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ON THE ANNIHILATOR IDEALS OF THE RADICAL OF A GROUP ALGEBRA

Dedicated to Professor K. Asano for his 60th birthday

YUKIO TSUSHIMA

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1. Introduction

Let $G$ be a finite group, and $k$ a field of characteristic $p$. Let $\pi$ denote the Jacobson radical of the group algebra $kG$, and $r(\pi)$ the right annihilator ideal of $\pi$. In this paper we shall show some connections between $r(\pi)$ and $p$-elements of $G$. One of them will state that $r(\pi)$ contains the sum of all $p$-elements of $G$ (including the identity). This may be regarded in a sense as a refinement of Maschke's theorem. In fact, if $p$ does not divide the order of $G$ then the identity is the only $p$-element, which implies $r(\pi) \ni 1$ and hence $\pi = 0$. On the other hand, as is easily seen from a theorem of T. Nakayama on Frobenius algebras (see §2), $r(\pi)$ is a principal ideal. We shall show that it is generated by an element which is left invariant by every automorphism of $kG$ induced by that of $G$. As an application of this fact, we shall give a lower bound for the first Cartan invariant in terms of the chief composition factors of $G$. The present study owes heavily to some general results on Frobenius algebras and symmetric algebras, which will be summarized in the next section.

NOTATION. If $A$ is a ring, rad$(A)$ will denote the Jacobson radical of $A$. For a subset $T$ of $A$, $r(T)$ and $l(T)$ will denote respectively the set of right annihilators and the set of left annihilators of $T$ in $A$. If $M$ is a subset of a finite group $G$, then $\Delta_M = \sum_{\sigma \in M} \sigma \in kG$.

2. Preliminary results

Let $A(\ni 1)$ be a finite dimensional algebra over a field $k$.

DEFINITION. A linear function $\lambda$ ($\in A^* = Hom_k(A, k)$) is called non-singular if its kernel contains no left or right ideals other than zero. While, $\lambda$ is called symmetric if $\lambda(ab) = \lambda(ba)$ for all $a, b \in A$.

If $\lambda$ is a linear function and $a \in A$, we denote by $\lambda_a$ the linear function defined
by $\lambda_a(x) = \lambda(ax), x \in A$. One may remark here that $f: a \to \lambda_a$ is a left $A$-homomorphism from $A$ into $A^*$. It is an (onto) isomorphism if and only if it is non-singular. $A$ is a Frobenius [symmetric] algebra if and only if it has (at least) one non-singular [symmetric, non-singular] linear function.

**Theorem A** (T. Nakayama [6], [8], see also [2]). Let $A$ be a Frobenius [symmetric] algebra, $\lambda$ a non-singular [symmetric, non-singular] linear function on $A$, and $\mathfrak{a}$ a two-sided ideal of $A$. If $A/\mathfrak{a}$ is Frobenius [symmetric], and $\mu$ a non-singular [symmetric, non-singular] linear function on $A/\mathfrak{a}$ then there exists an element [central element] $c \in A$ such that $\mu \psi = \lambda_c$ and $r(\mathfrak{a}) = cA$, where $\psi$ is the natural map $A \to A/\mathfrak{a}$. Conversely, if there exists an element [central element] $c \in A$ such that $r(\mathfrak{a}) = cA$ then $A/\mathfrak{a}$ is Frobenius [symmetric].

Proof. As was noted above, there exists an element $c \in A$ such that $\mu \psi = \lambda_c$. We shall show $r(\mathfrak{a}) = cA$. Since $\lambda(\mathfrak{a}c) = \lambda_c(\mathfrak{a}) = 0$ and $\lambda$ is non-singular, it follows at once $cA \subseteq r(\mathfrak{a})$. On the other hand, if $xc = 0$ then $\lambda_c(Ax) = \lambda(Axc) = 0$. Since $\lambda_c$ is non-singular as a linear function on $A/\mathfrak{a}$, it follows $x \in \mathfrak{a}$ and hence $l(cA) \subseteq \mathfrak{a}$. Recalling here that $A$ is Frobenius, we have then $cA = r(l(cA)) \supseteq r(\mathfrak{a})$. Now, suppose further both $\lambda$ and $\mu$ are symmetric. Then $\lambda(xy) = \lambda(x) = \lambda(x)(y) = cA \subseteq r(\mathfrak{a})$ and hence $\lambda_c(x) = \lambda(x) = \lambda_c(y) = \lambda(x) = \lambda_c(y)$ for all $x, y \in A$. Therefore, $yc = cy$ and $c$ is central. Next, we shall prove the converse. Suppose $r(\mathfrak{a}) = cA$. Then $\lambda_c$ gives rise to a non-symmetric linear function on $A/\mathfrak{a}$. If $c$ is central, the linear function is evidently symmetric.

