdot \Delta^2 \nabla u\right)\right\|_{\mathbf{L}^2} & \leq t^2 \|u\|_{\mathbf{L}^2} \left\|x \cdot \Delta^2 \nabla u\right\|_{\mathbf{L}^\infty} \\
& \leq t^2 \|u\|_{\mathbf{L}^2} \left\||x| \left|\left|\partial_x\right|^{3-\theta} \left(\left|\partial_x\right|^{1+\theta} \nabla u\right)\right|\right\|_{\mathbf{L}^\infty} \\
& \leq C \langle t \rangle^{\gamma_1} \varepsilon^{\frac{1}{2}} \left(\|u\|_{\mathbf{H}^{12}} + \sum_{|\alpha| \leq 1} \|\mathcal{J}^\alpha u\|_{\mathbf{H}^9} + \sum_{|\alpha| \leq 3} \|\mathcal{J}^\alpha u\|_{\dot{\mathbf{H}}^{2+\theta}}\right) \\
& \leq C \langle t \rangle^{\gamma_1 + 4\gamma_1} \varepsilon,
\end{aligned}$$

where $\theta > 0$ is small. Thus, we have

$$t^3 \left\|u \Delta^4 u\right\|_{\mathbf{L}^2} \leq C \langle t \rangle^{\gamma_3} \varepsilon,$$

where $\gamma_3 = 5\gamma_1$. Similarly, we get

$$\left\| \left(\Delta^2 u\right)^2 \right\|_{\mathbf{L}^2} \leq Ct^{-1} \left(\left\| u\Delta^2 u \right\|_{\mathbf{L}^2} + \left\| \Delta^2 u \nabla \cdot \mathcal{J} u \right\|_{\mathbf{L}^2} + \left\| \Delta^2 u \left(x \cdot \nabla u \right) \right\|_{\mathbf{L}^2} \right).$$

By Corollary 3.2.5 and Lemma 3.2.6, we get

$$\begin{aligned} t^2 \left\| \Delta^2 u \nabla \cdot \mathcal{J} u \right\|_{\mathbf{L}^2} &\leq Ct^2 \left\| \nabla \cdot \mathcal{J} u \right\|_{\mathbf{L}^2} \left\| \left|\partial_x\right|^{3-\theta} \left|\partial_x\right|^{\theta} u \right\|_{\mathbf{L}^\infty} \\ &\leq C\varepsilon \end{aligned}$$

and

$$\begin{aligned} t^2 \left\| \Delta^2 u \left(x \cdot \nabla u \right) \right\|_{\mathbf{L}^2} &= t^2 \left\| \left(x\Delta^2 u \right) \cdot \nabla u \right\|_{\mathbf{L}^2} \\ &\leq t^2 \left\| u \right\|_{\mathbf{H}^1} \left\| |x| \left| \left|\partial_x\right|^{3-\theta} \left(\left|\partial_x\right|^{1+\theta} u \right) \right| \right\|_{\mathbf{L}^\infty} \\ &\leq C \left\langle t \right\rangle^{\gamma_1} \varepsilon^{\frac{1}{2}} \left(\left\| u \right\|_{\mathbf{H}^{12}} + \sum_{|\alpha| \leq 1} \left\| \mathcal{J}^{\alpha} u \right\|_{\mathbf{H}^9} + \sum_{|\alpha| \leq 3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^{1+\theta}} \right) \\ &\leq C \left\langle t \right\rangle^{\gamma_3} \varepsilon, \end{aligned}$$

where $\theta > 0$ is small. Thus, we have

$$t^3 \left\| \left(\Delta^2 u\right)^2 \right\|_{\mathbf{L}^2} \leq C \left\langle t \right\rangle^{\gamma_3} \varepsilon.$$

Consequently, we find

$$\sum_{|\alpha|=3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^3} \leq C \left\langle t \right\rangle^{\gamma_3} \varepsilon.$$

Thus, we obtain

$$\sup_{t \in [0,T]} \sum_{j=0}^{3} \left\langle t \right\rangle^{-\gamma_j} \sum_{|\alpha|=3} \left\| \mathcal{J}^{\alpha} u \right\|_{\dot{\mathbf{H}}^j} \leq C\varepsilon. \tag{3.4.18}$$

Next we estimate $\left\| \left\langle \cdot \right\rangle^3 \mathcal{F}\mathcal{U}\left(-t\right) u \right\|_{\mathbf{L}^q}$. Applying $\mathcal{F}\mathcal{U}\left(-t\right)$ to both sides of (3.1.1), we get

$$i\partial_t \left(\mathcal{F}\mathcal{U}\left(-t\right) u \right) = \lambda \mathcal{F}\mathcal{U}\left(-t\right) \overline{u}^2.$$

