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## ON THE ANNIHILATOR IDEALS OF THE RADICAL OF A GROUP ALGEBRA

Dedicated to Professor K. Asano for his 60th birthday

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### 1. Introduction

Let  $G$  be a finite group, and  $k$  a field of characteristic  $p$ . Let  $\mathfrak{n}$  denote the Jacobson radical of the group algebra  $kG$ , and  $r(\mathfrak{n})$  the right annihilator ideal of  $\mathfrak{n}$ . In this paper we shall show some connections between  $r(\mathfrak{n})$  and  $p$ -elements of  $G$ . One of them will state that  $r(\mathfrak{n})$  contains the sum of all  $p$ -elements of  $G$  (including the identity). This may be regarded in a sense as a refinement of Maschke's theorem. In fact, if  $p$  does not divide the order of  $G$  then the identity is the only  $p$ -element, which implies  $r(\mathfrak{n}) \ni 1$  and hence  $\mathfrak{n} = 0$ . On the other hand, as is easily seen from a theorem of T. Nakayama on Frobenius algebras (see §2),  $r(\mathfrak{n})$  is a principal ideal. We shall show that it is generated by an element which is left invariant by every automorphism of  $kG$  induced by that of  $G$ . As an application of this fact, we shall give a lower bound for the first Cartan invariant in terms of the chief composition factors of  $G$ . The present study owes heavily to some general results on Frobenius algebras and symmetric algebras, which will be summarized in the next section.

NOTATION. If  $A$  is a ring,  $\text{rad}(A)$  will denote the Jacobson radical of  $A$ . For a subset  $T$  of  $A$ ,  $r(T)$  and  $l(T)$  will denote respectively the set of right annihilators and the set of left annihilators of  $T$  in  $A$ . If  $M$  is a subset of a finite group  $G$ , then  $\Delta_M = \sum_{\sigma \in M} \sigma \in kG$ .

### 2. Preliminary results

Let  $A (\ni 1)$  be a finite dimensional algebra over a field  $k$ .

DEFINITION. A linear function  $\lambda (\in A^* = \text{Hom}_k(A, k))$  is called *non-singular* if its kernel contains no left or right ideals other than zero. While,  $\lambda$  is called *symmetric* if  $\lambda(ab) = \lambda(ba)$  for all  $a, b \in A$ .

If  $\lambda$  is a linear function and  $a \in A$ , we denote by  $\lambda_a$  the linear function defined

by  $\lambda_a(x) = \lambda(xa)$ ,  $x \in A$ . One may remark here that  $f: a \rightarrow \lambda_a$  is a left  $A$ -homomorphism from  $A$  into  $A^*$ . It is an (onto) isomorphism if and only if it is non-singular.  $A$  is a Frobenius [symmetric] algebra if and only if it has (at least) one non-singular [symmetric, non-singular] linear function.

**Theorem A** (T. Nakayama [6], [8], see also [2]). *Let  $A$  be a Frobenius [symmetric] algebra,  $\lambda$  a non-singular [symmetric, non-singular] linear function on  $A$ , and  $\mathfrak{z}$  a two-sided ideal of  $A$ . If  $A/\mathfrak{z}$  is Frobenius [symmetric], and  $\mu$  a non-singular [symmetric, non-singular] linear function on  $A/\mathfrak{z}$  then there exists an element [central element]  $c \in A$  such that  $\mu\psi = \lambda_c$  and  $r(\mathfrak{z}) = cA$ , where  $\psi$  is the natural map  $A \rightarrow A/\mathfrak{z}$ . Conversely, if there exists an element [central element]  $c \in A$  such that  $r(\mathfrak{z}) = cA$  then  $A/\mathfrak{z}$  is Frobenius [symmetric].*