**Theorem B** (T. Nakayama [6]). If $\mathfrak{a}$ is a two-sided ideal of a symmetric algebra $A$ then $r(\mathfrak{a}) = l(\mathfrak{a})$.

Proof. Let $\lambda$ be a symmetric, non-singular linear function on $A$. Then $r(\mathfrak{a}) = \{x \in A | \lambda(\mathfrak{a}x) = 0\} = \{x \in A | \lambda(x\mathfrak{a}) = 0\} = l(\mathfrak{a})$.

3. The generator of $r(\mathfrak{n})$

From now on, $k$ will denote a field of characteristic $p$, and $G$ a finite group of order $|G| = p^ng_0$, where $(p, g_0) = 1$. Let $\nu_l(p)$ denote the exponent of $p$ in the primary decomposition of an integer $l$. Let $\mathfrak{n}$ be the radical of the group algebra $kG$ as before. To be easily seen, $kG$ is a symmetric algebra through the following linear function $\lambda$ which will be fixed throughout the subsequent study: $\lambda(\Sigma_{\sigma \in G} a_{\sigma}) = a_1$, where $a_{\sigma} \in k$ and $1$ denotes the identity of $G$.

**Remark 1.** Since there exists a splitting field for $G$ which is finite separable over $k$, $kG/\mathfrak{n}$ is a separable algebra over $k$. In particular, if $K$ is an arbitrary extension field of $k$ and $\mathfrak{n}_K$ denotes the radical of $KG$, then $\mathfrak{n}_K = K\mathfrak{n}$ and $\mathfrak{n}_K \cap kG = \mathfrak{n}$. Similar relations hold for $r(\mathfrak{n})$ and the right annihilator ideal of $\mathfrak{n}_K$ in $KG$. 

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Let $\phi_1, \phi_2, \ldots, \phi_r$ be the distinct irreducible characters over a suitable splitting field for $G$ containing $k$, and let $\bar{\phi} = \sum_{i=1}^{r} \phi_i$. Then, it is clear that $\phi(\sigma)$ is contained in the prime field for every $\sigma \in G$.

**Proposition 1.** If $v = \sum_{\sigma \in G} \phi(\sigma^{-1}) \sigma$ then $r(n) = (kG)v$.

Proof. By the above remark, we may assume that $k$ is a splitting field for $G$. Then, $kG/n$ is a direct sum of total matrix algebras over $k$: $kG/n = \bigoplus_{i=1}^{r} k^{n_i}$. Let $\psi$ and $p_i$ denote respectively the natural map $kG \to kG/n$ and the projection $kG/n \to (k)^{n_i}$. Since the trace map $\text{tr}_i: (k)^{n_i} \to k$ is a symmetric, non-singular linear function, so is $\mu = \sum_{i=1}^{r} \text{tr}_i p_i$ on $kG/n$. Therefore, by Theorem A, there exists a central element $v = \sum_{\sigma} a_\sigma \sigma \in kG$ such that $\mu v = \lambda v$ and $r(n) = v(kG)$. Noting here that $\mu v = \bar{\phi}$ on $G$, we obtain $a_\sigma = \lambda(\sigma^{-1}) = \lambda(\sigma) = \bar{\phi}(\sigma^{-1})$. This completes the proof.

**Remark 2.** Let $f$ be an arbitrary automorphism of $G$. Then, it permutes the irreducible characters by $\phi_i \to \phi_i^f$, where $\phi_i^f(\sigma) = \phi_i(\sigma^f), \sigma \in G$. In particular, it follows $\bar{\phi}(\sigma^f) = \phi^f(\sigma) = \bar{\phi}(\sigma)$ for all $\sigma \in G$. Hence, regarding $f$ naturally as an automorphism of $kG$, we obtain $v^f = \sum_{\sigma} \bar{\phi}(\sigma^{-1}) \sigma^f = \sum_{\sigma} \phi^f((\sigma^f)^{-1}) \sigma^f = \sum_\sigma \phi((\tau^{-1})^f) = v$. Now, let $H$ be a normal subgroup of $G$, and $m = \text{rad } kH$. Then $I = (kG)m = m(kG)$ is a nilpotent two-sided ideal of $kG$.

