Title	Towards Infrared Finite S-matrix in Quantum Field Theory
Author(s)	平井, 隼人
Citation	大阪大学, 2020, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/76372
rights	
Note	

Ph.D. thesis

Towards Infrared Finite S-matrix in Quantum Field Theory

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February 2020

Abstract

We study infrared dynamics in quantum electrodynamics to construct the well-defined S-matrix without infrared divergences. S-matrix is a fundamental quantity for the scattering theory of particles in quantum field theories. However, the conventional S-matrix for theories with massless particles is not well-defined due to the infrared divergences. This problem originates in the fact that the interactions mediated by low energy massless particles create infinitely long-range forces between charged particles. Therefore, the better understanding of the infrared dynamics is necessary for improving the S-matrix. In the first half of this thesis, we focus on the following subjects that capture the universal features of the infrared dynamics: asymptotic symmetry, soft theorem, and memory effect. We elucidate the fundamental properties of the charge conservation law associated with the asymptotic symmetry and also develop the new relations among the three subjects. In the last half, the proper asymptotic states for the infrared finite S-matrix is investigated. The Faddeev-Kulish(F-K) dressed state has been known as a candidate for such a state. However, there was an argument that the F-K dressed states are not gauge invariant. We resolve the problem by deriving a correct gauge invariant condition and showing that the F-K dressed state is a solution of the condition. We also discuss the relation between the asymptotic state and the asymptotic symmetry for QED.

Acknowledgement

First and foremost, I would like to thank my advisor Satoshi Yamaguchi for his support during my Ph.D. program. I have always been influenced by his very deep understanding of physics and also how to explain complicated concepts in a simple and clear way. I also thank him for allowing me to work freely on the topics that I am interested in, and for always supporting my research enormously. Second, I am very grateful to my collaborator, Sotaro Sugishita. This thesis is largely based on the two joint papers with him. I have always enjoyed discussing with him and I have learned so many things through the joint research with him. I would also like to express my appreciation to Kotaro Tamaoka for many discussions and his kind support in the last few years. Third, I would like to thank Masaharu Aoki, Minoru Tanaka, Satoshi Yamaguchi, Shinya Kanemura, and Tetsuya Onogi for reading this thesis carefully and giving me useful comments.

I am also deeply grateful to all the members of the Particle Physics Theory Group at Osaka University for providing me with a good environment during my Ph.D. program. I acknowledge financial support from the JSPS fellowship for the past year.

Last but not least, I would like to express my gratitude to my family, in particular my parents, for their constant support and encouragement.

Contents

Chapter 1

Introduction and Summary

1.1 Introduction: IR triangle and IR finite S-matrix

This thesis is devoted to a better understanding of infrared structures of scattering theory in gauge theories, in particular, *quantum electrodynamics (QED)* in four-dimensional asymptotically flat spacetime. The investigation of scattering phenomena in the gauge theories describing our world at low energy, such as QED, has a long history since the early nineteenth century. The comparison between theoretical predictions and experimental data of scattering cross-sections has been one of the main sources of ideas in developing the models explaining our world. Nowadays the Standard Model of particle physics has demonstrated tremendous successes in explaining the data provided by collider experiments with great accuracy, at least with the current experimental resolution.

However, the infrared dynamics in the gauge theories involving long-range forces has recently turned out to be worth reinvestigating, which was first triggered by the discovery of the *asymptotic symmetry* in Maxwell and Yang-Mills theory coupled to massless charged matters [1]. Moreover, the discovery of these new symmetries led to the discoveries of a triangle equivalence of seemingly unrelated following three subjects that concern the infrared dynamics in QED, QCD, gravity: *asymptotic symmetry, soft theorem*, and *memory effect*. This equivalence, called *infrared triangle*, is one of the main subjects in this thesis.

Asymptotic symmetry transformations, which I will define precisely later, refers to a gauge transformation which changes boundary conditions on asymptotic spacetime regions while keeping physically reasonable fall-offs behaviors of gauge fields. Although asymptotic symmetry is local symmetry, this is *physical symmetry*. The history of the asymptotic symmetry analysis goes back to the seminal work in gravity by Bondi, van der Burg, Metzner and Sachs (BMS) [2, 3] in 1962. They found the infinite-dimensional subgroup of diffeomorphisms of asymptotically flat spacetime that act non-trivially on the boundary data, which is now called the BMS group. On the other hand, as already mentioned, the asymptotic symmetries in gauge theories in asymptotically flat spacetime were revealed recently [1, 4, 5, 6]

The *soft theorems* describe the universal property of scattering amplitudes with external *soft* particles *i.e.* massless particles whose energies are much less than the energies of external charged matters. For example, the soft photon theorem [7, 8, 9, 10, 11, 12, 13, 14] in QED simply says that the scattering amplitude of the process $\alpha \to \beta$, say $\mathcal{M}_{\beta\alpha}$, and the one with an additional external soft photon of momentum $k^\mu = (\omega, \vec{k})$, say $\mathcal{M}_{\beta\alpha}(k)$, are proportional to each other as

$$\mathcal{M}_{\beta\alpha}(k) = \mathcal{M}_{\beta\alpha} \sum_{n}^{N} \frac{\eta_n e_n \epsilon(k) \cdot p_n}{p_n \cdot k} + \mathcal{O}(\omega^0), \tag{1.1.1}$$

where e_n and p_n are the charge and the four-momentum of n-th particle in the initial state α and the final state β, $\epsilon_\mu(k)$ is the polarization of the soft photon, and η_n is a sign factor which takes $+1$ for particles in β and -1 for particles in α. The factor of proportionality, called the (leading) soft factor, diverges as the energy of soft photon tends to zero, since it is order $\mathcal{O}(\omega^{-1})$. This fact reflects one of the important property of infrared dynamics; the slight acceleration of a charged particle results in the radiation of infinite number of low energy photons.

The *memory effect* is concerned with the detection of waves coming from faraway sources. It has been investigated mainly as a mechanism for the observation of gravitational waves, called the gravitational memory effect. It originates in a proposal in 1974 by Zel'dovich and Polnarev [15], and developed by many others [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. The gravitational memory effect claims that the passage of gravitational waves coming from faraway sources produces a permanent displacement of the metric and results in a permanent displacement of relative position of freely falling particles (viewed as detectors). Detection of the memory effect at LISA [27] and at LIGO [28] has been proposed recently. Although the gravitational memory effect has a long history, the electromagnetic analog of the memory effect was first studied recently [29, 23, 30]. The electromagnetic memory effect is a phenomenon: the total net charge that has passed through at an local angle is given by a permanent displacement of the gauge field at the corresponding angle, instead of the metric in gravity.

The infrared (IR) triangle reveals the surprising fact that the above three subjects, describing seemingly different aspects in infrared (long distance) physics, are just different perspective of one subject. The equivalence of those was first discussed in the Yang-Mills theory [1], and extended to gravity [31, 32, 33] and also to QED [4, 6, 5]. The equivalence has also been extended to higher orders of the soft expansion (*e.g.* [34, 35, 36, 37]) to higher spacetime dimensions (*e.g.* [38, 39]), and also to other theories (*e.g.* [40, 41, 42]) by many others [1]. The IR triangles are not just about the mathematical equivalence among the three things already known because the IR triangles are universal equivalences valid for many theories including massless particles and there were a few theories in which all corners of the triangle were fully understood. In fact, the insight from the viewpoint of the infrared triangle has led to the many discoveries of new corners in many different theories in the last several years, for example, see [35, 34, 42, 37, 36]. Furthermore, the application goes beyond just finding the new corners of the triangles. The investigation of asymptotic symmetries had led to many interesting applications to various directions: flat space holography [43, 44, 45, 46], black hole information paradox [33, 47, 48], and IR finite S-matrix [49, 50]. In particular the last one, construction of IR finite S-matrix in gauge theories, is one of our main motivation for pursuing the infrared physics.

The study of infrared dynamics involving the soft particles has played the important role in quantum field theory (QFT) and collider physics. As suggested by the soft theorems

[1] We have just referred to relatively old references here because there are too many. We will refer to the references directly related to our works in the later chapters.

in QED and (perturbative) gravity, the emissions of infinite number of soft photon and graviton are inevitable in any scattering process of charged particles. In other words, charged particles should be accompanied by the cloud of soft photons and gravitons in any scattering process. As a result, the S-matrix elements with a definite number of soft particles have *infrared divergences* in QED and gravity. Therefore, such S-matrix is not a well-defined object. This problem is sometimes referred as the *infrared problem.* There are two possible prescriptions for curing this problem: the *inclusive formalism* and the *dressed state formalism.* Here I will review the two prescriptions and their problems very briefly in the following.

The first one, the *inclusive formalism*, has been used as a standard prescription. In this formalism, observables are the *inclusive* transition probabilities that are given by sum over all the possible transition probabilities with emitted soft particles compatible with energy resolution of the detector. As is well known, the inclusive transition probabilities give finite results (at least for charged particles with definite momenta) and have been consistent with the experimental data so far. However, the S-matrix remains ill-defined in this formalism. Again, the origin of this problem would be the fact that charged particles are not accompanied by any clouds of soft particles. More concretely, we use free states, which are not surrounded by any cloud of soft particles, as the asymptotic states of charged particles. This seems unnatural or even illegal because the interactions mediated by photons and gravitons are long-range forces that cannot be ignored even when the charged particles are very far apart from each other.

The second one, the *dressed state formalism*, is a main subject in this thesis. This formalism tries to cure the S-matrix by using the modified asymptotic states surrounded by the cloud of soft particles, called *dressed states.* In other words, this formalism tries to compute the inclusive quantity at the level of S-matrix, instead of the transition probability. Our hope is that the S-matrix with properly constructed dressed states will give IR finite results and describes the scattering processes with hopefully higher precision. A dressed state was first introduced in the work by Chung [51] in 1965. The dressed state, which we call the *Chung's dressed state*, for a single electron with momentum $p^\mu = (E_p, \vec{p})$ is given by

$$|p\rangle_{Ch} = \exp\Big\{ -\frac{1}{2}\sum_{l=1,2}\int_\lambda \widetilde{d^3k}|S^{(l)}(k)|^2\Big\} \exp\Big\{ \sum_{l=1,2}\int_\lambda \widetilde{d^3k} S^{(l)}(k)a^{(l)\dagger}(k)\Big\} b^\dagger(p)|0\rangle \quad (1.1.2)$$

with

$$S^{(l)}(k) = \frac{ep\cdot\epsilon^{(l)}(k)}{k\cdot p} \quad (1.1.3)$$

where $k^\mu = (\omega, \vec{k})$ is a four-momentum of a photon and $\epsilon^{(l)}_\mu(k)(l = 1, 2)$ is the transverse polarization vectors of a photon. The measure $\widetilde{d^3k}$ is the Lorentz invariant measures for the integration of the spatial momentum which is defied as $\widetilde{d^3k} \equiv \frac{d^3k}{(2\pi)^3 2\omega_k}$, and λ is the photon mass introduced as an IR cutoff that is taken to zero at the end of the calculation. $a^{(l)\dagger}(k)$ and $b^\dagger(p)$ are the creation operators for a photon and an electron, respectively, and $|0\rangle$ is a Fock vacuum. The electron in Chung's dressed state is surrounded by the coherent photon cloud. He computed the S-matrix for Chung's dressed states and showed that all the IR divergences are canceled out at all orders of perturbative expansion in the S-matrix. This was a rather surprising observation. However, the function $S^{(l)}(k)$ for $k > 0$ can be chosen in any manner which makes the integral in (1.1.2) convergent as $k \to \infty$ because Chung's dressed states were given just by requiring to cancel out the infrared divergences. Therefore, these states do not have the ability to predict real scattering processes due to the non-specification of the hard momentum part of the dressing. After the Chung's discovery, Faddeev and Kulish in 1970 derived another version of dressed states by solving the infrared QED dynamics [52]. The dressed state, nowadays called the *Faddeev-Kulish (FK) dressed states*, is given by

$$|p_1, \cdots, p_N\rangle_{FK} = \lim_{t\to\pm\infty} e^{R(t)} e^{i\Phi(t,t_s)} |p_1, \cdots, p_N\rangle \tag{1.1.4}$$

with

$$R(t) \equiv \sum e \int \widetilde{d^3p}\, \rho(\vec{p}) \int \widetilde{d^3k} \frac{p^\mu}{p\cdot k} \left[a_\mu(\vec{k}) e^{i\frac{p\cdot k}{E_p}t} - a_\mu^\dagger(\vec{k}) e^{-i\frac{p\cdot k}{E_p}t} \right], \tag{1.1.5}$$

$$\Phi(t, t_s) = -\frac{e^2}{4\pi} \int \widetilde{d^3p}\ \widetilde{d^3q} : \rho(\vec{p})\rho(\vec{q}) : \frac{p\cdot q}{\sqrt{(p\cdot q)^2 - m^4}} \operatorname{sgn}(t) \ln \frac{|t|}{t_s}, \tag{1.1.6}$$

where $e\rho(\vec{p}) \equiv e(b^\dagger(\vec{p})b(\vec{p}) - d^\dagger(\vec{p})d(\vec{p}))$ is the charge density operator for electrons and positrons [2], and $|p_1, \cdots, p_N\rangle$ is a Fock state of electrons and positrons with the momenta $p_1, \cdots, p_N$. If we extract the soft momentum region $k \sim 0$ for the dressing operator (1.1.5), the operator takes the form

$$R_{\text{soft}} \sim \sum e \int \widetilde{d^3p}\, \rho(\vec{p}) \int_{\text{soft}} \frac{d^3k}{(2\pi)^3(2\omega)} \frac{p^\mu}{p\cdot k} \left[a_\mu(\vec{k}) - a_\mu^\dagger(\vec{k}) \right], \tag{1.1.7}$$

because $e^{i\frac{p\cdot k}{E_p}t} \sim 1$ at $k \sim 0$. This reproduces the cloud of soft photons used in Chung's dressed state (1.1.2). Once this approximation is justified, the IR finiteness of the S-matrix for the F-K dressed states follows from the proof by Chung. However, we need more careful analysis for the complete understanding of the contributions from the dressing factors. In addition, the contribution of the hard momentum region ($k > 0$) of the photon cloud (1.1.5) to physical observables is still not clear. Futhermore, Faddeev and Kulish argued in [52] that the dressed state in (1.1.4) is not gauge invariant because the state does not satisfy the Gupta-Bleuler condition. In order to make the state gauge invariant, they modified the dressed state by adding new terms in the dressing factor (1.1.5). However, such ad-hoc modification seemed unnatural because the dressed state (1.1.4) was derived from the QED dynamics. These problems in the dressed formalism should be resolved towards the complete formulation of the IR finite S-matrix theory.

Recently, one important relation between the IR divergences in S-matrix and the asymptotic symmetry was pointed out in [49]. In [49], it was argued that the appearance of IR divergences in the conventional S-matrix is a consequence of the conservation of the charges associated with the asymptotic symmetries. This relation strongly suggests that the asymptotic symmetry is one of the key ingredients for the construction of IR finite S-matrix.

[2] $b^{(\dagger)}(p)$ and $d^{(\dagger)}(p)$ is the annihilation (creation) operator for an electron and a positron, respectively.

1.2 Summary of the author's results

We present the following results in this thesis:

(I). **The asymptotic symmetry gives the charges that are conserved between the asymptotic future and past.** [in Chapter 2]

We derive the charges associated with the asymptotic symmetry in QED defined on the asymptotic Cauchy slices. We also show that the charges can have nontrivial values for physical solutions and the charges are conserved between the asymptotic future and past Cauchy slices by analyzing the asymptotic behaviors of classical fields and confirming that the contribution from the spatial infinity to the charge conservation vanishes.

(II). **The asymptotic symmetry is a physical symmetry in the canonical quantization and also in the BRST quantization.** [in Chapter 2]

We show that the charges that generate the asymptotic symmetry act on the physical Hilbert space nontrivially both in the canonical quantization and in the BRST quantization, although the symmetry is a local symmetry.

(III). **Equivalence between the soft photon theorem and the asymptotic symmetry at subleading order.** [in Chapter 2 & Chapter 4]

We find that the subleading photon theorem is equivalent to the charge conservation in massive scaler QED. We also find the relation between the conserved charge and the subleading component of the charge of asymptotic symmetry.

(IV). **The F-K dressed state is gauge invariant.** [in Chapter 5]

We show that the F-K dressed state (1.1.4) is gauge invariant by finding the BRST condition incorporating the asymptotic interaction and showing that the F-K dressed state satisfies the condition.

(V). **The F-K dressed state carries the asymptotic charges for the classical free charged particles with their relativistic Coulomb fields.** [in Chapter 5]

We show that the dressing operator in F-K dressed state carries asymptotic charge, and the eigenvalue is given by the classical asymptotic charge for the classical free charged particles with their relativistic Coulomb fields.This result leads to the quantum analog of the electromagnetic memory effect.

These results are mainly based on the following two author's works in collaboration with Sotaro Sugishita:

[53] H.Hirai and S.Sugishita,

"Conservation Laws from Asymptotic Symmetry and Subleading Charges in QED", *JHEP* **07** (2018) 122.

[54] H.Hirai and S.Sugishita,

"Dressed states from gauge invariance", *JHEP* **06** (2019) 023.

The result (I), (II), (III) are based on [53] and the result (IV) is based on [54].

1.3 Organization of this thesis

In the first three chapter, the fundamental results about the asymptotic symmetries, the soft theorems, and the memory effects are shown and their relationships are also discussed. In Chapter 2, we first explain the definition of the physical symmetry and the trivial symmetry, and then show that the asymptotic symmetry in QED is physical symmetry both in the classical theory and in the quantum theory. In Chapter 3, we review the definition of S-matrix, the soft photon theorem and the infrared divergence. Chapter 4 starts with the review of the equivalence between the asymptotic symmetry and the leading soft photon theorem. After that, we derive the charge whose conservation law is equivalent to the subleading soft photon theorem. In Chapter 5, we treat the construction and the properties of the asymptotic states for IR finite S-matrix and also discuss its relation to asymptotic symmetry. We conclude with further discussion in Chapter 6.

Chapter 2

Asymptotic symmetry

2.1 Physical symmetry and trivial symmetry

Symmetry is one of the most fundamental concepts in physics. Why are symmetries so important? Mostly, it is because symmetries constrain the property of dynamics or characterize the states of the system under consideration. For classical systems, the existence of a symmetry gives a conserved charge as a consequence of the Noether's theorem. The conservation law of the charge constrains the dynamics of the observables in the system. For example, a time translation symmetry in a system of multiple particles constrains the motion of the particles in such a manner that the total energy of the particles do not change during the time evolution, or in other words, the physical solutions can be characterized by the total energy. Also in quantum systems, a symmetry of Hamiltonian gives a conserved charge and it constrains the observables of the system. More generally, the constraint from a symmetry is given by the *Ward-Takahashi identity*. For example, if a system has a translation symmetry, the correlation functions are restricted to the functions that are invariant under the translation of the positions of fields. Once we recognize a symmetry in a system, it tells us nontrivial information of the observables and states in the system. In this sense, it would be natural to call a transformation a *physical* symmetry if it gives a nontrivial constraint to the theory. According to the definition, most of gauge (local) transformations are *not* physical symmetries both in classical theories and in quantum theories. Before explaining the reason for it, we first review some basic things about symmetry in classical theories.

Suppose we have a local conserved current J^μ in a theory. By integrating $\partial_\mu J^\mu = 0$ over a closed spacetime region B with boundary Σ ($\Sigma = \partial B$), we can define a charge $Q(\Sigma)$ as

$$Q(\Sigma) \equiv \int_B dV \partial_\mu J^\mu = \int_\Sigma dS_\mu J^\mu. \tag{2.1.1}$$

This charge is trivially zero because of $\partial_\mu J^\mu = 0$. Since we can freely change the spacetime region B in (2.1.1), the charge $Q(\Sigma)$ does not depend on the closed surface Σ, *i.e.*

$$Q(\Sigma) = Q(\Sigma') \tag{2.1.2}$$

holds for any closed surface Σ'. The conventional conservation of charges between different time slices follows from (2.1.1) by taking Σ as follows. Take Σ as a closed surface composed of the three surfaces, Σ_F, Σ_R, Σ_I. Σ_F and Σ_I are the spatial surfaces at $t = t_F$ and $t = t_I$ with the radius $0 \le r \le R$, respectively. Σ_R is the timelike surface at $r = R$ with $t_I \le t \le t_F$. Then the conservation of charge in (2.1.1) gives

$$0 = Q(\Sigma_F) + Q(\Sigma_R) - Q(\Sigma_I), \tag{2.1.3}$$

with $Q(\Sigma_a) = \int_{\Sigma_a} dS_\mu J^\mu$ where we choose dS^μ as the future-directed surface element when the surface is spacelike. In particular, $Q(\Sigma_R)$ is given by

$$Q(\Sigma_R) = R^2 \int_{t_I}^{t_F} dt \int d^2\Omega \; J^r(t, R, \hat{\Omega}), \tag{2.1.4}$$

where $\hat{\Omega}$ is a two-dimensional angular coordinate. Here, we usually drop this terms by assuming that the amplitude of the current falls off rapidly enough as $r \to \infty$. Under the assumption of $\lim_{R\to\infty} Q(\Sigma_R) = 0$, (2.1.3) gives

$$Q(t_F) = Q(t_I) \tag{2.1.5}$$

where $Q(t_F) = \lim_{R\to\infty} Q(\Sigma_F)$ and $Q(t_I) = \lim_{R\to\infty} Q(\Sigma_I)$ are the charges on the time slices at $t = t_F$ and $t = t_I$, respectively.

2.2 Classical asymptotic symmetry

Let us define the *classical asymptotic symmetry group (cASG)* as [1]

$$\text{cASG} = \frac{\text{classical allowed gauge transformations}}{\text{classical trivial gauge transformations}} \; . \tag{2.2.1}$$

Here, the *classical allowed gauge transformations* are any gauge transformations whose charges (generators) have well-defined and nontrivial values. The *classical trivial gauge transformations* are the ones whose charges (generators) are zero. Under this definition, we will see that nontrivial charges have one-to-one correspondence with the asymptotic symmetries.

Let's see what happens for the conservation of charges for gauge symmetries. Consider QED for concreteness. The Lagrangian is

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi - m\bar{\psi}\psi, \tag{2.2.2}$$

with $\not{D}\psi = \gamma^\mu(\partial_\mu \psi + ieA_\mu \psi)$. The gauge symmetry is the following local $U(1)$ transformation,

$$\psi(x) \to e^{ie\epsilon(x)}\psi(x) \; , \; \bar{\psi}(x) \to e^{-ie\epsilon(x)}\bar{\psi}(x) \; , \; A_\mu(x) \to A_\mu(x) + \partial_\mu \epsilon(x). \tag{2.2.3}$$

with an arbitrary function $\epsilon(x)$. The Noether current for this transformation is

$$J^\mu = F^{\mu\nu}\partial_\nu \epsilon + j^\mu_{mat}\epsilon, \tag{2.2.4}$$

where $j^\mu_{mat} = \bar{\psi}\gamma^0\psi$ is the matter current density for $U(1)$ global symmetry. The local current conservation $\partial_\mu J^\mu = 0$ is guaranteed by the equation of motion ($\partial_\mu F^{\mu\nu} = -j^\nu_{mat}$)

[1] Here, A/B is the quotient group of A by B.

and the conservation of the matter current ($\partial_\mu j^\mu_{mat} = 0$). If we set $\epsilon(x)$ to be constant, the the above current reduces to the $U(1)$ global current, since $\partial_\nu \epsilon$ vanishes. Using the equation of motion, we can write the current in the form of total derivative as

$$J^\mu = \partial_\nu (F^{\mu\nu}\epsilon). \tag{2.2.5}$$

Then the charge on a time slice Σ is given by

$$\begin{aligned} Q[\epsilon] &= \int d^3\Sigma^\mu J_\mu(x) \\ &= \int d^3x \left(\vec{\partial}\epsilon(x)\cdot\vec{E}(x) + \epsilon(x) j^0_{mat}(x)\right) && (2.2.6) \\ &= \int d^3x \partial_\nu \left(F^{0\nu}(x)\epsilon(x)\right) = \lim_{r\to\infty}\int d^2\Omega \; r^2 F^{0r}(x)\epsilon(x)\,. && (2.2.7) \end{aligned}$$

Here, we know that the radial component of the electric field, $F^{0r}(x)$, generally falls off like r^{-2} so as to have a finite amount of the charge. Therefore, if we choose $\epsilon(x)$ such that it falls off enough rapidly as $r\to\infty$, the charge vanishes, $Q[\epsilon]=0$, for *any* physical configurations. In this case, the conservation law is trivial in the sense that all physical solutions trivially satisfy the constraint. (In other words, we cannot use the charge to characterize physical solutions.) We call such gauge transformations *trivial* (or *small*) *gauge transformations*. In contrast, if $\epsilon(x)$ behaves like r^0 at spatial infinity, $Q[\epsilon]$ can have nontrivial finite value. We call such transformations *large gauge transformations*. The global symmetry is a special case of the large gauge symmetry because the charge in (2.2.7) reduces to the global $U(1)$ charge in the case of ϵ =constant.

In (2.2.7), we can see that the charge depends on the gauge parameter only by the value at $R=\infty$. Therefore, two large gauge transformations generated by two gauge parameters with the same values at $R=\infty$ are identified as the same asymptotic symmetry transformation, even though they have different values at $R<\infty$.

We have seen that the charge in (2.2.7) can have a nontrivial value for a large gauge parameter ϵ. However, it does not necessarily lead to the usual charge conservation between arbitrary different time slices. As already mentioned, $Q(\Sigma_R)=0$ with $R=\infty$ must vanish in order for the usual charge conservation (2.1.5) to hold. In the case of global $U(1)$ transformation, $Q(\Sigma_R)$ in (2.1.4) with $R=\infty$ for the current (2.2.4) is given by

$$\lim_{R\to\infty} Q(\Sigma_R) = \lim_{R\to\infty} R^2 \int_{t_I}^{t_F} dt \int d^2\Omega \;\; [F^{r\nu}\partial_\nu\epsilon + j^r_{mat}\epsilon] = 0\;. \tag{2.2.8}$$

Here, we have used $\lim_{R\to\infty} j^r_{mat} = 0$ because the matter current is localized at the position of the charged particles, and also $\partial_\nu\epsilon = 0$ for the global transformation. Thus

$$Q(t_F) = Q(t_I) \tag{2.2.9}$$

holds for the global symmetry. For the general large gauge transformations, we should be careful about the charge conservation laws because whether $Q(\Sigma_R)=0$ with $R=\infty$ vanishes or not is nontrivial. From now, we will study the charge conservation between infinitely past and future Cauchy slices because our purpose to study the asymptotic symmetry is to study the properties of S-matrix that is defined as the transition amplitude between two asymptotic states on infinite past and future. In order to do that, we will review the asymptotic structure of Minkowski spacetime in the next section and study the charge conservation between asymptotic regions in Section 2.4.

2.3 Asymptotic structure of Minkowski spacetime and useful coordinates

In this section, we review the asymptotic structures of Minkowski spacetime and introduce several kinds of coordinates that are useful to study the dynamics of massive and massless fields at the asymptotic past and future. To talk about the asymptotic regions, introducing the *Penrose diagram* is very useful. Because the Minkowski spacetime has infinite volume, the asymptotic regions like $t \to \pm\infty$ surface can not be visualized easily. Therefore, it is useful to scale the spacetime to some spacetime with finite volume so that we can draw the entire spacetime diagram on my thesis. The Penrose diagram of a spacetime is a diagram of a spacetime that is obtained by scaling the original spacetime in a way that the scaled spacetime has finite volume and all light-rays propagate at ± 45 degree in the diagram. The Penrose diagram of the Minkowski spacetime is given by the right figure in Fig.2.1. The red lines correspond to the trajectories of massless point particles (or the trajectories of the wave fronts of massless waves) and green lines correspond to the trajectory of a massive point particle (or of the wave front of massive wave). We typically have five kinds of asymptotic regions that have different qualitative features as follows;

- future timelike infinity (i^+): place where massive particles reach as $t \to \infty$.
- future null infinity ($\mathcal{I}^+$): place where massless particles reach as $t \to \infty$.
- spacelike infinity (i^0): place where any particles can not reach and only long range forces, like Coulomb force and gravitational force, can exist.
- past null infinity ($\mathcal{I}^-$): place where massless particles come from as $t \to -\infty$.
- past timelike infinity (i^-): place where massive particles come from as $t \to -\infty$.

In other words, the past/future timelike infinity is the Cauchy slice at the infinite past/future for the massive particles, and the past/future null infinity is the Cauchy slice at the infinite past/future for the massless particles.

The standard Minkowski coordinates are given by

$$ds^2 = -dT^2 + dR^2 + R^2 \gamma_{AB} d\Omega^A d\Omega^B \ , \tag{2.3.1}$$

where Ω^A $(A = 1, 2)$ can be any coordinate of a unit two-dimensional sphere with a metric γ_{AB}. This coordinate are not useful ones to analyze the physics around the asymptotic regions due to the following reasons. First, both $i^\pm$ and $\mathcal{I}^\pm$ are the region with $t = \pm\infty$, so we can not deal with them separately in the constant-T surface in (2.3.1). Secondly, $\mathcal{I}^\pm$ can not be parametrized properly by T and R because both R and T are infinite on $\mathcal{I}^+$. Therefore we will use the following coordinates to study the physics around each asymptotic region.

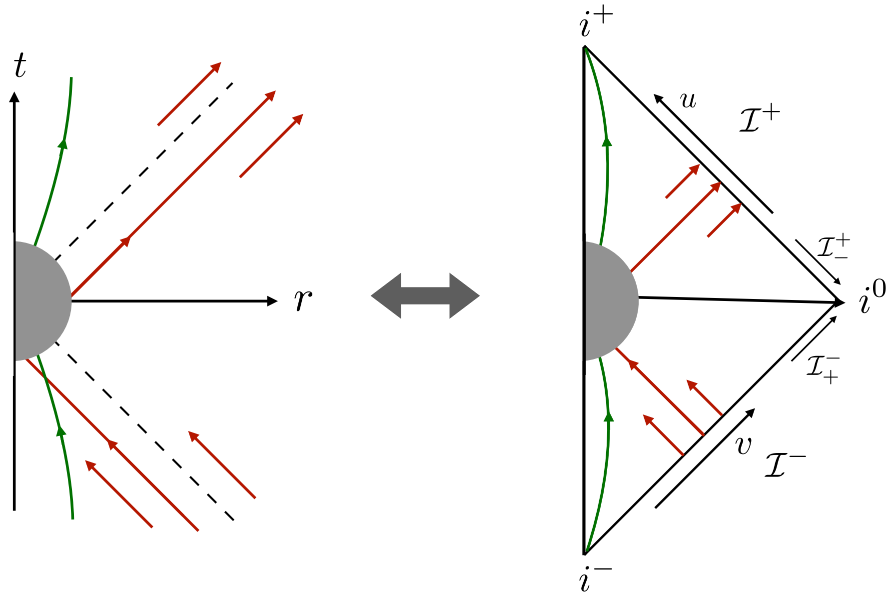

Figure 2.1: The left diagram is the usual Minkowski spacetime diagram, while the right diagram is the Penrose diagram of the Minkowski space. Every point in both diagrams corresponds to S^2 (except for $r = 0$), since the angular directions are not represented. In both diagrams, the green lines represent the trajectories of massive point particles, while the red lines represent the trajectories of massless point particles. The grey regions represent the effective regions where the scatterings occur.

The coordinates for $i^\pm$

• (τ, ρ, Ω^A) coordinates.

The coordinates (τ, ρ, Ω^A) with the Minkowski line element,

$$ds^2 = -d\tau^2 + \tau^2 \left[\frac{d\rho^2}{1+\rho^2} + \rho^2 \gamma_{AB} d\Omega^A d\Omega^B \right] , \tag{2.3.2}$$

are useful for focusing on the physics around the timelike infinity i^+ These coordinates can be obtained by the following coordinate transformation from the Minkowski coordinate in (2.3.1),

$$\tau^2 = T^2 - R^2 \ , \ \rho = \frac{R}{\sqrt{T^2 - R^2}} \, . \tag{2.3.3}$$

For $\tau^2 > 0$, the constant-τ hypersurfaces are 3-dimensional hyperbolic surfaces $\mathbb{H}_3$ (or Euclidean AdS_3). For $\tau^2 < 0$, the hypersurfaces are three-dimensional de Sitter space dS_3.

For later convenience, we introduce the following notation for $\tau > 0$,

$$ds^2 = -d\tau^2 + \tau^2 \, h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta, \tag{2.3.4}$$

where $\sigma^\alpha = (\rho, \Omega^A)$ are the coordinates of the unit 3-dimensional hyperbolic space $\mathbb{H}^3$

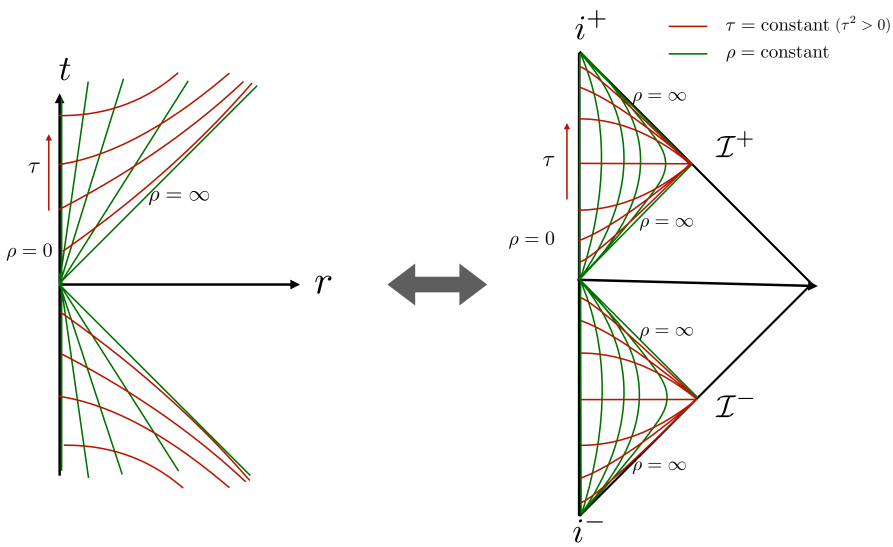

Figure 2.2: The left diagram is the usual Minkowski spacetime diagram, while the right diagram is the Penrose diagram of the Minkowski space. In both diagram, the green lines represent the constant-ρ surfaces, while the red lines represent the constant-τ surfaces.

with the line element

$$h_{\alpha\beta}d\sigma^\alpha d\sigma^\beta = \frac{d\rho^2}{1+\rho^2} + \rho^2\gamma_{AB}d\Omega^A d\Omega^B. \tag{2.3.5}$$

The reason why these coordinates are useful is the following. Solving (2.3.3) inversely for $\tau > 0$, we have

$$T = \pm\tau\sqrt{1+\rho^2}\ ,\ R = \tau\rho\ . \tag{2.3.6}$$

The second equation in (2.3.3) can be rewritten as

$$R = \pm\sqrt{\frac{\rho^2}{1+\rho^2}}T\ , \tag{2.3.7}$$

where $0 \le \sqrt{\frac{\rho^2}{1+\rho^2}} \le 1$. This means that the constant-ρ line corresponds to the trajectory of a massive particle with the constant velocity $v = \pm\sqrt{\frac{\rho^2}{1+\rho^2}}$, which is the asymptotic trajectory of massive particle as $|t| \to \infty$. In other words, ρ has one-to-one correspondence to the trajectory of a massive particle, therefore ρ is a natural coordinate for parametrizing the spacetime region that massive particles reach as $t \to \infty$. In (2.3.6), we can easily see that $\tau \to \infty$ with the fixed ρ corresponds to $T \to \infty$. Therefore $\tau = \infty$ surface spanned by (ρ, Ω^A) corresponds to the future timelike infinity. The constant-τ hypersurfaces and the constant-ρ hypersurfaces are presented in Figure 2.2.

The surface element on the constant-τ hypersurface is given by

$$d\Sigma_{i^+} \equiv d\rho d^2\Omega\sqrt{-g} = d^3\sigma\tau^3\sqrt{h} = d\rho d^2\Omega\frac{\tau^3\rho^2\sqrt{\gamma}}{\sqrt{1+\rho^2}} \tag{2.3.8}$$

We define an infinitesimal vector field $d\Sigma_{i^+\mu}$ as

$$d\Sigma_{i^+\mu} = n_\mu^{(i^+)} d\Sigma_{i^+} \quad \text{with} \quad n_\mu^{(i^+)} = \delta_{\mu\tau} \ , \tag{2.3.9}$$

where $n_\mu^{(i^+)}$ is defined as the unit vector orthogonal to the constant-τ surface. The metric in the matrix representation is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{\tau^2}{1+\rho^2} & 0 \\ 0 & 0 & \tau^2\rho^2\gamma_{AB} \end{pmatrix} \ , \ g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1+\rho^2}{\tau^2} & 0 \\ 0 & 0 & \frac{1}{\tau^2\rho^2}\gamma^{AB} \end{pmatrix} \ . \tag{2.3.10}$$

The nonzero components of the Christoffel symbols[2] are given by

$$\Gamma^\tau_{\rho\rho} = -\frac{\tau}{1+\rho^2} \ , \qquad \Gamma^\tau_{AB} = \tau\rho^2\gamma_{AB} \ , \ \Gamma^\rho_{\tau\rho} = \frac{1}{\tau} \ , \qquad \Gamma^\rho_{\rho\rho} = \frac{\rho}{1+\rho^2} \ , \tag{2.3.11}$$

$$\Gamma^\rho_{AB} = -(1+\rho^2)\rho\gamma_{AB} \ , \ \Gamma^A_{\tau B} = \frac{1}{\tau}\delta^A_{\ B} \ , \qquad \Gamma^A_{\rho B} = \frac{1}{\rho}\delta^A_{\ B} \ , \ \Gamma^{A'}_{AB} = \tilde{\Gamma}^{A'}_{AB}, \tag{2.3.12}$$

where $\tilde{\Gamma}^{A'}_{AB}$ is the Christoffel symbols on unit S^2 defined as

$$\Gamma^{A'}_{AB} \equiv \frac{1}{2}\gamma^{A'B'}\left(\partial_A\gamma_{B'B} + \partial_B\gamma_{B'A} - \partial_{B'}\gamma_{AB}\right). \tag{2.3.13}$$

The coordinates for $\mathcal{I}^\pm$

To study the asymptotic behavior of massless fields in Minkowski spacetime, it is useful to introduce the retarded coordinate u and the advanced coordinate v,

$$u = T - R \ , \ v = T + R \ . \tag{2.3.14}$$

The constant-u line corresponds to

$$R(T) = T - u \ , \tag{2.3.15}$$

which is the outgoing trajectory of a massless particle with $R(0) = -u$. Therefore, the retarded coordinate u is a natural coordinate that spans the future null infinity $\mathcal{I}^+$. Similarly, The $v =$ constant line corresponds to

$$R(T) = -T + v \ , \tag{2.3.16}$$

which is the incoming trajectory of a massless particle with $R(0) = v$. Then the advanced coordinate v spans the past null infinity $\mathcal{I}^-$.

We will use the following coordinates to study the physics around $\mathcal{I}^\pm$.

• (u, r, Ω^A) coordinates.

The line element is given by

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}d\Omega^A d\Omega^B \ . \tag{2.3.17}$$

This coordinates are related to the Minkowski coordinates in (2.3.1) by the coordinate transformations, $u = T - R \ , \ r = R \ .$

[2] $\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha_{\rho\beta}V^\rho, \nabla_\beta V_\alpha = \partial_\beta V_\alpha + \Gamma^\rho_{\alpha\beta}V_\rho$.

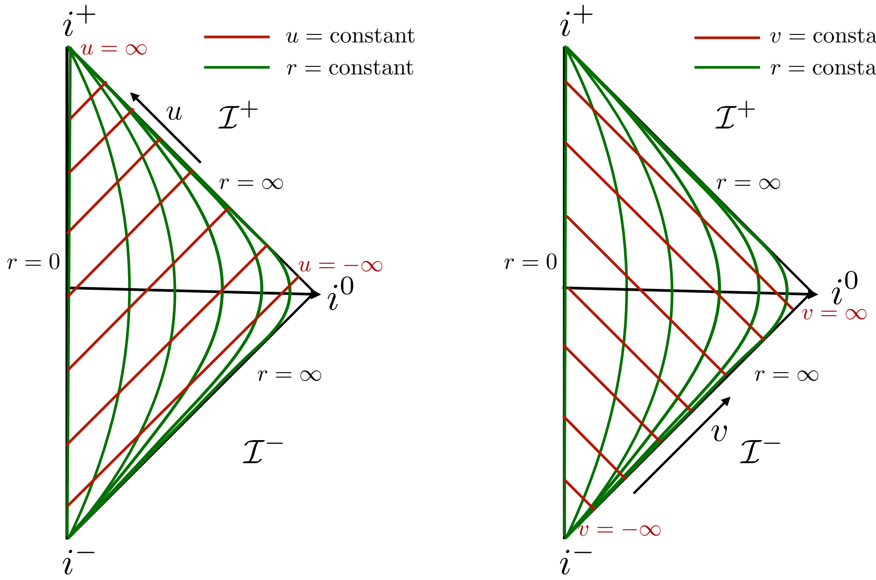

Figure 2.3: Both diagrams are the Penrose diagram of the Minkowski spacetime. The red lines represent the constant-u surfaces in the left diagram and constant-v surfaces in the right diagram. In both diagrams, the constant-r surfaces are represented by the green lines.