*Proof.* As was noted above, there exists an element  $c \in A$  such that  $\mu\psi = \lambda_c$ . We shall show  $r(\mathfrak{z}) = cA$ . Since  $\lambda(\mathfrak{z}c) = \lambda_c(\mathfrak{z}) = 0$  and  $\lambda$  is non-singular, it follows at once  $cA \subset r(\mathfrak{z})$ . On the other hand, if  $xc = 0$  then  $\lambda_c(Ax) = \lambda(Axc) = 0$ . Since  $\lambda_c$  is non-singular as a linear function on  $A/\mathfrak{z}$ , it follows  $x \in \mathfrak{z}$  and hence  $l(cA) \subset \mathfrak{z}$ . Recalling here that  $A$  is Frobenius, we have then  $cA = r(l(cA)) \supset r(\mathfrak{z})$ . Now, suppose further both  $\lambda$  and  $\mu$  are symmetric. Then  $\lambda(xyc) = \lambda_c(xy) = \lambda_c(yx) = \lambda(yxc) = \lambda(xcy)$  for all  $x, y \in A$ . Therefore,  $yc = cy$  and  $c$  is central. Next, we shall prove the converse. Suppose  $r(\mathfrak{z}) = cA$ . Then  $\lambda_c$  gives rise to a non-singular linear function on  $A/\mathfrak{z}$ . If  $c$  is central, the linear function is evidently symmetric.

**Theorem B** (T. Nakayama [6]). *If  $\mathfrak{z}$  is a two-sided ideal of a symmetric algebra  $A$  then  $r(\mathfrak{z}) = l(\mathfrak{z})$ .*

*Proof.* Let  $\lambda$  be a symmetric, non-singular linear function on  $A$ . Then  $r(\mathfrak{z}) = \{x \in A \mid \lambda(\mathfrak{z}x) = 0\} = \{x \in A \mid \lambda(x\mathfrak{z}) = 0\} = l(\mathfrak{z})$ .

### 3. The generator of $r(\mathfrak{n})$

From now on,  $k$  will denote a field of characteristic  $p$ , and  $G$  a finite group of order  $|G| = p^n g_0$ , where  $(p, g_0) = 1$ . Let  $\nu_p(l)$  denote the exponent of  $p$  in the primary decomposition of an integer  $l$ . Let  $\mathfrak{n}$  be the radical of the group algebra  $kG$  as before. To be easily seen,  $kG$  is a symmetric algebra through the following linear function  $\lambda$  which will be fixed throughout the subsequent study:  $\lambda(\sum_{\sigma \in G} a_\sigma \sigma) = a_1$ , where  $a_\sigma \in k$  and  $1$  denotes the identity of  $G$ .

REMARK 1. Since there exists a splitting field for  $G$  which is finite separable over  $k$ ,  $kG/\mathfrak{n}$  is a separable algebra over  $k$ . In particular, if  $K$  is an arbitrary extension field of  $k$  and  $\mathfrak{n}_K$  denotes the radical of  $KG$ , then  $\mathfrak{n}_K = K\mathfrak{n}$  and  $\mathfrak{n}_K \cap kG = \mathfrak{n}$ . Similar relations hold for  $r(\mathfrak{n})$  and the right annihilator ideal of  $\mathfrak{n}_K$  in  $KG$ .

Let  $\bar{\phi}_1, \bar{\phi}_2, \dots, \bar{\phi}_r$  be the distinct irreducible characters over a suitable splitting field for  $G$  containing  $k$ , and let  $\bar{\phi} = \sum_{i=1}^r \phi_i$ . Then, it is clear that  $\bar{\phi}(\sigma)$  is contained in the prime field for every  $\sigma \in G$ .

**Proposition 1.** *If  $v = \sum_{\sigma \in G} \bar{\phi}(\sigma^{-1})\sigma$  then  $r(\mathfrak{n}) = (kG)v$ .*

*Proof.* By the above remark, we may assume that  $k$  is a splitting field for  $G$ . Then,  $kG/\mathfrak{n}$  is a direct sum of total matrix algebras over  $k$ :  $kG/\mathfrak{n} = \sum_{i=1}^r (k)_{n_i}$ . Let  $\psi$  and  $p_i$  denote respectively the natural map  $kG \rightarrow kG/\mathfrak{n}$  and the projection  $kG/\mathfrak{n} \rightarrow (k)_{n_i}$ . Since the trace map  $tr_i: (k)_{n_i} \rightarrow k$  is a symmetric, non-singular linear function, so is  $\mu = \sum_{i=1}^r tr_i p_i$  on  $kG/\mathfrak{n}$ . Therefore, by Theorem A, there exists a central element  $v = \sum_{\sigma} a_{\sigma} \sigma \in kG$  such that  $\mu\psi = \lambda_v$  and  $r(\mathfrak{n}) = v(kG)$ . Noting here that  $\mu\psi = \bar{\phi}$  on  $G$ , we obtain  $a_{\sigma} = \lambda(\sigma^{-1}v) = \lambda_v(\sigma^{-1}) = \bar{\phi}(\sigma^{-1})$ . This completes the proof.