**Corollary 1.** Under the above notation, there holds the following:

1. $kG/I$ is a symmetric algebra over $k$.
2. Let $(I : \pi) = \{x \in kG | nx \subseteq I\}$. Then, for an arbitrary primitive idempotent $e$ of $kG$ there holds $(kG)e/Ie \cong (I : \pi)e/Ie$.

Proof. (1) Let $r_H(\pi)$ be the right annihilator ideal of $\pi$ in $kH$. Let $v$ be as in Proposition 1 applied to $kH$. Since $G$ induces an automorphism group on $H$, Remark 2 proves that $v$ lies in the center of $kG$. Hence, $r(I) = kG \cdot r_H(I) = (kG)v$ is a principal ideal generated by a central element. Theorem A proves therefore $kG/I$ is a symmetric algebra.

(2) Evidently, the residue class $e$ of $e$ modulo $I$ is a primitive idempotent of the symmetric algebra $kG/I$. Since $(I : \pi)/I$ is the right annihilator ideal of $\pi/I = \text{rad } kG/I$, there holds then $(kG/I)e/(\pi/I)e \cong ((I : \pi)/I)e$, which proves $(kG)e/Ie \cong (I : \pi)e/Ie$.

Under the above notation, if $H$ is a non-trivial $p$-group, then it is well-known that $m$ coincides with the augmentation ideal $I(H) = \{\sum_{\sigma \in H} a_\sigma \sigma | \sum_{\sigma} a_\sigma = 0\}$, so that $r_H(\pi) = kH \cdot \Delta_H = k \Delta_H$. We obtain therefore $r(\pi) \subset r(I) = kG \cdot \Delta_H \subset I$.

**Lemma 1.** Let $P_1 \supseteq H_1$ be normal subgroups of $G$ such that $P_1/H_1$ is a
Let $I_1 = kG \cdot \text{rad} kP_1$, and $\beta_1 = \ker (kG \to k(G/H_1))$. Then, there holds the following:

1. If $e$ is a primitive idempotent of $kG$ not contained in $\beta_1$, then $(I_1 + \beta_1 : n)e/(I_1 + \beta_1) \simeq (kG)e/ne$ and $(I_1 + \beta_1)e$ contains a submodule isomorphic to $(kG)e/ne$.

2. If $P_2 \cong H_2$ are normal subgroups of $G$ containing $P_1$ such that $P_2/H_2$ is a $p$-group then $I_1 + \beta_1 \supseteq (I_1 + \beta_1 : n)$, where $I_1 = kG \cdot \text{rad} kP_2$ and $\beta_2 = \ker (kG \to k(G/H_2))$.

**Proof.** (1) Since $I_1 + \beta_1 = k(G/H_1) \cdot \text{rad} k(P_1/H_1)$ and $P_1/H_1$ is a non-trivial $p$-group, the above remark proves $r(\text{rad} k(G/H_1)) \subseteq I_1 + \beta_1/I_1$. Let $\bar{e}$ be the residue class of $e$ modulo $\beta_1$. Then, it is still primitive by assumption, and the former is evident by Corollary 1. Further, noting that $(I_1 + \beta_1 : \beta_1) \neq 0$, it follows at once $0 \neq (I_1 + \beta_1 : \beta_1) \supseteq r(n)e = (kG)e/ne$, proving the latter.

(2) As in (1), we obtain $r(\text{rad} k(G/H_2)) \subseteq I_1 + \beta_2/I_1$. If $\beta_1 = \ker (kG \to k(G/P_1))$ then $(I_1 + \beta_1 : n) + \beta_1 \subseteq (I_1 + \beta_1 : n) + \beta_1 = r(\text{rad} k(G/P_1))$. Since the natural map $k(G/P_1) \to k(G/H_2)$ is an epimorphism, it sends $r(\text{rad} k(G/P_1))$ into $r(\text{rad} k(G/H_2))$. Therefore, $(I_1 + \beta_1 : n) + \beta_2/I_1 \supseteq r(\text{rad} k(G/H_2)) \subseteq I_1 + \beta_2/I_1$.