The matrix representation of the metric (2.3.17) is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2\gamma_{AB} \end{pmatrix} , \; g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & r^{-2}\gamma^{AB} \end{pmatrix} . \tag{2.3.18}$$

The nonzero components of the Christoffel symbols are given by

$$\Gamma^u_{AB} = r\gamma_{AB} \; , \; \Gamma^r_{AB} = -r\gamma_{AB}, \;\; \Gamma^A_{rB} = \frac{1}{r}\delta^A{}_B \; , \; \Gamma^{A'}_{AB} = \tilde{\Gamma}^{A'}_{AB} \; . \tag{2.3.19}$$

The future null infinity ($\mathcal{I}^+$) is parametrized by (u, Ω^A) with $r = \infty$. The infinitesimal surface element of the constant-r surface is given by $d\Sigma_{\mathcal{I}^+} \equiv r^2\sqrt{\gamma}dud^2\Omega$. We define a infinitesimal vector field $d\Sigma_{\mathcal{I}^+\mu}$ as

$$d\Sigma_{\mathcal{I}^+\mu} \equiv n^{(\mathcal{I}^+)}_\mu d\Sigma_{\mathcal{I}^+} \;\; \text{with} \;\; n^{(\mathcal{I}^+)}_\mu = \delta^r{}_\mu \; , \tag{2.3.20}$$

where $n^{(\mathcal{I}^+)}_\mu$ is a unit normal vector orthogonal to the constant-r surface.

• (v, r, Ω^A) coordinates.

The metric is give by

$$ds^2 = -dv^2 + 2dvdr + r^2\gamma_{AB}d\Omega^A d\Omega^B \; . \tag{2.3.21}$$

These coordinates are related to the Minkowski coordinates in (2.3.1) by the coordinate transformations, $v = T + R$, $r = R$. The matrix representation of the metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & r^2\gamma_{AB} \end{pmatrix} , \; g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & r^{-2}\gamma^{AB} \end{pmatrix} . \tag{2.3.22}$$

The past null infinity ($\mathcal{I}^-$) is parametrized by (v, Ω^A) with $r = \infty$. The infinitesimal surface element of the constant-r surface is given by $d\Sigma_{\mathcal{I}^-} \equiv r^2\sqrt{\gamma}dvd^2\Omega$. We define a infinitesimal vector field $d\Sigma_{\mathcal{I}^-\mu}$ as

$$d\Sigma_{\mathcal{I}^-\mu} \equiv n_\mu^{(\mathcal{I}^-)} d\Sigma_{\mathcal{I}^-} \quad \text{with} \quad n_\mu^{(\mathcal{I}^-)} = \delta^r{}_\mu \ , \tag{2.3.23}$$

where $n_\mu^{(\mathcal{I}^-)}$ is a unit normal vector orthogonal to the constant-r surface.

Finally, we introduce the two places $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ (see Figure 2.1). The $\mathcal{I}^+_-$ is defined as the *infinite past of future null infinity* which is parametrized by Ω^A with $u = -\infty, r = \infty$, and the $\mathcal{I}^-_+$ is defined as the *infinite future of past null infinity* which is parametrized by Ω^A with $v = +\infty, r = \infty$. Note that $\mathcal{I}^+_-$, $\mathcal{I}^-_+$, and i^0 are not the same places. For example, $\mathcal{I}^+_-$ can be reached by first taking the limit $r \to \infty$ with fixed u and then taking the limit $u \to -\infty$, whereas i^0 can be reached by taking the limit $r \to \infty$ with fixed T. In fact, we will see that the Coulomb field at $\mathcal{I}^+_-$ and the one at $\mathcal{I}^-_+$ are not the same but those are generally related to each other, and the relation is important for the conservation of charges associated with the asymptotic symmetry.

2.4 Classical asymptotic charge

In this section, we study the charge associated with the classical asymptotic symmetry on the asymptotic Cauchy slices, $i^+ \cup \mathcal{I}^+$ and $i^- \cup \mathcal{I}^-$, in QED.

We first study the charge on future infinity, which is given by integrating the local gauge current in (2.2.4) on $i^+ \cup \mathcal{I}^+$:

$$Q^+[\epsilon] \equiv Q^+_H[\epsilon] + Q^+_S[\epsilon] \ . \tag{2.4.1}$$

Here,

$$Q^+_H[\epsilon] \equiv \int_{i^+} d\Sigma_{i^+\mu} \ \left(F^{\mu\nu}\partial_\nu \epsilon + \epsilon j^\mu_{mat}\right) \ , \tag{2.4.2}$$

$$Q^+_S[\epsilon] \equiv \int_{\mathcal{I}^+} d\Sigma_{\mathcal{I}^+\mu} F^{\mu\nu}\partial_\nu \epsilon \ . \tag{2.4.3}$$

where we have dropped the matter current term in $Q^+_S[\epsilon]$ because the matter current does not reach the null infinity. We call $Q^+_H[\epsilon]$ the *hard charge* and $Q^+_S[\epsilon]$ the *soft charge*. Plugging (2.3.9) into (2.4.2), we have

$$Q^+_H[\epsilon] = \int_{i^+} d\Sigma_{i^+} \ \left(F^{\tau\nu}\partial_\nu \epsilon + \epsilon j^\tau_{mat}\right) \tag{2.4.4}$$

Plugging (2.3.23) into (2.4.3), we have

$$\begin{aligned} Q^+_S[\epsilon] &= \int_{\mathcal{I}^+} d\Sigma_{\mathcal{I}^+} F^{r\nu}\partial_\nu \epsilon \\ &= \lim_{r\to\infty} \int dud^2\Omega\sqrt{\gamma}r^2 \left[F^{ru}\partial_u \epsilon + F^{rB}\partial_B \epsilon\right] \ . \end{aligned} \tag{2.4.5}$$

Here, we study the fall-off behaviors of field strength for physical solutions. By the "physical solutions", we mean that field strength is a solution of Maxwell equation and have nonzero but finite amounts of conserved charges *e.g.* energy, angular momentum, and electric charge. The electromagnetic energy momentum tensor is given by

$$T^{\mu\nu} = \frac{-1}{2\pi}\left[F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}g^{\mu\nu}F^{\rho\lambda}F_{\rho\lambda}\right] \ . \tag{2.4.6}$$

The electromagnetic energy flux on $\mathcal{I}^+$ is then given by

$$\begin{aligned}\int_{\mathcal{I}^+} d\Sigma_{\mathcal{I}^+\mu}T^{\mu}{}_{\nu}\delta^{\nu}{}_{u} &= \lim_{r\to\infty}\int dud^2\Omega\sqrt{\gamma}r^2\ T^{r}{}_{u} \\ &= \lim_{r\to\infty}\int dud^2\Omega\sqrt{\gamma}\frac{\gamma^{AB}}{2\pi}\left[F_{uA}F_{uB} - F_{rA}F_{uB}\right]\end{aligned} \tag{2.4.7}$$

To have a finite amount of the energy flux, the field strength should fall off as

$$\gamma^{AB}\left[F_{uA}F_{uB} - F_{rA}F_{uB}\right] \sim \mathcal{O}(1) \quad \text{as} \quad r\to\infty \ . \tag{2.4.8}$$

The total momentum flux of the radial direction on $\mathcal{I}^+$ is then given by

$$\begin{aligned}\int_{\mathcal{I}^+} d\Sigma_{\mathcal{I}^+\mu}T^{\mu}{}_{\nu}\delta^{\nu}{}_{r} &= \lim_{r\to\infty}\int dud^2\Omega\sqrt{\gamma}r^2\ T^{r}{}_{r} \\ &= \lim_{r\to\infty}\int dud^2\Omega\sqrt{\gamma}\frac{r^2}{4\pi}\left[F_{ur}F_{ru} + \frac{\gamma^{AB}}{r^2}F_{rA}F_{rB}\right]\end{aligned} \tag{2.4.9}$$

To have finite amount of the momentum flux, the field strength should fall off as

$$F_{ur} \sim \mathcal{O}(r^{-1}) \ , \ F_{rA} \sim \mathcal{O}(1) \tag{2.4.10}$$

Moreover, in order to have finite amount of electric charge, the radial component of electric field should fall off as r^{-2},

$$F^{tr} = F^{ur} = -F_{ur} \sim \mathcal{O}(r^{-2}) \tag{2.4.11}$$

Here, we rewrite the above fall-off conditions for field strengths as the ones for gauge fields. We expand gauge fields by power series of r as

$$A_{\mu}(u,r,\Omega) = \sum_{n=0}^{\infty}\frac{A_{\mu}^{(n)}(u,\Omega)}{r^n} \tag{2.4.12}$$

From now, we work in the Lorenz gauge by imposing

$$\nabla^{\mu}A_{\mu} = 0 \ . \tag{2.4.13}$$

In this gauge, there are residual gauge transformations [3] that are generated by the gauge parameter $\epsilon(x)$ satisfying

$$\Box\epsilon(x) = 0 \, . \tag{2.4.14}$$

[3] The residual gauge transformations are defined as gauge transformations such that the transformed fields satisfy the gauge fixing condition.

This condition is written as follows in (u, r, Ω^A) coordinates,

$$\begin{aligned}
0 = \nabla_\mu A^\mu &= \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}A^\mu\right) \\
&= \partial_u A^u + \frac{1}{r^2}\partial_r\left(r^2 A^r\right) + \frac{1}{\sqrt{\gamma}}\partial_B\left(\sqrt{\gamma}A^B\right) \\
&= -\partial_u A_r + \partial_r(A_r - A_u) + \frac{2}{r}(A_r - A_u) + \frac{\gamma^{AB}}{r^2}D_A A_B \ , \qquad (2.4.15)
\end{aligned}$$

where we have used the metric given in (2.3.18). Plugging (2.4.12) into the above constraint, we have

$$\partial_u A_r^{(0)} = 0 \ , \ -\partial_u A_r^{(1)} + 2(A_r^{(0)} - A_u^{(0)}) = 0 \qquad (2.4.16)$$

Moreover, plugging (2.4.12) into (2.4.11), we have

$$\partial_u A_r^{(0)} = 0 \ , \ \partial_u A_r^{(1)} = 0 \ . \qquad (2.4.17)$$

Combining (2.4.16) and (2.4.17), we find

$$\partial_u A_r^{(0)} = 0 \ , \ \partial_u A_r^{(1)} = 0 \ , \ A_r^{(0)} - A_u^{(0)} = 0 \ . \qquad (2.4.18)$$

Further constraints do not come from the fall-off conditions (2.4.8) and (2.4.10). Now we express the soft charge defined in (2.4.5) in term of the gauge fields. Under the fall-off conditions of (2.4.18), the field strengths can be expanded as

$$\begin{aligned}
F^{rA} &= \frac{\gamma^{AB}}{r^2}\left(F_{uB} - F_{rB}\right) = \frac{\gamma^{AB}}{r^2}\left(\partial_u A_B - \partial_B A_u - \partial_r A_B + \partial_B A_r\right) \\
&= \frac{\gamma^{AB}}{r^2}\partial_u A_B^{(0)} + \mathcal{O}(r^{-3}) \ , \qquad (2.4.19)
\end{aligned}$$

where the second and the third terms cancel each other because of the third condition in (2.4.18). The residual gauge parameter can be expanded as (A.0.13):

$$\epsilon(u, r, \Omega) = \epsilon^{(0)}(\Omega) + \frac{u \log\frac{2r}{|u|}}{2r}\Delta_{\mathrm{S}^2}\epsilon^{(0)}(\Omega) + \mathcal{O}(r^{-1}). \qquad (2.4.20)$$

See Appendix A for the detail of the expansion. Then inserting (2.4.19) and (2.4.20) into the soft charge (2.4.5), we find

$$Q_S^+[\epsilon] = \int du d^2\Omega \sqrt{\gamma}\gamma^{AB} \ \partial_A \epsilon^{(0)} \partial_u A_B^{(0)} \ . \qquad (2.4.21)$$

This expression coincides with the soft charge that was first introduced in [4, 6].

2.5 Classical charge conservation associated with the asymptotic symmetry

In this section, we illustrate the existence of an infinite number of asymptotically conserved charges associated with large gauge transformations in the classical electromagnetism. (Precise meaning of "asymptotically conserved charges" will be given later.) We represent the matter current for massive charged particles by j^μ_{mat}, which is the source in Maxwell's equation $\partial_\nu F^{\nu\mu} = -j^\mu_{mat}$. Here, we assume that the charged particles behave as free particles except in a small region where they scatter, and we ignore the back-reaction. We also impose the initial condition that there is no radiation before the scattering of charged particles just for simplicity.

The conserved current for the gauge transformation with gauge parameter $\epsilon(x)$ is given by (2.2.4), which is

$$J^\mu = F^{\mu\nu}\partial_\nu \epsilon + j^\mu_{mat}\epsilon. \tag{2.5.1}$$

In order to study whether $Q(\Sigma_R) = 0$ at $R = \infty$ vanishes or not, we first consider a closed surface with a finite area in Minkowski spacetime and then take the limit that makes the area infinite. Consider the integration of the current conservation equation $\partial_\mu J^\mu = 0$ over the region represented in Fig. 2.4. The region is parametrized by two parameters T and U,

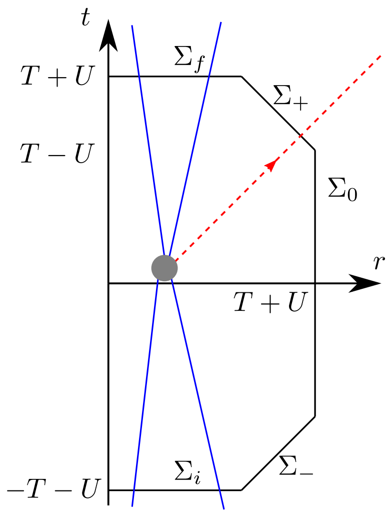

Figure 2.4: The region where we consider the current conservation. The directions along the two-dimensional sphere S^2 are suppressed in this figure. The blue lines represent trajectories of massive charged particles, which scatter at a small region. The red line represents a direction of the radiation emitted from this scattering. The region is parametrized by two parameters T and U. The parameter T is so large that all of the given massive particles go through the surface Σ_i and Σ_f, and U is also so large that any radiation coming from the scattering region passes through Σ_+.

and has five boundaries, $\Sigma_f, \Sigma_i, \Sigma_+, \Sigma_-, \Sigma_0$. Σ_f and Σ_i are time slices at the distant future and past, $t = \pm(T+U)$ with $0 \le r \le T - U$. Σ_0 is a timelike surface $r = \text{const.} = T + U$ with $-T + U < t < T - U$. $\Sigma_\pm$ are null surfaces where $\pm t + r = \text{const.} = 2T$. We set the parameter T to be so large that the massive matter current j^μ_{mat} vanishes on $\Sigma_\pm$ and Σ_0. We also set U to be so large that any electromagnetic radiation coming from the scattering region goes through the null surface Σ_+. Later, we take the limit first $T \to \infty$ and then $U \to \infty$ such that $\Sigma_\pm$ becomes the future and the past null infinities $\mathcal{I}^\pm$, where $\mathcal{I}^+$ ($\mathcal{I}^-$) is parametrized by the retarded (advanced) time $u = t - r$ ($v = t + r$) and the angular coordinates Ω^A. The ordering of the limits, $T \to \infty$ before $U \to \infty$, is crucial for our derivation of the memory effect formulae. The integration of the current conservation proves that the sum of the surface integral of the current over each boundary vanishes:

$$0 = \int dV\, \partial_\mu J^\mu = Q_f + Q_+ + Q_0 - Q_- - Q_i \tag{2.5.2}$$

with

$$Q_a \equiv \int_{\Sigma_a} dS^\mu J_\mu\,, \quad (a = f, i, 0, +, -)\,, \tag{2.5.3}$$

where we choose the surface element dS^μ to be future-directed when the surface is spacelike or null.

2.5.1 The asymptotic charge conservation and memory effect at leading order

We now see that Q_0, which is defined on Σ_0 as

$$Q_0 = \int_{-T+U}^{T-U} dt\, d^2\Omega \sqrt{\gamma} r^2 J^r|_{r=T+U}\,, \tag{2.5.4}$$

vanishes in the limit $T \to \infty$ without the limit $U \to \infty$. As the current can be represented as a total derivative $J^r = \partial_\mu(F^{r\mu}\epsilon)$, Q_0 splits into the integrations over two spheres on the future and the past boundaries of Σ_0 as

$$Q_0 = \int d^2\Omega \sqrt{\gamma} r^2 F_{tr}\epsilon|_{t=T-U, r=T+U} - \int d^2\Omega \sqrt{\gamma} r^2 F_{tr}\epsilon|_{t=-T+U, r=T+U}\,. \tag{2.5.5}$$

Since any radiation cannot reach Σ_0 due to our choice of the region, the field strength F_{tr} is the Coulombic electric field created by free-moving charged particles before the scattering. The behavior of the fields are analyzed in Appendix B. The leading term of F_{tr} in the large T expansion is $\mathcal{O}(T^{-2})$, and the leading terms are represented as

$$\lim_{T\to\infty} r^2 F_{tr}(t, r, \Omega)|_{t=T-U, r=T+U} \equiv F_{tr}^{+(2)}(\Omega)\,, \tag{2.5.6}$$

$$\lim_{T\to\infty} r^2 F_{tr}(t, r, \Omega)|_{t=-T+U, r=T+U} \equiv F_{tr}^{-(2)}(\Omega)\,. \tag{2.5.7}$$

The important facts are that they are independent of U and they satisfy the *antipodal matching condition* (see Appendix B.1),

$$F_{tr}^{+(2)}(\Omega) = F_{tr}^{-(2)}(\bar{\Omega})\,. \tag{2.5.8}$$

where $\bar{\Omega} = (\bar{\Omega}^1, \bar{\Omega}^2)$ is the antipodal angle of $\Omega = (\Omega^1, \Omega^2)$ [4]. We thus obtain

$$\lim_{T\to\infty} Q_0 = \int d^2\Omega \sqrt{\gamma} F_{tr}^{+(2)}(\Omega)\epsilon^{(0)}(\Omega) - \int d^2\Omega \sqrt{\gamma} F_{tr}^{-(2)}(\Omega)\epsilon^{(0)}(\bar{\Omega}) \tag{2.5.9}$$

[4] The importance of this condition in the context of the asymptotic symmetry was first emphasized in [31] (see also [55] for the detailed explanation).

where we used the fact that $\epsilon(x)$ approaches $\epsilon^{(0)}(\Omega)$ near $\mathcal{I}^+$ and $\epsilon^{(0)}(\bar{\Omega})$ near $\mathcal{I}^-$. Therefore, due to the matching condition (2.5.8), Q_0 vanishes in the limit $T \to \infty$:

$$\lim_{T\to\infty} Q_0 = \int d^2\Omega\sqrt{\gamma}[F_{tr}^{+(2)}(\Omega) - F_{tr}^{-(2)}(\bar{\Omega})]\epsilon^{(0)}(\Omega) = 0\,. \tag{2.5.10}$$

Thus in the limit $T \to \infty$, the conservation equation (2.5.2) indicates

$$\lim_{T\to\infty}(Q_f + Q_+) = \lim_{T\to\infty}(Q_i + Q_-)\,. \tag{2.5.11}$$

The Q_f and Q_i are given by

$$Q_f = \int_0^{T-U} drd^2\Omega\sqrt{\gamma}r^2(F^{ti}\partial_i\epsilon + j^t_{mat}\epsilon)|_{t=T+U}, \tag{2.5.12}$$

$$Q_i = \int_0^{T-U} drd^2\Omega\sqrt{\gamma}r^2(F^{ti}\partial_i\epsilon + j^t_{mat}\epsilon)|_{t=-T-U}. \tag{2.5.13}$$

The field strength F^{ti} in the above integrands is the Coulombic electric field created by free-moving charged particles, since the radiation does not reach Σ_f and Σ_i. Thus Q_f and Q_i are the usual gauge charges for free-moving charged particles and their Coulomb-like potential in the three-dimensional ball with radius $T - U$ at time $t = \pm(T + U)$. In particular, if ϵ is a constant, Q_f and Q_i equal to the total electric charges $(\times\epsilon)$. Eq. (2.5.11) just means that such gauge charges do not conserve for the nontrivial large gauge parameters $\epsilon(x)$ unless we include the contributions from the radiation $Q_\pm$.[5] Let us call the conservation laws (2.5.11) "asymptotic conservation", since it only holds in the limit $T \to \infty$. Since we can take arbitrary functions $\epsilon^{(0)}(\Omega)$ on unit two-sphere, we have an infinite number of the asymptotic conserved charges.

Using the retarded time $u = t - r$, we can write Q_+ as

$$Q_+ = \int_{-2U}^{2U} dud^2\Omega\sqrt{\gamma}r^2(F^{ru}\partial_u\epsilon + F^{rA}\partial_A\epsilon)|_{r=T-u/2}\,. \tag{2.5.14}$$

In the large T limit, the integration region becomes the subregion $-2U \le u \le 2U$ in $\mathcal{I}^+$. Using the condition (2.4.18) and expansion (2.4.20), we obtain

$$\begin{aligned}\lim_{T\to\infty} Q_+ &= \int_{-2U}^{2U} dud^2\Omega\sqrt{\gamma}\gamma^{AB}\partial_u A_B^{(0)}\partial_A\epsilon^{(0)} \\ &= \int d^2\Omega\sqrt{\gamma}\gamma^{AB}[A_B^{(0)}(u=2U) - A_B^{(0)}(u=-2U)]\partial_A\epsilon^{(0)}\,.\end{aligned} \tag{2.5.15}$$

On the other hand, for our setup or our initial condition, there is no radiation at Σ_-, and thus $\lim_{T\to\infty} Q_- = 0$. Therefore, we obtain

$$-\int d^2\Omega\sqrt{\gamma}\gamma^{AB}[A_B^{(0)}(u=2U) - A_B^{(0)}(u=-2U)]\partial_A\epsilon^{(0)} = \lim_{T\to\infty}(Q_f - Q_i)\,. \tag{2.5.16}$$

This equation holds for any function $\epsilon^{(0)}(\Omega)$ on the two-sphere. It implies that the shift of $A_B^{(0)}$ on $\mathcal{I}^+$ is related to the change of the gauge charges between Q_f and Q_i.[6] In particular,

[5] A similar statement was given for the case without massive charged fields in [56].

[6] The Ω-independent part of the shift cannot be determined from this formula.

when we perform the partial integration for the angular integral, (2.5.16) becomes

$$\int d^2\Omega\sqrt{\gamma}\gamma^{AB}\epsilon^{(0)}[\partial_A A_B^{(0)}(u=2U) - \partial_A A_B^{(0)}(u=-2U)] = \lim_{T\to\infty}(Q_f - Q_i) \tag{2.5.17}$$

Moreover, if we define the local electric charge at timelike past and future infinity, $\rho_i(\Omega)$ and $\rho_f(\Omega)$ for every angle Ω, by

$$Q_i \equiv \int d^2\Omega\sqrt{\gamma}\,\epsilon^{(0)}\rho_i(\Omega) \tag{2.5.18}$$

$$Q_f \equiv \int d^2\Omega\sqrt{\gamma}\,\epsilon^{(0)}\rho_f(\Omega)\,, \tag{2.5.19}$$

we find

$$\gamma^{AB}[\partial_A A_B^{(0)}(u=2U) - \partial_A A_B^{(0)}(u=-2U)] = \rho_f(\Omega) - \rho_i(\Omega) \tag{2.5.20}$$

This means that the total charge that has passed through the angle Ω after all particles have left is memorized by the shift of $\gamma^{AB}\partial_A A_B^{(0)}$ on $\mathcal{I}^+$. This is called the *electromagnetic memory effect formula* [57, 29, 30]. Note that the shift is gauge invariant although $\gamma^{AB}\partial_A A_B^{(0)}$ is not. Thus we have found the equivalence between charge conservation associated with the asymptotic symmetry and the electromagnetic memory effect.

If we take the limit $U\to\infty$, the charge from the radiation (2.5.15) becomes the soft charge given in (2.4.21),

$$Q_S^{\text{lead},+} \equiv -\int_{\mathcal{I}^+} du d^2\Omega\sqrt{\gamma}\gamma^{AB}\partial_u A_B^{(0)}\partial_A\epsilon^{(0)}. \tag{2.5.21}$$

Although $\lim_{T\to\infty} Q_-$ vanishes in our setup, if we consider the general situation where electromagnetic radiation exists initially, $\lim_{U\to\infty}\lim_{T\to\infty} Q_-$ becomes the soft charge $Q_S^{\text{lead},-}$ defined on $\mathcal{I}^-$. The Q_f and Q_i become the so called hard charges $Q_H^{\text{lead},\pm}$ in the limit. Note that the hard charges include not only matter currents but also the contributions from the Coulombic electric field produced by the charged matters. We will study the expression of those charges concretely in Section 2.8 and 2.9 (for example see (2.8.8) and (2.9.76).)

In the limit $U\to\infty$, we thus obtain

$$Q_S^{\text{lead},+} + Q_H^{\text{lead},+} = Q_S^{\text{lead},-} + Q_H^{\text{lead},-}\,. \tag{2.5.22}$$

We note again that the conservation laws (2.5.22) come from the fact that the charge on spacelike infinity, $\lim_{T\to\infty} Q_0$, vanishes due to the antipodal matching condition. In other words, the total asymptotic charge $Q_S^{\text{lead},+} + Q_H^{\text{lead},+}$ is equal to the integral on two-sphere at the past boundary of $\mathcal{I}^+$ at $u=-\infty$ as

$$Q_S^{\text{lead},+} + Q_H^{\text{lead},+} = -\int_{\mathcal{I}^+_-} d^2\Omega\sqrt{\gamma}F_{tr}^{+(2)}(\Omega)\epsilon^{(0)}(\Omega)\,, \tag{2.5.23}$$

because the current J^μ is a total derivative, and $Q_S^{\text{lead},-} + Q_H^{\text{lead},-}$ also equals to the integral on two-sphere at the future boundary of $\mathcal{I}^-$ as

$$Q_S^{\text{lead},-} + Q_H^{\text{lead},-} = -\int_{\mathcal{I}^-_+} d^2\Omega\sqrt{\gamma}F_{tr}^{+(2)}(\Omega)\epsilon^{(0)}(\bar{\Omega})\,, \tag{2.5.24}$$

which is equal to (2.5.23). We also remark that the surface $\Sigma_\pm \cup \Sigma_{f/i}$ becomes the Cauchy surface in the limit $U \to \infty$ after $T \to \infty$. Therefore, $Q_S^{\text{lead},\pm} + Q_H^{\text{lead},\pm}$ is actually a charge defined on the asymptotic Cauchy surface.

2.5.2 Asymptotic charge conservation and memory effect at subleading order

We have seen that the current conservation equation (2.5.2) leads to the formula of the leading memory effect (2.5.16) in the large T limit because the charge from spacelike infinity, Q_0, vanishes in the limit due to the antipodal matching. The subleading memory effect can also be obtained by considering the corrections in the large T expansion.

We first consider the correction to Q_+ defined as eq. (2.5.14). By using the formula (A.0.13), we can expand the gauge parameter $\epsilon(x)$ on Σ_+ in the large T limit as

$$\epsilon(u, r = T - u/2, \Omega) = \epsilon^{(0)}(\Omega) - \frac{u\log\frac{|u|}{2T}}{2T}\Delta_{S^2}\epsilon^{(0)}(\Omega) + \mathcal{O}(T^{-1})\,, \tag{2.5.25}$$

where Δ_{S^2} is the Laplacian on the unit two-sphere. Note that the correction to $\epsilon^{(0)}(\Omega)$ starts from $\mathcal{O}(T^{-1}\log T)$. In Appendix C, we give general expansions of radiation fields which are compatible with this large gauge parameters. Using the expansions (2.5.25), (C.0.1), (C.0.2) and (C.0.3), we find that the first correction to Q_+ is $\mathcal{O}(T^{-1}\log T)$, and Q_+ takes the form

$$Q_+ = -\int_{-2U}^{2U} dud^2\Omega\sqrt{\gamma}\partial_u A_B^{(0)}\gamma^{BA}\partial_A\epsilon^{(0)} - (Q_+^{\log} + Q_+^{\log\prime})\frac{\log T}{T} + \mathcal{O}(T^{-1})\,, \tag{2.5.26}$$

where the first term is just the leading soft charge (2.5.15), and the second term is the first correction. Here, $Q_+^{\log}$ and $Q_+^{\log\prime}$ are given by

$$\begin{aligned} Q_+^{\log} &= \frac{1}{2}\int_{-2U}^{2U} dud^2\Omega\sqrt{\gamma}u\partial_u A_B^{(0)}\gamma^{BA}\partial_A\Delta_{S^2}\epsilon^{(0)} \\ &= -\frac{1}{2}\int_{-2U}^{2U} dud^2\Omega\sqrt{\gamma}\epsilon^{(0)}u\partial_u\Delta_{S^2}\nabla^B A_B^{(0)}\,, \end{aligned} \tag{2.5.27}$$

$$Q_+^{\log\prime} = -\frac{1}{2}\int_{-2U}^{2U} dud^2\Omega\sqrt{\gamma}\left[\left(A_r^{(1)} + \nabla^B A_B^{(0)} + 2C_u^{(1)}\right)\Delta_{S^2}\epsilon^{(0)} + 2\gamma^{AB}\left(\partial_u C_A^{(1)} - \partial_A C_u^{(1)}\right)\partial_B\epsilon^{(0)}\right]\,, \tag{2.5.28}$$

where ∇_B denotes the covariant derivative on the unit two-sphere w.r.t. the metric γ_{AB}, and the derivative with the upper index is defined as $\nabla^A \equiv \gamma^{AB}\nabla_B$. The definition of $C_u^{(1)}$, $A_r^{(1)}$ and $C_A^{(1)}$ are given in (C.0.1), (C.0.2) and (C.0.3).

On the spatially distant surface Σ_0, the leading part of the charge Q_0 in the large T expansion is $\mathcal{O}(T^{-1})$ as shown in eq. (B.1.12) in Appendix B.1, and it does not have any $\mathcal{O}(T^{-1}\log T)$ term. Thus the coefficient of $T^{-1}\log T$ is also conserved without the contribution from Σ_0 like the leading memory. The contributions of the future and past timelike infinities are extracted as

$$Q^{\log}_{f,i} = \lim_{T\to\infty} \frac{T^2}{\log T}\frac{dQ_{f,i}}{dT}. \tag{2.5.29}$$

When we use these symbols, the finite U version of the subleading memory effect formula takes the form

$$-\frac{1}{2}\int_{-2U}^{2U} dud^2\Omega\sqrt{\gamma}\epsilon^{(0)}u\partial_u\Delta_{\mathrm{S}^2}\nabla^B A^{(0)}_B = -Q^{\log\prime}_{+} - Q^{\log}_{f} + Q^{\log}_{i}\,. \tag{2.5.30}$$

In the limit $U\to\infty$, the subleading radiation charge $Q^{\log}_{+}$ becomes

$$Q^{\mathrm{sub},+}_{S} \equiv \lim_{U\to\infty} Q^{\log}_{+} = -\frac{1}{2}\int_{\mathcal{I}^+} dud^2\Omega\sqrt{\gamma}\epsilon^{(0)}u\partial_u\Delta_{\mathrm{S}^2}\nabla^B A^{(0)}_B, \tag{2.5.31}$$

which agrees, up to a numerical factor, with the electric-type subleading soft charge in [36], and it is shown that the charge also agrees with that in [35] by taking their vector field Y^A on unit two-sphere as $Y^A \propto \nabla^A\epsilon^{(0)}$. We also represent the other contributions in the limit $U\to\infty$ including the initial radiation on Σ_- by[7]

$$Q^{\mathrm{sub},-}_{S} \equiv \lim_{U\to\infty} Q^{\log}_{-}\,,\quad Q^{\mathrm{sub},+}_{H} \equiv \lim_{U\to\infty}(Q^{\log\prime}_{+} + Q^{\log}_{f})\,,\quad Q^{\mathrm{sub},-}_{H} \equiv \lim_{U\to\infty}(Q^{\log\prime}_{-} + Q^{\log}_{i})\,. \tag{2.5.32}$$

We then obtain the subleading charge conservation

$$Q^{\mathrm{sub},+}_{S} + Q^{\mathrm{sub},+}_{H} = Q^{\mathrm{sub},-}_{S} + Q^{\mathrm{sub},-}_{H}\,. \tag{2.5.33}$$

Unlike the leading memory effect, this equation does not directly relate a shift of the radiation field to the change of the configuration of hard charges. Nevertheless, the existence of a nonzero shift $Q^{\mathrm{sub},+}_{S} - Q^{\mathrm{sub},-}_{S}$ is known for causing a permanent displacement of the position of a probe charged particle [58]. Thus this is the memory effect. It can also be detectable as a permanent change of the direction of the spin of a heavy probe particle with the magnetic moment [59].

Before closing this section, we briefly comment on the sub-subleading order. We have seen that the charge Q_0 on spacelike infinity is $\mathcal{O}(T^{-1})$, which is the same order as the sub-subleading corrections to Q_a ($a = +,-,f,i$). Therefore, we conclude that there is no asymptotic conservation at the sub-subleading order without including the contribution from Q_0 (see also similar discussions in [36, 60]).[8] This conclusion is probably related to the fact that there is no sub-subleading soft photon theorem in QED as we will see in Section 3.4(see also [61] for the discussion that the soft expansion of amplitudes is not associated with a universal soft factor at the sub-subleading order).

[7] As shown in Appendix B.2, $Q^{\log\prime}_{+}$ generally diverges in the limit $U\to\infty$, and $Q^{\log}_{f}$ also does. Nevertheless, the combination $(Q^{\log\prime}_{+} + Q^{\log}_{f})$ is finite as far as we know.

[8] The arguments that there is no subsubleading charge were given in [36, 60]. However, the reason seems to be different from ours. Their reason is that the sub-subleading charge has an inevitable divergence. On the other hand, from our construction, the charges for the large gauge transformations are "conserved" if we take account of the contributions from spacelike infinity.

2.6 Asymptotic symmetry in quantum field theory

In this section, we argue about the asymptotic symmetry and its charge in quantum field theories. Here, we define the asymptotic symmetry transformations (AST) in QFT as

$$\text{AST} = \frac{\text{allowed gauge transformations}}{\text{trivial gauge transformations}} \,. \tag{2.6.1}$$

The *allowed gauge transformations* are any gauge transformations generated by the well-defined charge. The *allowed gauge transformations* are the ones generated by the charge that acts trivially on any physical states in the given theory.

In classical cases, the charge associated with the gauge symmetry was just given by integrating the local gauge current on a Cauchy slice. In contrast, the procedure become subtle in quantum theories due to the following reason. In quantum case, we first need to quantize the theory. In the case of gauge theories, the quantization becomes subtle due to the existence of gauge (redundant) degrees of freedom, and we need to fix them to quantize the theory. There are two well-known procedures for the quantization of gauge theories: the canonical quantization and the BRST quantization. In both procedures, we need to add some new terms to the original Lagrangian to eliminate the gauge degrees of freedom. More concretely, we change the Lagrangian by introducing Lagrangian multipliers in the canonical quantization. In the BRST quantization, we add BRST exact terms. After the gauge fixing, the gauge symmetries of the original Lagrangian are no longer symmetries and there is no Noether current of gauge symmetries. Therefore we need different ways to find the conserved charge associated with the asymptotic symmetries. We explain asymptotic symmetries and their charges in QED, first in the canonical quantization in the next section, and in BRST quantization in Section 2.9.

2.7 Asymptotic symmetry in canonical quantization

Consider QED again for concreteness. The Lagrangian for QED (2.2.2) has the following two first class constraints (see Appendix D for the derivation),

$$\phi_0 \equiv \Pi^0(x) = 0, \tag{2.7.1}$$
$$\phi_1 \equiv \partial_i \Pi^i(x) - j^0_{mat}(x) = 0, \tag{2.7.2}$$

where $\Pi^0 = -F^{00} = 0$ and $\Pi^i = -F^{0i} = -E^i$ are the canonical conjugate variables for A_0 and A_i, respectively. In quantum theory, the first class constraints are imposed on any physical state $|\psi\rangle$ as

$$\Pi^0(x)|\psi\rangle = 0, \tag{2.7.3}$$
$$\left(\partial_i E^i(x) - j^0_{mat}(x)\right)|\psi\rangle = 0. \tag{2.7.4}$$

These conditions make the physical states gauge invariant because the first class constraints are the generators of gauge transformations. The second constraint (2.7.4) is called the *Gauss law constraint*. Multiplying (2.7.4) by $\epsilon(x)$ and integrating over space, we get

$$0 = \int d^3x\ \epsilon(x)\left(\partial_i E^i(x) - j^0_{mat}(x)\right)|\psi\rangle \tag{2.7.5}$$

$$= \lim_{r\to\infty}\int d^2\Omega\ r^2\epsilon(x)E^r(x)|\psi\rangle - \int d^3x\left(\partial_i\epsilon(x)E^i(x) + \epsilon(x)j^0_{mat}(x)\right)|\psi\rangle, \tag{2.7.6}$$

where we have performed a partial integration in the second equality. Thus we find for any physical state $|\psi\rangle$,

$$Q[\epsilon]|\psi\rangle = \lim_{r\to\infty}\int d^2\Omega\ r^2\epsilon(x)E^r(x)|\psi\rangle, \tag{2.7.7}$$

where $Q[\epsilon]$ was defined in (2.2.7).

The trivial (small) gauge transformations are the gauge transformations such that $\epsilon(x)$ falls off so rapidly that the righthand side in (2.7.7) vanishes:

$$Q[\epsilon]|\psi\rangle = 0 \tag{2.7.8}$$

The trivial gauge transformation actually does not act on anything, in other words, we can call it "do-nothing transformations". Therefore it is not the physical transformation. Once the physical Hilbert space is restricted to an invariant subspace for the trivial gauge transformations, physical observables are also restricted to the set of invariant operators under the transformations due to Elitzur's theorem. Therefore, the Ward-Takahashi identity for the trivial gauge transformation is

$$\langle Q[\epsilon](\cdots)\rangle = \langle\delta(\cdots)\rangle = 0, \tag{2.7.9}$$

where $(\cdots)$ can be any physical observables. All physical observables trivially satisfy this equation by definition, so the Ward-Takahashi identity actually constraints nothing.

On the contrary, the large gauge transformations can act on physical states by (2.7.7) nontrivially as

$$Q[\epsilon]|\psi\rangle = \lim_{r\to\infty}\int d^2\Omega\ r^2\epsilon(x)E^r(x)|\psi\rangle \neq 0 \tag{2.7.10}$$

Thus the physical states can be characterized by these charges. Note that we have the charges for each choice of the large gauge parameter ϵ. However these charge operators do not act on any observable $\mathcal{O}$ with a finite spatial support because

$$\langle\psi|[Q[\epsilon],\mathcal{O}]|\psi\rangle = \lim_{r\to\infty}\int d^2\Omega\ r^2\epsilon(x)\langle\psi|[E^r(x),\mathcal{O}]|\psi\rangle = 0\ , \tag{2.7.11}$$

where we have used the causality for observables in the second equality. Therefore, these charges can not be changed by acting any quasilocal operators. (See [62] for the rigorous proof.) Thus the charges associated with the large gauge transformations decompose the Hilbert space into superselection sectors. In the next section, we give more systematic definition for the asymptotic symmetries.