Therefore, we have

$$\mathcal{F}\mathcal{U}\left(-t\right) u = \hat{u_0} - i\lambda \int_0^t \mathcal{F}\mathcal{U}\left(-t\right) \overline{u}^2 d\tau.$$

By Lemma 3.2.7, we obtain

$$\begin{aligned}
&\left\| \langle\cdot\rangle^3 \mathcal{F}\mathcal{U}(-t)\, u \right\|_{\mathbf{L}^q} \\
&\leq \left\| \langle\cdot\rangle^3 \hat{u_0} \right\|_{\mathbf{L}^q} + C\int_0^1 \left\| \langle\cdot\rangle^3 \mathcal{F}\mathcal{U}(-t\tau)\, \overline{u}^2 \right\|_{\mathbf{L}^q} d\tau + C\int_1^t \left\| \langle\cdot\rangle^3 \mathcal{F}\mathcal{U}(-\tau)\, \overline{u}^2 \right\|_{\mathbf{L}^q} d\tau \\
&\leq C \left\| u_0 \right\|_{\mathbf{H}^{3,3}} + C\varepsilon + C\int_1^t \left\| \langle\cdot\rangle^4 \mathcal{F}\mathcal{U}(-\tau)\, \overline{u}^2 \right\|_{\mathbf{L}^\infty} d\tau \\
&\leq C\varepsilon.
\end{aligned}$$

Thus we get

$$\sup_{t\in[0,T]} \left\| \langle\cdot\rangle^3 \mathcal{F}\mathcal{U}(-t)\, u \right\|_{\mathbf{L}^q} \leq C\varepsilon. \tag{3.4.19}$$

Collecting estimates (3.4.1) - (3.4.3), (3.4.10), (3.4.11), (3.4.18) and (3.4.19), we obtain

$$\| u(t) \|_{\mathbf{X}'} \leq C\varepsilon < \varepsilon^{\frac{1}{2}}$$

for all $t \in [0,T]$. Then we get a contradiction. Therefore, we can let $T = \infty$. Let $0 \leq \alpha < 3$, $\frac{18}{6-\alpha} < r_1 < q$ and $\gamma \geq \frac{3}{2}\left(\frac{1}{r_1} - \frac{1}{q}\right) + \gamma_0$. Due to Corollary 3.2.5, we have

$$\begin{aligned}
\| u \|_{\dot{\mathbf{H}}^{\alpha}_q} &\leq C \langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{r_1}\right)-\frac{\alpha}{4}} \left(\| u \|_{\mathbf{H}^6} + \sum_{|\alpha|\leq 3} \| \mathcal{J}^\alpha u \|_{\mathbf{L}^2} \right) \\
&\leq C\varepsilon^{\frac{1}{2}} \langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{\alpha}{4}+\frac{3}{2}\left(\frac{1}{r_1}-\frac{1}{q}\right)+\gamma_0} \\
&\leq C \langle t\rangle^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{\alpha}{4}+\gamma}
\end{aligned}$$

for any $t \geq 0$. This completes the proof of Theorem 3.1.2. □

3.5 Proof of Corollary 3.1.3

By (3.1.1), we get

$$u = \mathcal{U}(t)\, u_0 - i\int_0^t \mathcal{U}(t-\tau)\, f(u,\overline{u})\, d\tau,$$

from which it follows that

$$\mathcal{U}(-t)\, u(t) - \mathcal{U}(-s)\, u(s) = i\int_s^t \mathcal{U}(-\tau)\, f(u,\overline{u})\, d\tau.$$

Using Corollary 3.2.5 and Lemma 3.2.6, we obtain

$$\lim_{t,s\to\infty}\|\mathcal{U}(-t)u(t)-\mathcal{U}(-s)u(s)\|_{\mathbf{H}^{12}\cap\mathbf{H}^{9,1}}=0. \tag{3.5.1}$$

By (3.4.4), we have

$$\sum_{|\alpha|=2}x^{\alpha}\mathcal{U}(-t)u$$
$$=\sum_{|\alpha|=2}x^{\alpha}u_0-t^2\mathcal{U}(-t)\mathcal{A}_{2,1}-i\int_0^t\mathcal{U}(-\tau)\left(\mathcal{A}_0+\mathcal{A}_1+\mathcal{R}_{\mathcal{A}}+2i\tau\mathcal{A}_{2,1}+\tau^2\mathcal{A}_{2,2}\right)d\tau.$$