**REMARK 2.** Let  $f$  be an arbitrary automorphism of  $G$ . Then, it permutes the irreducible characters by  $\bar{\phi}_i \rightarrow \bar{\phi}_i^f$ , where  $\bar{\phi}_i^f(\sigma) = \bar{\phi}_i(\sigma^f)$ ,  $\sigma \in G$ . In particular, it follows  $\bar{\phi}(\sigma^f) = \bar{\phi}^f(\sigma) = \bar{\phi}(\sigma)$  for all  $\sigma \in G$ . Hence, regarding  $f$  naturally as an automorphism of  $kG$ , we obtain  $v^f = \sum_{\sigma} \bar{\phi}(\sigma^{-1})\sigma^f = \sum_{\sigma} \bar{\phi}((\sigma^f)^{-1})\sigma^f = \sum_{\tau} \bar{\phi}(\tau^{-1})\tau = v$ .

Now, let  $H$  be a normal subgroup of  $G$ , and  $\mathfrak{m} = \text{rad } kH$ . Then  $\mathfrak{l} = (kG)\mathfrak{m} = \mathfrak{m}(kG)$  is a nilpotent two-sided ideal of  $kG$ .

**Corollary 1.** *Under the above notation, there holds the following :*

- (1)  $kG/\mathfrak{l}$  is a symmetric algebra over  $k$ .
- (2) Let  $(\mathfrak{l} : \mathfrak{n}) = \{x \in kG \mid x\mathfrak{n} \subset \mathfrak{l}\}$ . Then, for an arbitrary primitive idempotent  $e$  of  $kG$  there holds  $(kG)e/\mathfrak{n}e \cong (\mathfrak{l} : \mathfrak{n})e/\mathfrak{l}e$ .

*Proof.* (1) Let  $r_H(\mathfrak{m})$  be the right annihilator ideal of  $\mathfrak{m}$  in  $kH$ . Let  $v$  be as in Proposition 1 applied to  $kH$ . Since  $G$  induces an automorphism group on  $H$ , Remark 2 proves that  $v$  lies in the center of  $kG$ . Hence,  $r(\mathfrak{l}) = kG \cdot r_H(\mathfrak{l}) = (kG)v$  is a principal ideal generated by a central element. Theorem A proves therefore  $kG/\mathfrak{l}$  is a symmetric algebra.

(2) Evidently, the residue class  $\bar{e}$  of  $e$  modulo  $\mathfrak{l}$  is a primitive idempotent of the symmetric algebra  $kG/\mathfrak{l}$ . Since  $(\mathfrak{l} : \mathfrak{n})/\mathfrak{l}$  is the right annihilator ideal of  $\mathfrak{n}/\mathfrak{l} = \text{rad}(kG/\mathfrak{l})$ , there holds then  $(kG/\mathfrak{l})\bar{e}/(\mathfrak{n}/\mathfrak{l})\bar{e} \cong ((\mathfrak{l} : \mathfrak{n})/\mathfrak{l})\bar{e}$ , which proves  $(kG)e/\mathfrak{n}e \cong (\mathfrak{l} : \mathfrak{n})e/\mathfrak{l}e$ .

Under the above notation, if  $H$  is a non-trivial  $p$ -group, then it is well-known that  $\mathfrak{m}$  coincides with the augmentation ideal  $I(H) = \{\sum_{\sigma \in H} a_{\sigma} \sigma \mid \sum_{\sigma} a_{\sigma} = 0\}$ , so that  $r_H(\mathfrak{m}) = kH \cdot \Delta_H = k\Delta_H$ . We obtain therefore  $r(\mathfrak{n}) \subset r(\mathfrak{l}) = kG \cdot \Delta_H \subset \mathfrak{l}$ .

**Lemma 1.** *Let  $P_1 \cong H_1$  be normal subgroups of  $G$  such that  $P_1/H_1$  is a*

*p*-group. Let  $I_1 = kG \cdot \text{rad } kP_1$ , and  $\mathfrak{h}_1 = \ker(kG \rightarrow k(G/H_1))$ . Then, there holds the following :

(1) If  $e$  is a primitive idempotent of  $kG$  not contained in  $\mathfrak{h}_1$  then  $(I_1 + \mathfrak{h}_1 : n)e / (I_1 + \mathfrak{h}_1)e \cong (kG)e / ne$  and  $(I_1 + \mathfrak{h}_1)e$  contains a submodule isomorphic to  $(kG)e / ne$ .