**Theorem 1.** Let $m$ be the number of the chief composition factors of $G$ which are (non-trivial) $p$-groups. Then the first Cartan invariant $c_{11}$ is at least $m + 1$.

**Proof.** We take a primitive idempotent $e$ of $kG$ such that $(kG)e/ne$ is isomorphic to the trivial $kG$-module $k= kG$. Then, $e \Delta_0$ being non-zero, $e$ is not contained in $\ker (kG \to k(G/N))$ (= the ideal generated by $\{1 - \eta | \eta \in N\}$) for any normal subgroup $N$ of $G$. Hence, by Lemma 1, we can easily see that $(kG)e$ possesses at least $m + 1$ composition factors isomorphic to $k$, namely, $c_{11} \geq m + 1$.

4. **The sum of all $p$-elements**

First, we shall introduce some notations. Let $\bar{\mathbb{E}}$ be a primitive $g_0$-th root of unity over the prime field of characteristic $p$. In what follows, whenever we consider Brauer characters, it is assumed that there is defined (and fixed) a homomorphism $\mathbb{Z}[\bar{\mathbb{E}}] \to k[\bar{\mathbb{E}}]$ such that $\bar{\mathbb{E}}$ is the image of a primitive $g_0$-th root of unity $\mathbb{E}$ in the complex number field. As is well-known, there exists a unique (up to isomorphisms) indecomposable projective module $P_i$ such that $P_i/nP_i$ affords the irreducible character $\phi_i$. Let $\eta_i$ be the character of the representation afforded by $P_i$ and $\mathcal{U}_i$ the dimension of $P_i$. As is well-known, $\mathcal{U}_i$ is divisible by $p^t$. Let $u_i = p^t \eta_i$. We may assume, after a suitable change of index if necessary, the first $u_1, u_2, \ldots, u_t$ are all such that $\nu_p(u_i) = n$. Let $\phi_i$ and $\eta_i$ be the Brauer characters of $\phi_i$ and $\eta_i$, respectively.

Noting that $h_j = 0$ in $k$ for $t < j < r$ and $\phi_i(\sigma) = \phi_i(\sigma')$ for the $p$-regular part $\sigma'$ or $\sigma$, the orthogonality relation ([3] p. 561)
\[ \Sigma_{i=1} \eta_i(\sigma^{-1})\phi_i(\tau) = \begin{cases} |C_G(\sigma)| & \text{if } \sigma \text{ is conjugate to } \tau, \\ 0 & \text{otherwise} \end{cases} \]

implies

\[ \Sigma_{i=1} h_i \tilde{\phi}_i(\tau) = \begin{cases} g_0 & \text{if } \tau \text{ is a } p\text{-element}, \\ 0 & \text{otherwise}. \end{cases} \]

**Lemma 2.** Assume that \( k \) is a splitting field for \( G \). Let \( \mathfrak{s} = \Sigma_e (kG)e + \mathfrak{n} \), where \( e \) runs over the primitive idempotents such that \( \nu_p(\dim_k(kG)e) > n \). Let \( c \) be the sum of all \( p\)-elements of \( G \). Then there holds \( r(\mathfrak{s}) = (kG)c \).

**Proof.** We use the same notation as in the proof of Proposition 1. Then, \( kG/\mathfrak{n} = \Sigma_{i=1} (k)_{n_i} + \Sigma_{j=t+1} (k)_{n_j} \) (direct sum), where \( n_e = \dim_k P_i/\mathfrak{n} P_i \). It is clear that \( \mathfrak{s} \) is the inverse image of \( \mathfrak{b} \) by the natural map \( \psi \). Hence, it is a two-sided ideal and there holds \( kG/\mathfrak{b} = \Sigma_{i=1} (k)_{n_i} \). We set here \( \mu = \Sigma_{i=1} h_i tr_P P_i \). Since \( h_i \neq 0 \) in \( k \), \( \mu \) is a symmetric, non-singular linear function on \( kG/\mathfrak{b} \). Then, by Theorem A, there exists a central element \( \mathcal{c}' \in kG \) such that \( \mu_{\mathcal{c}'} = \lambda_{\mathcal{c}} \) and \( r(\mathfrak{s}) = (kG)c' \), where \( \mathcal{c} \) is the natural map \( kG \to kG/\mathfrak{b} \). Now, by making use of (\(*\)), we can prove \( \mathcal{c}' = g_{\mathcal{c}} c \) in the same way as in the last part of the proof of Proposition 1. This completes the proof.