In the same way of the proof of (3.4.5), we obtain

$$\sum_{|\alpha|=2}\|x^{\alpha}\mathcal{U}(-t)u(t)-x^{\alpha}\mathcal{U}(-s)u(s)\|_{\mathbf{L}^2}$$
$$\le C\left(\langle t\rangle^{-k}+\langle s\rangle^{-k}\right)+C\int_s^t\langle\tau\rangle^{-1-k}d\tau,$$

where $0<k<\frac{1}{8}-\gamma_0-\gamma_1$. Therefore, we find

$$\lim_{t,s\to\infty}\sum_{|\alpha|=2}\|x^{\alpha}\mathcal{U}(-t)u(t)-x^{\alpha}\mathcal{U}(-s)u(s)\|_{\mathbf{L}^2}=0. \tag{3.5.2}$$

Collecting (3.4.6)-(3.4.9), we have

$$\sum_{\substack{|\alpha|=2\\|\delta|=6}}\left\|\mathcal{U}(-t)\left(\partial_x^{\delta}\mathcal{J}^{\alpha}-\Delta^2Q^{\alpha}\right)u(t)-\mathcal{U}(-s)\left(\partial_x^{\delta}\mathcal{J}^{\alpha}-\Delta^2Q^{\alpha}\right)u(s)\right\|_{\mathbf{L}^2}$$
$$\le C\left(\langle t\rangle^{-k}+\langle s\rangle^{-k}\right),$$

where $0<k<\frac{1}{4}$. In the same way of proof of (3.5.2), we get

$$\sum_{|\alpha|=2}\left\|\mathcal{U}(-t)\Delta^2Q^{\alpha}u(t)-\mathcal{U}(-s)\Delta^2Q^{\alpha}u(s)\right\|_{\dot{\mathbf{H}}^6}$$
$$\le C\left(\langle t\rangle^{-k}+\langle s\rangle^{-k}\right)+C\int_s^t\langle\tau\rangle^{-1-k}d\tau,$$

where $0<k<\frac{1}{4}$. Thus, we find

$$\lim_{t,s\to\infty}\sum_{|\alpha|=2}\|x^{\alpha}\mathcal{U}(-t)u(t)-x^{\alpha}\mathcal{U}(-s)u(s)\|_{\dot{\mathbf{H}}^6}=0. \tag{3.5.3}$$

By (3.5.1)-(3.5.3), we have

$$\lim_{t,s\to\infty}\|\mathcal{U}(-t)u(t)-\mathcal{U}(-s)u(s)\|_{\mathbf{H}^{12}\cap\mathbf{H}^{9,1}\cap\mathbf{H}^{6,2}}=0.$$

Therefore, there exists a unique final state $u_+\in\mathbf{H}^{12}\cap\mathbf{H}^{9,1}\cap\mathbf{H}^{6,2}$, such that

$$\lim_{t\to\infty}\|\mathcal{U}(-t)u(t)-u_+\|_{\mathbf{H}^{12}\cap\mathbf{H}^{9,1}\cap\mathbf{H}^{6,2}}=0.$$

This completes the proof of Corollary 3.1.3.

Acknowledgments

First of all, my deepest gratitude goes to Professor Nakao Hayashi. He gave me valuable advise, supports and encouragements for six years as the supervisor.

I would like to express my sincere thanks to Professors Shin-ichi Doi, Takahisa Inui, Soichiro Katayama, Satoshi Masaki, Hayato Miyazaki, Haruya Mizutani, Tatsuo Nishitani, Hideaki Sungawa, Naohito Tomita and Kota Uriya for giving me helpful comments, warm encouragements and kindness.

I am deeply grateful to Professor Gaku Hoshino, Dr. Yuichiro Kawahara and Mr. Takahiro Ogino for their helpful suggestions and warm encouragements.

I would like to express my gratitude to Professor Pavel I. Naumkin for our joint work [2].

Finally, special thanks go to my family for supporting me for a long time.

Bibliography

[1] K. Aoki.: *Global existence of small solutions for a quadratic nonlinear fourth-order Schrödinger equation in six space dimensions*, Tohoku Mathematical Journal. to appear.