(2) If  $P_2 \cong H_2$  are normal subgroups of  $G$  containing  $P_1$  such that  $P_2/H_2$  is a *p*-group then  $I_2 + \mathfrak{h}_2 \supset (I_1 + \mathfrak{h}_1 : n)$ , where  $I_2 = kG \cdot \text{rad } kP_2$  and  $\mathfrak{h}_2 = \ker(kG \rightarrow k(G/H_2))$ .

Proof. (1) Since  $I_1 + \mathfrak{h}_1 = k(G/H_1) \cdot \text{rad } k(P_1/H_1)$  and  $P_1/H_1$  is a non-trivial *p*-group, the above remark proves  $r(\text{rad } k(G/H_1)) \subset I_1 + \mathfrak{h}_1/\mathfrak{h}_1$ . Let  $\bar{e}$  be the residue class of  $e$  modulo  $\mathfrak{h}_1$ . Then, it is still primitive by assumption, and the former is evident by Corollary 1. Further, noting that  $(I_1 + \mathfrak{h}_1/\mathfrak{h}_1)\bar{e} \neq 0$ , it follows at once  $0 \neq (I_1 + \mathfrak{h}_1)e \supset r(n)e \cong (kG)e / ne$ , proving the latter.

(2) As in (1), we obtain  $r(\text{rad } k(G/H_2)) \subset I_2 + \mathfrak{h}_2/\mathfrak{h}_2$ . If  $\mathfrak{p}_1 = \ker(kG \rightarrow k(G/P_1))$  then  $(I_1 + \mathfrak{h}_1 : n) + \mathfrak{p}_1/\mathfrak{p}_1 \subset (I_1 + \mathfrak{p}_1/\mathfrak{p}_1 : n + \mathfrak{p}_1/\mathfrak{p}_1) = r(\text{rad } k(G/P_1))$ . Since the natural map  $k(G/P_1) \rightarrow k(G/H_2)$  is an epimorphism, it sends  $r(\text{rad } k(G/P_1))$  into  $r(\text{rad } k(G/H_2))$ . Therefore,  $(I_1 + \mathfrak{h}_1 : n) + \mathfrak{h}_2/\mathfrak{h}_2 \subset r(\text{rad } k(G/H_2)) \subset I_2 + \mathfrak{h}_2/\mathfrak{h}_2$ .

**Theorem 1.** Let  $m$  be the number of the chief composition factors of  $G$  which are (non-trivial) *p*-groups. Then the first Cartan invariant  $c_{11}$  is at least  $m+1$ .

Proof. We take a primitive idempotent  $e$  of  $kG$  such that  $(kG)e / ne$  is isomorphic to the trivial  $kG$ -module  $k \cong k\Delta_G$ . Then,  $e\Delta_G$  being non-zero,  $e$  is not contained in  $\ker(kG \rightarrow k(G/N))$  ( $=$  the ideal generated by  $\{1 - \eta \mid \eta \in N\}$ ) for any normal subgroup  $N$  of  $G$ . Hence, by Lemma 1, we can easily see that  $(kG)e$  possesses at least  $m+1$  composition factors isomorphic to  $k$ , namely,  $c_{11} \geq m+1$ .

#### 4. The sum of all *p*-elements

First, we shall introduce some notations. Let  $\bar{\varepsilon}$  be a primitive  $g_0$ -th root of unity over the prime field of characteristic  $p$ . In what follows, whenever we consider Brauer characters, it is assumed that there is defined (and fixed) a homomorphism  $Z[\varepsilon] \rightarrow k[\bar{\varepsilon}]$  such that  $\bar{\varepsilon}$  is the image of a primitive  $g_0$ -th root of unity  $\varepsilon$  in the complex number field. As is well-known, there exists a unique (up to isomorphisms) indecomposable projective module  $P_i$  such that  $P_i/nP_i$  affords the irreducible character  $\bar{\phi}_i$ . Let  $\bar{\eta}_i$  be the character of the representation afforded by  $P_i$  and  $u_i$  the dimension of  $P_i$ . As is well-known,  $u_i$  is divisible by  $p^n$ . Let  $u_i = p^n h_i$ . We may assume, after a suitable change of index if necessary, the first  $u_1, u_2, \dots, u_r$  are all such that  $v_p(u_i) = n$ . Let  $\phi_i$  and  $\eta_i$  be the Brauer characters of  $\phi_i$  and  $\eta_i$ , respectively.