2.8 Charges associated with asymptotic symmetry in QED

The Gauss law constraint in (2.7.4) can be written in (τ, ρ, Ω^A) coordinates defined in (2.3.2) as

$$0 = \left(\nabla_i E^i - j^0_{mat}\right)|\psi\rangle \tag{2.8.1}$$

$$= \left(\nabla_\rho E^\rho + \frac{1}{\tau^2\rho^2}\nabla_A E^A - j^0_{mat}\right)|\psi\rangle\ , \tag{2.8.2}$$

where we used the metric in (2.3.10) and $\nabla_A E^A \equiv \gamma^{AB}\nabla_A E_B$. The Gauss law constraint in (2.7.4) can be written in the retarded coordinates as

$$0 = \left(\nabla_i E^i - j^0_{mat}\right)|\psi\rangle \tag{2.8.3}$$

$$= \left(\nabla_r E^r + \frac{1}{r^2}\nabla_A E^A - j^0_{mat}\right)|\psi\rangle\ , \tag{2.8.4}$$

where we have used $E_0 = g_{0\mu}E^\mu = -E^u$ in the final equality. Multiplying (2.8.3) by $\epsilon(x)$ and integrating it over $\mathcal{I}^+$, we have

$$0 = \lim_{\tau\to\infty}\int_{i^+} d^3\Sigma_{i^+}\ \epsilon\left(\nabla_\rho E^\rho + \frac{1}{\tau^2\rho^2}\nabla_A E^A - j^0_{mat}\right)|\psi\rangle$$
$$+ \lim_{r\to\infty}\int_{\mathcal{I}^+} d^3\Sigma_{\mathcal{I}^+}\epsilon\left(\nabla_r E^r + \frac{1}{r^2}\nabla_A E^A\right)|\psi\rangle \tag{2.8.5}$$

where we have dropped the matter current term in the second term, since the massive matter current asymptotically goes to the time-like infinity and can not reach to the null infinity. Performing the partial integrations and using $\nabla_r E^r = \partial_r E^r$, $\nabla_\rho E^\rho = \partial_\rho E^\rho + \frac{1}{\tau}E^\tau + \frac{\rho}{1+\rho^2}E^\rho$, we obtain

$$Q^+[\epsilon]|\psi\rangle$$
$$= \Bigg[\lim_{\rho\to\infty}\lim_{\tau\to\infty}\int d^2\Omega\ \frac{\rho^2\sqrt{\gamma(\Omega)}}{\sqrt{1+\rho^2}}\tau^3\ \epsilon E^\rho$$
$$+ \lim_{u\to\infty}\lim_{r\to\infty}\int d^2\Omega\sqrt{\gamma(\Omega)}\ r^2\epsilon E^r - \lim_{u\to-\infty}\lim_{r\to\infty}\int d^2\Omega\sqrt{\gamma(\Omega)}\ r^2\epsilon E^r\Bigg]|\psi\rangle\ , \tag{2.8.6}$$

where $Q^+[\epsilon]$ is defined as

$$Q^+[\epsilon] \equiv Q^+_H[\epsilon] + Q^+_S[\epsilon] \tag{2.8.7}$$

with

$$Q^+_H[\epsilon] \equiv \int_{i^+} d^3\Sigma_{i^+}\left((\partial_\rho\epsilon)E^\rho + \frac{1}{\tau^2\rho^2}(\partial_A\epsilon)E^A + \epsilon j^0_{mat}\right)\ , \tag{2.8.8}$$

$$Q^+_S[\epsilon] \equiv \int_{\mathcal{I}^+} d^3\Sigma_{\mathcal{I}^+}\left((\partial_r\epsilon)E^r + \frac{1}{r^2}(\partial_A\epsilon)E^A\right)\ . \tag{2.8.9}$$

This is the charge that generates the asymptotic symmetry with the gauge parameter ϵ on the future asymptotic infinity, which is the quantum analog of the classical asymptotic charge given in (2.4.1). In the right hand side of (2.8.6), the first term and the second term must be canceled out with each other because the boundary of i^+ and the boundary of $\mathcal{I}^+$ at $u = \infty$ are the same spacetime surfaces, and the outward vectors orthogonal to each surface have opposite direction. Therefore, (2.8.6) becomes

$$\begin{aligned} Q^+[\epsilon]|\psi\rangle &= -\lim_{u\to-\infty}\lim_{r\to\infty}\int d^2\Omega\sqrt{\gamma(\Omega)}\; r^2\epsilon E^r|\psi\rangle \\ &= -\int_{\mathcal{I}^+_-} d^2\Omega\sqrt{\gamma(\Omega)}\; r^2\epsilon^+ E^r|\psi\rangle\ , \end{aligned} \tag{2.8.10}$$

where we have defined $\epsilon^+(\Omega)$ as

$$\epsilon^+(\Omega) \equiv \lim_{u\to-\infty}\lim_{r\to\infty}\epsilon(u,r,\Omega)\ , \tag{2.8.11}$$

It is important that the action of the asymptotic charge on physical states, given by (2.8.10), *only* involves $\epsilon^+(\Omega)$. The charges with the gauge parameter $\epsilon^+(\Omega) = 0$ act on any physical state trivially and the charges with the gauge parameter $\epsilon^+(\Omega) \neq 0$ act on physical states nontrivially in general. Therefore, The latter ones are the physical asymptotic charges and the transformations generated by the charges are the asymptotic symmetries.

In the case of $\epsilon =$ constant, the asymptotic charge in (2.8.7) reduces to

$$Q^+[\epsilon] = \epsilon\lim_{\tau\to\infty}\int_{i^+} d\Sigma_{i^+}\; j^0_{mat}\ , \tag{2.8.12}$$

which is the usual global $U(1)$ charge. The right hand side in (2.8.10) also reduces to the total electric charge, since there exist only Coulomb forces at $\mathcal{I}^+_-$.

Repeating the same argument for the asymptotic past infinity, we have

$$\begin{aligned} Q^-[\epsilon]|\psi\rangle &= \lim_{v\to\infty}\lim_{r\to\infty}\int d^2\Omega\sqrt{\gamma(\Omega)}\;\epsilon r^2 E^r|\psi\rangle \\ &= \int_{\mathcal{I}^-_+} d^2\Omega\sqrt{\gamma(\Omega)}\;\epsilon r^2 E^r|\psi\rangle\ , \end{aligned} \tag{2.8.13}$$

where the asymptotic charge on the past infinity is given by

$$Q^-[\epsilon] \equiv Q^-_H[\epsilon] + Q^-_S[\epsilon] \tag{2.8.14}$$

with

$$Q^-_H[\epsilon] \equiv \int_{i^-} d^3\Sigma_{i^-}\left((\partial_\rho\epsilon)E^\rho + \frac{1}{\tau^2\rho^2}(\partial_A\epsilon)E^A + \epsilon j^0_{mat}\right)\ , \tag{2.8.15}$$

$$Q^-_S[\epsilon] \equiv \int_{\mathcal{I}^-} d^3\Sigma_{\mathcal{I}^-}\left((\partial_r\epsilon)E^r + \frac{1}{r^2}(\partial_A\epsilon)E^A\right)\ . \tag{2.8.16}$$

The conservation law for these asymptotic charges is

$$\langle\beta|Q^+[\epsilon]S|\alpha\rangle + \langle\beta|T\left[Q^0[\epsilon]S\right]|\alpha\rangle = \langle\beta|SQ^-[\epsilon]|\alpha\rangle \tag{2.8.17}$$

where $Q^0[\epsilon]$ is the charge defined on i^+. Here, we assume that the antipodal matching condition also holds in QED and the second term in the left hand side of (2.8.18) vanishes. We will see that this assumption has a support if we consider the dressed state formalism in Section 5.7. The asymptotic charges then conserve between the future infinity and the past infinity:

$$\langle\beta|Q^+[\epsilon]S|\alpha\rangle = \langle\beta|SQ^-[\epsilon]|\alpha\rangle\ . \tag{2.8.18}$$

We will see that this conservation law is equivalent to the soft photon theorem in the next chapter.

2.9 Asymptotic symmetry in BRST formalism

We have seen why the large gauge transformations are physical symmetries in the canonical quantization. However, we often quantize gauge theories by the BRST quantization rather than the canonical quantization, especially for computing S-matrixes. In this section, we argue that the asymptotic symmetry in QED is physical symmetry based on the BRST formalism, although they are naïvely parts of the gauge symmetry. We first review the covariant quantization of QED in the BRST formalism in Subsection 2.9.1. Next, we look at the asymptotic behaviors of gauge fields and ghost fields in Subsection 2.9.2. Finally, we see that the charges associated with asymptotic symmetry act on physical Hilbert space nontrivially under the BRST condition.

2.9.1 BRST quantization in QED and the correct BRST charge

In this subsection we review the covariant quantization of massive scalar QED[9] in the BRST formalism[10], and discuss the symmetries.

The Lagrangian in the Feynman gauge is given by

$$\mathcal{L}_{QED} = \mathcal{L}_{EM} + \mathcal{L}_{matter} + \mathcal{L}_{GF} + \mathcal{L}_{FP}, \tag{2.9.1}$$

where

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \quad \mathcal{L}_{matter} = -\frac{1}{2}D_\mu\phi^\dagger D_\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi\,, \tag{2.9.2}$$

$$\mathcal{L}_{GF} = -\frac{1}{2}\left(\nabla_\mu A^\mu\right)^2\ ,\ \mathcal{L}_{FP} = i\partial^\mu\bar{c}\partial_\mu c. \tag{2.9.3}$$

Here, ϕ is a massive charged scalar field[11] with charge e where $D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$, and

[9]Although the arguments of this section can be straightforwardly applied to any kind of charged particles such as massive or massless fermions and scalars, we explicitly write the expressions of a massive scalar because we use them in Section 4.1.

[10]In QED, the ghost sector is completely decoupled. Nevertheless, we use the BRST formalism to discuss what the physical states are.

[11]In this paper, we consider only a single species of charge for simplicity. The generalization to many species is straightforward.

c, $\bar{c}$ are ghost fields [12]. $\mathcal{L}_{GF}+\mathcal{L}_{FP}$[13] is a BRST exact term, which is added to the original Lagrangian to eliminate the gauge symmetry. The equation of motions are given by

$$\Box A^{\mu} = -j^{\mu}\ ,\ \ \Box c = 0\ ,\ \ \Box \bar{c} = 0\ . \tag{2.9.4}$$

The canonical conjugate fields for each $A^{\mu}, c, \bar{c}$ are given by

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^0)} = -\nabla_{\mu} A^{\mu}, \quad \Pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^i)} = -F^0{}_i, \tag{2.9.5}$$

$$\pi_{(c)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 c)} = -i\partial_0 \bar{c}, \quad \bar{\pi}_{(c)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{c})} = i\partial_0 c. \tag{2.9.6}$$

The Hamiltonian is then given by free part H_0 and the other interacting part V:

$$H = H_0 + V\,, \tag{2.9.7}$$

where

$$H_0 = H_{EM} + H_{matter} + H_{ghost}, \tag{2.9.8}$$

with[14]

$$H_{EM} = \int d^3x \left[\frac{1}{2}\Pi_{\mu}\Pi^{\mu} + (\partial_i \Pi_0)A^i + (\partial_i \Pi_i)A^0 + \frac{1}{4}F_{ij}F^{ij}\right], \tag{2.9.9}$$

$$H_{ghost} = i\int d^3x (\bar{\pi}_{(c)}\pi_{(c)} - \partial_i \bar{c}\partial_i c), \tag{2.9.10}$$

and H_{matter} is the free part of the Hamiltonian of matter fields. The equal-time (anti)commutation relations for the gauge fields and the ghost fields are given by

$$[A_{\mu}(\vec{x}), \Pi_{\nu}(\vec{y})] = i\eta_{\mu\nu}\delta^3(\vec{x}-\vec{y})\,, \quad \{c(\vec{x}), \pi_{(c)}(\vec{y})\} = \{\bar{c}(\vec{x}), \bar{\pi}_{(c)}(\vec{y})\} = i\delta^3(\vec{x}-\vec{y}). \tag{2.9.11}$$

In the interaction picture, the equation of motions for them are given by free ones; $\Box A^I_{\mu}(x) = 0$. The general solutions of the equation of motions that can be expanded by the Fourier expansions are given by

$$\begin{aligned} A^I_{\mu}(x) &= \int \widetilde{d^3k}\big(a_{\mu}(\vec{k})e^{ikx} + a^{\dagger}_{\mu}(\vec{k})e^{-ikx}\big), \\ c^I(\vec{x}) &= \int \widetilde{d^3k}\left[c(\vec{k})e^{ikx} + c^{\dagger}(\vec{k})e^{-ikx}\right], \\ \bar{c}^I(\vec{x}) &= \int \widetilde{d^3k}\left[\bar{c}(\vec{k})e^{ikx} + \bar{c}^{\dagger}(\vec{k})e^{-ikx}\right], \end{aligned} \tag{2.9.12}$$

where k^{μ} is massless on-shell momenta and $\widetilde{d^3k}$ is the Lorentz invariant measures for the integration of the spatial momentum which is defied as $\widetilde{d^3k} \equiv \frac{d^3k}{(2\pi)^3 2\omega_k}$. The mode expansions of the above operators in the Schrödinger picture can be given by taking $t = t_s$

[12] Both c and $\bar{c}$ are Hermitian fields. This Hermicity is required to make the Lagrangian Hermite.

[13] We already integrated out the Nakanishi-Lautrup field in $\mathcal{L}_{GF}$.

[14] H_{EM} given by (2.9.9) is different from the Hamiltonian obtained in a canonical way from Lagrangian (2.9.1) by a total derivative term. We have eliminated the boundary term, and then this H_{EM} commutes with the BRST charge without a boundary term. This difference is not important except in sec 5.7.

in the above expansion. The conjugate fields in (2.9.5) are then given by

$$\Pi_0^I(\vec{x}) = -i\int \widetilde{d^3k}\left[k^\mu a_\mu(\vec{k})e^{ikx} - k^\mu a_\mu^\dagger(\vec{k})e^{-ikx}\right], \tag{2.9.13}$$

$$\Pi_i^I(\vec{x}) = -i\int \widetilde{d^3k}\left[(k_i a_0(\vec{k}) + \omega a_i(\vec{k}))e^{ikx} - (k_i a_0^\dagger(\vec{k}) + \omega a_i^\dagger(\vec{k}))e^{-ikx}\right], \tag{2.9.14}$$

$$\pi_{(c)}^I(\vec{x}) = -\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[\bar{c}(\vec{k})e^{ikx} - \bar{c}^\dagger(\vec{k})e^{-ikx}\right], \tag{2.9.15}$$

$$\bar{\pi}_{(c)}^I(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[c(\vec{k})e^{ikx} - c^\dagger(\vec{k})e^{-ikx}\right], \tag{2.9.16}$$

The commutation relations (2.9.11) lead to the following equal-time (anti)commutation relations between annihilation and creation operators:

$$[a_\mu(\vec{k}), a_\nu^\dagger(\vec{k}')] = (2\pi)^3(2\omega_k)\eta_{\mu\nu}\delta^{(3)}(\vec{k}-\vec{k}')\ , \tag{2.9.17}$$

$$\{c(\vec{k}), \bar{c}^\dagger(\vec{k}')\} = i(2\omega)(2\pi)^3\delta^3(\vec{k}-\vec{k}'), \quad \{\bar{c}(\vec{k}), c^\dagger(\vec{k}')\} = -i(2\omega)(2\pi)^3\delta^3(\vec{k}-\vec{k}'). \tag{2.9.18}$$

The Hilbert space is just the Fock space $\mathcal{H}_{\text{Fock}}$.

This Lagrangian has two kinds of symmetries: the residual gauge symmetry and the *BRST symmetry* [63, 64] . The residual gauge symmetry is

$$\delta_g\phi(x) = ie\epsilon(x)\phi(x)\ ,\ \delta_g\phi^\dagger(x) = -ie\epsilon(x)\phi^\dagger(x)\ ,\ \delta_g A_\mu(x) = \partial_\mu\epsilon(x), \tag{2.9.19}$$

with

$$\Box\epsilon(x) = 0. \tag{2.9.20}$$

These residual symmetries are usually regarded as "gauge" redundancies, but we will argue in the next subsection 2.9.2 that parts of them associated with the large gauge parameters (A.0.4) are physical symmetries, which impose the nontrivial constraints on the S-matrices as the Ward-Takahashi identities. Using the Lagrangian (2.9.1), we have the Noether current associated with (2.9.19);

$$j_\epsilon^\mu(x) = j_S^\mu(x) + j_H^\mu(x), \tag{2.9.21}$$

where

$$j_S^\mu(x) = \nabla_\nu\epsilon(x)F^{\mu\nu}(x) + \nabla^\mu\epsilon(x)\nabla_\nu A^\nu(x)\,, \tag{2.9.22}$$

$$j_H^\mu(x) = \epsilon(x)j_{mat}^\mu(x)\,, \quad \text{with} \quad j_{mat}^\mu(x) = ie\big(D^\mu\phi^\dagger(x)\,\phi(x) - \phi^\dagger(x)\,D^\mu\phi(x)\big). \tag{2.9.23}$$

The current is different from (2.5.1) due to the gauge fixing term. The charge that generates the above symmetry on a spatial surface Σ is given by

$$Q[\epsilon] = \int_\Sigma dS^\mu\, j_\epsilon^\mu(x)\,. \tag{2.9.24}$$

The asymptotic charge on the future infinity is then given by

$$Q^+[\epsilon] = Q_{i^+}[\epsilon] + Q_{\mathcal{I}^+}[\epsilon] \tag{2.9.25}$$

with

$$Q_{i^+}[\epsilon] = \int_{i^+} d^3\Sigma_{i^+\mu}\, j^\mu_\epsilon(x)\ ,\ Q_{\mathcal{I}^+}[\epsilon] = \int_{\mathcal{I}^+} d^3\Sigma_{\mathcal{I}^+\mu}\, j^\mu_S(x)\,. \tag{2.9.26}$$

On the other hand, the BRST symmetry (in the case of scaler matter) is given by

$$\delta_g\phi(x) = iec(x)\phi(x)\ ,\ \delta_g\phi^\dagger(x) = -iec(x)\phi^\dagger(x)\ ,\ \delta_g A_\mu(x) = \partial_\mu c(x)\,. \tag{2.9.27}$$

We have the charge that generates the BRST symmetry called the BRST charge. The BRST charge on a Minkowski time slice ($x^0 = t$) is given by

$$\begin{aligned} Q_{BRST} &= \int d^3x\left[-(\partial_\mu c)\Pi^\mu + cj^0\right] \\ &= \int d^3x\left[i\bar{\pi}_{(c)}\Pi^0 - (\partial_i c)\Pi^i + cj^0\right]. \end{aligned} \tag{2.9.28}$$

It is important that the last term involving the matter current is needed to generate the BRST transformation (2.9.27) correctly, in particular,

$$[iQ_{BRST}, \phi] = iec\phi, \tag{2.9.29}$$

although the term has been sometimes missed. We will see that the term becomes important for the gauge invariance of the dressed state in Section 5.5. This BRST charge commutes with the total Hamiltonian and the matter current

$$[Q_{BRST}, H] = 0\,,\quad [Q_{BRST}, j^\mu(\vec{x})] = 0. \tag{2.9.30}$$

The BRST charge on general Cauchy slice Σ is given by

$$Q_{BRST} = \int_\Sigma dS_\mu\left[(\partial_0 c)g^{\mu 0}\nabla_\nu A^\nu + (\partial_i c)F^{\mu i} + cj^\mu\right]\ . \tag{2.9.31}$$

When we take Σ as the usual time slice ($t = const.$) and substitute (2.9.12) into (2.9.28), the BRST charge is expressed as

$$Q^I_{BRST}(t) = -\int\widetilde{d^3k}\left[c(\vec{k})\{k^\mu a^\dagger_\mu(\vec{k}) + e^{-i\omega t}\tilde{j}^{0I}(t,-\vec{k})\} + c^\dagger(\vec{k})\{k^\mu a_\mu(\vec{k}) + e^{i\omega t}\tilde{j}^{0I}(t,\vec{k})\}\right], \tag{2.9.32}$$

where $\tilde{j}^{0I}(t,\vec{k})$ is the Fourier transformation of $j^{0I}(t,\vec{x})$ defined as

$$\tilde{j}^I_\mu(t,\vec{k}) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}} j^I_\mu(t,\vec{x}). \tag{2.9.33}$$

In order to ensure that transition amplitudes are independent of the choice of gauge fixing, we need to impose the *BRST condition* [65] on the physical Hilbert space:

$$Q_{BRST}|\psi\rangle = 0\,, \tag{2.9.34}$$

where $Q_{BRST}(= Q^I_{BRST}(t_s))$ is the BRST charge in the Schrödinger picture and $|\psi\rangle$ is any physical state at $t = t_s$. We will see in Section 5.5 that the BRST condition for the past asymptotic state $|\psi\rangle_0$ for $|\psi\rangle$ [15] is given by

$$\lim_{t\to-\infty} Q^I_{BRST}(t)|\psi\rangle_0 = 0\,. \tag{2.9.35}$$

We usually assume that the ghost sector of physical Hilbert space is always the vacuum state that is annihilated by $c(\vec{k})$, since the ghost fields are completely decoupled from the other fields in QED. Moreover, in the conventional scattering theory, we assume that *the QED interaction can be ignored in the past and future infinities.* Under these assumption, the BRST condition for the photon sector is equivalent to the *Gupta-Bleuler* condition [66, 67]:

$$k^\mu a_\mu(\vec{k})\,|\psi\rangle = 0\,, \tag{2.9.36}$$

which ensures that the longitudinal modes of gauge fields do not contribute to the dynamics of QED. We will see in the next section that the above Gupta-Bleuler condition is modified once the effect of QED interaction in the asymptotic regions is incorporated.

The subspace annihilated by Q_{BRST} is represented by $\mathcal{H}_{closed}$. $\mathcal{H}_{closed}$ is not the same as the physical Hilbert space from the following reason. For any state $|\xi\rangle \in \mathcal{H}_{\text{Fock}}$, $Q_{BRST}|\xi\rangle$ is included in $\mathcal{H}_{closed}$ because of $Q^2_{BRST} = 0$. The set of such states is called the BRST closed subspace, which is defined as

$$\mathcal{H}_{closed} \equiv \{Q_{BRST}|\xi\rangle \mid |\xi\rangle \in \mathcal{H}_{\text{Fock}}\}\,. \tag{2.9.37}$$

However, all the state in $\mathcal{H}_{closed}$ are orthogonal to any state in $\mathcal{H}_{closed}$ because of the BRST condition (2.9.34). Therefore, two closed states that differs by a BRST closed state as

$$|\psi'\rangle = |\psi\rangle + i\lambda Q_{BRST}|\xi\rangle\,, \tag{2.9.38}$$

have the same inner products with all the physical states. It means that such two states are physically indistinguishable. The true physical space is obtained by identifying such equivalent states and thus given by the cohomology of Q_{BRST}:

$$\mathcal{H}_{phys} \equiv \mathcal{H}_{closed}/\mathcal{H}_{exact}\,. \tag{2.9.39}$$

Therefore, the BRST transformation,

$$|\psi\rangle \to |\psi'\rangle = e^{i\lambda Q_{BRST}}\,|\psi\rangle = |\psi\rangle + i\lambda Q_{BRST}\,|\psi\rangle\,, \tag{2.9.40}$$

is a trivial transformation, since it just transforms a physical state into an identical physical state.

In order for observable to be independent of the above BRST transformation, any physical operator $\mathcal{O}$ acting on $\mathcal{H}_{phys}$ must satisfy the BRST invariant condition, namely

$$\delta_{BRST}\mathcal{O} = \big[i\lambda Q_{BRST}, \mathcal{O}\big] = 0\,. \tag{2.9.41}$$

We have observed that the BRST symmetry is not physical symmetry but trivial symmetry because the BRST charge acts on any physical state and observable trivially.

[15] The definition of the asymptotic state is explained in Chapter 3.

2.9.2 Large r expansion of fields in QED

In this subsection, we consider the asymptotic behaviors of gauge fields to study the expression of the BRST charge and the charge associated with large gauge symmetry on the future and past infinities.

To investigate the asymptotic behaviors near future null infinity $\mathcal{I}^+$, we use the retarded coordinates (u, r, Ω^A). Near the asymptotic region $\mathcal{I}^+$, the radiation fields $A_\mu(x)$ would be well approximated by the free field which has the free wave expansion (2.9.12). The free field can be expressed in (u, r, θ, φ) coordinates as

$$\begin{aligned} A_\mu(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a_\mu(\vec{k}) e^{-i\omega_k u - i\omega_k r(1-\hat{k}\cdot\hat{x})} + (h.c.) \right] \\ &= \frac{1}{16\pi^3} \int_0^\infty d\omega_k \omega_k \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \left[a_\mu(\vec{k}) e^{-i\omega_k u - i\omega_k r(1-\cos\theta)} + (h.c.) \right] , \end{aligned} \tag{2.9.42}$$

where we have used the polar coordinates $(\omega_k, \theta, \varphi)$ for $\vec{k}$ and performed the integration of φ in the second line. Here we use the saddle point approximation at large r with fixed u. The saddle point in θ space is given by

$$0 = \frac{\partial}{\partial\theta} \left(\omega_k u - \omega_k r(1 - \cos\theta) \right) = -\omega_k r \sin\theta \;\Rightarrow\; \theta = 0, \pi \, . \tag{2.9.43}$$

At the saddle point at $\theta = \pi$, the phase factor in (2.9.42) becomes $\omega_k u - 2\omega_k r$. In this case, the phase factor diverges as $r \to \infty$ and the integration over ω_k is expected to be zero due to the Riemann-Lebesgue Lemma. The fact that the saddle point at large r is given at $\theta = 0$ justifies the intuitive picture that every wave can be seen as coming from the center of space when we observe the waves at $r = \infty$. Then by expanding the integrand in (2.9.42) around $\theta = 0$ and we find

$$\begin{aligned} A_\mu(x) &= \bar{A}_\mu^{(0)} + \frac{1}{8\pi^2} \sum_{\alpha=\pm} \int_0^\infty d\omega_k \omega_k \left[a_\mu\left(\omega_k \hat{x}\right) e^{-i\omega_k u} \int_0^\pi d\theta \theta e^{-i\omega_k r^2/2} + (h.c.) \right] + \mathcal{O}\left(r^{-2}\right) \\ &= \bar{A}_\mu^{(0)} - \frac{i}{8\pi^2 r} \int_0^\infty d\omega \left[a_\mu(\omega\hat{x}) e^{-i\omega u} - (h.c.) \right] + \mathcal{O}(r^{-2}) \, , \end{aligned} \tag{2.9.44}$$

where $\bar{A}_\mu^{(0)}$ is the exact zero mode which is given by

$$\bar{A}_\mu^{(0)} \equiv \lim_{\omega\to 0} \frac{1}{16\pi^3} \int d^2\Omega_k \sqrt{\gamma(\Omega_k)} \left[\omega_k a_\mu(\vec{k}) + (h.c.) \, . \right] \tag{2.9.45}$$

We have separated the zero mode from other parts because the saddle point approximation at large r is not valid for $\omega_k = 0$ mode in (2.9.42) [16]. Accordingly, each component of the

[16] One may consider $\bar{A}_\mu^{(0)} = 0$, since it contains $\lim_{\omega\to 0} \omega_k a_\mu(\vec{k})$. However, we should be careful because $a_\mu(\vec{k})$ is not a c-number but an operator. In fact, $a_\mu(\vec{k})$ gives a factor of order $\frac{1}{\omega_k}$ when it acts on out-states as we can see in the leading soft photon theorem (3.4.19). In this case, $\lim_{\omega\to 0} \omega_k a_\mu(\vec{k})$ gives an order-one contribution in the S-matrix.

gauge field in the (u, r, Ω^B) coordinates is obtained as follows:

$$A_u(x) = \frac{\partial x^\mu}{\partial u} A_\mu(x) = \bar{A}_u^{(0)} + \frac{A_u^{(1)}(u,\Omega)}{r} + \mathcal{O}(r^{-2}\log r), \tag{2.9.46}$$

$$A_r(x) = \frac{\partial x^\mu}{\partial r} A_\mu(x) = \bar{A}_r^{(0)} + \frac{A_r^{(1)}(u,\Omega)}{r} + \mathcal{O}(r^{-2}\log r), \tag{2.9.47}$$

$$A_B(x) = \frac{\partial x^\mu}{\partial \Omega^B} A_\mu(x) = r A_B^{(-1)}(\Omega) + A_B^{(0)}(u,\Omega) + \mathcal{O}(r^{-1}\log r)\,, \tag{2.9.48}$$

with

$$\begin{aligned}
&A_B^{(-1)}(\Omega) = \partial_B \hat{x}^i \bar{A}_i^{(0)}\ ,\ A_u^{(0)} = A_t^{(0)}\ ,\ A_r^{(0)} = A_t^{(0)} + \hat{x}^i A_i^{(0)}\ ,\\
&A_u^{(1)}(u,\Omega) = -\frac{i}{8\pi^2}\int_0^\infty d\omega \big[a_u(\omega\hat{x})e^{-i\omega u} - (h.c.)\big],\\
&A_r^{(1)}(u,\Omega) = -\frac{i}{8\pi^2}\int_0^\infty d\omega\ \big[a_r(\omega\hat{x})e^{-i\omega_k u} - (h.c.)\big],\\
&A_B^{(0)}(u,\Omega) = -\frac{i}{8\pi^2}\int_0^\infty d\omega\ \big[a_B(\omega\hat{x})e^{-i\omega u} - (h.c.)\big],
\end{aligned} \tag{2.9.49}$$

where each annihilation operator is defined as

$$a_u(\vec{k}) = a_t(\vec{k})\ ,\ a_r(\vec{k}) = q^\mu a_\mu(\vec{k})\ ,\ a_B(\vec{k}) = \partial_B \hat{x}^i a_i(\vec{k})\,, \quad q^\mu \equiv (1,\hat{x})\,. \tag{2.9.50}$$

Note that the first term in (2.9.48) diverge as $r \to \infty$. However, it gives the finite contribution when we compute the gauge (BRST) invariant operator like F_{rB}.

We now see that the leading $\mathcal{O}(1)$ components $A_B^{(0)}$ constitute the Cauchy data on the future null infinity. In other words, the two components correspond to the two physical degrees of freedom of photons.

We now consider the fall-off condition for the ghost field. Since the ghost field satisfies the free equation of motion

$$\Box c(x) = 0, \tag{2.9.51}$$

the saddle point approximation at large r for (2.9.12) can be used in the same way for the gauge fields. We then obtain

$$c(u,r,\Omega) = c^{(0)} + \frac{c^{(1)}(u,\Omega)}{r} + \mathcal{O}(r^{-2}\log r)\,, \tag{2.9.52}$$

where

$$c^{(0)} = \lim_{\omega\to 0}\frac{1}{16\pi^3}\int_0^\pi d^2\Omega_k \sqrt{\gamma(\Omega_k)}\,\Big[\omega_k c(\vec{k}) + (h.c.)\Big] \tag{2.9.53}$$

$$c^{(1)}(u,\Omega) = -\frac{i}{8\pi^2}\int_0^\infty d\omega_k \big[c(\omega_k\hat{x})e^{-i\omega_k u} - (h.c.)\big] + \mathcal{O}(r^{-2}\log r)\,. \tag{2.9.54}$$

2.9.3 Asymptotic BRST condition and asymptotic symmetry

Now we study the BRST charge on the future null infinity. Writing the BRST charge (2.9.31) on the future null infinity, we obtain

$$Q^{(+)}_{BRST} = Q^{(+)}_{BRST,H} + Q^{(+)}_{BRST,S} \,. \tag{2.9.55}$$

Here, the $Q^{(+)}_{BRST,H}$ is given by

$$Q^{(+)}_{BRST,H} = \int_{i^+} d\Sigma_{i^+\mu}\ \left[(\partial_0 c) g^{\mu 0}\nabla_\nu A^\nu + (\partial_i c)F^{\mu i} + c j^\mu_{mat}\right] . \tag{2.9.56}$$

If we adopt the assumption that the QED interaction can be ignored in the past and future infinities, any electromagnetic field does not exist in i^+. Then we can drop the first term and second terms in (2.9.56) and we obtain

$$Q^{(+)}_{BRST,H} = \int_{i^+} d\Sigma_{i^+} c j^\tau_{mat}. \tag{2.9.57}$$

The $Q^{(+)}_{BRST,S}$ is given by

$$\begin{aligned} Q^{(+)}_{BRST,S} &= \int_{\mathcal{I}^+} d\Sigma_{\mathcal{I}^+\mu}\ \left[(\partial_0 c) g^{\mu 0}\nabla_\nu A^\nu + (\partial_i c)F^{\mu i} + c j^\mu_{mat}\right] \\ &= \int_{\mathcal{I}^+} d\Sigma_{\mathcal{I}^+}\ \left[\partial_u c \nabla_\nu A^\nu + (\partial_i c)F^{ri}\right] , \end{aligned} \tag{2.9.58}$$

where we have dropped the term cj^μ in the first line since the matter fields do no reach $\mathcal{I}^+$ and have also used $d\Sigma_{\mathcal{I}^+\mu} = d\Sigma_{\mathcal{I}^+}\delta^r{}_\mu$ and the metric given in (2.3.18) in the second line. Here, $\nabla_\nu A^\nu$ and $(\partial_i c)F^{ri}$ can be written as

$$\nabla_\nu A^\nu = -\partial_u A_r + \partial_r(A_r - A_u) + \frac{2}{r}(A_r - A_u) + \frac{\gamma^{AB}}{r^2} D_A A_B \tag{2.9.59}$$

$$(\partial_i c)F^{ri} = g^{i\mu}(\partial_i c)F^r{}_\mu = \frac{\gamma^{AB}}{r^2}\partial_A c\,(F_{rB} - F_{uB}) \,. \tag{2.9.60}$$

Inserting above ones with the expansions (2.9.46)-(2.9.47) and (2.9.52) into (2.9.58), we have

$$\begin{aligned} Q^{(+)}_{BRST,S} &= \int dud^2\Omega\sqrt{\gamma(\Omega)}\ \left[\partial_u c^{(1)}\left(\partial_u A^{(1)}_r + 2(A^{(0)}_r - A^{(0)}_u)\right)\right] \\ &= \frac{1}{64\pi^2}\int d\omega d^2\Omega\sqrt{\gamma(\Omega)}\omega^2\ \left[c(\omega\hat{x})a^\dagger_r(\omega\hat{x}) + c^\dagger(\omega\hat{x})a_r(\omega\hat{x})\right] \\ &\quad - \frac{1}{4\pi^2}(A^{(0)}_r - A^{(0)}_u)\int d\omega d^2\Omega\sqrt{\gamma(\Omega)}\lim_{\omega\to 0}\left[c(\omega\hat{x}) + c^\dagger(\omega\hat{x})\right] \end{aligned} \tag{2.9.61}$$

The BRST condition (2.9.34) for the photon sector is then equivalent to

$$\partial_u A^{(+)(1)}_r\left|\psi\right\rangle = 0\ ,\ \left(A^{(0)}_r - A^{(0)}_u\right)\left|\psi\right\rangle = 0\ . \tag{2.9.62}$$

where $A_r^{(+)(1)}$ is the part of $A_r^{(1)}$ which includes the annihilation operators. The first condition is equivalent to

$$0 = a_r(\omega\hat{x})\,|\psi\rangle = \omega q^\mu a_\mu(\omega\hat{x})\,|\psi\rangle \ , \tag{2.9.63}$$

which is the Gupta-Bleuler condition with the identification of the photon momentum with $q^\mu (= (1, \hat{x}))$.

The BRST condition for the matter sector is given by

$$0 = Q^{(+)}_{BRST,H}\,|\psi\rangle = \int_{i^+} d\Sigma_{i^+} c j^\tau_{mat}\,|\psi\rangle \ , \tag{2.9.64}$$

where $Q^{(+)}_{BRST,H}$ was given in (2.9.57). Now we focus on the falloff behaviors of c and j^τ_{mat} near i^+. By (A.0.19), the ghost field c at i^+ is given by

$$c_{\mathbb{H}^3}(\rho, \Omega) \equiv \lim_{\tau\to\infty} c(\tau, \rho, \Omega) = c^{(0)} \, . \tag{2.9.65}$$

The fall-off behavior of j^τ_{mat} is given by

$$j^\tau_{mat}(\tau, \rho, \Omega) = \frac{j^{\tau(3)}(\rho, \Sigma)}{\tau^3} + \mathcal{O}(\tau^{-3-\delta}) \, , \quad (\delta > 0) \, . \tag{2.9.66}$$

This fall-off behavior is required to have a finite amount of electric charge because the global $U(1)$ charge can be written as

$$Q = \int_{i^+} d\Sigma_{i^+} j^\tau_{mat} = \lim_{\tau\to\infty} \int_{\mathbb{H}_3} d^3\sigma \tau^3 \sqrt{h}\, j^\tau_{mat} \, . \tag{2.9.67}$$

The fall-off behavior (2.9.66) can be derived by the saddle point approximation and the $j^{\tau(3)}$ can be determined (see Appendix E). Then plugging (2.9.65) and (2.9.66) into (2.9.64), we obtain the BRST condition for the matter sector as

$$0 = Q^{(+)}_{BRST,H}\,|\psi\rangle = \int_{\mathbb{H}_3} d^3\sigma \sqrt{h}\, c^{(0)} j^{\tau(3)}_{mat}\,|\psi\rangle \ . \tag{2.9.68}$$

To satisfy this condition, we have two choices: $c^{(0)}\,|\psi\rangle = 0$ or $j^{\tau(3)}_{mat}\,|\psi\rangle = 0$. However, the latter condition is too strong because it restricts all the physical states onto the sector with zero $U(1)$ charge. We thus impose

$$c^{(0)}\,|\psi\rangle = 0 \, , \tag{2.9.69}$$

on the physical Hilbert space and the matter sector is not restricted by any nontrivial BRST condition.

On the other hand, the charge associated with the large gauge transformation was given by (2.9.25), which can be written as

$$Q^+[\epsilon] = Q_{i^+}[\epsilon] + Q^+_{\mathcal{I}^+}[\epsilon] \tag{2.9.70}$$

with

$$Q_{i^+}[\epsilon] = \int_{i^+} d\Sigma_{i^+} \epsilon j^\tau_{mat} \ , \quad Q_{\mathcal{I}^+}[\epsilon] = \int_{\mathcal{I}^+} d\Sigma_{\mathcal{I}^+} \left[\partial^r \epsilon \nabla_\nu A^\nu + (\partial_\mu \epsilon) F^{r\mu}\right] \, . \tag{2.9.71}$$

Let us rewrite the expansion of the gauge parameter (2.4.20) as

$$\epsilon(u,r,\Omega) = \epsilon^{(0)}(\Omega) + \frac{1}{r}\epsilon^{(1)} + \mathcal{O}(r^{-2}) \ , \tag{2.9.72}$$

where $\epsilon^{(1)}$ takes the form of $\epsilon^{(1)} = \epsilon^{(1)'}(u,\Omega)\log r + \epsilon^{(1)''}(u,\Omega)$. Plugging (2.9.59) and (2.9.60) into (2.9.71), with the expansions (2.9.46)-(2.9.48), (2.9.52), and (2.9.72), we obtain

$$Q_{i^+}[\epsilon] = \int_{\mathbb{H}_3} d^3\sigma\sqrt{h}\,\epsilon_{\mathbb{H}^3} j^{\tau(3)}_{mat} \tag{2.9.73}$$

$$Q_{\mathcal{I}^+}[\epsilon] = \int_{\mathcal{I}^+} dud^2\Omega\sqrt{\gamma(\Omega)}\,\left[2\partial_u\epsilon^{(1)}\left(A^{(0)}_u - A^{(0)}_r\right) + \gamma^{AB}\partial_A\epsilon\left(F_{rB} - F_{uB}\right)\right] \tag{2.9.74}$$

where $\epsilon_{\mathbb{H}^3}(\sigma)$ is the gauge parameter on i^+ represented as (A.0.18). With the BRST condition (2.9.62), $Q^{(+)}[\epsilon]$ acts on the physical Hilbert space as

$$Q^{(+)}[\epsilon]\,|\psi\rangle = \left(Q^{\text{lead},+}_H[\epsilon^{(0)}] + Q^{\text{lead},+}_S[\epsilon^{(0)}]\right)|\psi\rangle \tag{2.9.75}$$

where

$$Q^{\text{lead},+}_H[\epsilon^{(0)}] = Q_{i^+}[\epsilon] = \int_{\mathbb{H}_3} d^3\sigma\sqrt{h}\,\epsilon_{\mathbb{H}^3} j^{\tau(3)}_{mat}\ , \tag{2.9.76}$$

$$Q^{\text{lead},+}_S[\epsilon^{(0)}] = -\int_{\mathcal{I}^+} dud^2\Omega\sqrt{\gamma(\Omega)}\,\gamma^{AB}\partial_A\epsilon^{(0)}\partial_u A^{(0)}_B\ . \tag{2.9.77}$$

We thus have obtained the leading *hard charge* $Q^{\text{lead},+}_H[\epsilon^{(0)}]$ and the leading *soft charge* $Q^{\text{lead},+}_S[\epsilon^{(0)}]$, which were first introduced in [4, 6, 5]. We also have found in the BRST formalism that the charge associated with the large gauge symmetry acts on physical states nontrivially and the charge can be represented on the physical Hilbert space in the same form as the asymptotic charge in the classical case (2.4.21). Therefore, the asymptotic symmetry is also physical symmetry in the BRST formalism and gives the quantum version of the classical asymptotic charge conservation. Performing the integration over u in (2.9.77) gives the delta function of ω, then the soft charge can be expressed as [17] (2.9.49)

$$Q^{\text{lead},+}_S[\epsilon^{(0)}] = \frac{1}{8\pi}\lim_{\omega\to 0}\int d^2\Omega\sqrt{\gamma(\Omega)}\,\gamma^{AB}\partial_A\epsilon^{(0)}\left[\omega a_B(\omega\hat{x}) + \omega a^\dagger_B(\omega\hat{x})\right]\ . \tag{2.9.78}$$

We will use this expression when we show the equivalence between the leading soft theorem and the asymptotic charge conservation in Section 4.1.