[2] K. Aoki, N. Hayashi, P. I. Naumkin.: *Global existence of small solutions for the fourth-order nonlinear Schrödinger equation*, Nonlinear Differ. Equ. Appl. 23:65 (2016)

[3] M. Ben-Artzi, H. Koch, J. C. Saut.: *Dispersion estimates for fourth order Schrödinger equations*, C.R.A.S. Série 1 **330**. 87–92 (2000)

[4] S. Cohn.: *Global existence for the nonresonant Schrödinger equation in two space dimensions*, Canadian Applied Mathematical Quarterly, **2**. 257–282 (1994)

[5] R. R. Coifman, Y. Meyer.: *Au delà des opèrateurs pseudodiffèrentieles*, Astèrique, **57**, Socièté Mathèmatique de France (1978)

[6] K. B. Dysthe.: *Note on a modification to the nonlinear Schrödinger equation for application to deep water waves*, Proc. R. Soc. Lond. Ser. A **369**, 105–114 (1979)

[7] M.V. Fedoryuk, Asymptotic Methods in Analysis, in: Analysis. I. Integral representations and asymptotic methods. Encyclopaedia of Mathematical Sciences, 13. Springer-Verlag, Berlin, 1989. vi+238 pp.

[8] C. Guo, S. Cui.: *Global existence of solutions for a fourth-order nonlinear Schrödinger equation*, Appl. Math. Lett. **19**, 706–711 (2006)

[9] N. Hayashi.: *Global existence of small solutions to quadratic nonlinear Schrödinger equations*, Commun. Partial Differ. Equ. **18**, 1109–1124 (1993)

[10] N. Hayashi, Jesus A. Mendez-Navarro and P.I. Naumkin, *Scattering of solutions to the fourth-order nonlinear Schrödinger equation.* Commun. Contemp. Math., **18** (2016), no. 3, 1550035, 24 pp.

[11] N. Hayashi, Jesus A. Mendez-Navarro and P.I. Naumkin, *Asymptotics for the fourth-order nonlinear Schrödinger equation in the critical case*, Journal of Differential Equations. **261**(9), 5144–5179 (2016)

[12] N. Hayashi, T. Mizumachi, P. I. Naumkin.: *Time decay of small solutions to quadratic nonlinear Schrödinger equations in 3D*, Differential Integral Equations. **16**, 159–179 (2003)

[13] N. Hayashi, P. I. Naumkin.: *On the quadratic nonlinear Schrödinger equation in three space dimensions*, International Mathematics Research Notices. **3**, 115–132 (2000)

[14] N.Hayashi and P. Naumkin, *Domain and range of the modified wave operator for Schrodinger equations with a critical nonlinearity*, Commun. Math. Phys., **267** (2006), 477-492

[15] N.Hayashi and P. Naumkin, *On the new critical exponent for the nonlinear Schrödinger equations*, NoDEA Nonlinear Differential Equations Appl.,**21**(2014),no.3, 415-440.

[16] Z. Huo, Y. Jia.: *The Cauchy problem for the fourth-order nonlinear Schrödinger equation related to the vortex filament*, J. Differ. Equ. **214**, 1–35 (2005)

[17] Z. Huo, Y. Jia.: *Well-posedness for the fourth-order nonlinear derivative Schrödinger equation in higher dimension*, J. Math. Pures Appl. **96**(2), 190–206 (2011)

[18] Y. Kawahara.: *Global existence and asymptotic behavior of small solutions to nonlinear Schrödinger equations in 3D*, Differential Integral Equations. **18**, 169–194 (2005)

[19] J. Segata.: *Well-posedness for the fourth-order nonlinear Schrödinger type equation related to the vortex filament*, Differ. Integral Equ. **16**(7), 841–864 (2003)

[20] J. Shatah.: *Normal form and quadratic nonlinear Klein-Gordon equations*, Commun. Pure Appl. Math. **38**, 685–696 (1985)

[21] W. A. Strauss.: *Nonlinear scattering theory at low energy*, J. Funct. Anal. **41**, 110–133 (1981)

[22] N. Tomita.: *A Hörmander type multiplier theorem for multilinear operators*, J. Funct. Anal. **259**, 2028–2044 (2010)

[23] Y. Wang.: *Global well-posedness for the generalized fourth-order Schrödinger equation*, Bull. Aust. Math. Soc. **85**(3), 371–379 (2012)

List of the author's papers cited in this thesis

1. K. Aoki.: *Global existence of small solutions for a quadratic nonlinear fourth-order Schrödinger equation in six space dimensions*, Tohoku Mathematical Journal. to appear.

2. K. Aoki, N. Hayashi, P. I. Naumkin.: *Global existence of small solutions for the fourth-order nonlinear Schrödinger equation*, Nonlinear Differ. Equ. Appl. 23:65 (2016)