Noting that  $h_j = 0$  in  $k$  for  $t < j \leq r$  and  $\phi_i(\sigma) = \phi_i(\sigma')$  for the *p*-regular part  $\sigma'$  or  $\sigma$ , the orthogonality relation ([3] p. 561)

$$\sum_{i=1}^r \eta_i(\sigma^{-1})\phi_i(\tau) = \begin{cases} |C_G(\sigma)| & \text{if } \sigma \text{ is conjugate to } \tau, \\ 0 & \text{otherwise} \end{cases}$$

implies

$$(*) \quad \sum_{i=1}^t h_i \bar{\phi}_i(\tau) = \begin{cases} g_0 & \text{if } \tau \text{ is a } p\text{-element,} \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.** *Assume that  $k$  is a splitting field for  $G$ . Let  $\mathfrak{z} = \sum_e (kG)e + \mathfrak{n}$ , where  $e$  runs over the primitive idempotents such that  $\nu_p(\dim_k(kG)e) > n$ . Let  $c$  be the sum of all  $p$ -elements of  $G$ . Then there holds  $r(\mathfrak{z}) = (kG)c$ .*

*Proof.* We use the same notation as in the proof of Proposition 1. Then,  $kG/\mathfrak{n} = \sum_{i=1}^t (k)_{n_i} + \sum_{j=t+1}^r (k)_{n_j}$  (direct sum), where  $n_i = \dim_k P_i/\mathfrak{n}P_i$ . It is clear that  $\mathfrak{z}$  is the inverse image of  $\sum_{j=t+1}^r (k)_{n_j}$  by the natural map  $\psi$ . Hence, it is a two-sided ideal and there holds  $kG/\mathfrak{z} \cong \sum_{i=1}^t (k)_{n_i}$ . We set here  $\mu = \sum_{i=1}^t h_i tr_i p_i$ . Since  $h_i \neq 0$  in  $k$ ,  $\mu$  is a symmetric, non-singular linear function on  $kG/\mathfrak{z}$ . Then, by Theorem A, there exists a central element  $c' \in kG$  such that  $\mu\psi' = \lambda_{c'}$  and  $r(\mathfrak{z}) = (kG)c'$ , where  $\psi'$  is the natural map  $kG \rightarrow kG/\mathfrak{z}$ . Now, by making use of (\*), we can prove  $c' = g_0 c$  in the same way as in the last part of the proof of Proposition 1. This completes the proof.

Since  $\mathfrak{z} \supset \mathfrak{n}$ , we obtain in particular  $r(\mathfrak{n}) \ni c$ . However, in virtue of Remark 1, this holds without assuming that  $k$  is a splitting field. Thus, we have shown

**Theorem 2.**  *$r(\mathfrak{n})$  contains the sum of all  $p$ -elements of  $G$ .*

**Corollary 2.** *Let  $x = \sum_{\sigma} a_{\sigma} \sigma$  be an element of  $kG$ , and  $x(p)$  the sum of the coefficient  $a_{\sigma}$  of  $p$ -elements  $\sigma$ . If  $x$  is in  $\mathfrak{n}$  then  $x(p) = 0$ .*

*Proof.* Note that  $x(p)$  is equal to the coefficient of 1 in  $xc$ . If  $x$  is in  $\mathfrak{n}$  then  $xc = 0$ , and hence  $x(p) = 0$ .

If  $e$  is a primitive idempotent of  $kG$ , then  $r(\mathfrak{n})e$  is a minimal left ideal of  $kG$  and contains  $(kG)ce$ .

**Corollary 3.** *Let  $e$  be a primitive idempotent of  $kG$ . If  $\nu_p(\dim_k(kG)e) = n$  then  $r(\mathfrak{n})e = (kG)ce$ . The converse holds, provided  $k$  is a splitting field for  $G$ .*

*Proof.* First, we assume that  $k$  is a splitting field. By Lemma 2, if  $ce = 0$ , or what is the same, if  $e \in l(c) = \mathfrak{z}$ , then  $\nu_p(\dim_k(kG)e) > n$ , and conversely. We have seen therefore that  $ce \neq 0$  and  $\nu_p(\dim_k(kG)e) = n$  are equivalent.