[17] We use $\int_0^\infty f(\omega)\delta(\omega) = \frac{1}{2}\lim_{\omega\to 0} f(\omega)$.

Chapter 3

Infrared divergences

In this chapter, we first review the S-matrix in quantum field theories. We then review the soft photon theorems up to the subleading order. In the last section, we review the infrared divergences of the conventional S-matrix in QED.

3.1 S-matrix

The S-matrix is the most fundamental quantity in scattering theories in quantum mechanics. In this section, we review how to construct the S-matrix in general.

First of all, in order to construct physical quantities which can be compared with experimental outputs, we should ask what we are supposed to observe in the scattering experiment of elementary particles. In typical scattering experiments, the basic situation is the following. We first prepare some spatially separated incoming particles at a initial time $t = t_i$. After that, the particles approach to each other and scatter to various directions due to the interactions between them. Finally, the particle detectors located far enough away from where the scattering occurred, measure the momentum of each outgoing particle at final time $t = t_f$. By repeating the same scattering event, we construct the probability distribution of momentum of each outgoing particle for a given initial state. In the real experiments, the momenta of particles are basically determined by observing the trajectory of each particle under an homogeneous magnetic field. The solution of the classical equation of motion for a point particle with the electric charge e gives $p = eRB$ where R is the curvature radius of the trajectory and B is the strength of the magnetic field, therefore observing the trajectories of each particle leads to determining the momentum of each particle.

In these sequential procedures, there are important assumptions about the dynamics. A crucial assumption is

Assumption I *Asymptotically well localized wave*;
The wave functions of incoming and scattered particles, enough before or after the collision, are localized enough in space and far way from each other so that the interaction between them can be neglected effectively.

When considering how to measure the momentum of particles in the real scattering experiment, we need a stronger assumption,

Assumption II *Asymptotically classical point particle*;

The wave function of incoming and scattered particles, enough before or after the collision, are well localized in both spatial and momentum space (= uncertainty between position and momentum is enough small when compared to the resolution scale of the detector used in the experiment), and its expectation values of position and momentum obey the classical equation of motions.

The assumption I is needed to observe the quantum wave effectively as a collection of individual particles in which each particle has the dispersion relation of a single particle in the form of $E_p = \sqrt{m^2 + \vec{p}^2}$. Assumption II justifies the classical mechanism to determine the momentum of the particles which was already mentioned above.

On the other hand, the dynamics of elementary particles is described by *quantum field theory* in which the fundamental dynamical degrees of freedoms are *fields*. The fields behave as waves and of course they are not identical to the *particles* that satisfy the above assumptions. Therefore, we should specify the states describing such particles. The physical system can be specified by all possible states and their dynamics, in other words, the Hilbert space and the Hamiltonian of the system, $(\mathcal{H}, H)$. We then usually define particles as the eigenstates of the Hamiltonian H and spatial momentum operators $\vec{P}$ with eigenvalues $p^\mu = (E_\alpha, \vec{p}_\alpha)$,

$$H|\alpha\rangle = E_\alpha|\alpha\rangle \ , \ \vec{P}|\alpha\rangle = \vec{p}_\alpha|\alpha\rangle \ , \tag{3.1.1}$$

where α stands for all parameters characterizing the state: total momentum, relative momentum of different d.o.f, spins, charges, etc. However, the quantum states with definite momentum are not spatially localized waves, rather spread out all over the space due to the uncertainty principle. Therefore, they are not the particles observed in the scattering experiments because they do not satisfy the assumption I. Moreover, they do not change at all during the time evolution because the time evolution is generated by $U(t) = e^{-iHt}$ and it just gives the irrelevant over all phase $e^{iE_\alpha t}$ to $|\alpha\rangle$. Therefore, they are not the states describing the scattering processes that generally involve complicated time evolutions. To satisfy the assumption, the wave must be the *wave packet* that is the superposition of many waves with different momentum(=wave length).

$$|f\rangle \equiv \int d\alpha f(\alpha)|\alpha\rangle \ . \tag{3.1.2}$$

where $f(\alpha)$ is a smooth function of α. In Schrödinger picture, the time evolution of this state is given by

$$|f, t\rangle = U(t, t_s)|f\rangle = \int d\alpha e^{-iE_\alpha(t-t_s)} f(\alpha)|\alpha\rangle \ . \tag{3.1.3}$$

where t_s is an arbitrary finite time at which the Schrödinger operators are defined. Assumption I is equivalent to assuming the dynamics such that the effect of interaction in $U(t)$ becomes weaker as the wave packets become well spatially separated from each other as $t \to \pm\infty$. Let us express $U_{as}(t)$ as a time evolution operator that generates the effective time evolution with the weaker interaction after a long time has passed since the collision and each wave packet become well separated. Let us define the *asymptotic Hamiltonian* $H_{as}(t)$ as the Hamiltonian which generates $U_{as}(t)$ by $i\partial_t U_{as}(t) = H_{as}(t)U_{as}(t)$, and the *asymptotic Hilbert space* $\mathcal{H}_{as}$ as a Hilbert space within which $U_{as}(t)$ can act.

Then Assumption I means the existence of the *in-state* $|f\rangle_{in}$ and the *out-state* $|f\rangle_{out}$ such that they approach the well-separated wave packets evolved by $U_{as}(t, t_s)$ as $t \to \pm\infty$:

$$\lim_{t\to-\infty} U(t,t_s)|f\rangle_{in} = \lim_{t\to-\infty} \int d\alpha f(\alpha) U_{as}(t,t_s)|\alpha\rangle_{as} \ , \tag{3.1.4}$$

$$\lim_{t\to+\infty} U(t,t_s)|f\rangle_{out} = \lim_{t\to+\infty} \int d\beta f(\beta) U_{as}(t,t_s)|\beta\rangle_{as} \ , \tag{3.1.5}$$

where $|\alpha\rangle_{as} \in \mathcal{H}_{as}$. By (3.1.2) and(3.1.3), the in-state and out-state can be written in terms of the asymptotic states as

$$\int d\alpha f(\alpha)|\alpha\rangle_{in} = \lim_{t\to-\infty} U^\dagger(t,t_s)U_{as}(t,t_s) \int d\alpha f(\alpha)|\alpha\rangle_{as} \, , \tag{3.1.6}$$

$$\int d\beta f(\beta)|\beta\rangle_{out} = \lim_{t\to\infty} U^\dagger(t,t_s)U_{as}(t,t_s) \int d\beta f(\beta)|\beta\rangle_{as} \tag{3.1.7}$$

For brevity, we formally represent (3.1.6) and (3.1.7) as

$$|\alpha\rangle_{in} = \lim_{t_i\to-\infty} \Omega(t_i)|\alpha\rangle_{as} \ , \ |\beta\rangle_{out} = \lim_{t_f\to\infty} \Omega(t_f)|\beta\rangle_{as} \tag{3.1.8}$$

where $\Omega(t)$ is the Møller [1]operator defined as

$$\Omega(t) \equiv U^\dagger(t,t_s)U_{as}(t,t_s) \, . \tag{3.1.9}$$

Since what we observe as incoming and outgoing particles in the scattering experiments is supposed to be the asymptotic wave packets in the form of

$$\lim_{t\to-\infty} |f,t_i\rangle_0 \equiv \lim_{t\to-\infty} \int d\alpha f(\alpha) U_{as}(t,t_s)|\alpha\rangle_0 \tag{3.1.10}$$

$$\lim_{t\to\infty} |g,t_f\rangle_0 \equiv \lim_{t\to\infty} \int d\beta g(\beta) U_{as}(t,t_s)|\beta\rangle_0 \, , \tag{3.1.11}$$

the transition amplitude that the incoming particle evolves into the outgoing particle is given by

$$_{out}\langle g|f\rangle_{in} = \int d\beta \int d\alpha g(\beta)^* f(\alpha)_{out}\langle\beta|\alpha\rangle_{in} \, . \tag{3.1.12}$$

Therefore, the *S-matrix* defined as

$$S_{\beta\alpha} \equiv \ _{out}\langle\beta|\alpha\rangle_{in} \tag{3.1.13}$$

is the fundamental constitute of the transition amplitudes of scattering particles. Defining the *S-operator* as

$$S \equiv \lim_{t_f\to\infty} \lim_{t_i\to-\infty} \Omega^\dagger(t_f)\Omega(t_i) = \lim_{t_f\to\infty} \lim_{t_i\to-\infty} U^\dagger_{as}(t_f,t_s)U(t_f,t_s)U_{as}(t_i,t_s) \, , \tag{3.1.14}$$

we can write the S-matrix in terms of the asymptotic states as

$$S_{\beta\alpha} \equiv \ _{out}\langle\beta|\alpha\rangle_{in} = \ _{as}\langle\beta|S|\alpha\rangle_{as} \, . \tag{3.1.15}$$

[1]Usually the Møller operator is defined as $\lim_{t\to\infty}\Omega(t)$. However, let us call $\Omega(t)$ Møller operator.

3.2 Conventional S-matrix

In the conventional scattering theory for the QED and the low energy gravity in four dimensional Minkowski spacetime, we make an important assumption that the interaction between charged particles can be neglected in asymptotic regions. This means that the asymptotic dynamics is generated by the free Hamiltonian H_0, namely

$$U_{as}(t,t_s) = U_0(t,t_s) = e^{-iH_0(t-t_s)}\,, \tag{3.2.1}$$

and the asymptotic Hilbert space is the Fock space $\mathcal{H}_0$. The incoming particle and the outgoing particle in scattering experiment are assumed to be the wave packet evolved by the free dynamics generated by H_0. In this case, the in-state and out-state in(3.1.8) can be expressed as

$$|\alpha\rangle_{in} = \lim_{t_i\to-\infty}\Omega(t_i)|\alpha\rangle_0\ ,\ |\beta\rangle_{out} = \lim_{t_f\to\infty}\Omega(t_f)|\beta\rangle_0\,, \tag{3.2.2}$$

where $|\alpha\rangle_0$ and $|\beta\rangle_0$ are the eigenstates of H_0 and the Møller operator is given by

$$\Omega(t) = U^\dagger(t,t_s)U_0(t,t_s) = e^{iH(t-t_s)}e^{-iH_0(t-t_s)}\,. \tag{3.2.3}$$

The S-matrix is defined as

$$S_{\beta\alpha} = {}_{out}\langle\beta|\alpha\rangle_{in} = {}_0\langle\beta|S|\alpha\rangle_0\,, \tag{3.2.4}$$

where S-operator is

$$S = \lim_{t_f\to\infty}\lim_{t_i\to-\infty}\Omega^\dagger(t_f)\Omega(t_i) = \lim_{t_f\to\infty}\lim_{t_i\to-\infty}e^{iH_0(t_f-t_s)}e^{-iH(t_f-t_i)}e^{-iH_0(t_i-t_s)}\,. \tag{3.2.5}$$

If we define

$$S(t,t') \equiv e^{iH_0(t-t')}e^{-iH(t-t')}e^{-iH_0(t-t')}\,, \tag{3.2.6}$$

and divide the Hamiltonian into two parts as $H = H_0 + V$, the time derivative of $S(t,t')$ is then given by

$$i\frac{\partial}{\partial t}S(t,t_i) = V^I(t)S(t,t_i) \tag{3.2.7}$$

where $V^I(t)$ is the interaction operator in the interaction picture,

$$V^I(t) \equiv U_0(t,t_s)^{-1}\,V\,U_0(t,t_s). \tag{3.2.8}$$

The solution of (3.2.6) can be represented as the Dyson series [68]

$$S(t,t') = \mathrm{T}\exp\left(-i\int_{t_i}^{t}dt'\,V^I(t')\right)\,. \tag{3.2.9}$$

where the symbol T represents the time-ordered product. Thus the S-operator is given by

$$S = \mathrm{T}\exp\left(-i\int_{-\infty}^{+\infty}dt'\,V^I(t')\right)\,. \tag{3.2.10}$$

However, this S-matrix is not well defined due to the infrared(IR) divergences. We will see in Section 3.5 that when computing the S-matrix elements by the standard perturbation theory, we encounter the IR divergences. This problem is sometimes referred to as *infrared problem*. There are two possible prescriptions for this problem; the *inclusive formalism* and the *dressed state formalism*. The inclusive formalism has been the conventional prescription as seen in QFT textbooks such as [69, 70]. In this formalism, we give up the well-defined S-matrix and compute IR finite inclusive cross section. On the other hand, we try to construct the well-defined S-matrix in the dressed state formalism by seeking the proper asymptotic states which are not the free wave packets. The dressed state formalism is the main subject in Chapter 5.

3.3 Leading soft theorem

We first study the amplitudes for the emission of a single soft photon. Let us define the scattering amplitude $M_{\beta\alpha}$ of the process $\alpha \to \beta$ as

$$\begin{aligned} S_{\beta\alpha} &= {}_0\langle\beta|S|\alpha\rangle_0 \\ &\equiv \delta(\alpha-\beta) - 2i\pi\delta^3\left(\vec{p}_\beta - \vec{p}_\alpha\right)\delta\left(E_\beta - E_\alpha\right)M_{\beta\alpha} \end{aligned} \tag{3.3.1}$$

We can drop the first term, since we are interested in the case of $\alpha \neq \beta$. The amplitude $M_{\beta\alpha}$ can be computed by the usual Feynman diagram in the standard perturbation theory. Let us also write $M_{\beta\alpha}(k)$ as the amplitude with an additional external soft photon with the outgoing momentum $k^\mu = (\omega, \vec{k})$ and the polarization mode λ as

$${}_0\langle\beta|a_\lambda(k)S|\alpha\rangle_0 \equiv -2i\pi\delta^3\left(\vec{p}_\beta - \vec{p}_\alpha\right)\delta\left(E_\beta - E_\alpha\right)M_{\beta\alpha}(k) \tag{3.3.2}$$

where $a_\lambda(k)$ is an annihilation operator of the soft photon. Since $a_\lambda(k)$ can be decomposed as $a_\lambda(k) = \epsilon^{\lambda *}_\mu(k)a^\mu(\vec{k})$, $M_{\beta\alpha}(k)$ can also be decomposed as

$$M_{\beta\alpha}(k) = \epsilon^{\lambda *}_\mu(k)M^\mu_{\beta\alpha}(k)\,. \tag{3.3.3}$$

where $M^\mu_{\beta\alpha}(k)$ is defined as

$${}_0\langle\beta|a^\mu(k)S|\alpha\rangle_0 \equiv -2i\pi\delta\left(\mathbf{p}_\beta - \mathbf{p}_\alpha\right)\delta\left(E_\beta - E_\alpha\right)M^\mu_{\beta\alpha}(k) \tag{3.3.4}$$

To compute $M^\mu_{\beta\alpha}(k)$, we just need to attach one external soft photon line to the Feynman diagram associated with $M_{\beta\alpha}$. Consider attaching one external soft photon line to the external line of a charged particle with charge e, momentum p and spinor mode $u(p,\sigma)$, then it gives an additional internal line of the charged particle with momentum $p+k$ together with a new vertex, as in Figure 3.1. Such a new contribution is given by

$$\bar{u}(p,\sigma)(-ie\gamma^\mu)\frac{i(\not{p}+\not{k})+m}{(p+k)^2+m^2-i\epsilon} = \bar{u}(p,\sigma)(-ie\gamma^\mu)\frac{i\not{p}+m}{2p\cdot k - i\epsilon} + \mathcal{O}(\omega^0) \tag{3.3.5}$$

$$= \frac{ep^\mu}{p\cdot k - i\epsilon}\bar{u}(p,\sigma) + \mathcal{O}(\omega^0)\ , \tag{3.3.6}$$

where we have used $E_p \gg \omega_k$ [2] and the on-shell conditions for p, k in the first equality, $i\not{p}+m = \Sigma_\sigma u(p,\sigma)\bar{u}(p,\sigma)$ and $\bar{u}(p,\sigma)\gamma^\mu u(p,\sigma') = 2\delta_{\sigma\sigma'}p^\mu$ in the second equality. On the

[2] We do not need $p^\mu \gg k^\mu$ for the following reason. When we calculate the S-matrix element, (3.3.5) is

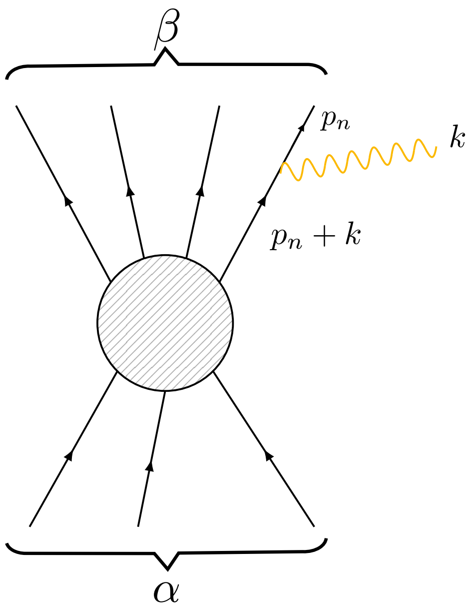

Figure 3.1: This is the Feynman diagram for the scattering process in which the one of the external charged particle emits a soft photon with momentum k.

other hand, attaching the soft photon line to the internal line of a charged particle gives an additional term,

$$(-ie\gamma^\mu)\frac{i(\not{p} \pm \not{k}) + m}{(p+k)^2 + m^2 - i\epsilon} = (-ie\gamma^\mu)\frac{i(\not{p} \pm \not{k}) + m}{p^2 + m^2 \pm 2p \cdot k - i\epsilon}. \tag{3.3.7}$$

where p is the off-shell momentum of the internal charged particle. This kind of terms does not give the additional contributions of order $\mathcal{O}(\omega^{-1})$ to the amplitude since the denominator in (3.3.7) does not have a pole at $\omega = 0$, called a *soft pole*. If we attach the soft photon line to an incoming charged particle of momentum p, the internal line of charged particle after emitting the soft photon has the momentum $p - k$, instead of $p + k$ in the previous case, the new contribution is given by

$$(-ie\gamma^\mu)\frac{i(\not{p} - \not{k}) + m}{(p-k)^2 + m^2 - i\epsilon}u(p,\sigma) = u(p,\sigma)\frac{ep^\mu}{-p \cdot k - i\epsilon} + \mathcal{O}(\omega^0)\ . \tag{3.3.8}$$

Therefore, the amplitude for emitting a single soft photon with the momentum k and the polarization $\epsilon_\mu(k)$ in the process $\alpha \to \beta$ is given by

$$M^\mu_{\beta\alpha}(k) = M_{\beta\alpha}\sum_{n\in\{\alpha,\beta\}}\frac{\eta_n e_n p_n^\mu}{p_n \cdot k - i\eta_n\epsilon} + \mathcal{O}(\omega^0), \tag{3.3.9}$$

where e_n and p_n are the charge and the four-momentum of n-th particle in the initial state α and the final state β, and η_n is a sign factor which takes $+1$ for particles in β and -1 for particles in α. The result (3.3.9) is called the *leading soft photon theorem* (for a single soft

contracted with another spinor and it gives a scalar function, like $\bar{u}(p,\sigma)(-ie\gamma^\mu)\frac{i(\not{p}+\not{k})+m}{(p+k)^2+m^2-i\epsilon}\cdots u(p',\sigma')$. In this scalar function, the contribution coming from $\not{k}$ can be neglected if $m \gg \omega_k$. (More precise condition is $E_{min} \gg \omega_k$ where $E_{min} \equiv \min\{E_{p_1}, \cdots, E_{p_N}\}$).

photon). The factor of proportionality, called (leading) soft factor, diverges as the energy of soft photon goes to zero, since it is order ω^{-1}. This fact reflects one of the important property of infrared dynamics; the slight acceleration of a charged particle results in the radiation of infinite number of low energy photons. Another important property is that the leading soft photon theorem is *universal* in the sense that the soft factor is independent of the details of scattering process and only depends on the information of the initial and final states. After some technical calculations, the leading soft theorem can be generalized to the case for emissions of multiple soft photons. The amplitudes $M_{\beta\alpha}^{\mu_1\cdots\mu_N}(k_1,\cdots,k_N)$ is for emitting N number of soft photon with momentum $k_1,\cdots,k_N$ and polarization indices $\mu_1,\cdots,\mu_N$ in the process $\alpha\to\beta$ is given by

$$M_{\beta\alpha}^{\mu_1\cdots\mu_N}(k_1,\cdots,k_N)=M_{\beta\alpha}\prod_{r=1}^{N}\left(\sum_{n\in\{\alpha,\beta\}}\frac{\eta_n e_n p_n^{\mu_r}}{p_n\cdot k_r-i\eta_n\epsilon}\right)+\mathcal{O}(\omega^0)\ . \tag{3.3.10}$$

In the next section, we will study the subleading term.

3.4 Subleading soft photon theorem

In this section, we review the derivation of the subleading soft theorem based on [71].

The scattering amplitude of a single soft photon with momentum k and N charged scalar particles with the momentum p_n and the charge e_n $(n=1,\cdots N)$ can be expanded in terms of soft photon energy as

$$M^\mu(k;p_1,\cdots,p_N)=\sum_{n=1}^{N}\frac{\eta_n e_n p_n^\mu}{p_n\cdot k-i\eta_n\epsilon}M_N(p_1,\cdots,p_n+k,\cdots,p_N)+B^\mu(k;p_1,\cdots,p_N)\ . \tag{3.4.1}$$

We have used the leading soft photon theorem (3.3.9) in the first and second terms is the subleading term which is order $\mathcal{O}(\omega_k^0)$. We now use the Ward-Takahashi identity for $U(1)$ gauge symmetry (for example see subsec. 10.5 in [70]):

$$0=k_\mu M^\mu(k;p_1,\cdots,p_N)\ . \tag{3.4.2}$$

Plugging (3.4.1) into (3.4.2), we have

$$0=\sum_{n=1}^{N}\eta_n e_n M_N(p_1,\cdots,p_n+k,\cdots,p_N)+k_\mu B^\mu(k;p_1,\cdots,p_N)\ . \tag{3.4.3}$$

If we set $\omega_k=0$, the above reduces to

$$0=\left(\sum_{n=1}^{N}\eta_n e_n\right)M_N(p_1,\cdots,p_N)\ . \tag{3.4.4}$$

This ensures the conservation of the total electric charge, $\sum_{n=1}^{N}\eta_n e_n=0$. Under the assumption that $M_N(p_1,\cdots,p_n+k,\cdots,p_N)$ and $B^\mu(k;p_1,\cdots,p_N)$ are analytic around

$k^\mu = 0$, we expand them around $k^\mu = 0$ in (3.4.3), and we obtain

$$
\begin{aligned}
0 &= \sum_{n=1}^{N} \eta_n e_n \left[M_N(p_1, \cdots, p_n, \cdots, p_N) + k_\mu \frac{\partial}{\partial p_{n\mu}} M_N(p_1, \cdots, p_n, \cdots, p_N) \right] \\
&\quad + k_\mu B^\mu(k=0; p_1, \cdots, p_N) + \mathcal{O}(\omega_k^2) \qquad (3.4.5) \\
&= \sum_{n=1}^{N} \eta_n e_n k_\mu \frac{\partial}{\partial p_{n\mu}} M_N(p_1 \cdots, p_N) + k_\mu B^\mu(k=0; p_1, \cdots, p_N) + \mathcal{O}(\omega_k^2) \qquad (3.4.6)
\end{aligned}
$$

where we have used (3.4.4) in the second equality. Thus we have

$$
k_\mu B^\mu(0; p_1, \cdots, p_N) = -\sum_{n=1}^{N} \eta_n e_n k_\mu \frac{\partial}{\partial p_{n\mu}} M_N(p_1, \cdots, p_N) \qquad (3.4.7)
$$

By solving this constraint, $B^\mu(0; p_1, \cdots, p_N)$ can be written as

$$
B^\mu(0; p_1, \cdots, p_N) = -\sum_{n=1}^{N} \eta_n e_n \frac{\partial}{\partial p_{n\mu}} M_N(p_1, \cdots, p_N) + C^\mu(p_1, \cdots, p_N) \ , \qquad (3.4.8)
$$

where $C^\mu(p_1, \cdots, p_N)$ is a function that satisfies $k_\mu C^\mu(p_1, \cdots, p_N) = 0$. However, such $C^\mu(p_1, \cdots, p_N)$ actually does not exist because it is just a local (analytic) function of $p_1, \cdots, p_N$. Plugging this into (3.4.1) gives

$$
\begin{aligned}
&M^\mu(k; p_1, \cdots, p_N) \\
&= \sum_{n=1}^{N} \frac{\eta_n e_n p_n^\mu}{p_n \cdot k - i\eta_n \epsilon} M_N(p_1, \cdots, p_n + k, \cdots, p_N) - \sum_{n=1}^{N} \eta_n e_n \frac{\partial}{\partial p_{n\mu}} M_N(p_1, \cdots, p_N) + \mathcal{O}(\omega_k) \qquad (3.4.9) \\
&= \sum_{n=1}^{N} \frac{\eta_n e_n p_n^\mu}{p_n \cdot k - i\eta_n \epsilon} M_N(p_1, \cdots, p_N) - i \sum_{n=1}^{N} \frac{\eta_n e_n k_\nu J_n^{\nu\mu}}{p_n \cdot k - i\eta_n \epsilon} M_N(p_1, \cdots, p_N) + \mathcal{O}(\omega_k) \qquad (3.4.10)
\end{aligned}
$$

where

$$
J_n^{\nu\mu} \equiv i \left(p_n^\mu \frac{\partial}{\partial p_n^\nu} - p_n^\nu \frac{\partial}{\partial p_n^\mu} \right) \qquad (3.4.11)
$$

is the orbital angular momentum operator of the n-th charged particle. Multiplying both the sides of (3.4.9) by the polarization vector $\epsilon_\mu^\lambda(k)$ of the soft photon, we thus find the *Low-Burnett-Kroll-Goldberger-Gell-Mann* (or shortly *Low's*) *soft photon theorem* [8, 9, 10, 14]:

$$
\langle \beta | a_\lambda(\vec{k}) S | \alpha \rangle = \left(J_\lambda^{(0)} + J_\lambda^{(1)} \right) \langle \beta | S | \alpha \rangle + \mathcal{O}(\frac{\omega_k}{m}) \ , \qquad (3.4.12)
$$

where

$$
J_\lambda^{(0)} = \sum_{n=1}^{N} \eta_n e_n \frac{\epsilon_\mu^{\lambda *} p_n^\mu}{p_n \cdot k - i\eta_n \epsilon} \ , \ J_\lambda^{(1)} = -i \sum_{n=1}^{N} e_n \frac{\epsilon_\mu^{\lambda *} k_\nu J_n^{\nu\mu}}{p_n \cdot k - i\eta_n \epsilon} \ . \qquad (3.4.13)
$$

It is important that the subleading soft factor $J_\lambda^{(1)}$ is also universal in the sense that it only depends on the information of the external particles and independent of the detail of the scattering processes.

If we expand (3.4.3) up to the second order, we obtain

$$\frac{1}{2}\sum_{i=1}^{N} e_i k_\mu k_\nu \frac{\partial^2}{\partial p_{i\mu}\partial p_{i\nu}} M_N\left(p_1,\ldots,p_n\right) + k_\mu k_\nu \frac{\partial B^\mu}{\partial k_\nu}\left(0;k_1,\ldots,k_n\right) = 0\,. \tag{3.4.14}$$

This can be rewritten as

$$\sum_{i=1}^{N} e_i k_\mu k_\nu \frac{\partial^2}{\partial p_{i\mu}\partial p_{i\nu}} M_N\left(p_1,\ldots,p_n\right) + k_\mu k_\nu \left[\frac{\partial B^\mu}{\partial k_\nu} + \frac{\partial B^\nu}{\partial k_\mu}\right]\left(0;k_1,\ldots,k_n\right) = 0\,. \tag{3.4.15}$$

Therefore, the only symmetric part of the $\frac{\partial B^\nu}{\partial k_\mu}\left(0;k_1,\ldots,k_n\right)$ can be fixed by the gauge invariance. It implies that the antisymmetric part depends on the detail of the scattering. In this sense, the sub-subleading soft photon theorem does not exist.

Now let us rewrite the Low's soft photon theorem (3.4.12) for later convenience. Applying $\lim_{\omega\to 0}(1+\omega\partial_\omega)$ to the both sides of (3.4.12) and using the properties below,

$$(1+\omega\partial_\omega)\omega^{-1} = 0\ ,\ \ (1+\omega\partial_\omega)1 = 1\ ,\ \ \lim_{\omega\to 0}(1+\omega\partial_\omega)\mathcal{O}(\omega) = 0\ , \tag{3.4.16}$$

we have

$$\lim_{\omega\to 0}\langle\beta|(1+\omega\partial_\omega)a_B(\omega\hat{x})S|\alpha\rangle = J_\lambda^{(1)}\langle\beta|S|\alpha\rangle\ . \tag{3.4.17}$$

This is what we call the *subleading soft photon theorem* in scaler QED.

As we have already seen in subsec 2.9.2, in the mode expansion of the gauge fields $A_\mu(x)$, the mode with $k^\mu = q^\mu(=(1,\hat{x}))$ only remains on the null infinity. In this case, the transverse modes of polarization vectors can be chosen as

$$\epsilon_B^\mu(\hat{x}) = \partial_B q^\mu\,. \tag{3.4.18}$$

With these bases, the leading soft photon theorem can be rewritten as

$$\lim_{\omega\to 0}\ {}_0\langle\mathrm{out}|\,\omega a_B(\omega\hat{x})\mathcal{S}\,|\mathrm{in}\rangle_0 = \left[\sum_{k\in out}\frac{e_k\vec{p}_k\cdot\partial_B\hat{x}}{p_k\cdot q} - \sum_{k\in in}\frac{e_k\vec{p}_k\cdot\partial_B\hat{x}}{p_k\cdot q}\right]{}_0\langle\mathrm{out}|\,\mathcal{S}\,|\mathrm{in}\rangle_0\ , \tag{3.4.19}$$

where we omit the $i\epsilon$ factors. The subleading soft photon theorem can also be rewritten as

$$\lim_{\omega\to 0}\ {}_0\langle\mathrm{out}|\,(1+\omega\partial_\omega)a_B(\omega\hat{x})\mathcal{S}\,|\mathrm{in}\rangle_0 = -i\sum_{k\in in,out}\frac{e_k q^\mu J^k_{\mu B}}{p_k\cdot q}\,{}_0\langle\mathrm{out}|\,\mathcal{S}\,|\mathrm{in}\rangle_0\ . \tag{3.4.20}$$

3.5 Infrared divergences in S-matrix

The radiative loop corrections involving soft particles give universal contributions to scattering amplitudes. Here, we see that the loop diagrams involving virtual soft photons cause infrared divergences.

We now calculate all the orders of loop-corrections involving virtual soft particles in the process $\alpha \to \beta$. Such loop diagrams can be given by attaching additional photon propagators to the Feynman diagram of $\alpha \to \beta$. If we attach at least one of the ends of a photon propagator to internal line of a charged particle, such diagrams do not have infrared divergences because such attachment does not give the new internal line with a soft pole, as already mentioned in the below of (3.3.7). (We will see it more concretely later.) Therefore, we focus on the diagrams with the loops of virtual soft photons exchanged only among the external charged particles. We evaluate the leading infrared contributions of such loops by using the soft photon theorem in (3.3.10). Since the theorem is valid only when every soft photon momentum $k_1, \cdots, k_N$ is smaller than the minimum of the energy of the charged particles in the initial and final states, $p_1, \cdots, p_N$, we introduce an energy scale $\Lambda \lesssim \min\{E_{p_1}, \cdots, E_{p_N}\}$. The upper bound cut-off Λ is just a convenient dividing scale chosen low enough to justify the soft photon theorem and then a soft photon is defined as the one with energy smaller than Λ. We also introduce the low bound energy scale λ of photon momentum to regulate the infrared divergent integrals. The dependence of λ must be removed at the end of calculations by taking $\lambda \to 0$.

We can evaluate the diagram with additional M photon loops connecting the external charged particles from a given amplitude $M_{\beta\alpha}$ as follows. In (3.3.10) with $N = 2M$, we first make M pairs among $2M$ soft factors and set the soft momenta in s-th pair to be the same momentum, say k_s. Secondly, we multiply it with N photon propagators,

$$\prod_{s=1}^{N} \frac{-i\eta_{\mu_s\mu_s'}}{k_s^2 - i\epsilon}, \tag{3.5.1}$$

and contract photon polarizations indices and integrate over the soft photon four-momenta. Then each pair gives the contribution,

$$e_n e_m \eta_n \eta_m J_{nm} \equiv \int_{\lambda \le |\vec{k}| \le \Lambda} \frac{d^4 k}{(2\pi)^4} \frac{-ie_n e_m \eta_n \eta_m p_n \cdot p_m}{[k^2 - i\epsilon]\,[p_n \cdot k - i\eta_n \epsilon]\,[-p_m \cdot k - i\eta_m \epsilon]}, \tag{3.5.2}$$

where p_n and p_m are the momenta of the two charged particles exchanging the virtual soft photons in each pair. Summing over all the contributions from all the possible combinations of the pairs, the effect of adding N virtual soft photon loops to the diagram of $M_{\beta\alpha}$ ends up with multiplying $M_{\beta\alpha}$ by a factor

$$\frac{1}{N!2^N} \left[\sum_{n,m \in \{\alpha,\beta\}} e_n e_m \eta_n \eta_m J_{nm} \right]^N \tag{3.5.3}$$

where $N!2^N$ is the symmetric factor of the diagram. This symmetric factor comes from the fact that rearranging N virtual photon lines ($N!$ combinations) and exchanging the two ends of the each line (2^N combinations) are duplicated by a part of rearranging the vertices. Summing over N, we find that the amplitude $M^{\lambda}_{\beta\alpha}$ for a process including any number of photon loops with momenta $|\vec{k}| \ge \lambda$ is given by

$$M^{\lambda}_{\beta\alpha} = M^{\Lambda}_{\beta\alpha} \exp\left[\frac{1}{2} \sum_{n,m \in \{\alpha,\beta\}} e_n e_m \eta_n \eta_m J_{nm} \right] \tag{3.5.4}$$

where $M^{\Lambda}_{\beta\alpha}$ is the amplitude including any number of virtual photons only with the momenta greater than Λ. Note that the contributions of virtual soft photons do not depend on the details of the scattering process because of the universality of the soft theorem. The dimension analysis tells us that J_{nm} in (3.5.2) has a logarithmic divergence as $\lambda \to 0$ [3]. In fact, performing the integration (see Appendix F for the details) gives

$$J_{nm} = -\frac{1}{8\pi^2 \beta_{nm}} \ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right) \ln\left(\frac{\lambda}{\Lambda}\right) + \frac{i\delta_{\eta_n \eta_m}}{4\pi\beta_{nm}} \ln\left(\frac{\lambda}{\Lambda}\right) \tag{3.5.5}$$

where β_{nm} is the relativistic relative velocity of particles n and m:

$$\beta_{nm} \equiv \sqrt{1 - \frac{m^4}{(p_n \cdot p_m)^2}} \ . \tag{3.5.6}$$

Then the matrix element can be written as

$$M^{\lambda}_{\beta\alpha} = \left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2} e^{iP_{\beta\alpha}} M^{\Lambda}_{\beta\alpha} \tag{3.5.7}$$

where

$$A_{\beta,\alpha} = -\sum_{n,m\in\{\alpha,\beta\}} \frac{e_n e_m \eta_n \eta_m}{8\pi^2 \beta_{nm}} \ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right) \ , \tag{3.5.8}$$

$$P_{\beta\alpha} \equiv \sum_{n,m\in\{\alpha,\beta\}} \delta_{\eta_n \eta_m} \frac{e_n e_m \eta_n \eta_m}{8\pi\beta_{nm}} \ln\left(\frac{\lambda}{\Lambda}\right) \ . \tag{3.5.9}$$

Because of the non-negativity of $A_{\beta,\alpha}$ (proved in appendix F), most matrix elements go to zero as $\lambda \to 0$ [4] except in the case where the initial state and the final state are identical [72]. This is what we call *infrared suppression*. The $P_{\beta\alpha}$ gives an infrared-divergent phase factor for the matrix elements. We do not usually pay attention to the phase factor because it drops out when we take the absolute value of the matrix element with definite momenta so as to compute the transition rate.

[3] Then attaching an end point of a virtual soft photon line to an internal line of a charged particle gives no infrared divergent contributions.

[4] Whether the matrix elements really go to zero as $\lambda \to 0$ is not perfectly clear because the perturbation series are not a convergent series but a asymptotic series.

Chapter 4

Equivalence between soft theorem and asymptotic symmetry

As we have already explained in Chapter 1, the infrared triangle equivalence refers to the mathematical equivalence among the asymptotic symmetry, the soft theorem, and the memory effect. The equivalence of those was first discussed in the Yang-Mills theory [1], and extended to gravity [31, 32, 33] and also to QED [4, 6, 5]. The equivalence has been also extended to higher orders of the soft expansion, to higher spacetime dimensions, and also to other theories by many others. In particular for QED, it is known in [4, 6] that the Ward-Takahashi identities for the large transformations result in the leading soft photon theorem. The similar analysis for the subleading soft theorem was done in [35, 36, 60] for massless scalar QED. In [36, 60] it was found that the symmetries are nothing but the large gauge transformations [1]. In [53], we extended the discussions to massive scalar QED, and obtained the expression of the charges associated with the subleading soft theorem. Since we have already seen the equivalence between the charge conservation associated with the asymptotic symmetry and the electromagnetic effect, we focus on the equivalence between the asymptotic symmetry and the soft theorem.

First, we review the equivalence between the leading soft photon theorem and the asymptotic symmetry in Section 4.1 based on [53] [2]. We then proceed to the subleading order, and review the soft part of the subleading charges along the work [35] in subsection 4.2. In subsection 4.2.1, we derive the expression of the hard part of the subleading charges defined on the future (or past) timelike infinity.

[1]The large gauge transformations are slightly different in two papers [36, 60]. In [36], the gauge parameter is $\mathcal{O}(r)$ at $\mathcal{I}^+$, and thus the generator is divergent but includes the subleading finite part, which is relevant to the subleading soft theorem. On the other hand, in [60], it is shown that the subleading part of $\mathcal{O}(1)$ gauge parameter is related to the subleading soft theorem. Our argument is similar to the latter, although the gauge fixing condition is different.

[2]The equivalence between the asymptotic symmetry and the soft theorem was derived by using the specific angular coordinates in [4, 6], and we generalized it more covariantly in [53].

4.1 Soft photon theorem ⇔ Asymptotic charge conservation, at leading order

In this section, we review the equivalence between the leading soft photon theorem and the asymptotic symmetry based on [53]. For simplicity, we concentrate on future infinities and omit the analysis for past infinities because it is just a repeat of the similar argument.

The charge associated with the asymptotic symmetry defined on $\mathcal{I}^+$ is given by (2.9.75);

$$Q^{(\text{lead},+)}[\epsilon]\,|\psi\rangle = Q_H^+[\epsilon^{(0)}] + Q_S^{\text{lead},+}[\epsilon^{(0)}] \tag{4.1.1}$$

where

$$Q_H^{\text{lead},+}[\epsilon^{(0)}] = Q_{i^+}[\epsilon] = \int_{\mathbb{H}_3} d^3\sigma\sqrt{h}\,\epsilon_{\mathbb{H}^3} j^{\tau(3)}_{mat}\,, \tag{4.1.2}$$

$$Q_S^{\text{lead},+}[\epsilon^{(0)}] = -\int_{\mathcal{I}^+} dud^2\Omega\sqrt{\gamma(\Omega)}\,\partial_A\epsilon^{(0)}\partial_u A_B^{(0)}\,. \tag{4.1.3}$$

Here, $\epsilon_{\mathbb{H}^3}(\sigma)$ is a limit of large gauge parameter $\epsilon(x)$ on i^+ given by (A.0.18) , which is defined as[3]

$$\epsilon_{\mathbb{H}^3}(\sigma) \equiv \lim_{\tau\to\infty}\epsilon(\tau,\rho,\Omega) = \int d^2\Omega'\sqrt{\gamma(\Omega')}\,G_{\mathbb{H}^3}(\sigma;\Omega')\epsilon^{(0)}(\Omega') \tag{4.1.4}$$

with

$$G_{\mathbb{H}^3}(\sigma;\Omega') = \frac{1}{4\pi\left[-\sqrt{1+\rho^2}+\rho\,\hat{q}(\Omega')\cdot\hat{x}(\Omega)\right]^2}\,. \tag{4.1.5}$$

Besides, $j^{\tau(3)}_{mat}(\sigma)$ in (4.1.2) is defined as

$$j^{\tau(3)}_{mat}(\sigma) \;\equiv \lim_{\tau\to\infty}\tau^3 : j^{\tau}_{mat}(\tau,\sigma) :\,, \tag{4.1.6}$$

which is the leading coefficient at large τ of the matter current with the normal ordering. It is given by

$$j^{\tau(3)}_{mat}(\sigma) = \frac{em^2}{2(2\pi)^3}\left[b^\dagger(\vec{p})b(\vec{p}) - d^\dagger(\vec{p})d(\vec{p})\right]|_{\vec{p}=m\rho\hat{x}(\Omega)}\,. \tag{4.1.7}$$

See Appendix E for our convention of the quantization of the scalar field. From this expression, one can easily find that the hard charge operator $Q_H^{\text{lead},+}[\epsilon^{(0)}]$ acts on an asymptotic future state ${}_0\langle\text{out}|$ containing charged particles with momenta $\vec{p}_k = m\rho_k\hat{y}(\tilde{\Omega}_k)$ and charge e_k as[4]

$${}_0\langle\text{out}|\,Q_H^{\text{lead},+}[\epsilon^{(0)}] = \sum_{k\in out} e_k\epsilon_{\mathbb{H}^3}(\rho_k,\tilde{\Omega}_k)\,{}_0\langle\text{out}|\;. \tag{4.1.8}$$

[3] In fact, in the coordinates (τ,ρ,σ), Green's function (A.0.6) does not depend on τ. Hence, $\epsilon_{\mathbb{H}^3}(\sigma) = \epsilon(\tau,\sigma)$.