Secondly, we assume that  $\nu_p(\dim_k(kG)e) = n$ . Let  $K$  be a splitting field for  $G$  containing  $k$ , and  $e = \sum_j e_j$  a decomposition of  $e$  into orthogonal primitive idempotents of  $KG$ . Then, by assumption there exists at least one  $e_i$  such that  $\nu_p(\dim_k(KGe_i)) = n$ . Since  $ce_i \neq 0$  by the first step, we have  $ce \neq 0$ , completing the proof.

Now, if  $G$  is  $p$ -solvable then  $\nu_p(\dim_k(kG)e) = n$  for every primitive idempotent  $e$  (P. Fong [5]), whence it follows  $\mathfrak{z} = n$  and therefore  $r(n) = r(\mathfrak{z}) = (kG)c$ .

**Corollary 4.** *If  $G$  is  $p$ -solvable then  $r(n) = (kG)c$ .*

### 5. Some nilpotent ideal of $kG$

The present section is independent of the preceding ones. Let  $T$  be a subgroup of  $G$ , and  $m$  a left nilpotent ideal of  $kT$ . Let  $m^\sigma = \{\sigma^{-1}x\sigma \mid x \in m\}$  for  $\sigma \in G$ , and  $r_T(m)$  the right annihilator ideal of  $m$  in  $kT$ .

**Proposition 2.** *Let  $\tilde{m} = \bigcap_{\sigma \in G} kG \cdot m^\sigma$ . Then there hold the following :*

- (1)  $\tilde{m}$  is a nilpotent two-sided ideal of  $kG$ .
- (2)  $r(\tilde{m}) = \sum_{\sigma \in G} r_T(m)^\sigma kG$ .
- (3) If  $m$  is a two-sided ideal of  $kT$ , then  $\tilde{m} = \bigcap_{\sigma \in G} m^\sigma \cdot kG$ .

*Proof.* (1) For every  $\tau \in G$ , there holds  $\tilde{m}\tau \subset \bigcap_{\sigma \in G} kG \cdot m^{\sigma\tau} = \tilde{m}$ , and hence  $\tilde{m}$  is a two-sided ideal. Accordingly,  $m^2 \subset m(kG \cdot m) = m \cdot m \subset kG \cdot m^2$ , so that  $\tilde{m}^t \subset kG \cdot m^t$  for every positive integer  $t$ . We see therefore  $\tilde{m}$  is nilpotent.

(2) Since  $kT$  is Frobenius, there holds  $m = l_T(r_T(m))$ . Then, one will easily see that  $kG \cdot m = l(r_T(m) \cdot kG)$  and  $kG \cdot m^\sigma = l(r_{T\sigma}(m^\sigma) \cdot kG) = l(r_T(m)^\sigma \cdot kG)$ . Hence,  $\tilde{m} = \bigcap_{\sigma} l(r_T(m)^\sigma \cdot kG) = l(\sum_{\sigma} r_T(m)^\sigma \cdot kG)$ . Since  $kG$  is Frobenius, our assertion is clear by the last.

(3) Using freely the fact that the left annihilator ideal of a two-sided ideal in a symmetric algebra coincides with the right one (Theorem B), we obtain  $\tilde{m} = l(\sum_{\sigma} r_T(m)^\sigma \cdot kG) = r(\sum_{\sigma} l_T(m)^\sigma \cdot kG) = r(\sum_{\sigma} kG \cdot l_T(m)^\sigma) = \bigcap_{\sigma} m^\sigma \cdot kG$ .

**Theorem 3.** *Let  $\Omega$  be the set of all Sylow  $p$ -subgroups of  $G$ . Then,  $r(n) \subset \sum_{S \in \Omega} \Delta_S \cdot kG$ .*

*Proof.* In proposition 2, we set  $T = S \in \Omega$  and  $m = \text{rad}(kS) = I(S)$ . Since every Sylow  $p$ -subgroup is conjugate to each other, Proposition 2 proves that  $\tilde{m} = \bigcap_{S \in \Omega} kG \cdot I(S)$  is contained in  $n$ . We obtain therefore  $r(n) \subset r(\tilde{m}) = \sum_{S \in \Omega} \Delta_S \cdot kG$ .

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