[4] $\hat{y}(\tilde{\Omega})$ is a unit three-dimensional vector parametrized by spherical coordinates $\tilde{\Omega}^A$, and e_k is $+e$ for particles and $-e$ for antiparticles.

Similarly, the past hard charge $Q_H^{\text{lead},-}[\epsilon^{(0)}]$ acts on asymptotic past states as

$$Q_H^{\text{lead},-}[\epsilon^{(0)}]\,|\text{in}\rangle_0 = \sum_{k\in in} e_k \epsilon_{\mathbb{H}^3}(\rho_k, \tilde{\Omega}_k)\,|\text{in}\rangle_0 \ . \tag{4.1.9}$$

As explained in section 2.5, the leading charges associated with the large gauge transformations are "asymptotically conserved". Therefore, we should have the following Ward-Takahashi identity for the physical S-matrix:

$${}_0\langle\text{out}|\left[\left(Q_S^{\text{lead},+} + Q_H^{\text{lead},+}\right)\mathcal{S} - \mathcal{S}\left(Q_S^{\text{lead},-} + Q_H^{\text{lead},-}\right)\right]|\text{in}\rangle_0 = 0. \tag{4.1.10}$$

We now show that the conservation law (4.1.10) is equivalent to the leading soft photon theorem (3.4.19);

$$\lim_{\omega\to 0}\, {}_0\langle\text{out}|\,\omega a_B(\omega\hat{x})\mathcal{S}\,|\text{in}\rangle_0 = \left[\sum_{k\in out}\frac{e_k\vec{p}_k\cdot\partial_B\hat{x}}{p_k\cdot q} - \sum_{k\in in}\frac{e_k\vec{p}_k\cdot\partial_B\hat{x}}{p_k\cdot q}\right]{}_0\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle_0 \ , \tag{4.1.11}$$

where $q^\mu = (1, \hat{x})$.

Integrating the l.h.s. of the soft theorem (4.1.11) w.r.t. the direction $\hat{x}(\Omega)$ of the momentum of the soft photon, we obtain a S-matrix element with the insertion of soft charge (2.9.78) as follows:

$$\lim_{\omega\to 0}\frac{1}{4\pi}\int d^2\Omega\sqrt{\gamma}\gamma^{AB}\partial_A\epsilon^{(0)}\,{}_0\langle\text{out}|\,\omega a_B(\omega\hat{x})\mathcal{S}\,|\text{in}\rangle_0 = {}_0\langle\text{out}|\left(Q_S^{lead,+}\mathcal{S} - \mathcal{S}Q_S^{lead,-}\right)|\text{in}\rangle_0 \ , \tag{4.1.12}$$

where we have used the fact

$$\lim_{\omega\to 0}\,{}_0\langle\text{out}|\,\omega a_B(\omega\hat{x})\mathcal{S}\,|\text{in}\rangle_0 = -\lim_{\omega\to 0}\,{}_0\langle\text{out}|\,\mathcal{S}\omega a_B^\dagger(\omega\hat{x})\,|\text{in}\rangle_0 \ . \tag{4.1.13}$$

The soft theorem (4.1.11) equates (4.1.12) with

$$\frac{1}{4\pi}\int d^2\Omega\sqrt{\gamma}\gamma^{AB}\partial_A\epsilon^{(0)}\sum_k\frac{\eta_k e_k\vec{p}_k\cdot\partial_B\hat{x}}{p_k\cdot q}\,{}_0\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle_0 \ , \tag{4.1.14}$$

where we have introduced the symbol η_k which is $+1$ (-1) for $k\in$ out ($k\in$ in). Performing a partial integration and using the formula

$$\nabla_A\left[\gamma^{AB}\frac{\vec{p}_k\cdot\partial_B\hat{x}(\Omega)}{p_k\cdot q(\Omega)}\right] = 4\pi G_{\mathbb{H}^3}(\rho_k,\tilde{\Omega}_k;\Omega) - 1 \quad \text{with} \quad \vec{p}_k \equiv m\rho_k\hat{y}(\tilde{\Omega}_k)\,, \tag{4.1.15}$$

we then have

$$\begin{aligned}\frac{1}{4\pi}\int d^2\Omega\sqrt{\gamma}\gamma^{AB}\partial_A\epsilon^{(0)}\sum_k\frac{\eta_k e_k\vec{p}_k\cdot\partial_B\hat{x}}{p_k\cdot q} &= -\sum_k\eta_k e_k\int d^2\Omega\sqrt{\gamma}\epsilon^{(0)}(\Omega)\left[G_{\mathbb{H}^3}(\rho_k,\tilde{\Omega}_k;\Omega) - \frac{1}{4\pi}\right]\\ &= -\sum_k\eta_k e_k\epsilon_{\mathbb{H}^3}(\rho_k,\tilde{\Omega}_k) + \frac{1}{4\pi}\left(\sum_k\eta_k e_k\right)\int d^2\Omega\sqrt{\gamma}\epsilon^{(0)}\,. \end{aligned}\tag{4.1.16}$$

Since $\sum_k \eta_k e_k = 0$ due to the total electric charge conservation, we finally obtain

$$(4.1.12) = -\sum_k \eta_k e_k \epsilon_{\mathbb{H}^3}(\rho_k, \tilde{\Omega}_k)\, {}_0\langle \text{out}| \mathcal{S} |\text{in}\rangle_0 = -\, {}_0\langle \text{out}| \left(Q_H^{\text{lead},+} \mathcal{S} - \mathcal{S} Q_H^{\text{lead},-} \right) |\text{in}\rangle_0 \tag{4.1.17}$$

where we have used (4.1.8) and (4.1.9). Therefore, we have confirmed that we can obtain the Ward-Takahashi identity (4.1.10) from the soft theorem (4.1.11), and vice versa because (4.1.10) holds for any $\epsilon^{(0)}$.

4.2 Soft photon theorem ⇔ Asymptotic charge conservation, at subleading order

Like the leading soft theorem (4.1.11), the subleading soft photon theorem gives the following relation between an amplitude containing a soft photon and an amplitude without that:

$$\lim_{\omega \to 0} {}_0\langle \text{out}| (1 + \omega \partial_\omega) a_B(\omega \hat{x}) \mathcal{S} |\text{in}\rangle_0 = S_B^{(\text{sub})}\, {}_0\langle \text{out}| \mathcal{S} |\text{in}\rangle_0 \quad \text{with} \quad S_B^{(\text{sub})} \equiv -i \sum_k \frac{e_k q^\mu J^k_{\mu B}}{p_k \cdot q}\,, \tag{4.2.1}$$

where the sum in $S_B^{(\text{sub})}$ is taken for all of the incoming and outgoing charged particles which are labeled by k, and $J^k_{\mu\nu}$ is the total angular momentum operator of k-th particle (with momentum $\vec{p}_k$ and charge e_k) defined as

$$J^k_{\mu\nu} = -i \left(p_{k\mu} \frac{\partial}{\partial p_k^\nu} - p_{k\nu} \frac{\partial}{\partial p_k^\mu} \right)\,, \tag{4.2.2}$$

and $q^\mu = (1, \hat{x})$ represents the direction of the soft photon.

On the other hand, we have already seen that the asymptotic symmetry leads to the subleading asymptotic charge conservation (2.5.33):

$${}_0\langle \text{out}| \left[\left(Q_S^{\text{sub},+} + Q_H^{\text{sub},+} \right) \mathcal{S} - \mathcal{S} \left(Q_S^{\text{sub},-} + Q_H^{\text{sub},-} \right) \right] |\text{in}\rangle_0 = 0. \tag{4.2.3}$$

As noted below eq. (2.5.31), the soft part is given by

$$Q_S^{\text{sub},+} = -\frac{1}{2} \int_{\mathcal{I}^+} du d^2\Omega \sqrt{\gamma} \epsilon^{(0)} u \partial_u \Delta_{\mathrm{S}^2} \nabla^B A_B^{(0)}, \tag{4.2.4}$$

and by using eq. (2.9.49), we can write as

$$Q_S^{\text{sub},+} = -\frac{i}{16\pi} \lim_{\omega \to 0} \int d^2\Omega \sqrt{\gamma}\, \Delta_{\mathrm{S}^2} \epsilon^{(0)} \nabla^B \left[(1 + \omega \partial_\omega)(a_B(\omega \hat{x}) - a_B^\dagger(\omega \hat{x})) \right]\,. \tag{4.2.5}$$

The soft part of the past charge, $Q_S^{\text{sub},-}$, also takes the same expression. Hence, $Q_S^{\text{sub},\pm}$ contains $(1 + \omega \partial_\omega)[a_B(\omega \hat{x}) - a_B^\dagger(\omega \hat{x})]$ which corresponds to the subleading soft photons.

The subleading soft theorem (4.2.1) thus states that

$$ {}_0\langle\text{out}|\left(Q_S^{\text{sub},+}\mathcal{S}-\mathcal{S}Q_S^{\text{sub},-}\right)|\text{in}\rangle_0 = -\frac{1}{8\pi}\int d^2\Omega\sqrt{\gamma}\,\Delta_{\mathrm{S}^2}\epsilon^{(0)}\sum_k \nabla^B\left[\frac{e_k q^\mu\partial_B\hat{x}^i}{p_k\cdot q}J_{\mu i}^k\right]{}_0\langle\text{out}|\mathcal{S}|\text{in}\rangle_0 \,. \tag{4.2.6}$$

If operators $Q_H^{\text{sub},\pm}$ exist such that the r.h.s. of (4.2.6) is equal to

$$ -{}_0\langle\text{out}|\left(Q_H^{\text{sub},+}\mathcal{S}-\mathcal{S}Q_H^{\text{sub},-}\right)|\text{in}\rangle_0\,, \tag{4.2.7}$$

then we can establish the equivalence between the subleading soft theorem and the subleading charge conservation (4.2.3).

For massless QED, such hard operators $Q_H^{\text{sub},\pm}$ were obtained [35, 36, 60], where the operators are defined on the future and past null infinities $\mathcal{I}^\pm$. What we want to do is to obtain the expression of $Q_H^{\text{sub},\pm}$ for massive charged particles. This is the subject of the next subsection.

4.2.1 Hard part of the subleading charges

Unlike massless QED, $Q_H^{\text{sub},\pm}$ is an operator on timelike infinities $i^\pm$ acting on the asymptotic states of massive particles. Thus like the leading case (4.1.2), it should be expressed as an integral over three-dimensional hyperbolic space $\mathbb{H}^3$ with gauge parameter $\epsilon_{\mathbb{H}^3}(\sigma)$ on the space. We now obtain such an expression for the future part $Q_H^{\text{sub},+}$.

First, let us parametrize an on-shell momentum by $(p,\tilde{\Omega}^A)$ as $p^\mu=(E_p,p\hat{y}(\tilde{\Omega}))$ where $E_p=\sqrt{p^2+m^2}$ and $\hat{y}\cdot\hat{y}=1$. With this parametrization, the angular momentum operators are expressed as

$$ J_{0i} = i\left[\hat{y}^i E_p\partial_p + \frac{E_p}{p}\tilde{\gamma}^{AB}(\tilde{\partial}_A\hat{y}^i)\partial'_B\right] \tag{4.2.8}$$

$$ J_{ij} = -i\left[\hat{y}^i(\tilde{\partial}_A\hat{y}^j)-\hat{y}^j(\tilde{\partial}_A\hat{y}^i)\right]\tilde{\gamma}^{AB}\partial'_B, \tag{4.2.9}$$

where $\tilde{\partial}_A\equiv\frac{\partial}{\partial\tilde{\Omega}^A}$ is the derivative with respect to the direction of on-shell momentum of massive particle. Here, $\tilde{\gamma}^{AB}$ is the inverse of the induced metric $\tilde{\gamma}_{AB}\equiv(\tilde{\partial}_A\hat{y})\cdot(\tilde{\partial}_B\hat{y})$. Note that if we parametrize the on-shell momentum as $\vec{p}=m\rho\vec{y}(\tilde{\Omega})$, $E_p=m\sqrt{1+\rho^2}$, the angular momentum operators can also be represented as

$$ J_{0i} = i\sqrt{1+\rho^2}\left[\hat{y}^i\partial_\rho+\frac{1}{\rho}\tilde{\gamma}^{AB}(\tilde{\partial}_A\hat{y}^i)\,\tilde{\partial}_B\right], \tag{4.2.10}$$

$$ J_{ij} = -i\left[\hat{y}^i(\tilde{\partial}_A\hat{y}^j)-\hat{y}^j(\tilde{\partial}_A\hat{y}^i)\right]\tilde{\gamma}^{AB}\tilde{\partial}_B. \tag{4.2.11}$$

We then define the following operator $Q_B^{\text{sub}}(\Omega)$ with angular index B as

$$ Q_B^{\text{sub}}(\Omega) = \frac{e}{2}\int\frac{d^3p}{(2\pi)^3 2E_p}\frac{q^\mu(\Omega)\partial_B\hat{x}^i(\Omega)}{p\cdot q(\Omega)} \times\left[(J_{\mu i}b^\dagger(\vec{p}))b(\vec{p})-b^\dagger(\vec{p})(J_{\mu i}b(\vec{p}))-(J_{\mu i}d^\dagger(\vec{p}))d(\vec{p})+d^\dagger(\vec{p})(J_{\mu i}d(\vec{p}))\right]. \tag{4.2.12}$$

One can confirm, by performing some partial integrations[5], that the first term and the second term in (4.2.12) are the same when they act on the physical states. The third term and the forth term are also the same. Accordingly, one can find that $Q_B^{\rm sub}$ acts on the 1-particle state as

$$Q_B^{\rm sub}(\Omega)\,|p_k\rangle = e_k \frac{q^\mu(\Omega)\partial_B \hat{x}^i(\Omega)}{p_k\cdot q(\Omega)} J^k_{\mu i}\,|p_k\rangle\,, \tag{4.2.13}$$

$$\langle p_k|\,Q_B^{\rm sub}(\Omega) = -e_k \frac{q^\mu(\Omega)\partial_B \hat{x}^i(\Omega)}{p_k\cdot q(\Omega)} J^k_{\mu i}\,\langle p_k|\,. \tag{4.2.14}$$

Therefore, if one defines the hard charge operator as

$$Q_H^{\rm sub,\pm} = -\frac{1}{8\pi}\int d^2\Omega\sqrt{\gamma}\,\Delta_{\rm S^2}\epsilon^{(0)}\,\nabla^B Q_B^{sub}(\Omega)\,, \tag{4.2.15}$$

it satisfies the desired property

$${}_0\langle{\rm out}|\,(Q_H^{\rm sub,+}\mathcal{S} - \mathcal{S}Q_H^{\rm sub,-})\,|{\rm in}\rangle_0 = \frac{1}{8\pi}\int d^2\Omega\sqrt{\gamma}\,\Delta_{\rm S^2}\epsilon^{(0)}\,\sum_k \nabla^B\left[\frac{e_k q^\mu\partial_B\hat{x}^i}{p_k\cdot q} J^k_{\mu i}\right]\,{}_0\langle{\rm out}|\,\mathcal{S}\,|{\rm in}\rangle_0\,. \tag{4.2.16}$$

Next, we now express $Q_B^{\rm sub}$ in terms of the local matter current of charged particles in the asymptotic region i^+. The matter current j_μ^{mat} asymptotically decays as $\mathcal{O}(\tau^{-3})$ with τ-dependent oscillations. Assuming that the charged scalar is free in the asymptotic region, one can extract τ-independent finite parts of j_μ^{mat} (see appendix E) as

$$I_\alpha^{mat}(\tilde\sigma) \equiv \lim_{\tau\to\infty}\left(\frac{1}{4m^2}\partial_\tau^2+1\right)\tau^3 : j_\alpha^{mat}(\tau,\tilde\sigma): \tag{4.2.17}$$

$$= \frac{iem}{4(2\pi)^3}\left[\partial_\alpha b^\dagger(\vec{p})b(\vec{p}) - b^\dagger(\vec{p})\partial_\alpha b(\vec{p}) - \partial_\alpha d^\dagger(\vec{p})d(\vec{p}) + d^\dagger(\vec{p})\partial_\alpha d(\vec{p})\right]|_{\vec{p}=m\rho\hat{y}(\tilde\Omega)}\,, \tag{4.2.18}$$

where $\tilde\sigma^\alpha = (\rho,\tilde\Omega^A)$ are the coordinates on $\mathbb{H}^3$. In addition, using this and also eqs. (4.2.10), (4.2.11), one can obtain the following equations:

$$[(J_{0i}b^\dagger(\vec{p}))b(\vec{p}) - b^\dagger(\vec{p})(J_{0i}b(\vec{p})) - (J_{0i}d^\dagger(\vec{p}))d(\vec{p}) + d^\dagger(\vec{p})(J_{0i}d(\vec{p}))]|_{\vec{p}=m\rho\hat{y}(\tilde\Omega)}$$
$$= \frac{4(2\pi)^3}{em}\sqrt{1+\rho^2}[\hat{y}^i I_\rho^{mat}(\rho,\tilde\Omega) + \frac{1}{\rho}\tilde\gamma^{AB}\tilde\partial_A\hat{y}^i I_B^{mat}(\rho,\tilde\Omega)]\,, \tag{4.2.19}$$

$$[(J_{ij}b^\dagger(\vec{p}))b(\vec{p}) - b^\dagger(\vec{p})(J_{ij}b(\vec{p})) - (J_{ij}d^\dagger(\vec{p}))d(\vec{p}) + d^\dagger(\vec{p})(J_{ij}d(\vec{p}))]|_{\vec{p}=m\rho\hat{y}(\tilde\Omega)}$$
$$= -\frac{4(2\pi)^3}{em}(\hat{y}^i\tilde\partial_A\hat{y}^j - \hat{y}^j\tilde\partial_A\hat{y}^i)\tilde\gamma^{AB} I_B^{mat}(\rho,\tilde\Omega). \tag{4.2.20}$$

[5] These partial integrations involve not only the creation and annihilation operators but also soft factors and the integration measure.

From these equations, (4.2.12) can be rewritten as

$$Q_B^{\rm sub}(\Omega) = \int_{\mathbb{H}^3} d^3\tilde{\sigma}\sqrt{\tilde{h}}\left[\frac{\sqrt{1+\rho^2}\partial_B\hat{x}(\Omega)\cdot\hat{y}(\tilde{\Omega})}{q\cdot Y} I_\rho^{mat}(\rho,\tilde{\Omega})\right.$$
$$\left.+\frac{1}{q\cdot Y}\left\{\frac{\sqrt{1+\rho^2}}{\rho}\partial_B\hat{x}\cdot\tilde{\partial}_A\hat{y} - (\hat{x}\cdot\hat{y})(\partial_B\hat{x}\cdot\tilde{\partial}_A\hat{y}) + (\hat{x}\cdot\tilde{\partial}_A\hat{y})(\partial_B\hat{x}\cdot\hat{y})\right\}\tilde{\gamma}^{AC}I_C^{mat}(\rho,\tilde{\Omega})\right], \tag{4.2.21}$$

where $d^3\tilde{\sigma}\sqrt{\tilde{h}} = d\rho d^2\tilde{\Omega}\frac{\rho^2}{\sqrt{1+\rho^2}}\sqrt{\tilde{\gamma}}$ and $Y^\mu = (\sqrt{1+\rho^2}, \rho\hat{y}(\tilde{\Omega}))$.

Therefore, the hard charge $Q_H^{\rm sub,+}$ can be expressed in terms of the asymptotic matter current I_α^{mat} by inserting (4.2.21) into (4.2.15). However, the expression seems to be unnatural because $Q_H^{\rm sub,+}$ is given by an integral over S^2 with parameter function $\epsilon^{(0)}$, not $\epsilon_{\mathbb{H}^3}$. Since $Q_H^{\rm sub,+}$ is associated with the large gauge transformation acting on massive particles, $Q_B^{\rm sub}(\Omega)$ should be written as an integral over the surface at timelike infinity $\mathbb{H}^3$ with parameter function $\epsilon_{\mathbb{H}^3}$ on that surface, like the leading case (4.1.2). In fact, after some computations (see Appendix G), one can express $Q_H^{\rm sub,+}$ in such an integral as follows:

$$Q_H^{\rm sub,+} = \frac{1}{2}\int_{\mathbb{H}^3} d^3\sigma\sqrt{h}\frac{\sqrt{1+\rho^2}}{\rho}\left[\rho^2 h^{\alpha\beta}(\nabla_\alpha^{(h)}\nabla_\rho^{(h)}\epsilon_{\mathbb{H}^3})I_\beta^{mat} + 2\rho h^{\alpha\beta}(\nabla_\alpha^{(h)}\epsilon_{\mathbb{H}^3})I_\beta^{mat}\right], \tag{4.2.22}$$

where $\nabla_\alpha^{(h)}$ denotes the covariant derivative compatible with the metric $h_{\alpha\beta}$ on $\mathbb{H}^3$. The obtained charge is written as an integral over the three-dimensional hyperbolic space at timelike infinity $\mathbb{H}^3$, and the integrand takes a local form and contains the components of matter current I_α^{mat} which is defined in (4.2.17). This is a similar form to the massless case [35, 36, 60].

Chapter 5

Towards the IR finite S-matrix

5.1 Motivations for seeking IR finite S-matrix

As we have already seen in Chapter 3, the conventional S-matrix is not well defined due to the infrared divergences, and this infrared problem can be avoided by computing only the inclusive transition probability. However, the fact that the inclusive amplitudes are IR finite does not mean that the infrared problem is perfectly resolved because the S-matrix is still not well-defined in the inclusive formalism. Moreover, there are several points which seem unnatural or even problematic in the inclusive formalism. The first point is about the assumption that the effect of the interaction becomes zero at the large time limit and the asymptotic states are just the Fock states with the free dynamics. This assumption may be too strong because the interaction mediated by massless particles create infinitely long-ranged force between charged particles. In fact, it is known in quantum mechanics that the asymptotic state with the Coulomb potential is not a free state. Based on the principle of quantum theories that all physically possible history are superposed at the level of wave function, it would be natural to expect that the asymptotic state is not a free state but a superposition of charged states surrounded by emitted soft particles in all possible ways.

The *dressed state formalism* is the approach to the infrared problem based on the expectation mentioned above. This formalism tries to construct the IR finite S-matrix by using the asymptotic states surrounded by soft particles. The history of the dressed state formalism goes back to the 60's. In 1965, Chung first proposed a asymptotic state of electrons surrounded by a cloud of coherent soft photon and showed that the infrared divergences are canceled out in all order perturbation theory at the level of S-matrix if we employ the state as the asymptotic state of the standard S-matrix [51]. We call the dressed state the *Chung's dressed state*. The scattering theory with the asymptotic state state dressed by the coherent soft cloud was developed by others in [73, 74, 75, 76, 52]. In particular, Faddeev and Kulish [52] in 1970 proposed another version of dressed states, which is the same as Chung's dressed states except for oscillating phase factors, by solving the infrared QED dynamics. The dressed states are nowadays called the *Faddeev-Kulish (FK) dressed states*.

Although the dressed state formalism was proposed many years ago, it has recently been reconsidered in the connection with the asymptotic symmetry (see, e.g., [77, 78, 49, 79, 50, 72, 80]). It was pointed out in [49] that the vanishing of the amplitudes due to the infrared suppression explained in Section 3.5 is related to the asymptotic symmetry of QED. More concretely, they argued that the initial and the final states used in the conventional S-matrix generally belong to different sectors with respect to the asymptotic symmetry. Therefore, the amplitude between them should vanish, otherwise it breaks the conservation law, and we need dressed states in order to obtain non-vanishing amplitudes. Moreover, it was argued in [81, 82, 72] that the inclusive and the dressed state formalism yields different answer to the cross section if the incoming state is a wave packet and the difference is related to the conservation of asymptotic charge.

Motivated by these facts, we investigated the dressed state formalism in [54]. In particular, we revisited the gauge invariance of the dressed state. We argued that there is a problem on the gauge invariant condition in [52], and resolved the problem. In our method, the dressed states are obtained just from the appropriate gauge invariant condition. We will also discuss the $i\epsilon$ prescription for the dressed states. In addition, the relation between the dressed state formalism and the asymptotic symmetry is also discussed.

The outline of this chapter is as follows. We first briefly review Chung's dressed states and the IR finiteness of S-matrix for the states in Section 5.2. We then review the derivation of the F-K state based on [52]. Section 5.4 explains the problem of the gauge invariance for F-K dressed state and we resolve the problem in Section 5.5 based on [54]. in Section 5.6, the physical interpretation of the F-K dressed state is discussed. Section 5.7, we reveal the relation between the F-K dressed state and the asymptotic symmetry.

5.2 Chung's dressed state

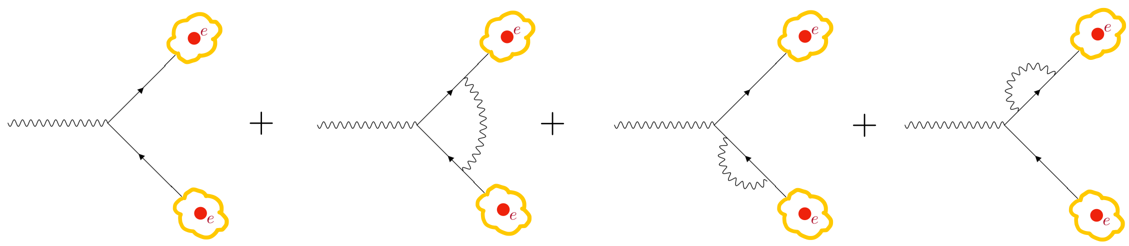

Figure 5.1: The tree level and its radiative corrections up to one-loop order under the usual Feynman rule. The soft photon contributions in the one-loop integrals create infrared divergences.

In [51], Chung introduced the following state, which we call *Chung's dressed state*;

$$|p_1, \cdots, p_n\rangle_{Ch} = \exp\left\{ \sum_{l=1,2} \int d^3\widetilde{p}\,\rho(p) \int_\lambda^\Lambda d^3\widetilde{k} \left[S^{(l)}(k) a^{(l)\dagger}(k) - S^{(l)*}(k) a^{(l)}(k) \right\}\right] |p_1, \cdots, p_n\rangle_0 \quad (5.2.1)$$

with

$$S^{(l)}(k) = \frac{e p \cdot \epsilon^{(l)}(k)}{k \cdot p} \quad (5.2.2)$$

where $|p_1, \cdots, p_n\rangle_0$ is an eigenstate of the free Hamiltonian (Fock state) with momentum $p_1, \cdots, p_n$. Here, λ is the infrared cutoff which we will take the limit $\lambda \to 0$ in the end of the calculations, and Λ is an arbitrary constant that has the dimension of energy. These states describe the states of electrons and positrons with the superpositions of no soft photon state, one soft photon state, two soft photons state, $\cdots$. In other words, Chung's dressed states are the states of electrons and positrons surrounded by a cloud of indefinite number of soft photons.

Then it was shown that the new "S-matrix" given by

$$ {}_{Ch}\langle p_1', \cdots, p_n'|S|p_1, \cdots, p_m\rangle_{Ch} \tag{5.2.3}$$

where S is the Dyson's S-operator (3.2.10), has *no* infrared divergence in the all orders of perturbation series in QED. The structure for cancelling out the infrared divergences is as follows. For concreteness, let us consider a single electron scattering under a external potential. The tree level diagram and the radiative corrections by virtual photons to the diagram up to the second order (e^2) in the usual Feynman rule are represented in Figure 5.1. These one-loop virtual photon integrals give infrared divergences as we have seen in Section 3.5. However, in the new S-matrix in (5.2.3), there are creation and annihilation operators of soft photons in the cloud of soft photons in the asymptotic state (5.2.1). Therefore, the creation and annihilation operators of soft photons in the cloud can also be Wick-contracted with the creation operators in the interaction vertexes, and such Wick-contractions give new one-loop corrections represented in Section 5.2. These new diagrams describe the processes of the emissions or absorptions of real soft photons. In [51], Chung showed that the infrared divergences coming from the new diagrams cancel out the usual infrared divergences at any order of perturbation series for any QED processes. More precisely, the λ dependence coming from Chung's dressed state (5.2.1) cancels out the λ dependence in the form of $\left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2}$ in the usual S-matrix (3.5.7) [1]. The parameter Λ (5.2.1) is not fixed in this discussion because it is not relevant for the IR finiteness. We will comment this arbitrariness in Section 6.1.

5.3 Derivation of the Faddeev-Kulish dressed state

In this section, we review the derivation of the F-K dressed state based on [52]. The QED Hamiltonian is given by

$$ H = H_0 + V \quad \text{with} \quad V = -e\int d^3x A_\mu(\vec{x}) j^\mu_{mat}(t,\vec{x}) \,. \tag{5.3.1}$$

Here, H_0 is the free Hamiltonian given by (2.9.8) with the matter free Hamiltonian

$$ H_{matter} = \int d^3x\, \overline{\psi}\left(-i\gamma^i\partial_i + m\right)\psi \tag{5.3.2}$$

[1]The infrared divergent phase factor in in the usual S-matrix (3.5.7) is not eliminated in the new S-matrix (5.2.3).

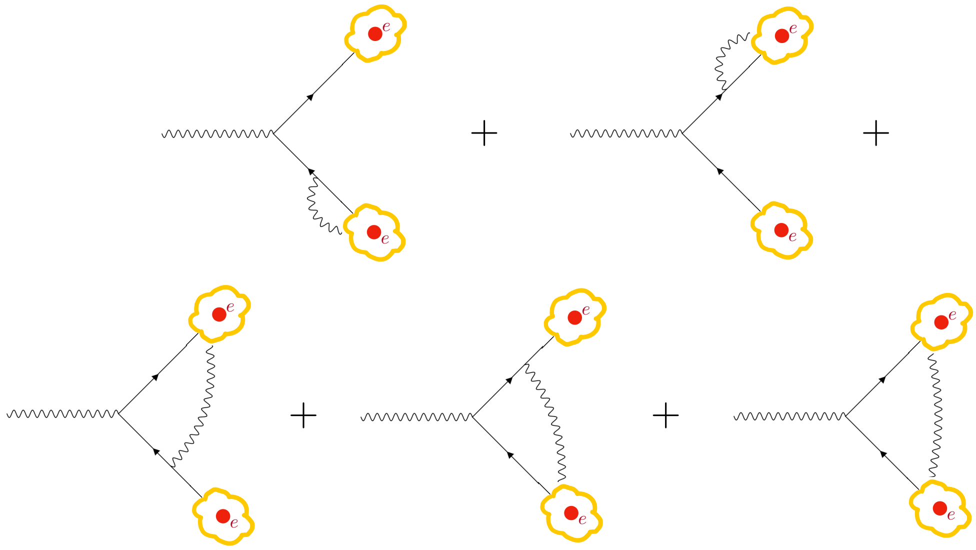

Figure 5.2: The new one-loop corrections coming due to exchange of soft photons between a vertex and a cloud or a cloud and another cloud.

where $\psi(x)$ and $\overline{\psi}(x)$ are the usual Dirac and conjugate Dirac fields. These fields can be expanded as

$$\psi(x) = \sum_{s=\pm} \int \widetilde{d^3p} \left[b_s(\vec{p}) u_s(\vec{p}) e^{ipx} + d_s^\dagger(\vec{p}) v_s(\vec{p}) e^{-ipx} \right] \tag{5.3.3}$$

$$\overline{\psi}(x) = \sum_{s=\pm} \int \widetilde{d^3p} \left[d_s(\vec{p}) \overline{v}_s(\vec{p}) e^{ipx} + b_s^\dagger(\vec{p}) \overline{u}_s(\vec{p}) e^{-ipx} \right] , \tag{5.3.4}$$

where $\widetilde{d^3p}$ is the Lorentz invariant measure defined by $\widetilde{d^3p} \equiv \frac{d^3p}{(2\pi)^3 2E_p}$. Here $b_s(\vec{p})$, $d_s(\vec{p})$ (, $b_s^\dagger(\vec{p})$, $d_s^\dagger(\vec{p})$) are the annihilation (creation) operators of the electron and the positron with the commutation relations;

$$\left\{ b_s(\vec{p}), b_{s'}^\dagger(\vec{q}) \right\} = (2\pi)^3 2E_p \delta^3 (\vec{p} - \vec{q}) \, \delta_{ss'} \ , \ \left\{ d_s(\vec{p}), d_{s'}^\dagger(\vec{q}) \right\} = (2\pi)^3 2E_p \delta^3 (\vec{p} - \vec{q}) \, \delta_{ss'} \ ,$$
$$\text{others} = 0 \, . \tag{5.3.5}$$

The spinors and barred spinors satisfy

$$\bar{u}_{s'}(\vec{p}) \gamma^\mu u_s(\vec{p}) = 2p^\mu \delta_{s's} \ , \bar{v}_{s'}(\vec{p}) \gamma^\mu v_s(\vec{p}) = 2p^\mu \delta_{s's} \tag{5.3.6}$$

The mode expansion of the gauge fields is given in (2.9.12). The matter local current $j^\mu_{mat}(t, \vec{x})$ is given by

$$j^\mu_{mat}(t, \vec{x}) =: \overline{\psi}(x) \gamma^\mu \psi(x) : \tag{5.3.7}$$

where : $\mathcal{O}$: is the normal ordering of the operator $\mathcal{O}$. Plugging the mode expansions (5.3.3) and (2.9.12) into the interaction operator in (5.3.1), we have

$$V^I(t) = -\sum_{s=\pm}\sum_{s'=\pm}\int d^3x \int \widetilde{d^3k}\int \widetilde{dp}\int \widetilde{dq}\left[a_\mu(\vec{k})e^{ikx} + a^\dagger_\mu(\vec{k})e^{-ikx}\right]$$
$$: \left[d_s(\vec{p})\overline{v}_s(\vec{p})e^{ipx} + b^\dagger_s(\vec{p})\overline{u}_s(\vec{p})e^{-ipx}\right]\gamma^\mu\left[b_{s'}(\vec{q})u_{s'}(\vec{q})e^{iqx} + d^\dagger_{s'}(\vec{q})v_{s'}(\vec{q})e^{-iqx}\right] : \tag{5.3.8}$$

Here we focus on the phase factor of each term. The spatial integration produces the delta function in each term. For example, in the terms that involve $b^\dagger_s(\vec{p})b_{s'}(\vec{q})$, the phase factor is given by

$$\exp\left(-i\left(E_p - E_q \pm \omega_k\right)t\right)\delta^3(\vec{p}-\vec{q}\pm\vec{k}) \tag{5.3.9}$$
$$= \exp\left(-i\left(\sqrt{\vec{p}^{\,2}-m^2} - \sqrt{(\vec{p}\pm\vec{k})^2-m^2} \pm \omega_k\right)t\right)\delta^3(\vec{p}-\vec{q}\pm\vec{k}) \tag{5.3.10}$$

For large time $t \to \pm\infty$, the interaction operator acts on the asymptotic state in the S-matrix and becomes c-number. The phase highly oscillates for large t and so one may expect that the integration over $\vec{k}$ in (5.3.8) gives zero as $t \to \pm\infty$ if the integrand is a smooth function. However, when the soft photon creation or annihilation operator with $b^\dagger_s(\vec{p})b_s(\vec{q})$ or $d^\dagger_s(\vec{p})d_s(\vec{q})$ acts on the asymptotic state, it corresponds to the Feynman diagram in which a charged external line emits a photon and gives a singular contribution by the soft factor as we have already seen in the soft theorem (3.4.12). Thus we expect that for large t, the leading contribution in the QED interaction comes from the terms with $b^\dagger_s(\vec{p})b_s(\vec{q})$ or $d^\dagger_s(\vec{p})d_s(\vec{q})$ multiplied by the photon creation and the annihilation operators in (5.3.8). By extracting only such terms from the the QED interaction, we obtain [2]

$$V^I_{as}(t) = \int \widetilde{d^3k}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E_p}\left[a_\mu(\vec{k})e^{i\frac{p\cdot k}{E_p}t} + a^\dagger_\mu(\vec{k})e^{-i\frac{p\cdot k}{E_p}t}\right]\rho(\vec{p}) \tag{5.3.11}$$

where $\rho(\vec{p})$ is the electric charge density given by

$$\rho(\vec{p}) = -\frac{e}{(2\pi)^3 2E_p}\sum_{s=\pm}\left[b^\dagger_s(\vec{p})b_{s'}(\vec{q}) - d_s(\vec{p})d^\dagger_{s'}(\vec{q})\right]. \tag{5.3.12}$$

To get the expression (5.3.11), we have used (5.3.6) and

$$\sqrt{\vec{p}^{\,2}-m^2} - \sqrt{(\vec{p}\pm\vec{k})^2-m^2} \pm \omega_k = \pm\frac{p\cdot k}{E_p} + \mathcal{O}\left(\frac{\omega_k^2}{|\vec{p}|^2}\right). \tag{5.3.13}$$

If we define

$$j^\mu_{as}(t,-\vec{k}) \equiv \int \frac{d^3p}{(2\pi)^3}\rho(\vec{p})\,\frac{p^\mu}{E_p}e^{i\frac{\vec{p}\cdot\vec{k}}{E_p}t}, \tag{5.3.14}$$

[2] The original explanation in [52] to derive (5.3.11) is different. In [52], it is argued that only the soft modes of diagonal parts in the interaction contribute at $t \to \infty$ because the phase factor (5.3.9) highly oscillates. (The phase of the non-diagonal parts (involving $d^\dagger_s(\vec{p})b_s(\vec{q})$ and $b^\dagger_s(\vec{p})d_s(\vec{q})$) do not become zero even at $\omega_k = 0$.) However, $\omega_k = 0$ is not the stationary point of the phase, so the original explanation seems not valid.

we can express (5.3.11) as

$$V^I_{as}(t) = \int \widetilde{d^3k} \left[a_\mu(\vec{k})e^{-i\omega_k t} + a^\dagger_\mu(-\vec{k})e^{i\omega_k t}\right] j^\mu_{as}(t,-\vec{k})\,, \tag{5.3.15}$$

which we call the *asymptotic interaction.* The expression of (5.3.14) in the position space is

$$\begin{aligned} j^\mu_{as}(t,\vec{x}) &\equiv \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} j^\mu_{as}(t,\vec{k}) = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{p^\mu}{E_p} e^{i\vec{k}\cdot(\vec{x}-\frac{\vec{p}}{E_p}t)} \rho(\vec{p}) \\ &= \int \frac{d^3k}{(2\pi)^3} \rho(\vec{p}) \frac{p^\mu}{E_p} \delta^3(\vec{x} - \frac{\vec{p}}{E_p}t) \end{aligned} \tag{5.3.16}$$

This expression is the same as the classical current for the point particle with the constant velocity $\vec{v} = \frac{\vec{p}}{E_p}$. Using this current, we can express (5.3.15) in the position space as

$$V^I_{as}(t) = -\int d^3x A_\mu(\vec{x}) j^\mu_{as}(t,\vec{x}), \tag{5.3.17}$$

We then expect that the time-evolution operator of the asymptotic state is given by

$$i\frac{\partial}{\partial t} U_{as}\left(t,t_s\right) = H^I_{as}(t) U_{as}\left(t,t_s\right) \tag{5.3.18}$$

where $H_{as}(t) = H_0 + V^I_{as}(t)$. If we write $U_{as}\left(t,t_s\right)$ as

$$U_{as}\left(t,t_s\right) = e^{-iH_0(t-t_s)} Z(t,t_s) \tag{5.3.19}$$

(5.3.18) is equivalent to

$$i\frac{\partial}{\partial t} Z_{as}\left(t,t_s\right) = V^{(I)}_{as}(t) Z_{as}\left(t,t_s\right) \tag{5.3.20}$$

The solution is given by

$$Z_{as}(t,t_s) \equiv \mathrm{T}\exp\left[-i\int_{t'}^{t} dt' V^I_{as}(t_s)\right]. \tag{5.3.21}$$

Then $U_{as}\left(t,t_s\right)$ is given by

$$U_{as}\left(t,t_s\right) = e^{-iH_0(t-t_s)}\, \mathrm{T}\exp\left[i\int_{t'}^{t} dt' V^I_{as}(t')\right] \tag{5.3.22}$$

Furthermore, since the commutator $[V^I_{as}(t_1), V^I_{as}(t_2)]$ commutes with $V^I_{as}(t)$ for any t, we obtain

$$U_{as}(t,t_s) = U_0(t,t_s)\, e^{-i\int_{t_s}^{t} dt'\, V^I_{as}(t')} e^{-\frac{1}{2}\int_{t_s}^{t} dt_1 \int_{t_s}^{t_1} dt_2\, [V^I_{as}(t_1), V^I_{as}(t_2)]}, \tag{5.3.23}$$

and by performing the t-integral, we have

$$-i\int_{t_s}^{t} dt'\, V^I_{as}(t') = R(t) - R(t_s) \tag{5.3.24}$$

with

$$R(t) \equiv \sum e \int \widetilde{d^3p}\, \rho(\vec{p}) \int \widetilde{d^3k} \frac{p^\mu}{p\cdot k} \left[a_\mu(\vec{k}) e^{i\frac{p\cdot k}{E_p}t} - a^\dagger_\mu(\vec{k}) e^{-i\frac{p\cdot k}{E_p}t} \right] . \tag{5.3.25}$$

The exponent including the commutator $[V^I_{as}(t_1), V^I_{as}(t_2)]$ in (5.3.23) is a c-number function, and we represent it as $i\Phi(t, t_s)$ where

$$\Phi(t, t_s) \equiv \frac{i}{2} \int_{t_s}^{t} dt_1 \int_{t_s}^{t_1} dt_2\, [V^I_{as}(t_1), V^I_{as}(t_2)] . \tag{5.3.26}$$

Plugging (5.3.11) into (5.3.26) and performing the integration, we find

$$\Phi(t, t_s) = -\frac{e^2}{4\pi} \int \widetilde{d^3p}\; \widetilde{d^3q}\, \rho(\vec{p})\rho(\vec{q}) \frac{p\cdot q}{\sqrt{(p\cdot q)^2 - m^4}}\, \mathrm{sgn}(t) \ln \frac{|t|}{t_s}, \tag{5.3.27}$$

We show that the stationary point of this phase at the large time limit leads to the classical trajectories of point charged particles under the relativistic Coulomb forces in Appendix H. In [52], $R(t_s)$ in (5.3.24) was deleted by a requirement for an initial condition. Under this assumption, The Møller operator defined in (3.1.9) is given by[3]

$$\Omega_{as}(t_i) = U^\dagger(t, t_s) U_{as}(t, t_s) = e^{iH(t-t_s)} e^{-iH_0(t-t_s)} e^{R(t)} e^{i\Phi(t,t_s)} . \tag{5.3.28}$$

The S-operator defined in (3.1.14) is then given by

$$\begin{aligned} S &= \lim_{t_f\to\infty,\, t_i\to-\infty} \Omega_{as}(t_f)^\dagger\, \Omega_{as}(t_i) \\ &= \lim_{t_f\to\infty,\, t_i\to-\infty} e^{-R(t_f)} e^{-i\Phi(t_f,t_s)} \left[\mathrm{T}\exp\left(-i \int_{t_i}^{t_f} dt'\, V^I(t') \right) \right] e^{R(t_i)} e^{i\Phi(t_i,t_s)}. \end{aligned} \tag{5.3.29}$$

We also assume that the asymptotic Hilbert space $\mathcal{H}_{as}$ is the Fock space $\mathcal{H}_{Fock}$. As a result, this S-matrix differs from the usual Dyson's one (3.2.10) only in the dressing factors e^R and $e^{i\Phi}$. Thus if we formally introduce a dressed Hilbert space $\mathcal{H}_{FK}$ as

$$\mathcal{H}_{FK} = \lim_{t\to-\infty} e^{R(t)} e^{i\Phi(t,t_s)} \mathcal{H}_{Fock}, \tag{5.3.30}$$

the S-matrix on $\mathcal{H}_{FK}$ is given by the usual one (3.2.10)[4].

The F-K dressing operator (5.3.25) is the same as the dressing operator in Chung's dressed state (5.2.1) up to the phase factor $e^{\pm i\frac{p\cdot k}{E_p}t}$. If we see the $k^\mu = 0$ mode in The F-K dressing operator, it is exactly the same as Chung's one. Thus F-K dressed state is believed to give IR finite S-matrix based on the Chung's proof. in Section 6.1, we will comment on a subtlety of the proof of IR finiteness.

[3] we have renamed the Møller operator to $\Omega_{as}(t)$ to distinguish the conventional one (3.2.3).

[4] Even on $\mathcal{H}_{FK}$, the notion of particles for charged fields is still valid because $j^\mu_{cl}(t, \vec{x})$ in the dressing factor $e^{R(t)}$ is a diagonal operator on the Fock space. However, the standard interpretation of photons on the Fock space seems to be lost because the dressing factor excites an infinite number of photons. As we will see in subsec. 6.1, the energy of the excited photons by the dressing factor is soft in the limit $t \to \pm\infty$. Hence, the particle notion for hard photons may be valid.

5.4 Problem on the gauge invariance of Faddeev-Kulish dressed state and our approach to this problem

We have to restrict $\mathcal{H}_{FK}$ to the subspace by imposing a gauge invariant condition in order to make sure that the physical observables are independent of the gauge choice. However, the treatment for the gauge invariance in [52] seems inappropriate. The Gupta-Bleuler condition (2.9.36) was imposed on $\mathcal{H}_{FK}$ as the physical state condition, *i.e.*, any physical state, $|\psi\rangle \in \mathcal{H}_{FK}$, was required to satisfy

$$k^\mu a_\mu(\vec{k})\,|\psi\rangle = 0 \quad \text{for any } \vec{k}. \tag{5.4.1}$$

However, we can easily see that the F-K state does not satisfy the condition because

$$k^\mu a_\mu(\vec{k})\, e^{R(t)} e^{i\Phi(t,t_s)}|\alpha\rangle_0 \neq 0 \tag{5.4.2}$$

In [52], to satisfy (5.4.1), the dressing operator R in (5.3.25) was modified by introducing a null vector $c^\mu(\vec{k})$ satisfying $k_\mu c^\mu = 1$. More concretely, the dressing operator was altered by shifting the coefficient $\frac{p^\mu}{p\cdot k}$ in (5.3.25) to $\frac{p^\mu}{p\cdot k} - c^\mu$.

We will see that the artificial vector c_μ is not needed if we impose an appropriate gauge invariant condition. Our claim is that the contributions of long-range interactions should also be incorporated into the gauge invariant condition, as the F-K dressed states are obtained by taking account of such interactions. The Gupta-Bleuler condition (5.4.1) is not adequate for the dressed states. In the next section, we will present the appropriate condition.

Furthermore, we will show that the dressed Hilbert space can be obtained just by requiring the gauge invariant condition. In our approach, it turns out that we do not need to solve the dynamics of the asymptotic Hamiltonian H_{as} as we reviewed in subsec 5.3. In fact, although the Dyson's S-matrix (3.2.10) is not a good operator on the usual Fock space $\mathcal{H}_{Fock}$, it may be well-defined on the dressed space $\mathcal{H}_{FK}$.[5] The asymptotic Hamiltonian H_{as} is just an approach to deriving the dressing factor $e^{R(t)}$. We think that the using the gauge invariant condition incorporating the asymptotic interaction is a simpler approach to obtaining the factor, and the interpretation is clear. The condition essentially just says that if there is a charged particle, there should exist electromagnetic fields around it by Gauss's law. The fields around the charge indeed make up the dress.

in Section 5.6, as a support of this interpretation and also another justification that we do not have to introduce c_μ, we will discuss the meaning of the original dressing operator $R(t)$ in eq. (5.3.25). As shown in [83], the dressing factor for a charged particle with momentum $\vec{p}$ corresponds to the Liénard-Wiechert potential for the uniformly moving charge with momentum $\vec{p}$. We will reconfirm this fact especially taking care of the $i\epsilon$ prescription.

Besides, our method allows a variety of dressing factors, and $\mathcal{H}_{FK}$ given by (5.3.30) is just one of them. We will see that in our gauge invariant condition, the physical Hilbert space $\mathcal{H}_{as}$ on which Dyson's S-matrix (3.2.10) acts takes the form

$$\mathcal{H}_{as} = e^{R_{as}}\mathcal{H}_{free}, \tag{5.4.3}$$

[5] In this paper, we do not take care of problems at ultraviolet regions. We assume that they can be resolved by a standard renormalization procedure.

where $e^{R_{as}}$ is a dressing factor, and $\mathcal{H}_{free}$ is a subspace of the Fock space $\mathcal{H}_{Fock}$ such that the Gupta-Bleuler condition is satisfied ($k^\mu a_\mu(\vec{k})\,|\psi\rangle = 0$, $|\psi\rangle \in \mathcal{H}_{free}$). The operator R_{as} can be $R(t) + i\Phi(t, t_s)$, but not necessarily. We will discuss the relation between the ambiguity of dressing and the asymptotic symmetry of QED in the subsection 5.7.

5.5 Gauge invariant asymptotic states

The BRST condition (2.9.34) is imposed on the physical Hilbert space $\mathcal{H}_{phys}$ as

$$Q^s_{BRST}|\alpha\rangle_{in} = 0 \ , \quad |\alpha\rangle_{in} \in \mathcal{H}_{phys} \tag{5.5.1}$$

where Q^s_{BRST} and $|\alpha\rangle_{in}$ are the BRST operator in (2.9.28) and the in-state in Schrödinger picture defined at t_s. Using (3.2.2), we have the BRST condition for an asymptotic state $|\beta\rangle_0$ as

$$0 = Q^s_{BRST}\,|\beta\rangle_{in} = \lim_{t_i\to-\infty} Q^s_{BRST}\Omega(t_i)\,|\beta\rangle_0 \,. \tag{5.5.2}$$

where $\Omega(t_i)$ is the Møller operator defined in (3.2.3). Since Q^s_{BRST} commutes with the exact Hamiltonian H^s, we have

$$Q^s_{BRST}\Omega(t) = U(t_s, t)Q^s_{BRST}U_0(t, t_s) = \Omega(t)Q^I_{BRST}(t), \tag{5.5.3}$$

where $Q^I_{BRST}(t)$ is the BRST operator in the interaction picture (2.9.32):

$$\begin{aligned} Q^I_{BRST}(t) &\equiv U_0(t, t_s)^{-1}\, Q^s_{BRST}\, U_0(t, t_s) \\ &= -\int \widetilde{d^3k}\left[c(\vec{k})\{k^\mu a^\dagger_\mu(\vec{k}) + e^{-i\omega t}\tilde{j}^{0I}(t, -\vec{k})\} + c^\dagger(\vec{k})\{k^\mu a_\mu(\vec{k}) + e^{i\omega t}\tilde{j}^{0I}(t, \vec{k})\}\right]. \end{aligned} \tag{5.5.4}$$

Therefore, $|\beta\rangle_0$ satisfies

$$\lim_{t_i\to-\infty} Q^I_{BRST}(t_i)\,|\beta\rangle_0 = 0. \tag{5.5.5}$$

By restricting the ghost-sector to the ghost-vacuum, this condition becomes

$$\lim_{t_i\to-\infty}\left[k^\mu a_\mu(\vec{k}) + e^{i\omega t_i}\tilde{j}^{0I}(t_i, \vec{k})\right]|\beta\rangle_0 = 0. \tag{5.5.6}$$

This means that the states satisfying the free Gupta-Bleuler condition $k^\mu a_\mu(\vec{k})\,|\psi\rangle = 0$ are generally not the physical asymptotic states. Thus the charged 1-particle states in the standard Fock space, such as $b^\dagger(\vec{p})\,|0\rangle$, cannot be the asymptotic physical states if there is the interaction.

We will show below that the states satisfying the condition (5.5.6) are dressed states. In fact, if there is an anti-Hermitian operator $\tilde{R}(t)$ such that

$$[k^\mu a_\mu(\vec{k}), \tilde{R}(t)] = -e^{i\omega t}\tilde{j}^{0I}(t, \vec{k}), \quad [\tilde{j}^{0I}(t, \vec{k}), \tilde{R}(t)] = 0, \tag{5.5.7}$$

then the states in $e^{\tilde{R}(t)}\mathcal{H}_{free}$ are annihilated by $k^\mu a_\mu(\vec{k}) + e^{i\omega t}\tilde{j}^{0I}(t, \vec{k})$ where $\mathcal{H}_{free}$ is a subspace of $\mathcal{H}_{Fock}$ satisfying the free Gupta-Bleuler condition. Thus the Hilbert space satisfying (5.5.6) is given by

$$\lim_{t_i \to -\infty} e^{\tilde{R}(t_i)} \mathcal{H}_{free}. \tag{5.5.8}$$

There are various choices of the dressing operator satisfying (5.5.7). One example is

$$\tilde{R}(t) = \int \widetilde{d^3k} \frac{1}{2\omega^2} \left[e^{i\omega t} \tilde{j}^{0I}(t, \vec{k}) \tilde{k}^\mu a^\dagger_\mu(\vec{k}) - e^{-i\omega t} \tilde{j}^{0I}(t, -\vec{k}) \tilde{k}^\mu a_\mu(\vec{k}) \right], \tag{5.5.9}$$

where $\tilde{k}^\mu = (\omega, -\vec{k})$.

Although one may use such a dressing operator $\tilde{R}(t)$, we can simplify it by recognizing that the current operator j^{0I} can be approximated in the asymptotic regions ($t \sim \pm\infty$) by the classical current operator j^0_{as} given by (5.3.16). For the time component of the current, we can straightforwardly obtain (see Appendix E for details)[6]

$$\lim_{\tau \to \pm\infty} j^{0I}(t, \vec{x}) = \lim_{\tau \to \pm\infty} j^0_{as}(t, \vec{x}). \tag{5.5.11}$$

Therefore, we can rewrite the condition (5.5.6) as

$$\lim_{t_i \to -\infty} \left[k^\mu a_\mu(\vec{k}) + e^{i\omega t_i} \tilde{j}^0_{as}(t_i, \vec{k}) \right] |\beta\rangle_0 = 0. \tag{5.5.12}$$

For later convenience, we represent the operator in (5.5.12) by $\hat{G}(t, \vec{k})$ as

$$\hat{G}(t, \vec{k}) \equiv k^\mu a_\mu(\vec{k}) + e^{i\omega t} \tilde{j}^0_{as}(t, \vec{k}). \tag{5.5.13}$$

Noting that the momentum representation of the classical current operator is given by (5.3.14), and a trivial equation $e^{-i\frac{p\cdot k}{E_p}t} = e^{i\omega t} e^{-i\frac{\vec{p}\cdot\vec{k}}{E_p}t}$, we can easily confirm that the Faddeev-Kulish dressing operator $R(t)$ in (5.3.25) satisfies

$$\hat{G}(t, \vec{k}) e^{R(t)} = e^{R(t)}\, k^\mu a_\mu(\vec{k}). \tag{5.5.14}$$

Thus an asymptotic physical Hilbert space satisfying (5.5.12) is given by

$$\lim_{t_i \to -\infty} e^{R(t_i)}\, \mathcal{H}_{free}. \tag{5.5.15}$$

Since the phase operator Φ in (5.3.26) commutes with $\hat{G}(t, \vec{k})$ and $R(t)$, Φ is not relevant for the gauge invariance (5.5.12). Therefore, the Faddeev-Kulish dressed space $\mathcal{H}_{FK}$ in (5.3.30) is gauge invariant without introducing a vector c_μ, if we restrict $\mathcal{H}_{Fock}$ to the subspace $\mathcal{H}_{free}$.

Besides the phase operator, there are other choices of the dressing operator $R_{as}(t)$ satisfying

$$\hat{G}(t, \vec{k}) e^{R_{as}(t)} = e^{R_{as}(t)}\, k^\mu a_\mu(\vec{k}). \tag{5.5.16}$$

One example of $R_{as}(t)$ other than the Faddeev-Kulish dressing operator (5.3.25) is ob-

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[6] Other components of the current also satisfy similar equations:

$$\lim_{\tau \to \pm\infty} j^{iI}_{free}(t, \vec{x}) = \lim_{\tau \to \pm\infty} j^i_{as}(t, \vec{x}). \tag{5.5.10}$$

Here, the subscript $free$ means that the current is that of the free theory.

tained by replacing $\tilde{j}^{0I}$ with $\tilde{j}^{0}_{as}$ in (5.5.9). Then we can define another asymptotic physical Hilbert space:

$$\lim_{t_i \to -\infty} e^{R_{as}(t_i)} \mathcal{H}_{free}, \tag{5.5.17}$$

which is a solution of the gauge invariant condition (5.5.12). Although the question that what types of dressing operators cancel the IR divergences in the S-matrix is beyond the scope of this paper, we will discuss in subsection 5.7 that the existence of many choices is natural from the viewpoint of asymptotic symmetry.

5.6 Interpretation of the Faddeev-Kulish dresses

It is shown in [83] that the Faddeev-Kulish dressing factor for a charged particle with momentum p^μ corresponds to the classical Liénard-Wiechert potential around the particle. This fact supports our statement that the Gauss's law requires the dressing factor. In this section, we will reconfirm this fact with taking care of the $i\epsilon$ prescription, and see that we should use different prescriptions for initial and final states, which might be useful for the explicit computation of scattering amplitudes.

5.6.1 Coulomb potential by point charges in the asymptotic region

Here, we will recall the expression of the electromagnetic potential created by a charged point particle with momentum p^μ. The classical equation of motion for the gauge field in the Lorenz gauge is given by

$$\Box A_\mu(x) = -j_\mu(x) \ , \ j_\mu(x) = e\int_{-\infty}^{\infty} d\tau \frac{dy_\mu(\tau)}{d\tau}\delta^4(x - y(\tau)), \tag{5.6.1}$$

where $y_\mu(\tau) = \frac{p_\mu}{m}\tau = \frac{p_\mu}{E_p}t$ is the trajectory of the charged particle, which is supposed to pass through the origin at $t = 0$. The position at $t = 0$ is not relevant when we consider the asymptotic region.[7] With the use of the retarded Green's function for the Klein-Gordon equation,

$$G_{ret}(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^0 - \omega + i\epsilon)(k^0 + \omega + i\epsilon)} e^{ik\cdot x}, \tag{5.6.2}$$

the general solutions of (H.1.1) are given by

$$\begin{aligned} A_\mu(x) &= A^{in}_\mu(x) + \int d^4x' G_{ret}(x - x') j_\mu(x') \\ &= A^{in}_\mu(x) + ie\frac{p_\mu}{E_p}\int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{t} dt' \frac{1}{2\omega}\left(e^{-i\omega(t-t')} - e^{i\omega(t-t')}\right)e^{-\epsilon(t-t')}e^{i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t'\right)} \\ &= A^{in}_\mu(x) - e\int \frac{d^3k}{(2\pi)^3(2\omega)}\left[\frac{p_\mu}{p\cdot k + i\epsilon}e^{i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)} + \frac{p_\mu}{p\cdot k - i\epsilon}e^{-i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)}\right]. \end{aligned} \tag{5.6.3}$$

[7]However, the position at $t = 0$ can contribute to subleading orders, and it was shown in [59] that the position is important for the subleading memory effect.

where $A_\mu^{in}(x)$ is the incoming free wave, which is specified at $t \to -\infty$, and the second term is the Liénard-Wiechert potential created by the particle with momentum p^μ and charge e. We represent this second term by $A_\mu^{ret}(x;\vec{p})$ as

$$A_\mu^{ret}(x;\vec{p}) \equiv -e \int \frac{d^3k}{(2\pi)^3(2\omega)} \Big[\frac{p_\mu}{p\cdot k + i\epsilon} e^{i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)} + \frac{p_\mu}{p\cdot k - i\epsilon} e^{-i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)} \Big]. \tag{5.6.4}$$

5.6.2 Coulomb potential from dressed states with $i\epsilon$ prescription

Let's consider a dressed state of a single incoming electron with momentum p^μ defined by

$$\| p(t) \rangle\rangle \equiv e^{R_{in}(t)} b^\dagger(\vec{p}) |0\rangle , \tag{5.6.5}$$

where $R_{in}(t)$ is an operator dressing the incoming single particle state. The gauge field in the interaction picture can be written as

$$A_\mu^I(x) = \int \frac{d^3k}{(2\pi)^3(2\omega)} \left(a_\mu(\vec{k}) e^{ik\cdot x} + a_\mu^\dagger(\vec{k}) e^{-ik\cdot x} \right). \tag{5.6.6}$$

Then we demand that its expectation value for the above dressed state[8] match the classical gauge field (5.6.4) created by a charged point particle with momentum p^μ as

$$\begin{aligned} &\langle\langle p(t) \| A_\mu^I(x) \| p(t) \rangle\rangle \\ &= -e \int \frac{d^3k}{(2\pi)^3(2\omega)} \Big(\frac{p_\mu}{p\cdot k + i\epsilon} e^{i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)} + \frac{p_\mu}{p\cdot k - i\epsilon} e^{-i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)} \Big) \langle\langle p(t) \| p(t) \rangle\rangle. \end{aligned} \tag{5.6.7}$$

We can easily check that the following dressing operator satisfies the above condition,

$$R_{in}(t) = e \int \frac{d^3p}{(2\pi)^3(2E_p)} \rho(\vec{p}) \int \frac{d^3k}{(2\pi)^3(2\omega)} \Big(\frac{p^\mu}{p\cdot k - i\epsilon} a_\mu(\vec{k}) e^{i\frac{p\cdot k}{E_p}t} - \frac{p^\mu}{p\cdot k + i\epsilon} a_\mu^\dagger(\vec{k}) e^{-i\frac{p\cdot k}{E_p}t} \Big). \tag{5.6.8}$$

This operator matches the dressing operator (5.3.25) up to the $i\epsilon$ insertion. How to insert $i\epsilon$ in the dressing operator is determined by how the initial condition of gauge fields is specified. Thus the dressed states stand for the states of (anti-)electrons surrounded by relativistic Coulomb fields created by themselves. We have considered a single charged particle state (5.6.5). The generalization to multi-particle states is trivial, and the expectation value of A_μ is given by the superposition of the Coulomb field created by each particle. In other words, in the dressed state, the charged particles are properly dressed by electromagnetic fields in the asymptotic region where the particles move at almost constant velocities. This result is natural because our dressed states are obtained by solving the BRST (gauge invariant) condition without ignoring the interaction in the asymptotic regions. We also would like to comment that this expectation value changes if we modify the dressing operator by introducing a vector c_μ as in [52]. This is another reason for thinking that such a modification is unnatural. Note also that R_{in} is anti-Hermitian ($R_{in}^\dagger = -R_{in}$). Thus the dressing factor $e^{-R_{in}}$ is unitary.[9]

[8] More precisely, we should use a wave-packet, since the state (5.6.5) is not normalized.

[9] If we write $e^{-R_{in}}$ in the normal ordering, the normalization factor has an IR divergence if we set $\epsilon = 0$. Thus it is often said (see, e.g., [52]) that the dressing factor is not a unitary operator on the Fock space in a rigorous sense. However, it does not matter if we keep ϵ nonzero. After computing IR finite

Similarly, we can fix the $i\epsilon$ prescription for the dressing operator $R_{out}(t)$ for outgoing states. We consider a dressed outgoing state

$$ {}_{out}\langle\langle p(t) \| \equiv \langle 0|\, b(\vec{p}) e^{-R_{out}(t)}, \tag{5.6.9}$$

and require that the expectation value of $A^I_\mu(x)$ agree with the advanced potential for the point particle, which is given by

$$A^{adv}_\mu(x;\vec{p}) = -e \int \frac{d^3k}{(2\pi)^3(2\omega)} \Big[\frac{p_\mu}{p\cdot k - i\epsilon} e^{i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)} + \frac{p_\mu}{p\cdot k + i\epsilon} e^{-i\vec{k}\cdot\left(\vec{x}-\frac{\vec{p}}{E_p}t\right)} \Big]. \tag{5.6.10}$$

The requirement

$$ {}_{out}\langle\langle p(t) \| A^I_\mu(x) \| p(t) \rangle\rangle_{out} = A^{adv}_\mu(x;\vec{p})\ {}_{out}\langle\langle p(t) \| p(t) \rangle\rangle_{out} \tag{5.6.11}$$

can be satisfied by the following dressing operator

$$R_{out}(t) = e \int \frac{d^3p}{(2\pi)^3(2E_p)} \rho(\vec{p}) \int \frac{d^3k}{(2\pi)^3(2\omega)} \Big(\frac{p^\mu}{p\cdot k + i\epsilon} a_\mu(\vec{k}) e^{i\frac{p\cdot k}{E_p}t} - \frac{p^\mu}{p\cdot k - i\epsilon} a^\dagger_\mu(\vec{k}) e^{-i\frac{p\cdot k}{E_p}t} \Big). \tag{5.6.12}$$

Thus the sign of $i\epsilon$ terms is opposite to that in the initial dressing operator R_{in} given by (5.6.8).[10] This R_{out} is also anti-Hermitian ($R^\dagger_{out} = -R_{out}$), and the dressing factor $e^{-R_{out}}$ is thus unitary.

The unitarity of the dressing factors, $e^{R_{in}}$ and $e^{-R_{out}}$, guarantees that the asymptotic Hilbert space is positive definite. The asymptotic dressed states are given by acting on the states satisfying the free Gupta-Bleuler condition with the unitary dressing factors. The dressed states thus have a positive norm, because the states satisfying the free Gupta-Bleuler condition are positive definite and any unitary transformation preserves the positive definiteness. Here, we also give a formal proof of the unitarity of S-matrix. Including the dressing factors, the S-matrix acting on the Fock space takes the form (up to phase operators)

$$S = \lim_{t_f\to\infty, t_i\to-\infty} S(t_f,t_i) \quad \text{with} \quad S(t_f,t_i) = e^{-R_{out}(t_f)} S_0(t_f,t_i) e^{R_{in}(t_i)}, \tag{5.6.13}$$

where S_0 denotes the usual (finite time) S-matrix:

$$S_0(t_f,t_i) = \mathrm{T}\exp\left(-i\int_{t_i}^{t_f} dt'\, V^I(t') \right) = U^\dagger_0(t_f,t_s) U(t_f,t_i) U_0(t_i,t_s). \tag{5.6.14}$$

The unitarity of $S_0(t_f,t_i)$ simply follows from the expression of eq.(5.6.14). Since R_{in} and R_{out} are anti-Hermitian, we can show the unitarity of $S(t_f,t_i)$ as

$$\begin{aligned} S^\dagger(t_f,t_i)S(t_f,t_i) &= e^{R^\dagger_{in}(t_i)} S^\dagger_0(t_f,t_i) e^{-R^\dagger_{out}(t_f)} e^{-R_{out}(t_f)} S_0(t_f,t_i) e^{R_{in}(t_i)} \\ &= e^{-R_{in}(t_i)} S^\dagger_0(t_f,t_i) e^{R_{out}(t_f)} e^{-R_{out}(t_f)} S_0(t_f,t_i) e^{R_{in}(t_i)} \\ &= e^{-R_{in}(t_i)} S^\dagger_0(t_f,t_i) S_0(t_f,t_i) e^{R_{in}(t_i)} \\ &= 1. \end{aligned} \tag{5.6.15}$$

physical quantities, we can take ϵ to 0.

[10] This difference between the $i\epsilon$ prescriptions for initial states and for final states may be related to the prescription used to define in-out and in-in propagators in nonstationary spacetime [84].

Therefore, the S-matrix is unitary.

5.7 Asymptotic symmetry in the dressed state formalism

We now discuss the relation between the asymptotic symmetry in QED and the dressed states (see also [77, 78, 49, 79, 50, 72, 80] for related discussions). We will show that the Faddeev-Kulish dressed states carry the charges associated with the asymptotic symmetry, and investigate the conservation law of the asymptotic charges for the S-matrix in the dressed state formalism.

The charge associated with the asymptotic symmetry was given by

$$Q^s_{as}[\epsilon] = \int d^3x \left[-\Pi^{0s}\partial_0\epsilon - \Pi^{is}\partial_i\epsilon + j^{0s}\epsilon \right], \tag{5.7.1}$$

where the gauge parameter $\epsilon(x)$ satisfies $\Box\epsilon = 0$ and the script s refers to the Schrödinger picture. This charge $Q^s_{as}[\epsilon]$ is BRST exact up to the boundary term:

$$Q^s_{as}[\epsilon] = -\int d^3x\, \partial_i(\Pi^{is}\epsilon) + \left\{ Q^s_{BRST}, \int d^3x(-\bar{c}^s\partial_0\epsilon + i\pi^s_{(c)}\epsilon) \right\}. \tag{5.7.2}$$

Therefore, if the gauge parameter $\epsilon(x)$ is the large gauge parameter, the charge does not vanish in the asymptotic regions as we have already seen. All of the asymptotic charges commute with the BRST charge:

$$[Q^s_{as}, Q^s_{BRST}] = 0, \tag{5.7.3}$$

and they commute with the Hamiltonian H^s up to the BRST exact term [11] :

$$\begin{aligned} [Q^s_{as}, H^s] &= -i\int d^3x \left[\partial_i\epsilon\, \partial_i\Pi^{0s} + \partial_0\epsilon\, (\partial_i\Pi^{is} + j^{0s}) \right] \\ &= \left\{ Q^s_{BRST}, -i\int d^3x \left(\partial_i\epsilon\partial_i\bar{c}^s + i\partial_0\epsilon\, \pi^s_{(c)} \right) \right\}. \end{aligned} \tag{5.7.4}$$

Therefore, the spectrum of the physical Hilbert space is infinitely degenerated.

This fact naturally leads us to classify the asymptotic states by Q^I_{as} in the interaction picture. We now see how Q^I_{as} acts on the initial dressing operator $R_{in}(t)$ given in eq. (5.6.8). As in (5.7.2), the asymptotic charge Q^I_{as} in the interaction picture takes the following form up to the BRST exact part:

$$Q^I_{as}[\epsilon] = -\int d^3x\, \partial_i(\Pi^{iI}\epsilon) = -\int d^3x\, [\Pi^I_i\partial^i\epsilon + (\partial_i\Pi^{iI})\epsilon]. \tag{5.7.5}$$

[11] This is the reason why we adopted the Hamiltonian (2.9.9). As mentioned in footnote 14, the canonical Hamiltonian H^s_{can} has extra boundary terms: $H^s_{can} = H^s - \int d^3x\, \partial_i(\Pi^s_0 A^{is} + \Pi^{is}A^{0s})$. The boundary terms affect the commutator (5.7.4) as $[Q^s_{as}, \int d^3x\, \partial_i(\Pi^s_0 A^{is} + \Pi^{is}A^{0s})] = -i\int d^3x\, \partial_i(\Pi^{0s}\partial_i\epsilon + \Pi^{is}\partial_0\epsilon) = \{Q^s_{BRST}, -i\int d^3x\, \partial_i(\bar{c}^s\partial_i\epsilon)\} - i\int d^3x\, \partial_i(\Pi^{is}\partial_0\epsilon)$. Since $\partial_0\epsilon = \mathcal{O}(r^{-1})$ at $r \to \infty$, we can neglect the effect of boundary terms if the radial component of the electric field operator, $\hat{x}^i\Pi^i$, decays as $\mathcal{O}(r^{-2})$. This condition is probably satisfied for physical scattering states in a reasonable setup.

The commutator of Π^I_i and $R_{in}(t)$ is given by

$$\begin{aligned}&[\Pi^I_i(t,\vec{x}),R_{in}(t)]\\&=ie\int\frac{d^3p}{(2\pi)^3(2E_p)}\rho(\vec{p})\int\frac{d^3k}{(2\pi)^3(2\omega)}\frac{E_pk_i-\omega p_i}{p\cdot k}\Big(e^{-i\vec{k}\cdot(\vec{x}-\frac{\vec{p}}{E_p}t)}-e^{i\vec{k}\cdot(\vec{x}-\frac{\vec{p}}{E_p}t)}\Big),\end{aligned}\tag{5.7.6}$$

where we have set $\epsilon=0$ because the integrand is not singular at $\vec{k}=0$. On the other hand, the classical electric field for the classical configuration $A^{ret}_\mu(x;\vec{p})$ given by (5.6.4) is computed as

$$\partial_0A^{ret}_i(x;\vec{p})-\partial_iA^{ret}_0(x;\vec{p})=ie\int\frac{d^3k}{(2\pi)^3(2\omega)}\frac{E_pk_i-\omega p_i}{p\cdot k}\Big(e^{-i\vec{k}\cdot(\vec{x}-\frac{\vec{p}}{E_p}t)}-e^{i\vec{k}\cdot(\vec{x}-\frac{\vec{p}}{E_p}t)}\Big),\tag{5.7.7}$$

where we have also set $\epsilon=0$. Hence, one can say that the commutator of Π^{iI} and $R_{in}(t)$ is given by the "classical operator" which represents the classical Liénard-Wiechert electric field as

$$[\Pi^I_i(t,\vec{x}),R_{in}(t)]=\int\frac{d^3p}{(2\pi)^3(2E_p)}\rho(\vec{p})[\partial_0A^{ret}_i(x;\vec{p})-\partial_iA^{ret}_0(x;\vec{p})]\equiv F^{cl}_{0i}(x).\tag{5.7.8}$$

Similarly, the commutator of $\partial_i\Pi^{iI}$ and $R_{in}(t)$ is given by the classical current as

$$[\partial_i\Pi^{iI}(t,\vec{x}),R_{in}(t)]=-e\int\widetilde{d^3p}\,\rho(\vec{p})\,\delta^3(\vec{x}-\vec{p}t/E_p)=-j^0_{as}(x).\tag{5.7.9}$$

Therefore, the asymptotic charge $Q^I_{as}[\epsilon]$ in (5.7.5) acts on $e^{R_{in}}$ as

$$[Q^I_{as}[\epsilon],e^{R_{in}}]=e^{R_{in}}\int d^3x\,[F^{0i}_{cl}\partial_i\epsilon+j^0_{as}\epsilon].\tag{5.7.10}$$

The integral

$$Q^{cl}_{as}[\epsilon]\equiv\int d^3x\,[F^{0i}_{cl}\partial_i\epsilon+j^0_{as}\epsilon]\tag{5.7.11}$$

is in fact the asymptotic charge operator on the Fock space of charged particles. In the limit $t\to\pm\infty$, the eigenvalues agree with the classical leading hard charges (2.4.4). The leading hard charges are the contributions to the asymptotic charges from uniformly moving charged particles and their Coulomb-like electric fields. Therefore, eq. (5.7.10) represents that the charged Fock particles with the dressing operator (5.3.25) carry the asymptotic charges for the classical free charged particles with their Liénard-Wiechert electric fields. This result is natural because the dressing corresponds to creating the Liénard-Wiechert potential as we have seen in Section 5.6. The result also suggests that the antipodal matching condition (2.5.8) also holds in QED.

At the classical level, the conservation of asymptotic charges lead to the electromagnetic memory effect. Let us see the implication at the quantum level. To make our discussion simple, we suppose that the radiation sector is given by eigenstates of asymptotic charges at $t=\pm\infty$; that is, we consider the asymptotic states $|\Lambda_{in}\rangle$ and $\langle\Lambda_{out}|$ such that they contain only transverse photons and satisfy

$$Q^{I,-}_{as}[\epsilon]\,|\Lambda_{in}\rangle=\Lambda_{in}[\epsilon^0]\,|\Lambda_{in}\rangle\,,\quad\langle\Lambda_{out}|\,Q^{I,+}_{as}[\epsilon]=\langle\Lambda_{out}|\,\Lambda_{out}[\epsilon^0],\tag{5.7.12}$$

where $Q^{I,\pm}_{as}[\epsilon] = \lim_{t\to\pm\infty} Q^{I}_{as}[\epsilon]$, and Λ_{in}, Λ_{out} are arbitrary (c-number) functionals of ϵ^0 to which ϵ asymptotically approaches. We then prepare the following dressed states by exciting charged particles on $|\Lambda_{in}\rangle$ and $\langle\Lambda_{out}|$ as

$$|in\rangle = e^{R_{in}(t=-\infty)}\hat{\Psi}^{\dagger}_{in}|\Lambda_{in}\rangle\,, \quad \langle out| = \langle\Lambda_{out}|\,\hat{\Psi}_{out}e^{-R_{out}(t=+\infty)}, \tag{5.7.13}$$

where $\hat{\Psi}^{\dagger}_{in}$ is an arbitrary product of creation operators $b^{\dagger}, d^{\dagger}$ of charged particles and $\hat{\Psi}_{out}$ is any product of annihilation operators b, d. The asymptotic symmetry implies

$$\langle out|\,(Q^{I,+}_{as}S_0 - S_0 Q^{I,-}_{as})\,|in\rangle = 0, \tag{5.7.14}$$

where S_0 is given by (5.6.14) with limits $t_f\to\infty, t_i\to-\infty$. From (5.7.10) and a similar computation for $e^{-R_{out}}$, we have

$$Q^{I,-}_{as}|in\rangle = (Q^{-}_{H} + \Lambda_{in})\,|in\rangle\,, \quad \langle out|\,Q^{I,+}_{as} = \langle out|\,(Q^{+}_{H} + \Lambda_{out}). \tag{5.7.15}$$

Here, Q^{-}_{H} and Q^{+}_{H} represent the hard charge eigenvalues for the states $\hat{\Psi}^{\dagger}_{in}|0\rangle$ and $\langle 0|\,\hat{\Psi}_{out}$ respectively as

$$\left(\lim_{t\to-\infty} Q^{cl}_{as}\right)\hat{\Psi}^{\dagger}_{in}|0\rangle = Q^{-}_{H}\hat{\Psi}^{\dagger}_{in}|0\rangle\,, \quad \langle 0|\,\hat{\Psi}_{out}\left(\lim_{t\to\infty} Q^{cl}_{as}\right) = \langle 0|\,\hat{\Psi}_{out}Q^{+}_{H}. \tag{5.7.16}$$

Thus (5.7.14) becomes

$$(Q^{+}_{H} + \Lambda_{out} - Q^{-}_{H} - \Lambda_{in})\,\langle out|\,S_0\,|in\rangle = 0. \tag{5.7.17}$$

This means that the S-matrix elements can take non-zero values only when the asymptotic charges are conserved,

$$Q^{+}_{H} + \Lambda_{out} = Q^{-}_{H} + \Lambda_{in}, \tag{5.7.18}$$

between the out states and the in states [49]. We would also be able to interpret it as the quantum analog of the classical memory effect. In a scattering event, if the hard charges are not conserved $Q^{+}_{H} \neq Q^{-}_{H}$, there should be a change in the radiation sector $|\Lambda_{in}\rangle \to |\Lambda_{out}\rangle$ so that (5.7.18) holds for any ϵ^0. Conversely, a change in the radiation sector, $\Lambda_{out} - \Lambda_{in}$, is memorized in the change of the hard charges $Q^{+}_{H} - Q^{-}_{H}$.

We here comment on the possibility of other dressing operators. The standard Fock vacuum is not the eigenstate of Q^{I}_{as}.[12] Roughly speaking, the eigenstates in (5.7.12) would consist of clouds of soft photons without charged particles. However, the Faddeev-Kulish dressing operator $R(t)$ in (5.3.25) makes a photon cloud only when there are charged particles. Thus we need other dressing operators than Faddeev-Kulish's in order to prepare eigenstates (5.7.12). As we have already argued in subsection 5.5, the dressing operators are not uniquely fixed from the gauge invariant condition. We think that this variety is related to the asymptotic symmetry, and leave it for a future work to classify gauge invariant dressed states in terms of the asymptotic charges.

[12] The asymptotic symmetries for general gauge parameters ϵ are spontaneously broken in the standard Fock vacuum [4].

Chapter 6

Conclusion and further discussion

In Chapter 2, we have derived the classical charges associated with the asymptotic symmetry up to the subleading order. We have also shown that the contribution from the charge in the spatial infinity to the conservation law vanishes and therefore the charges are conserved between the asymptotic future and past time slices. We have then derived the quantum leading charge associated with the asymptotic symmetry in the BRST formalism and shown that the charge has the same expression as the one of the leading classical charge if the BRST condition is imposed on the physical Hilbert space. In Chapter 3, we have reviewed the S-matrix theory, the soft photon theorems, and the infrared divergences of the conventional S-matrix. In Chapter 4, we have derived the subleading charge conservation law from the subleading photon theorem in massive scaler QED. In Chapter 5, we have shown that the Faddeev-Kulish dressed states can be obtained just from the gauge-invariant condition without solving the asymptotic dynamics. In addition, we have shown the possibility of other types of gauge-invariant dressed states. We have also shown that the Faddeev-Kulish dressing operator carry the charges associated with the asymptotic symmetry and the S-matrix obeys the conservation law of the charges.

In the analysis in Chapter 2 and Chapter 4, we have assumed that massive particles are the free particles in the asymptotic region. Therefore the contributions from electromagnetic potential created by the massive charged particles to the hard charges have been neglected in the analysis. However, the hard charges should contain such contributions because the effect of long-range interactions would not be ignored even in the asymptotic region. In fact, the asymptotic charge carried by the dressed charged particles (5.7.10) contains not only the contribution from the charge density of massive particles but also the contribution from the electromagnetic potential created by the charged particles. Once the asymptotic interactions in the asymptotic regions are taken into consideration, the subleading charge and also the soft theorems would be modified. Therefore it would be interesting to study the subleading charge of asymptotic symmetry, the soft theorems, and their relations in the dressed state formalism because they may give some important constraints to the IR finite S-matrix.

We close this chapter with further discussion and comments on future directions.

6.1 Softness of dresses and infrared scales

In the derivation of the F-K dressed state, there were several assumptions and subtle points. We assumed that the interaction in the dynamics of the asymptotic state is generated by the asymptotic interaction in (5.3.15) [1]. We also assumed that the asymptotic Hilbert space is the Fock space. This assumption seems not so natural, since electrons and photons in the asymptotic state interact with each other by the asymptotic interaction in the dressed state formalism. The difficulty originates from the fact that the asymptotic interaction is time-dependent even in the Schrödinger picture. The time-dependence itself would be natural because the short range modes in the QED interaction among charged particles are decoupled as the distance of the wave packets of the charged particles increases in the time evolution. However, the notion of "particle" and vacuum states become subtle due to the time-dependence because the stable state does not exist for the time-dependent Hamiltonian.

The infrared finiteness of the F-K dressed state is based on Chung's analysis reviewed in Section 5.2. The argument is as follows. If we extract soft momentum region $k \sim 0$ for the dressing operator (5.3.25) in the F-K dressed state, the operator takes the form

$$R_{\text{soft}} \sim \sum e \int \widetilde{d^3p}\, \rho(\vec{p}) \int_{\text{soft}} \frac{d^3k}{(2\pi)^3(2\omega)} \frac{p^\mu}{p\cdot k} \left[a_\mu(\vec{k}) - a^\dagger_\mu(\vec{k})\right], \tag{6.1.1}$$

because $e^{i\frac{p\cdot k}{E_p}t} \sim 1$ at $k \sim 0$. Roughly, this is the dressing operator in Chung's dressed state (5.2.1). Since $\omega > 0$ region is not relevant to the proof of the IR finiteness, the behavior of the dressing operator was specified only at $\omega \to 0$ in [51]

However, we should pay attention to the contribution of the nonzero soft momentum in (5.3.25) because it would affect the physical observables. We can make a rough argument that the contribution vanishes in the limit $|t| \to \infty$ as follows. First, note that $p \cdot k = -\omega(E_p - \vec{p}\cdot\hat{k})$ can be zero only when $\omega = 0$ because p^μ is an on-shell momentum of a massive particle ($E_p > |\vec{p}|$). Then owing to the oscillating factor $e^{i\frac{p\cdot k}{E_p}t}$, the contributions from nonzero momenta ($\omega > 0$) can be ignored in the limit $t \to \pm\infty$. This statement can be made more rigorous with the use of ϵ-inserted dressing operator eq. (5.6.8) or eq. (5.6.12). We use the following identity as a distribution:

$$\lim_{\epsilon\to 0}\lim_{t\to\pm\infty} \frac{e^{i\alpha t}}{\alpha \pm i\epsilon} = \mp i\pi\delta(\alpha). \tag{6.1.2}$$

From this identity, we have

$$\lim_{\epsilon\to 0}\lim_{t\to-\infty} R_{in}(t) = -\frac{i\pi}{2}\sum e \int \widetilde{d^3p}\, \rho(\vec{p}) \int \frac{d^3k}{(2\pi)^3} \frac{p^\mu}{p\cdot k} \left[a_\mu(\vec{k}) + a^\dagger_\mu(\vec{k})\right] \delta(\omega), \tag{6.1.3}$$

$$\lim_{\epsilon\to 0}\lim_{t\to\infty} R_{out}(t) = \frac{i\pi}{2}\sum e \int \widetilde{d^3p}\, \rho(\vec{p}) \int \frac{d^3k}{(2\pi)^3} \frac{p^\mu}{p\cdot k} \left[a_\mu(\vec{k}) + a^\dagger_\mu(\vec{k})\right] \delta(\omega). \tag{6.1.4}$$

Therefore, we can say that the only soft photons constitute the dresses in the asymptotic limit $t \to \pm\infty$. However, the above limit is dangerous or even nonsense in the following sense.

[1] The time component of the asymptotic current (5.3.16) can be derived more precisely as in (E.0.10).

First, the large time limit for the F-K-type S-matrix is given by

$$\lim_{t_f\to\infty,\, t_i\to-\infty} e^{-R_{out}(t_f)}\,\mathrm{T}\exp\left(-i\int_{t_i}^{t_f} dt'\, V^I(t')\right) e^{R_{in}(t_i)}. \tag{6.1.5}$$

Thus we should first compute the finite time conventional S-matrix element and then take the limits $t_f \to \infty$ and $t_i \to -\infty$ with the dressing factors. In addition, the phase operator such as (5.3.26) might be needed to make the S-matrix well-defined because the conventional S-matrix in (3.5.7) suffers from an infinitely oscillating phase factor.

Secondly, t_f and t_i are finite in the real scattering experiments. It may be needed to introduce the infrared scale $1/t$ with $t = t_i, t_f$ in the infrared finite S-matrix. The soft expansion used to derive the soft theorem (3.3.9) is valid for $\omega \lesssim m$ (ω: soft photon energy, m: electron mass.) In this sense, it is expected that the "soft region" in the dressing operator (5.3.25) is $\omega \lesssim m$. Moreover, the energy range of the soft photons that effectively contribute in the dressing factor to be roughly restricted to $\omega \lesssim 1/t$ due to the oscillating factor $\frac{1}{p\cdot k}\exp(i\frac{p\cdot k}{E_p}t)$. For these infrared scales, we usually have the hierarchy $1/t \ll m$ in the real experiment [2].

In order to solve the above issues and construct the proper IR finite S-matrix, we need to further study the physical Hilbert space of asymptotic state at the appropriate large time limit.

6.2 Extension to Gravity

It is important to extend our analysis to other theories. In the perturbative gravity, there are soft graviton theorems up to sub-subleading order [13, 85, 86, 87, 34, 88, 89]. Our analysis about the derivation of asymptotic charge conservation and its relations to soft theorems would be extended to gravity. It may be more interesting to work in the (dynamical) black hole backgrounds [48, 90].

The dressed state formalism for the perturbative gravity was also developed in [91] (see also [79, 50]). However, a tensor $c_{\mu\nu}$, which is an analog of a vector c_μ in [52], was introduced by imposing a free "gauge invariant" condition which is a gravitational counterpart of the free Gupta-Bleuler condition (5.4.1). As in QED, we should impose an appropriate physical condition on the physical Hilbert space, and we expect that the tensor $c_{\mu\nu}$ is unnecessary.

6.3 Other future directions

We would like to comment on other future directions.

Mandelstam developed a manifestly gauge-independent formalism of gauge theories [92, 93]. In the formalism, the dynamical variables of QED are the field strength $F_{\mu\nu}$ and path-dependent charged fields such as

$$\phi(x;\Gamma) \equiv e^{-ie\int_\Gamma^x d\xi^\mu A_\mu(\xi)}\phi(x). \tag{6.3.1}$$

[2]For example, the time for moving $1\mu m$ at the speed of light gives the infrared energy scale $\hbar c/10^{-6} = 2\times 10^{-7}$ MeV , which is much smaller than the electron mass $m_e = 0.5$ MeV.

Such fields attached with Wilson lines are also considered in the context of the bulk reconstruction in the AdS/CFT correspondence (see e.g. [94, 95, 96]). A similarity between Mandelstam's formalism and the dressed state formalism was discussed in [97]. However, the dressing operator constructed in [97] has the additional terms depending on the choice of the path Γ. Thus the dressing operator seems not to be related directly to Faddeev-Kulish's one (5.3.25). Furthermore, we should also investigate the relation to the asymptotic symmetry. As explained in [98] for gravitational theories in AdS, the operators like (6.3.1) are transformed under the asymptotic symmetry, and the behavior of the path Γ near the asymptotic boundary is important in determining the transformation law of the symmetry. In [92, 93], the behavior of Γ near the asymptotic region was not specified. Thus it is interesting to understand more precisely the relations among Mandelstam's formalism, the dressed state formalism and the asymptotic symmetry.

We hope to come back to these issues in the non-asymptotic future.

Appendix A

Asymptotic behavior of the residual gauge parameter

The residual gauge transformations in Lorenz (Feynman) gauge are generated by the gauge parameter $\epsilon(x)$ satisfying $\nabla_\mu \nabla^\mu \epsilon = 0$. In the retarded (u, r, Ω) coordinates, the condition can be written as

$$0 = \left[\partial_r^2 - 2\partial_u \partial_r + \frac{2}{r} \left(-\partial_u + \partial_r \right) + \frac{1}{r^2} \Delta_{S^2} \right] \epsilon \, . \tag{A.0.1}$$

We assume that the gauge parameter is $O(1)$ at large r and we expand it as

$$\epsilon = \epsilon^{(0)}(u, \Omega) + \mathcal{O}(r^{-1}) \, . \tag{A.0.2}$$

Inserting this expansion to (A.0.1), we have $\partial_u \epsilon^{(0)} = 0$. This means that the leading boundary condition at $r \to \infty$ is given by u-independent function, and the residual gauge parameter can expanded as

$$\epsilon = \epsilon^{(0)}(\Omega) + \mathcal{O}(r^{-1}) \, . \tag{A.0.3}$$

The solution can be expressed by the following integral form [6, 55]:

$$\epsilon(x) = \int d^2\Omega \sqrt{\gamma(\Omega)} \, G(x; \Omega) \epsilon^{(0)}(\Omega) \, , \tag{A.0.4}$$

where $G(x; \Omega')$ is the Green function which satisfies

$$\Box G(x; \Omega') = 0 \ , \quad \lim_{\substack{r \to \infty \\ u \text{ fixed}}} G(x; \Omega') = \delta^2(\hat{x} - \hat{q}) \, . \tag{A.0.5}$$

This Green function is given by

$$G(x; \Omega') = -\frac{1}{4\pi} \frac{x^\mu x_\mu}{(q^\mu x_\mu)^2} \quad \text{with} \quad q^\mu = (1, \hat{q}(\Omega')), \tag{A.0.6}$$

where $\hat{q}(\Omega)$ is a three-dimensional unit vector parametrized by Ω^A[1]. The first property in (A.0.5) is easily checked as

$$\begin{aligned}\Box G(x;\Omega') &= -\frac{1}{4\pi}\partial_\nu\left[\frac{2x^\nu}{(q\cdot x)^2} - \frac{2x^\mu x_\mu}{(q\cdot x)^3}q^\nu\right]\\ &= -\frac{1}{4\pi}\left[\frac{8}{(q\cdot x)^2} - \frac{4}{(q\cdot x)^2} - \frac{4}{(q\cdot x)^2} + \frac{6x^\mu x_\mu}{(q\cdot x)^4}q^2\right] = 0\,. \end{aligned} \tag{A.0.7}$$

The second property is also shown as follows. In (u,r,Ω) coordinate, the Green function can be written as

$$\begin{aligned}G(u,r,\Omega;\Omega') &= -\frac{1}{4\pi}\frac{x^\mu x_\mu}{(q^\mu x_\mu)^2} = -\frac{1}{4\pi}\frac{t^2-r^2}{(t-r\hat{q}\cdot\hat{x})^2}\\ &= -\frac{1}{4\pi}\frac{u(u+2r)}{(u+r(1-\hat{x}\cdot\hat{q}))^2}\,. \end{aligned} \tag{A.0.8}$$

We then easily find

$$\lim_{\substack{r\to\infty\\ u\text{ fixed}}} G(x,\hat{q}) = -\lim_{\substack{r\to\infty\\ u\text{ fixed}}}\frac{1}{4\pi}\frac{2ur}{r^2(1-\hat{q}\cdot\hat{x})^2} = 0 \quad \text{for} \quad \hat{q}\cdot\hat{x}\neq 1\,, \tag{A.0.9}$$

and

$$\begin{aligned}\int d^2\Omega\sqrt{\gamma(\Omega)}G(x;\Omega) &= -\frac{1}{4\pi}\int d^2\Omega\sqrt{\gamma(\Omega)}\frac{u(u+2r)}{(u+r(1-\hat{q}\cdot\hat{x}))^2}\\ &= -\frac{u(u+2r)}{2}\int_{-1}^{1} d(\cos\theta)\frac{1}{(u+r(1-\cos\theta))^2}\\ &= \frac{u(u+2r)}{2}\left[\frac{1}{r(u+r(1-x))}\right]\Bigg|_{-1}^{1} = 1\,. \end{aligned} \tag{A.0.10}$$

The Green function thus satisfies the second condition in (A.0.5).

For the Green function (A.0.8) in the limit that $r\to\infty$ with $u=t-r$ fixed, we have

$$\int d^2\Omega'\sqrt{\gamma(\Omega')}G(u,r,\Omega;\Omega')Y_{\ell m}(\Omega') = Y_{\ell m}(\Omega) + \frac{\ell(\ell+1)u\log\frac{|u|}{2r} + s_\ell u}{2r}Y_{\ell m}(\Omega) + \mathcal{O}(r^{-1-\varepsilon})\,, \tag{A.0.11}$$

where $Y_{\ell m}(\Omega)$ are the spherical harmonics, and the coefficients s_ℓ are[2]

$$s_\ell = \frac{1}{2^\ell}\sum_{j=0}^{\lfloor \ell/2\rfloor}\frac{(-1)^j(2\ell-2j)!}{j!(\ell-j)!(\ell-2j)!}c_{\ell-2j} \quad \text{with} \quad c_n = -1+(-1)^n + n\left(4\sum_{k=1}^{n}\frac{1}{k} - 2\sum_{k=1}^{\lfloor n/2\rfloor}\frac{1}{k}\right)\,. \tag{A.0.12}$$

As a result, the large gauge parameter $\epsilon(u,r,\Omega)$ has the following large-r expansion,

$$\epsilon(u,r,\Omega) = \epsilon^{(0)}(\Omega) + \frac{u\log\frac{2r}{|u|}}{2r}\Delta_{S^2}\epsilon^{(0)}(\Omega) + \mathcal{O}(r^{-1}). \tag{A.0.13}$$

[1] More precisely, Green's function is defined as $G(x;\tilde{\Omega}) = -\frac{1}{8\pi}\lim_{\varepsilon\to 0}\left[\frac{x^\mu x_\mu}{(q^\mu x_\mu - i\varepsilon)^2} + \frac{x^\mu x_\mu}{(q^\mu x_\mu + i\varepsilon)^2}\right]$.

[2] $c_0 = 0, c_1 = 2$.

Similarly, in the limit that $r \to \infty$ with $v = t + r$ fixed, it is expanded as

$$\epsilon(v, r, \Omega) = \epsilon^{(0)}(\bar{\Omega}) - \frac{v \log \frac{2r}{|v|}}{2r} \Delta_{\mathrm{S}^2} \epsilon^{(0)}(\bar{\Omega}) + \mathcal{O}(r^{-1}), , \tag{A.0.14}$$

where $\bar{\Omega}$ is the antipodal angle of Ω. Therefore, if we define the coefficients of large-r expansion of $\epsilon(x)$ as

$$\lim_{r\to\infty, u:\text{fixed}} \epsilon(x) = \epsilon^{(0)}(\Omega) + \epsilon^{(log,+)}(u, \Omega) \frac{\log r}{r} + \mathcal{O}(r^{-1}), \tag{A.0.15}$$

$$\lim_{r\to\infty, v:\text{fixed}} \epsilon(x) = \epsilon^{(0)}(\bar{\Omega}) + \epsilon^{(log,-)}(v, \Omega) \frac{\log r}{r} + \mathcal{O}(r^{-1}), \tag{A.0.16}$$

then $\epsilon^{(log,+)}(u = -2U, \Omega) = \epsilon^{(log,-)}(v = 2U, \bar{\Omega})$ holds.

Now we study the behavior of the residual gauge parameter on timelike infinity i^+. In (τ, ρ, Ω^A) coordinates, the Green function can be written as

$$G(\tau, \rho, \Omega; \Omega') = -\frac{1}{4\pi} \frac{1}{\left[\sqrt{1+\rho^2} - \rho \hat{x} \cdot \hat{q}\right]^2} . \tag{A.0.17}$$

Note that $G(\tau, \rho, \Omega; \Omega')$ have turned out to be independent of τ. It means that the residual gauge parameter (A.0.4) is independent of τ. Then the residual gauge parameter at i^+ is given by

$$\epsilon_{\mathbb{H}^3}(\sigma) \equiv \lim_{\tau\to\infty} \epsilon(\tau, \rho, \Omega) = \int d^2\Omega' \sqrt{\gamma(\Omega')}\, G_{\mathbb{H}^3}(\sigma; \Omega') \epsilon^{(0)}(\Omega') \tag{A.0.18}$$

with $G_{\mathbb{H}^3}(\sigma; \Omega') \equiv G(\tau, \rho, \Omega; \Omega')$. If we take $\epsilon^{(0)}(\Omega') = \epsilon^{(0)}$(=constant), it reduces to

$$\epsilon_{\mathbb{H}^3}(\sigma) = \epsilon^{(0)} . \tag{A.0.19}$$

Appendix B

Electromagnetic Fields in null infinity

B.1 Electromagnetic fields by uniformly moving charges

If there is a charge e moving with a constant velocity $\vec{v}$ as

$$\vec{x} = \vec{x}_0 + \vec{v}(t - t_0)\,, \tag{B.1.1}$$

the gauge potential produced by the charge in the Lorenz gauge is

$$A^0(x) = \frac{e}{4\pi\,\ell(x)}\,, \quad \vec{A}(x) = \frac{e\vec{v}}{4\pi\,\ell(x)}\,, \tag{B.1.2}$$

where

$$\ell(x) = \sqrt{(1-|\vec{v}|^2)(|\vec{x}-\vec{x}_0|^2 - [\hat{v}\cdot(\vec{x}-\vec{x}_0)]^2) + (\hat{v}\cdot(\vec{x}-\vec{x}_0) - |\vec{v}|(t-t_0))^2} \tag{B.1.3}$$

with

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}. \tag{B.1.4}$$

At the point $t = T - U, r = T + U$ with large T, the electric flux F_{tr} is expanded as

$$F_{tr}(t,r,\Omega)|_{t=T-U,r=T+U} \tag{B.1.5}$$

$$= -\frac{e(1-|\vec{v}|^2)}{4\pi(1-\vec{v}\cdot\hat{x}(\Omega))^2T^2} + \frac{e(1-|\vec{v}|^2)f(U,\Omega;\vec{v},t_0,\vec{x}_0)}{4\pi(1-\vec{v}\cdot\hat{x}(\Omega))^4T^3} + \mathcal{O}(T^{-4}) \tag{B.1.6}$$

with

$$\begin{aligned} f(U,\Omega;\vec{v},t_0,\vec{x}_0) \equiv\, & 2U(1-|\vec{v}|^2 - 2|\vec{v}_\perp|^2) + [1-\vec{v}\cdot\hat{x}(\Omega) - 3(1-|\vec{v}|^2)]\vec{x}_0\cdot\hat{x}(\Omega) \\ & + 3[1-\vec{v}\cdot\hat{x}(\Omega)]\vec{v}\cdot\vec{x}_0 + [2(1-|\vec{v}|^2) - 2(1-\vec{v}\cdot\hat{x}(\Omega)) - |\vec{v}_\perp|^2]t_0\,, \end{aligned} \tag{B.1.7}$$

where $\vec{v}_\perp \equiv \vec{v} - [\vec{v}\cdot\hat{x}(\Omega)]\hat{x}(\Omega)$. At the point $t = -T+U, r = T+U$ with large T, the electric flux F_{tr} is

$$F_{tr}(t,r,\Omega)|_{t=-T+U,r=T+U} \tag{B.1.8}$$

$$= -\frac{e(1-|\vec{v}|^2)}{4\pi(1+\vec{v}\cdot\hat{x}(\Omega))^2T^2} + \frac{e(1-|\vec{v}|^2)f(U,\Omega;-\vec{v},-t_0,\vec{x}_0)}{4\pi(1+\vec{v}\cdot\hat{x}(\Omega))^4T^3} + \mathcal{O}(T^{-4}). \tag{B.1.9}$$

Thus the *antipodal matching condition* eq. (2.5.8) at the leading order

$$F_{tr}^{+(2)}(\Omega) = F_{tr}^{-(2)}(\bar{\Omega}) \tag{B.1.10}$$

holds where $\bar{\Omega}$ denotes the antipodal point of Ω. Thus if we have initially charges e_n moving as

$$\vec{x}^{(n)} = \vec{x}_0^{(n)} + \vec{v}_n(t-t_0^{(n)})\,, \tag{B.1.11}$$

Q_0 given by eq. (2.5.5) is computed as

$$Q_0 = \sum_n \frac{e_n(1-|\vec{v}_n|^2)}{2\pi T}\int d^2\Omega\sqrt{\gamma(\Omega)}\frac{\epsilon^{(0)}(\Omega)}{(1-\vec{v}_n\cdot\hat{x}(\Omega))^4}\Big\{[1-\vec{v}_n\cdot\hat{x}(\Omega)-3(1-|\vec{v}_n|^2)]\vec{x}_0^{(n)}\cdot\hat{x}(\Omega)$$
$$+3[1-\vec{v}_n\cdot\hat{x}(\Omega)]\vec{v}_n\cdot\vec{x}_0^{(n)} + [2(1-|\vec{v}_n|^2)-2(1-\vec{v}_n\cdot\hat{x}(\Omega))-|\vec{v}_{n\perp}|^2]t_0^{(n)}\Big\} + \mathcal{O}(T^{-2})\,. \tag{B.1.12}$$

Therefore, we have confirmed that the $\log T/T$ term doesn't appear in Q_0 at least in this setup.

B.2 Computation of memories

Here, we check the memory effect formulae (2.5.16) and (2.5.30) for a concrete example. We consider the following trajectory of a charged particle with charge e such as it first rests at $\vec{x}_0$ and moves with a constant velocity $\vec{v}$ after a time t_0:

$$\vec{x} = \vec{x}_0 + \Theta(t-t_0)\vec{v}(t-t_0). \tag{B.2.1}$$

We represent the matter current for this trajectory by j^μ_{mat}, which is the source in Maxwell's equation $\partial_\nu F^{\nu\mu} = -j^\mu_{mat}$. The retarded electromagnetic field created by this particle is written in the Lorenz gauge $\partial_\mu A^\mu = 0$ as

$$A^0(x) = \Theta(|\vec{x}-\vec{x}_0|-t+t_0)\frac{e}{4\pi|\vec{x}-\vec{x}_0|} + \Theta(-|\vec{x}-\vec{x}_0|+t-t_0)\frac{e}{4\pi\,\ell(x)}\,, \tag{B.2.2}$$

$$\vec{A}(x) = \Theta(-|\vec{x}-\vec{x}_0|+t-t_0)\frac{e\vec{v}}{4\pi\,\ell(x)}\,, \tag{B.2.3}$$

where $\ell(x)$ is given by eq. (B.1.3).

We first consider the charge Q_f. It is given by

$$Q_f = \int d^2\Omega\sqrt{\gamma}\left(r^2F^{tr}\epsilon\right)|_{t=T+U,r=T-U}. \tag{B.2.4}$$

At $t=T+U, r=T-U$ with large T, electric field F^{tr} is expanded as

$$F^{tr}|_{t=T+U,r=T-U} = \frac{e(1-|\vec{v}|^2)}{4\pi(1-\vec{v}\cdot\hat{x})^2T^2} + \mathcal{O}(T^{-3})\,. \tag{B.2.5}$$

Since our gauge parameter has the expansion as eq. (2.5.25), Q_f is expanded as

$$Q_f = \frac{e(1-|\vec{v}|^2)}{4\pi}\int d^2\Omega\sqrt{\gamma}\frac{\epsilon^{(0)}}{(1-\vec{v}\cdot\hat{x})^2} + \frac{U\log T}{T}\frac{e(1-|\vec{v}|^2)}{4\pi}\int d^2\Omega\sqrt{\gamma}\frac{\Delta_{\mathrm{S}^2}\epsilon^{(0)}}{(1-\vec{v}\cdot\hat{x})^2} + \mathcal{O}(T^{-1})\,. \tag{B.2.6}$$

Thus we have

$$\lim_{T\to\infty} Q_f[\epsilon^{(0)}] = \frac{e(1-|\vec{v}|^2)}{4\pi}\int d^2\Omega\sqrt{\gamma}\frac{\epsilon^{(0)}}{(1-\vec{v}\cdot\hat{x})^2}\,, \tag{B.2.7}$$

$$Q_f^{\log}[\epsilon^{(0)}] = -\frac{Ue(1-|\vec{v}|^2)}{4\pi}\int d^2\Omega\sqrt{\gamma}\frac{\Delta_{\mathrm{S}^2}\epsilon^{(0)}}{(1-\vec{v}\cdot\hat{x})^2} = -U\lim_{T\to\infty} Q_f[\Delta_{\mathrm{S}^2}\epsilon^{(0)}]\,. \tag{B.2.8}$$

Note that $Q_f^{\log}$ diverges in the limit $U\to\infty$ although $Q_f^{\log} + Q_+^{\log\prime}$ is finite as we will see later. Similarly, the charge Q_i is computed as

$$\lim_{T\to\infty} Q_i[\epsilon^{(0)}] = \frac{e}{4\pi}\int d^2\Omega\sqrt{\gamma}\epsilon^{(0)}\,, \tag{B.2.9}$$

$$Q_i^{\log}[\epsilon^{(0)}] = -\frac{Ue}{4\pi}\int d^2\Omega\sqrt{\gamma}\Delta_{\mathrm{S}^2}\epsilon^{(0)} = 0\,. \tag{B.2.10}$$

Next, we compute the future null infinity charge $\lim_{T\to\infty} Q_+$ given by (2.5.15). Since the angular components of the gauge field is expanded as

$$A_B(x) = \Theta(u+\vec{x}_0\cdot\hat{x}-t_0)\frac{e\vec{v}\cdot\partial_B\hat{x}}{4\pi(1-\vec{v}\cdot\hat{x})} + \mathcal{O}(T^{-1})\,, \tag{B.2.11}$$

the charge is given by

$$\lim_{T\to\infty} Q_+ = \int d^2\Omega\sqrt{\gamma}\epsilon^{(0)}\gamma^{AB}\nabla_A\left[\frac{e\vec{v}\cdot\partial_B\hat{x}}{4\pi(1-\vec{v}\cdot\hat{x})}\right]\,. \tag{B.2.12}$$

With the use of the formula

$$\gamma^{AB}\nabla_A\left[\frac{\vec{v}\cdot\partial_B\hat{x}}{1-\vec{v}\cdot\hat{x}}\right] = \frac{-2\vec{v}\cdot\hat{x}}{1-\vec{v}\cdot\hat{x}} + \frac{|\vec{v}|^2-(\vec{v}\cdot\hat{x})^2}{(1-\vec{v}\cdot\hat{x})^2} = 1 - \frac{1-|\vec{v}|^2}{(1-\vec{v}\cdot\hat{x})^2}\,, \tag{B.2.13}$$

the charge has the form

$$\lim_{T\to\infty} Q_+ = \frac{e}{4\pi}\int d^2\Omega\sqrt{\gamma}\epsilon^{(0)} - \frac{e(1-|\vec{v}|^2)}{4\pi}\int d^2\Omega\sqrt{\gamma}\frac{\epsilon^{(0)}}{(1-\vec{v}\cdot\hat{x})^2} = -\lim_{T\to\infty}(Q_f - Q_i)\,. \tag{B.2.14}$$

This certainly agrees with the leading memory effect (2.5.16).

Finally, we compute the subleading charges $Q_+^{\log}$ and $Q_+^{\log\prime}$. Since we now have

$$\partial_u A_B^{(0)} = \delta(u+\vec{x}_0\cdot\hat{x}-t_0)\frac{e\vec{v}\cdot\partial_B\hat{x}}{4\pi(1-\vec{v}\cdot\hat{x})}\,, \tag{B.2.15}$$

the charge $Q_+^{\log}$ given by eq. (2.5.27) is computed as

$$\begin{aligned}Q_+^{\log} &= -\frac{1}{2}\int d^2\Omega\sqrt{\gamma}\gamma^{AB}(\vec{x}_0\cdot\hat{x}-t_0)\frac{e\vec{v}\cdot\partial_B\hat{x}}{4\pi(1-\vec{v}\cdot\hat{x})}\nabla_A\Delta_{S^2}\epsilon^{(0)} \\ &= \frac{e}{8\pi}\int d^2\Omega\sqrt{\gamma}\left[(\vec{x}_0\cdot\hat{x}-t_0)\left(1-\frac{1-|\vec{v}|^2}{(1-\vec{v}\cdot\hat{x})^2}\right)+\frac{\vec{x}_0\cdot\vec{v}-(\vec{v}\cdot\hat{x})(\vec{x}_0\cdot\hat{x})}{1-\vec{v}\cdot\hat{x}}\right]\Delta_{S^2}\epsilon^{(0)}\,.\end{aligned} \tag{B.2.16}$$

Note that this does not depend on U. As shown in [59], this charge is related to the soft factor in the subleading soft photon theorem. The momentum of the charged particle is initially $p^\mu = m(1,0)$ and finally $p'^\mu = \omega(1,\vec{v})$ with $\omega = m/\sqrt{1-|\vec{v}|^2}$. The angular momentum is initially $J^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu$ and finally $J'^{\mu\nu} = x_0^\mu p'^\nu - x_0^\nu p'^\mu$. They read $p^u = m = -q\cdot p$, $p'^u = \omega(1-\vec{v}\cdot\hat{x}) = -q\cdot p'$, $p'_B = \omega\vec{v}\cdot\partial_B\vec{x}$, $J^u{}_B = -p^u\vec{x}_0\cdot\partial_B\vec{x}$ and $J'^u{}_B = -(\vec{x}_0\cdot\hat{x}-t_0)p'_B - p'^u\vec{x}_0\cdot\partial_B\vec{x}$ where $q^\mu = (1,\hat{x})$. Using them, we have

$$(\vec{x}_0\cdot\hat{x}-t_0)\frac{\vec{v}\cdot\partial_B\hat{x}}{1-\vec{v}\cdot\hat{x}} = \frac{1}{r}\left[\frac{J'^u{}_B}{q\cdot p'}-\frac{J^u{}_B}{q\cdot p}\right]. \tag{B.2.17}$$

Thus $Q_+^{\log}$ can also be written as

$$Q_+^{\log} = -\frac{e}{8\pi}\int d^2\Omega\sqrt{\gamma}\lim_{r\to\infty}\left(\frac{rJ'^{uA}}{q\cdot p'}-\frac{rJ^{uA}}{q\cdot p}\right)\nabla_A\Delta_{S^2}\epsilon^{(0)}\,. \tag{B.2.18}$$

The charge $Q_+^{\log\prime}$ given by (2.5.28) is computed as follows. The radial component A_r in (u,r,Ω) coordinates is

$$A_r = -\frac{e}{4\pi r}+\mathcal{O}(r^{-2}), \tag{B.2.19}$$

and we thus have $A_r^{(1)} = -e/(4\pi)$, which does not contribute to the charge $Q_+^{\log\prime}$ because

$$\int d^2\Omega\sqrt{\gamma}A_r^{(1)}\Delta_{S^2}\epsilon^{(0)} = -\frac{e}{4\pi}\int d^2\Omega\sqrt{\gamma}\Delta_{S^2}\epsilon^{(0)} = 0\,. \tag{B.2.20}$$

We also have $C_u^{(1)} = 0$, $C_A^{(1)} = 0$ and

$$\begin{aligned}\nabla^B A_B^{(0)} &= \frac{e}{4\pi}\Theta(u+\vec{x}_0\cdot\hat{x}-t_0)\left[1-\frac{1-|\vec{v}|^2}{(1-\vec{v}\cdot\hat{x})^2}\right] \\ &\quad+\frac{e}{4\pi}\delta(u+\vec{x}_0\cdot\hat{x}-t_0)\left[\frac{\vec{x}_0\cdot\vec{v}-(\vec{v}\cdot\hat{x})(\vec{x}_0\cdot\hat{x})}{1-\vec{v}\cdot\hat{x}}\right]\,.\end{aligned} \tag{B.2.21}$$

Therefore,

$$\begin{aligned}Q_+^{\log\prime} &= -\frac{e}{8\pi}\int d^2\Omega\sqrt{\gamma}(2U+\vec{x}_0\cdot\hat{x}-t_0)\left[1-\frac{1-|\vec{v}|^2}{(1-\vec{v}\cdot\hat{x})^2}\right]\Delta_{S^2}\epsilon^{(0)} \\ &\quad-\frac{e}{8\pi}\int d^2\Omega\sqrt{\gamma}\left[\frac{\vec{x}_0\cdot\vec{v}-(\vec{v}\cdot\hat{x})(\vec{x}_0\cdot\hat{x})}{1-\vec{v}\cdot\hat{x}}\right]\Delta_{S^2}\epsilon^{(0)}\,,\end{aligned} \tag{B.2.22}$$

and for any $\epsilon^{(0)}(\Omega)$, we have

$$
\begin{aligned}
Q_+^{\log\prime} + Q_f^{\log} &= -\frac{e}{8\pi}\int d^2\Omega\sqrt{\gamma}(\vec{x}_0\cdot\hat{x}-t_0)\left[1-\frac{1-|\vec{v}|^2}{(1-\vec{v}\cdot\hat{x})^2}\right]\Delta_{S^2}\epsilon^{(0)} \\
&\quad -\frac{e}{8\pi}\int d^2\Omega\sqrt{\gamma}\left[\frac{\vec{x}_0\cdot\vec{v}-(\vec{v}\cdot\hat{x})(\vec{x}_0\cdot\hat{x})}{1-\vec{v}\cdot\hat{x}}\right]\Delta_{S^2}\epsilon^{(0)} && \text{(B.2.23)} \\
&= -Q_+^{\log}\,. && \text{(B.2.24)}
\end{aligned}
$$

This is the subleading memory effect.

Appendix C

Asymptotic expansion of radiation fields

We here investigate the large-r expansion of radiation fields in the Lorenz gauge $\partial_\mu A^\mu = 0$. We suppose that gauge fields are generally expanded as follows:[1]

$$A_u = \frac{\log\frac{|u|}{2r}}{r} C_u^{(1)}(u,\Omega) + \frac{1}{r} A_u^{(1)}(u,\Omega) + \frac{\log\frac{|u|}{2r}}{r^2} C_u^{(2)}(u,\Omega) + \frac{1}{r^2} A_u^{(2)}(u,\Omega) + \cdots , \quad \text{(C.0.1)}$$

$$A_r = \frac{1}{r} A_r^{(1)}(u,\Omega) + \frac{\log\frac{|u|}{2r}}{r^2} C_r^{(2)}(u,\Omega) + \frac{1}{r^2} A_r^{(2)}(u,\Omega) + \cdots , \quad \text{(C.0.2)}$$

$$A_B = A_B^{(0)}(u,\Omega) + \frac{\log\frac{|u|}{2r}}{r} C_B^{(1)}(u,\Omega) + \frac{1}{r} A_B^{(1)}(u,\Omega) + \cdots . \quad \text{(C.0.3)}$$

Inserting them into the Lorenz gauge condition

$$-\partial_u A_r + \partial_r(-A_u + A_r) + \frac{2}{r}(-A_u + A_r) + \frac{1}{r^2}\nabla^B A_B = 0, \quad \text{(C.0.4)}$$

we find

$$\partial_u A_r^{(1)} = 0 , \quad C_u^{(1)} + \partial_u C_r^{(2)} = 0 , \quad -\partial_u A_r^{(2)} - \frac{1}{u} C_r^{(2)} + C_u^{(1)} - A_u^{(1)} + A_r^{(1)} + \nabla^B A_B^{(0)} = 0 , \quad \text{(C.0.5)}$$

where ∇_B is the covariant derivative associated with the two-sphere metric γ_{AB}, and $\nabla^B = \gamma^{BA}\nabla_A$.

Eq. (2.5.28) is obtained by expanding F^{rB} and F^{ru} in (2.5.14) with (C.0.1), (C.0.2)

[1] The expansion is more general than that in [55], because we allow $\log r$ terms like the gauge parameter $\epsilon(x)$ [see (2.5.25)]. In particular, $\log r$ terms with coefficients $C_u^{(1)}$, $C_r^{(2)}$ and $C_B^{(1)}$ are included because these terms are generated by the large gauge transformation with the gauge parameter (2.5.25).

and (C.0.3). With the use of the above expansion, F^{rB} and F^{ru} are computed as

$$\begin{aligned} F^{rB} &= -\partial_u A^B + \partial_r A^B + \frac{\gamma^{BC}}{r^2}\partial_C(A_u - A_r) \\ &= -\frac{1}{r^2}\gamma^{BC}\partial_u A_C^{(0)} - \frac{\log\frac{|u|}{2r}}{r^3}\gamma^{BA}\left(\partial_u C_A^{(1)} - \partial_A C_u^{(1)}\right) + \mathcal{O}(r^{-3})\,, \qquad \text{(C.0.6)} \\ F^{ru} &= \partial_u A_r - \partial_r A_u = \frac{1}{r^2}\left(A_r^{(1)} + \nabla^B A_B^{(0)} + 2C_u^{(1)}\right) + \mathcal{O}(r^{-2-\varepsilon})\,, \qquad \text{(C.0.7)} \end{aligned}$$

and the expansions lead to (2.5.28).

The free equations of motion $\Box A_\mu = 0$ in the Lorenz gauge can be written in the retarded coordinates as

$$\left[\partial_r^2 - 2\partial_u\partial_r + \frac{2}{r}(-\partial_u + \partial_r) + \frac{1}{r^2}\Delta_{S^2}\right]A_u = 0\,, \qquad \text{(C.0.8)}$$

$$\left[\partial_r^2 - 2\partial_u\partial_r + \frac{2}{r}(-\partial_u + \partial_r) + \frac{1}{r^2}\Delta_{S^2}\right]A_r - \frac{2}{r^2}(-A_u + A_r) - \frac{2}{r^3}\nabla^B A_B = 0\,, \qquad \text{(C.0.9)}$$

$$\left[\partial_r^2 - 2\partial_u\partial_r\right]A_B + \frac{1}{r^2}\Delta_{S^2}A_B + \frac{2}{r}\partial_B(-A_u + A_r) = 0\,. \qquad \text{(C.0.10)}$$

Inserting the expansions (C.0.1), (C.0.2) and (C.0.3), we obtain

$$\partial_u C_u^{(1)} = 0, \quad \partial_u C_u^{(2)} = -\frac{1}{2}\Delta_{S^2}C_u^{(1)}, \quad \partial_u A_u^{(2)} + \frac{1}{u}C_u^{(2)} = -\frac{1}{2}C_u^{(1)} + \frac{1}{2}\Delta_{S^2}\left(C_u^{(1)} - A_u^{(1)}\right)\,, \qquad \text{(C.0.11)}$$

$$\partial_u C_r^{(2)} = -C_u^{(1)}\,, \quad \partial_u A_r^{(2)} + \frac{1}{u}C_r^{(2)} = C_u^{(1)} - A_u^{(1)} + A_r^{(1)} - \frac{1}{2}\Delta_{S^2}A_r^{(1)} + \nabla^B A_B^{(0)}\,, \qquad \text{(C.0.12)}$$

$$\partial_u C_B^{(1)} = \partial_B C_u^{(1)}\,, \quad \partial_u A_B^{(1)} + \frac{1}{u}C_B^{(1)} = -\partial_B(C_u^{(1)} - A_u^{(1)} + A_r^{(1)}) - \frac{1}{2}\Delta_{S^2}A_B^{(0)}\,. \qquad \text{(C.0.13)}$$

Using the condition (C.0.5), we find that $A_r^{(1)}$ is a constant.

Appendix D

Gauss law constraint in canonical quantization of QED

In this appendix, we briefly review how the Gauss law constraint appears in the context of the canonical quantization of QED. The following argument holds without specifying the concrete metric.

We consider the following QED Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu + \mathcal{L}_{mat} \ , \tag{D.0.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{D.0.2}$$

and $\mathcal{L}_{mat}$ is the kinetic term of the matter sector and j^μ is the current of global $U(1)$ charge. Here we try to quantize the gauge fields in the covariant manner. The conjugates are given by

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} \tag{D.0.3}$$

Here we find a primary constraint,

$$\Pi^0 = 0 \ . \tag{D.0.4}$$

The Poisson brackets are given by

$$[A_\mu(x), \Pi^\nu(y)]_P = i\delta^\nu_\mu \delta^3(\vec{x} - \vec{y}) \ , \tag{D.0.5}$$

$$[A_\mu(x), A_\nu(y)]_P = [\Pi^\mu(x), \Pi^\nu(y)]_P = 0 \tag{D.0.6}$$

where $x^0 = y^0$. We define a Hamiltonian H on a time slice Σ as

$$\begin{aligned} H &= \int d^3\Sigma \left(\Pi^k \dot{A}_k - \mathcal{L}\right) \\ &= \int d^3\Sigma \left(\Pi^k (F_{0k} + \nabla_k A_0) + \frac{1}{2} F_{0k} F^{0k} + \frac{1}{4} F_{ij} F^{ij} - A_\mu j^\mu\right) \\ &= \int d^3\Sigma \left(\frac{1}{2}\Pi^k \Pi_k - \nabla_k \Pi^k A_0 + \frac{1}{4} F_{ij} F^{ij} - A_\mu j^\mu\right) \ , \end{aligned} \tag{D.0.7}$$

where we have performed a partial integration and dropped out the boundary term $\int dS_k \Pi^k A_0$ in the final line. To impose the primary constraint (D.0.4), we define a new Hamiltonian $\widetilde{H}$ by adding a new degree of freedom $\lambda(x)$ as a Lagrangian multiplier:

$$\widetilde{H} \equiv H + \int d^3\Sigma \ \lambda(x) \Pi^0(x) \ . \tag{D.0.8}$$

Because the constraint (D.0.4) must hold during the time evolution, we have a following secondary constraint as a consistency condition,

$$0 = i\dot{\Pi}^0 = [\Pi^0, \widetilde{H}]_P = \nabla_k \Pi^k + j^0 \ . \tag{D.0.9}$$

Thus we find (the local) *Gauss law*:

$$-\nabla_k F^{0k} + j^0 = 0 \ , \tag{D.0.10}$$

as a secondary constraint. The consistency condition for the Gauss law constraint is given by

$$0 = [-\nabla_k F^{0k} + j^0, \widetilde{H}]_P = \nabla_i \nabla_k F^{ik} \ , \tag{D.0.11}$$

which is trivially satisfied by the antisymmetry of the indices of F^{ik}. Therefore, further constraints do not come up. We have the Poisson bracket relation for the constraints,

$$[-\nabla_k F^{0k} + j^0, \Pi^0]_P = 0 \ . \tag{D.0.12}$$

This means that the both constraints, (D.0.4) and (D.0.10), are the *first class constraints*, which generate the flows along the gauge orbits. When we quantize the theory, we need to impose the first class constraints on any physical state $|\psi\rangle$ as

$$\Pi^0 |\psi\rangle = 0 \ , \tag{D.0.13}$$

$$\left(-\nabla_k F^{0k} + j^0\right) |\psi\rangle = 0 \ , \tag{D.0.14}$$

to make the physical states gauge invariant. The constraint (D.0.14) is called the *Gauss law constraint.*

Appendix E

Asymptotic behaviors of the massive particles

In this appendix, we show the concrete expressions of the matter current of a massive scalar in the asymptotic regions.

A free massive complex scalar $\phi(x)$ can be expressed as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2E_p} \left(b(\vec{p}) e^{ipx} + d^\dagger(\vec{p}) e^{-ipx} \right), \tag{E.0.1}$$

where $b(\vec{p})$ and $d(\vec{p})$ are the annihilation operators for particles and antiparticles, respectively. The nonzero commutation relations of the creation and annihilation operators are given by

$$[b(\vec{p}), b^\dagger(\vec{p}')] = [d(\vec{p}), d^\dagger(\vec{p}')] = (2\pi)^3 (2E_p) \delta^{(3)}(\vec{p} - \vec{p}'). \tag{E.0.2}$$

All massive particles go to the future timelike infinity i^+, not the null infinity in the asymptotic future time. When we work around the timelike infinity, it is convenient to use the coordinates (2.3.2):

$$\tau^2 = t^2 - r^2 \ , \ \rho = \frac{r}{\sqrt{t^2 - r^2}}. \tag{E.0.3}$$

The Minkowski line element then takes the form

$$ds^2 = -d\tau^2 + \tau^2 \, h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta, \tag{E.0.4}$$

where $\sigma^\alpha = (\rho, \Omega^A)$ are coordinates of the unit three-dimensional hyperbolic space $\mathbb{H}^3$ with the line element

$$h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = \frac{d\rho^2}{1+\rho^2} + \rho^2 \gamma_{AB} d\Omega^A d\Omega^B. \tag{E.0.5}$$

In the large τ limit ($\tau \to +\infty$), using the saddle point approximation [6], we can obtain the asymptotic form of the scalar field as

$$\phi(\tau, \rho, \Omega) = \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} \left(b(\vec{p}) e^{-im\tau - 3\pi i/4} + d^\dagger(\vec{p}) e^{im\tau + 3\pi i/4} \right)|_{\vec{p} = m\rho\hat{x}(\Omega)} + \mathcal{O}(\tau^{-\frac{3}{2} - \varepsilon}). \tag{E.0.6}$$

Therefore, in the asymptotic region, $\phi(\tau,\rho,\Omega)$ only creates (or annihilates) the (anti-)particle with localized momentum,

$$\vec{p} = m\rho\hat{x}(\Omega)\ ,\ E_p = m\sqrt{1+\rho^2} \tag{E.0.7}$$

at the leading order.

Then if we ignore the interaction near the timelike infinity, the global U(1) current of the massive charged scalar with the normal ordering is given by

$$\begin{aligned} j^{mat}_\mu(\tau,\rho,\Omega) &= ie : \big(\partial_\mu\phi^\dagger(x)\phi(x) - \phi^\dagger(x)\partial_\mu\phi(x)\big) : && \text{(E.0.8)}\\ &= \frac{j^{(3)}_\mu(\tau,\rho,\Omega)}{\tau^3} + \mathcal{O}(\tau^{-3-\varepsilon}), && \text{(E.0.9)} \end{aligned}$$

where

$$j^{(3)}_\tau(\tau,\rho,\Omega) = j^{(3)}_\tau(\sigma) = -\frac{em^2}{2(2\pi)^3}\big(b^\dagger b - d^\dagger d\big), \tag{E.0.10}$$

$$j^{(3)}_\rho(\tau,\rho,\Omega) = \frac{iem}{4(2\pi)^3}\big[\big(\partial_\rho b^\dagger\, b - b^\dagger\partial_\rho b\big) + i\big(b^\dagger\partial_\rho d^\dagger e^{2im\tau} - \partial_\rho b\, de^{-2im\tau}\big) - (b\leftrightarrow d)\big], \tag{E.0.11}$$

$$j^{(3)}_A(\tau,\rho,\Omega) = \frac{iem}{4(2\pi)^3}\big[\big(\partial_A b^\dagger\, b - b^\dagger\partial_A b\big) + i\big(b^\dagger\partial_A d^\dagger e^{2im\tau} - \partial_A b\, de^{-2im\tau}\big) - (b\leftrightarrow d)\big]. \tag{E.0.12}$$

Here, we have represented $b = b(m\rho\hat{x}(\Omega))$, $d = d(m\rho\hat{x}(\Omega))$ for brevity. Then one can extract the diagonal parts from the $j^{(3)}_\rho$ and $j^{(3)}_A$ by multiplying the projection operator $\frac{1}{4m^2}(\partial^2_\tau + 4m^2)$,

$$\partial_\rho b^\dagger b - b^\dagger\partial_\rho b - \partial_\rho d^\dagger d + d^\dagger\partial_\rho d = \frac{-i(2\pi)^3}{em^3}(\partial^2_\tau + 4m^2)j^{(3)}_\rho, \tag{E.0.13}$$

$$\partial_A b^\dagger b - b^\dagger\partial_A b - \partial_A d^\dagger d + d^\dagger\partial_A d = \frac{-i(2\pi)^3}{em^3}(\partial^2_\tau + 4m^2)j^{(3)}_A. \tag{E.0.14}$$

Since $\frac{1}{4m^2}(\partial^2_\tau + 4m^2)j^{(3)}_\rho$ and $\frac{1}{4m^2}(\partial^2_\tau + 4m^2)j^{(3)}_A$ are independent of τ, we represent them by $I^{mat}_\alpha(\sigma)$ as

$$\begin{aligned} I^{mat}_\alpha(\sigma) &\equiv \lim_{\tau\to\infty}\left[\frac{1}{4m^2}\partial^2_\tau + 1\right]\tau^3 j^{mat}_\alpha(\tau,\sigma) && \text{(E.0.15)}\\ &= \frac{iem}{4(2\pi)^3}\left[\partial_\alpha b^\dagger b - b^\dagger\partial_\alpha b - \partial_\alpha d^\dagger d + d^\dagger\partial_\alpha d\right]\ . && \text{(E.0.16)} \end{aligned}$$

Appendix F

Integration of J_{nm} and non-negativity of $A_{\beta\alpha}$

In this appendix, we perform the integration of J_{nm} in (3.5.2) and also show the positivity of $A_{\beta\alpha}$ in (3.5.8).] The proof of the positivity of $A_{\beta\alpha}$ is based on [81].

The I_{nm} is defined as

$$J_{nm} \equiv \int_{\lambda \le |\vec{k}| \le \Lambda} \frac{d^4k}{(2\pi)^4} \frac{-ip_n \cdot p_m}{(k^2 - i\epsilon)\,(p_n \cdot k - i\eta_n \epsilon)\,(-p_m \cdot k - i\eta_m \epsilon)} . \tag{F.0.1}$$

We first perform the integration of k^0 in this integral. The integrant has poles at

$$k^0 = |\vec{k}| - i\epsilon \ , \qquad k^0 = -|\vec{k}| + i\epsilon \ , \tag{F.0.2}$$

$$k^0 = \vec{v}_n \cdot \vec{k} - i\eta_n \epsilon \ , \ k^0 = \vec{v}_m \cdot \vec{k} + i\eta_m \epsilon \ , \tag{F.0.3}$$

where $\vec{v}_n \equiv \vec{p}_n / p_n^0$. We can calculate the integral over k^0 by picking up the contributions from a subset of the above four poles as follows. In the case of $\eta_n = -\eta_m = \pm 1$, we can close the contour in either upper half or lower half k^0-plane to avoid the contributions from poles at $k^0 = \vec{v}_n \cdot \vec{k} - i\eta_n \epsilon$, $\vec{v}_m \cdot \vec{k} + i\eta_m \epsilon$ and then pick up a single residue at either $k^0 = |\vec{k}| - i\epsilon$ or $-|\vec{k}| + i\epsilon$. In this case, we have

$$\begin{aligned} J_{nm} &= \int_{\lambda \le |\vec{k}| \le \Lambda} \frac{d^3k}{(2\pi)^3} \frac{p_n \cdot p_m}{(2|\vec{k}| - i\epsilon)(p_n \cdot k - i\eta_n \epsilon)\,(-p_m \cdot k - i\eta_m \epsilon)} \Bigg|_{k^0 = |\vec{k}| - i\epsilon} \\ &= \frac{1}{2} \int_{\lambda \le |\vec{k}| \le \Lambda} \frac{d^3k}{(2\pi)^3} \frac{p_n \cdot p_m}{|\vec{k}|^3 (p_n \cdot \hat{k})(p_m \cdot \hat{k})} = \ln\left(\frac{\lambda}{\Lambda}\right) a_{nm} \ . \end{aligned} \tag{F.0.4}$$

In the final line, we have defined

$$a_{nm} \equiv \frac{1}{16\pi^3} \int d^2\hat{k} \frac{p_n \cdot p_m}{(p_n \cdot \hat{k})(p_m \cdot \hat{k})} \tag{F.0.5}$$

where $\hat{k}^\mu \equiv k^\mu / |\vec{k}|$ and $d^2\hat{k}$ is the integral measure on unit 2-dimensional sphere.

In the case of $\eta_n = \eta_m = \pm 1$, both the upper half and the lower half planes have two poles, so we close the contour in either the upper half or the lower half plane and pick up those two residues. In this case, we have

$$\begin{aligned} J_{nm} = \int_{\lambda\le|\vec{k}|\le\Lambda} \frac{d^3k}{(2\pi)^3} \Bigg[& \frac{p_n\cdot p_m}{(2|\vec{k}|-i\epsilon)(p_n\cdot k - i\eta_n\epsilon)\,(-p_m\cdot k - i\eta_m\epsilon)}\Bigg|_{k^0=|\vec{k}|-i\epsilon} \\ & + \frac{p_n\cdot p_m}{(k^2-i\epsilon)\,(-p_n^0)\,(-p_m\cdot k - i\eta_m\epsilon)}\Bigg|_{k^0=\vec{v}_n\cdot\vec{k}-i\eta_n\epsilon}\Bigg] \\ = \ln\left(\frac{\lambda}{\Lambda}\right)\eta_n\eta_m e_n e_m (a_{nm}+ib_{nm}) \end{aligned} \tag{F.0.6}$$

where we have defined

$$\begin{aligned} b_{nm} &\equiv \frac{-i}{16\pi^3}\int d^2\hat{k}|\vec{k}|^2 \frac{p_n\cdot p_m}{(k^2-i\epsilon)\,p_n^0\,(-p_m\cdot k - i\eta_m\epsilon)}\Bigg|_{k^0=\vec{v}_n\cdot\vec{k}-i\eta_n\epsilon} \\ &= \frac{i}{16\pi^3}\frac{p_n\cdot p_m}{p_n^0 p_m^0}\int d^2\hat{k}\frac{1}{\left(1-(\vec{v}_n\cdot\hat{\vec{k}})^2\right)\left(\vec{v}_n\cdot\hat{\vec{k}}-\vec{v}_m\cdot\hat{\vec{k}}-i\eta_m\epsilon\right)}\ . \end{aligned} \tag{F.0.7}$$

We have left the $i\epsilon$ in the last term to make the integral well-defined in the case of $m = n$. This integral gives a real-valued result because the complex conjugate of b_{nm} with the change of the variable $\hat{\vec{k}} \to -\hat{\vec{k}}$ goes back to itself. Combining the results (F.0.4) and (F.0.6), the integration (F.0.1) is given by

$$J_{nm} = \ln\left(\frac{\lambda}{\Lambda}\right)(a_{nm} + i\delta_{\eta_n\eta_m} b_{nm}) \tag{F.0.8}$$

$$\begin{aligned} A_{\beta\alpha} &= \sum_{n,m\in\{\alpha,\beta\}} \eta_n\eta_m e_n e_m a_{nm} = \frac{1}{16\pi^3}\sum_{n,m\in\{\alpha,\beta\}}\int d^2\hat{k}\frac{\eta_n\eta_m e_n e_m p_n\cdot p_m}{(p_n\cdot\hat{k})(p_m\cdot\hat{k})} \\ &= \frac{1}{16\pi^3}\int d^2\hat{k}\ t(\hat{k})\cdot t(\hat{k})\ , \end{aligned} \tag{F.0.9}$$

where we have defined

$$t^\mu(\hat{k}) \equiv \sum_{n\in\{\alpha,\beta\}} \frac{\eta_n e_n p_n^\mu}{p_n\cdot\hat{k}} \tag{F.0.10}$$

This vector is orthogonal to $\hat{k}$:

$$t^\mu(\hat{k})\cdot\hat{k} = \sum_{n\in\{\alpha,\beta\}}\eta_n e_n = 0 \tag{F.0.11}$$

where the final equality is ensured by the conservation of the $U(1)$ global charge. Then $t^\mu(\hat{k})$ can be expanded by the three orthogonal vectors perpendicular to the null vector $\hat{k}^\mu$. We can choose such three vectors as $\hat{k}^\mu$ itself, and $\hat{k}_\perp^{(1)\mu}$, $\hat{k}_\perp^{(2)\mu}$ that are normalized spacial vectors. Then we have

$$t^\mu(\hat{k}) = c_0(\hat{k})\hat{k} + \sum_{i=1,2} c_i(\hat{k})\hat{k}_\perp^{(i)\mu} \tag{F.0.12}$$

with some real coefficients $c_0(\hat{k}), c_1(\hat{k}), c_2(\hat{k})$. Putting this decomposition (F.0.12) back to (F.0.9), we find

$$A_{\beta\alpha} = \frac{1}{16\pi^3} \int d^2\hat{k} \ \left(|c_2(\hat{k})|^2 + |c_2(\hat{k})|^2\right) \geq 0 \ . \tag{F.0.13}$$

Appendix G

Derivation of eq. (4.2.22)

In this appendix, we explain some details of computation to derive (4.2.22). Inserting (4.2.21) into (4.2.15), $Q_H^{\text{sub},+}$ is written as a sum of two parts $Q_H^{\text{sub},(\rho)}$ and $Q_H^{\text{sub},(\varphi)}$ as follows:

$$Q_H^{\text{sub},+} = Q_H^{\text{sub},(\rho)} + Q_H^{\text{sub},(\varphi)} \tag{G.0.1}$$

$$Q_H^{\text{sub},(\rho)} \equiv -\frac{1}{8\pi}\int d^2\Omega\sqrt{\gamma}\int_{\mathbb{H}^3} d^3\tilde{\sigma}\sqrt{\tilde{h}}\,\Delta_{\mathrm{S}^2}\epsilon^{(0)}\, I^{mat}_\rho(\rho,\tilde{\Omega})\nabla^B\left[\frac{\sqrt{1+\rho^2}\,\partial_B\hat{x}(\Omega)\cdot\hat{y}(\tilde{\Omega})}{q\cdot Y}\right], \tag{G.0.2}$$

$$\begin{aligned} Q_H^{\text{sub},(\varphi)} \equiv & -\frac{1}{8\pi}\int d^2\Omega\sqrt{\gamma}\int_{\mathbb{H}^3} d^3\tilde{\sigma}\sqrt{\tilde{h}}\,\Delta_{\mathrm{S}^2}\epsilon^{(0)}\,\tilde{\gamma}^{CD} I^{mat}_D(\rho,\tilde{\Omega}) \\ & \times\nabla^B\left[\frac{1}{q\cdot Y}\left\{\frac{\sqrt{1+\rho^2}}{\rho}\partial_B\hat{x}\cdot\tilde{\partial}_C\hat{y} - (\hat{x}\cdot\hat{y})(\partial_B\hat{x}\cdot\tilde{\partial}_C\hat{y}) + (\hat{x}\cdot\tilde{\partial}_C\hat{y})(\partial_B\hat{x}\cdot\hat{y})\right\}\right], \end{aligned} \tag{G.0.3}$$

where $d^3\tilde{\sigma}\sqrt{\tilde{h}} = d\rho d^2\tilde{\Omega}\frac{\rho^2}{\sqrt{1+\rho^2}}\sqrt{\tilde{\gamma}}$. We now show that

$$Q_H^{\text{sub},(\rho)} = \frac{1}{2}\int_{\mathbb{H}^3} d^3\sigma\sqrt{h}\frac{\sqrt{1+\rho^2}}{\rho}\left[\rho^2 h^{\rho\rho}(\nabla^{(h)}_\rho\nabla^{(h)}_\rho\epsilon_{\mathbb{H}^3})I^{mat}_\rho + 2\rho h^{\rho\rho}(\nabla^{(h)}_\rho\epsilon_{\mathbb{H}^3})I^{mat}_\rho\right], \tag{G.0.4}$$

$$Q_H^{\text{sub},(\varphi)} = \frac{1}{2}\int_{\mathbb{H}^3} d^3\sigma\sqrt{h}\frac{\sqrt{1+\rho^2}}{\rho}\left[\rho^2 h^{AB}(\nabla^{(h)}_A\nabla^{(h)}_\rho\epsilon_{\mathbb{H}^3})I^{mat}_B + 2\rho h^{AB}(\nabla^{(h)}_A\epsilon_{\mathbb{H}^3})I^{mat}_B\right], \tag{G.0.5}$$

where $\nabla^{(h)}_\alpha$ denotes the covariant derivative compatible with the metric $h_{\alpha\beta}$ on $\mathbb{H}^3$. If these (G.0.4) and (G.0.5) are obtained, eq. (4.2.22) is obvious.

In the following calculations, the formulae

$$\partial_A\hat{x}\cdot\partial_B\hat{x} = \gamma_{AB}, \quad \gamma^{AB}\partial_A\hat{x}_i\,\partial_B\hat{x}_j = \delta_{ij} - \hat{x}_i\,\hat{x}_j, \quad \Delta_{\mathrm{S}^2}\hat{x}_i = -2\hat{x}_i \tag{G.0.6}$$

are useful.

We first derive eq. (G.0.4). The key equation is

$$\nabla^B \left[\frac{\partial_B \hat{x}(\Omega)\cdot \hat{y}(\tilde{\Omega})}{q(\Omega)\cdot Y(\rho,\tilde{\Omega})} \right] = \frac{4\pi}{\rho} G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega) - \frac{1}{\rho}\,, \tag{G.0.7}$$

where $G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega)$ was defined by eq. (4.1.5). Furthermore, $G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega)$ satisfies the following property

$$\Delta_{S^2} G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega) = \tilde{\Delta}_{S^2} G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega), \tag{G.0.8}$$

since $G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega)$ depends on angle Ω^A only through the inner product $\hat{x}(\Omega)\cdot\hat{y}(\tilde{\Omega})$. We thus have

$$\int d^2\Omega \sqrt{\gamma}\,[\Delta_{\mathrm{S}^2}\epsilon^{(0)}(\Omega)]\, G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega) = \tilde{\Delta}_{\mathrm{S}^2} \int d^2\Omega\sqrt{\gamma}\,\epsilon^{(0)}(\Omega)\, G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega) = \tilde{\Delta}_{\mathrm{S}^2}\epsilon_{\mathbb{H}^3}(\rho,\tilde{\Omega}), \tag{G.0.9}$$

where $\epsilon_{\mathbb{H}^3}$ was defined by (4.1.4). By virtue of the above equations, $Q_H^{\mathrm{sub},(\rho)}$ can be written as

$$Q_H^{\mathrm{sub},(\rho)} = -\frac{1}{2}\int_{\mathbb{H}^3} d^3\tilde{\sigma}\sqrt{\tilde{h}}\frac{\sqrt{1+\rho^2}}{\rho} I_\rho^{mat}(\rho,\tilde{\Omega}) \int d^2\Omega\sqrt{\gamma}\,\Delta_{\mathrm{S}^2}\epsilon^{(0)} \left[G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega) - \frac{1}{4\pi}\right] \tag{G.0.10}$$

$$= -\frac{1}{2}\int_{\mathbb{H}^3} d^3\tilde{\sigma}\sqrt{\tilde{h}}\frac{\sqrt{1+\rho^2}}{\rho} I_\rho^{mat}(\rho,\tilde{\Omega})\tilde{\Delta}_{\mathrm{S}^2}\epsilon_{\mathbb{H}^3}(\rho,\tilde{\Omega})\,. \tag{G.0.11}$$

where we have used (G.0.9) and $\int d^2\Omega\sqrt{\gamma}\,\Delta_{\mathrm{S}^2}\epsilon^{(0)} = 0$ in the second equality. In addition, since $\epsilon_{\mathbb{H}^3}(\sigma)$ is a solution of the Laplace equation on $\mathbb{H}^3$ as $\Delta_{\mathbb{H}^3}\epsilon_{\mathbb{H}^3}(\sigma) = 0$, it satisfies

$$\Delta_{\mathrm{S}^2}\epsilon_{\mathbb{H}^3} = -(1+\rho^2)\rho^2\nabla_\rho^{(h)}\nabla_\rho^{(h)}\epsilon_{\mathbb{H}^3} - 2(1+\rho^2)\rho\nabla_\rho^{(h)}\epsilon_{\mathbb{H}^3}. \tag{G.0.12}$$

Using this equation and noting that $h^{\rho\rho} = 1+\rho^2$, we can obtain eq. (G.0.4).

We next consider eq. (G.0.5). In (G.0.3), performing a partial integration, one encounters the following quantity:

$$\Delta_{\mathrm{S}^2}\nabla^A \left[\frac{1}{q\cdot Y} \left\{ \frac{\sqrt{1+\rho^2}}{\rho}\partial_A\hat{x}\cdot\tilde{\partial}_C\hat{y} - (\hat{x}\cdot\hat{y})(\partial_A\hat{x}\cdot\tilde{\partial}_C\hat{y}) + (\hat{x}\cdot\tilde{\partial}_C\hat{y})(\partial_A\hat{x}\cdot\hat{y}) \right\}\right]. \tag{G.0.13}$$

Performing the derivative, we have

$$(\mathrm{G.0.13}) = \Delta_{\mathrm{S}^2}\left[\frac{\hat{x}\cdot\tilde{\partial}_C\hat{y}}{(q\cdot Y)^2}\left(\frac{2}{\rho} + \rho - \sqrt{1+\rho^2}\hat{x}\cdot\hat{y}\right)\right]. \tag{G.0.14}$$

Performing the Laplacian, it further becomes

$$\begin{aligned}(\mathrm{G.0.14}) &= -\frac{2\hat{x}\cdot\tilde{\partial}_C\hat{y}}{(q\cdot Y)^4}\left(\frac{2}{\rho} - \rho + \sqrt{1+\rho^2}\hat{x}\cdot\hat{y}\right) \\ &= -4\pi\frac{\sqrt{1+\rho^2}}{\rho}\left[\nabla_\rho^{(h)}\tilde{\nabla}_C^{(h)} G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega) + \frac{2}{\rho}\tilde{\nabla}_C^{(h)} G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega)\right].\end{aligned} \tag{G.0.15}$$

Therefore, (G.0.3) can be written as

$$
\begin{aligned}
Q_H^{\mathrm{sub},(\varphi)} &= \frac{1}{2}\int_{\mathbb{H}^3} d^3\tilde{\sigma}\sqrt{\tilde{h}}\frac{\sqrt{1+\rho^2}}{\rho}\tilde{\gamma}^{CD} I_D^{mat}(\rho,\tilde{\Omega}) \\
&\qquad \times \int d^2\Omega\sqrt{\gamma}\epsilon^{(0)}\left[\nabla_\rho^{(h)}\tilde{\nabla}_C^{(h)} G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega) + \frac{2}{\rho}\tilde{\nabla}_C^{(h)} G_{\mathbb{H}^3}(\rho,\tilde{\Omega};\Omega)\right] \\
&= \frac{1}{2}\int_{\mathbb{H}^3} d^3\tilde{\sigma}\sqrt{\tilde{h}}\frac{\sqrt{1+\rho^2}}{\rho}\tilde{\gamma}^{CD} I_D^{mat}(\rho,\tilde{\Omega})\left[\nabla_\rho^{(h)}\tilde{\nabla}_C^{(h)}\epsilon_{\mathbb{H}^3}(\rho,\tilde{\Omega}) + \frac{2}{\rho}\tilde{\nabla}_C^{(h)}\epsilon_{\mathbb{H}^3}(\rho,\tilde{\Omega})\right] \\
&= \frac{1}{2}\int_{\mathbb{H}^3} d^3\sigma\sqrt{h}\frac{\sqrt{1+\rho^2}}{\rho} h^{CD} I_D^{mat}(\rho,\Omega)\left[\rho^2\nabla_\rho^{(h)}\nabla_C^{(h)}\epsilon_{\mathbb{H}^3}(\rho,\Omega) + 2\rho\nabla_C^{(h)}\epsilon_{\mathbb{H}^3}(\rho,\Omega)\right],
\end{aligned}
\tag{G.0.16}
$$

where we have just renamed the integration variables and also used $\gamma^{AB} = \rho^2 h^{AB}$ in the last line. Thus we have obtained eq. (G.0.5).

Appendix H

Particle trajectory under the Coulomb force and phase factor in F-K dressed state

H.1 Particle trajectory under the Coulomb force

The relativistic equation of motion (e.o.m) for a point particle with its trajectory $y^\mu(\tau)$ with the electric field F^μ_ν is

$$m\frac{d^2y^\mu}{d\tau^2} - eF^\mu_\nu \frac{dy^\nu}{d\tau} = 0 \ . \tag{H.1.1}$$

We consider the case where the particle is moving in the static Coulomb potential,

$$A_\mu(x) = -\frac{Q}{4\pi}\frac{1}{r}\delta_{\mu 0} \ . \tag{H.1.2}$$

Then we drive the asymptotic trajectory at $t \to \pm\infty$ by solving the e.o.m. First, the electric fields is given by

$$F^0_i = \partial^0 A_i - \partial_i A^0 = \frac{Q}{4\pi}\frac{r_i}{r^3} \ . \tag{H.1.3}$$

Then we decompose $y^\mu(\tau)$ into two parts as

$$y^\mu(\tau) = y^\mu_0(\tau) + \delta y^\mu(\tau) \tag{H.1.4}$$

where

$$y^\mu_0(\tau) = a^\mu + v^\mu \tau \tag{H.1.5}$$

with arbitrary constant vectors a^μ and $v^\mu (= p^\mu/m)$. $y^\mu_0(\tau)$ is the trajectory of the free particle that is the solution of the equation of motion without the Coulomb potential. Since it is expected that the trajectory $\vec{y}(\tau)$ approaches the free trajectory $\vec{y}(\tau)$ in large-τ region, we assume $\frac{|\delta\vec{y}(\tau)|}{|\vec{y}_0(\tau)|} \to 0$ as $\tau \to \infty$. Plugging (H.1.5) into (H.1.1), the time component of e.o.m is given by

$$
\begin{aligned}
m\frac{d^2\delta y^0}{d\tau^2} &= -\frac{eQ}{4\pi}\frac{1}{r^2}\frac{\vec{r}}{r}\cdot(\vec{v}+\delta\dot{\vec{y}})\bigg|_{\vec{r}=\vec{y}(\tau)} \\
&= -\frac{eQ}{4\pi}\frac{1}{(\vec{y}_0+\delta\vec{y})^2}\frac{\vec{y}_0+\delta\vec{y}}{\sqrt{(\vec{y}_0+\delta\vec{y})^2}}\cdot(\vec{v}+\delta\dot{\vec{y}}) \\
&= -\frac{eQ}{4\pi}\frac{1}{(\vec{y}_0)^2}\frac{\vec{y}_0\cdot\vec{v}}{\sqrt{(\vec{y}_0)^2}}+\mathcal{O}(\frac{|\delta\vec{y}(\tau)|}{|\vec{y}_0(\tau)|}) \\
&= -\frac{eQ}{4\pi}\frac{1}{|\vec{v}|\tau^2}+\mathcal{O}(\frac{|\delta\vec{y}(\tau)|}{|\vec{y}_0(\tau)|}) \ .
\end{aligned}
\tag{H.1.6}
$$

We can easily solve this equation and the solution is given by

$$
\delta y^0(\tau) = \frac{eQ}{4\pi m|\vec{v}|}\ln\tau + c\tau + d \ , \tag{H.1.7}
$$

with two integration constants c and d. Since we used $\frac{|\delta\vec{y}(\tau)|}{|\vec{y}_0(\tau)|} \to 0$ as $\tau \to \infty$, c is set to be zero and d can also be neglected as $\tau \to \infty$. Then we have found the leading solution in asymptotic region,

$$
\delta y^0(\tau) = \frac{eQ}{4\pi m|\vec{v}|}\ln\tau \ . \tag{H.1.8}
$$

The e.o.m of the spatial component is

$$
\begin{aligned}
m\frac{d^2\delta\vec{y}}{d\tau^2} &= -\frac{eQ}{4\pi}\frac{1}{r^2}\frac{\vec{r}}{r}(v^0+\delta\dot{y}^0)\bigg|_{\vec{r}=\vec{y}(\tau)} \\
&= -\frac{eQ}{4\pi}\frac{v^0\vec{v}}{|\vec{v}|^3\tau^2}+\mathcal{O}(\frac{|\delta\vec{y}(\tau)|}{|\vec{y}_0(\tau)|}) \ .
\end{aligned}
\tag{H.1.9}
$$

The solution is

$$
\delta\vec{y}(\tau) = \frac{eQv^0\vec{v}}{4\pi m|\vec{v}|^3}\ln\tau \ , \tag{H.1.10}
$$

where we have already set the integration constant to zero for the same reason as the solution of time component. Thus we found the asymptotic trajectory of a point particle in the static Coulomb potential:

$$
y^\mu(\tau) = v^\mu\tau + \frac{eQ}{4\pi m|\vec{v}|^3}\begin{pmatrix}|\vec{v}|^2\vec{v}\\ v^0\vec{v}\end{pmatrix}\ln\tau + \mathcal{O}(\frac{\ln\tau}{\tau}) \ . \tag{H.1.11}
$$

The subleading term means that the particle is pulled off to the source if the charges have the opposite signs ($eQ > 0$) and the particle is repulsed away from the source if the charges have the same signs ($eQ < 0$).

Now we try to find the fully covariant solution of the asymptotic trajectory. Since $y^\mu(\tau)$ is a covariant four-vector, it should be composed of the covariant vectors existing in our system; the velocity four-vector of the particle v^μ and the velocity four-vector of the source particle w^μ. In previous case, since we supposed that the source particle with the charge Q is at rest, w^μ is given by $w^\mu = (1, \vec{0})$. Using this vector, we can write $|\vec{v}|$

and v^0 in Lorentz invariant forms as

$$|\vec{v}| = \sqrt{v^2 + (v \cdot w)^2} = \sqrt{(v \cdot w)^2 - 1} \ , \tag{H.1.12}$$

$$v^0 = -v \cdot w \ . \tag{H.1.13}$$

With the use of these expressions, the four-vectors appeared in (H.1.11) can be written as

$$\begin{pmatrix} |\vec{v}|^2 \\ v^0 \vec{v} \end{pmatrix} = \begin{pmatrix} (v \cdot w)^2 - 1 \\ -(v \cdot w)\vec{v} \end{pmatrix} = -(v \cdot w) v^\mu - w^\mu \ . \tag{H.1.14}$$

Plugging (H.1.12) and (H.1.14) back into (H.1.11), we find the covariant solution,

$$y^\mu(\tau) = v^\mu \tau - \frac{eQ}{4\pi} \frac{(v \cdot w)\, v^\mu + w^\mu}{\left((v \cdot w)^2 - 1\right)^{3/2}} \ln \tau + \mathcal{O}(\frac{\ln \tau}{\tau}) \ . \tag{H.1.15}$$

We can trivially generalize this solution to multi-particles case. The result is

$$y_i^\mu(\tau) = v_i^\mu \tau_i - \sum_{j=1 (j \neq i)}^{N} \frac{e_i e_j}{4\pi m_i} \frac{(v_i \cdot v_j)\, v_i^\mu + v_j^\mu}{\left((v_i \cdot v_j)^2 - 1\right)^{3/2}} \ln \tau_i + \mathcal{O}(\frac{\ln \tau_i}{\tau_i}) \ , \tag{H.1.16}$$

where τ_i, $y_i^\mu(\tau)$ and v_i^μ are the proper time, the position and leading velocity four-vector of the i-th particle with the charge e_i.

Now let us write this spacial trajectory in terms of the reference time t. The time component of (H.1.16) is

$$t(\tau) = v_i^0 \tau_i - \sum_{j=1 (j \neq i)}^{N} \frac{e_i e_j}{4\pi m_i} \frac{(v_i \cdot v_j)\, v_i^0 + v_j^0}{\left((v_i \cdot v_j)^2 - 1\right)^{3/2}} \ln \tau_i + \mathcal{O}(\frac{\ln \tau_i}{\tau_i}) \ . \tag{H.1.17}$$

Solving this equation in terms of τ_i, the proper time for i-th particle then is given by

$$\tau_i = \frac{t}{v_i^0} + \sum_{j=1 (j \neq i)}^{N} \frac{e_i e_j}{4\pi m_i v_i^0} \frac{(v_i \cdot v_j)\, v_i^0 + v_j^0}{\left((v_i \cdot v_j)^2 - 1\right)^{3/2}} \ln t + \mathcal{O}(\frac{\ln t}{t}) \ . \tag{H.1.18}$$

Plugging (H.1.18) into the spatial component of (H.1.16), we have

$$\begin{aligned} \vec{y}_i(\tau) &= \vec{v}_i \left(\frac{t}{v_i^0} + \sum_{j=1 (j \neq i)}^{N} \frac{e_i e_j}{4\pi m_i v_i^0} \frac{(v_i \cdot v_j)\, v_i^0 + v_j^0}{\left((v_i \cdot v_j)^2 - 1\right)^{3/2}} \ln t \right) \\ &\quad - \sum_{j=1 (j \neq i)}^{N} \frac{e_i e_j}{4\pi m_i} \frac{(v_i \cdot v_j)\, \vec{v}_i + \vec{v}_j}{\left((v_i \cdot v_j)^2 - 1\right)^{3/2}} \ln t + \mathcal{O}(\frac{\ln t}{t}) \\ &= \frac{\vec{v}_i}{v_i^0} t + \sum_{j=1 (j \neq i)}^{N} \frac{e_i e_j}{4\pi m_i v_i^0} \frac{v_j^0 \vec{v}_i - v_i^0 \vec{v}_j}{\left((v_i \cdot v_j)^2 - 1\right)^{3/2}} \ln t + \mathcal{O}(\frac{\ln t}{t}) \end{aligned} \tag{H.1.19}$$

H.2 Stationary trajectory of phase factor in F-K dressed state

In this section, we consider the wave function with the dressing phase factor (5.3.27) in the F-K state, and show that the stationary point of the phase of the wave function corresponds to the trajectory of electrons and positrons under the Coulomb force (H.1.19).

For simplicity, let us consider the wave packet of two charged particle with the dressing phase factor (5.3.27) in QED in the one-dimensional space, which is given by

$$|\psi(t)\rangle = \int dp_1 dp_2 f(p_1,p_2)\ e^{i\Phi(t,t_0)}|p_1,s_1;p_2,s_2\rangle \tag{H.2.1}$$

where $|p_1,s_1;p_2,s_2\rangle = b^\dagger_{s_1}(\vec{p_1})b^\dagger_{s_2}(\vec{p_2})|0\rangle$, and $f(p_1,p_2)$ is a smooth function of p_1,p_2. We consider the wave function in the position space which is given by

$$\begin{aligned}\psi(x_1,x_2) &= \int dp_1 dp_2 f(p_1,p_2)\langle 0|\psi(x_1)\psi(x_2)e^{i\Phi(t,t_0)}|p_1,p_2\rangle \\ &\equiv \int dp_1 dp_2 f(p_1,p_2)\left[u_{s_1}(\vec{p_1})u_{s_2}(\vec{p_2})\exp\left(iF(t;x_1,p_1;x_2,p_2)\right) + (p_1\leftrightarrow p_2)\right]\end{aligned} \tag{H.2.2}$$

where $\psi(x)$ is the free Dirac field operator (5.3.3) and $F(t;x_1,p_1;x_2,p_2)$ is given by

$$\begin{aligned}&F(t;x_1,p_1;x_2,p_2)\\ &= ip_1x + ip_2x_2 + i\Phi(p_1,p_2;t,t_0)\\ &= \left(p_1^0+p_2^0\right)(t-t_0) - \vec{p}_1\cdot\vec{x}_1 - \vec{p}_2\cdot\vec{x}_2 + \frac{e_ie_j}{4\pi}\frac{-p_1\cdot p_2}{\left((p_1\cdot p_2)^2 - m_1^2m_2^2\right)^{1/2}}\log\frac{t}{t_0}\end{aligned} \tag{H.2.3}$$

The stationary phase is then given by

$$0 = \frac{\partial}{\partial\vec{p_1}}F(t;x_1,p_1;x_2,p_2) = \frac{\vec{p_1}}{p_1^0}(t-t_0) - \vec{x}_1 + \frac{e_ie_j}{4\pi}\frac{m_1^2m_2^2\left(\frac{p_2^0}{p_1^0}\vec{p}_1 - \vec{p}_2\right)}{\left((p_1\cdot p_2)^2-m_1^2m_2^2\right)^{3/2}}\log\frac{t}{t_0}\,. \tag{H.2.4}$$

This solution corresponds to the following particle trajectry,

$$\vec{x}_1 = \frac{\vec{p_1}}{p_1^0}(t-t_0) + \frac{e_ie_j}{4\pi p_1^0}\frac{m_1^2m_2^2\left(p_2^0\vec{p}_1 - p_1^0\vec{p}_2\right)}{\left((p_1\cdot p_2)^2-m_1^2m_2^2\right)^{3/2}}\log\frac{t}{t_0} \tag{H.2.5}$$

$$= \frac{\vec{v_1}}{v_1^0}t + \frac{e_1e_2}{4\pi m_1v_1^0}\frac{v_2^0\vec{v}_1 - v_1^0\vec{v}_2}{\left((v_1\cdot v_2)^2-1\right)^{3/2}}\ln t + \mathcal{O}(\frac{1}{t}) \tag{H.2.6}$$

where we have used $p_i^\mu = m_iv_i^\mu$ in the final equality. We can trivially generalize this trajectory to the trajectory in N charged particle case as

$$\vec{x}_i = \frac{\vec{v_i}}{v_i^0}t + \sum_{j=1(j\neq i)}^N \frac{e_ie_j}{4\pi m_iv_i^0}\frac{v_j^0\vec{v}_i - v_i^0\vec{v}_j}{\left((v_i\cdot v_j)^2-1\right)^{3/2}}\ln t + \mathcal{O}(\frac{1}{t}) \tag{H.2.7}$$

This trajectory is exactly the same as the asymptotic trajectory of point particles with the relativistic Coulomb interactions (H.1.19). This means that if the wave packet is effectively localized at $p_1, \cdots, p_N$ in the momentum space ($f(P_1, \cdots, P_N)$ has the peaks at $p_1, \cdots, p_N$), the wave function of the dressed state is effectively localized at the classical trajectories (H.1.19) of point particles under the Coulomb force at the large time limit.

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