Since \( \mathfrak{s} \supseteq \mathfrak{n} \), we obtain in particular \( r(\mathfrak{n}) \supseteq c \). However, in virtue of Remark 1, this holds without assuming that \( k \) is a splitting field. Thus, we have shown

**Theorem 2.** \( r(\mathfrak{n}) \) contains the sum of all \( p\)-elements of \( G \).

**Corollary 2.** Let \( x = \Sigma_{\sigma} a_{\sigma} \sigma \) be an element of \( kG \), and \( x(p) \) the sum of the coefficient \( a_{\sigma} \) of \( p\)-elements \( \sigma \). If \( x \) is in \( \mathfrak{n} \) then \( x(p) = 0 \).

**Proof.** Note that \( x(p) \) is equal to the coefficient of 1 in \( xc \). If \( x \) is in \( \mathfrak{n} \) then \( xc = 0 \), and hence \( x(p) = 0 \).

If \( e \) is a primitive idempotent of \( kG \), then \( r(\mathfrak{n})e \) is a minimal left ideal of \( kG \) and contains \( (kG)ce \).

**Corollary 3.** Let \( e \) be a primitive idempotent of \( kG \). If \( \nu_p(\dim_k(kG)e) = n \) then \( r(\mathfrak{n})e = (kG)ce \). The converse holds, provided \( k \) is a splitting field for \( G \).

**Proof.** First, we assume that \( k \) is a splitting field. By Lemma 2, if \( ce = 0 \), or what is the same, if \( e \in l(e) = \mathfrak{s} \), then \( \nu_p(\dim_k(kG)e) > n \), and conversely. We have seen therefore that \( ce = 0 \) and \( \nu_p(\dim_k(kG)e) = n \) are equivalent.

Secondly, we assume that \( \nu_p(\dim_k(kG)e) = n \). Let \( K \) be a splitting field for \( G \) containing \( k \), and \( e = \Sigma_j e_j \) a decomposition of \( e \) into orthogonal primitive idempotents of \( KG \). Then, by assumption there exists at least one \( e_j \) such that \( \nu_p(\dim_k(KGe_j)) = n \). Since \( ce_i = 0 \) by the first step, we have \( ce_i = 0 \), completing the proof.
Now, if $G$ is $p$-solvable then $\nu_p(\dim_k (kG)e)=n$ for every primitive idempotent $e$ (P. Fong [5]), whence it follows $\xi=n$ and therefore $r(n)=r(\xi)=(kG)c$.

**Corollary 4.** If $G$ is $p$-solvable then $r(n)=(kG)c$.

5. Some nilpotent ideal of $kG$

The present section is independent of the preceding ones. Let $T$ be a subgroup of $G$, and $m$ a left nilpotent ideal of $kT$. Let $m^\sigma=\{\sigma^{-1}x\sigma \mid x \in m\}$ for $\sigma \in G$, and $r_T(m)$ the right annihilator ideal of $m$ in $kT$.

**Proposition 2.** Let $\tilde{m}=\bigcap_{\sigma \in G} kG \cdot m^\sigma$. Then there hold the following:
1. $\tilde{m}$ is a nilpotent two-sided ideal of $kG$.
2. $r(\tilde{m})=\Sigma_{\sigma \in G} r_T(m)^\sigma kG$.
3. If $m$ is a two-sided ideal of $kT$, then $\tilde{m}=\bigcap_{\sigma \in G} m^\sigma \cdot kG$.

Proof. (1) For every $\tau \in G$, there holds $\tilde{m} \tau \subseteq \bigcap_{\sigma} kG \cdot m^\sigma = \tilde{m}$, and hence $\tilde{m}$ is a two-sided ideal. Accordingly, $m^t \subseteq m (kG \cdot m)=m \cdot m \subseteq kG \cdot m^t$, so that $\tilde{m}^t \subseteq kG \cdot m^t$ for every positive integer $t$. We see therefore $\tilde{m}$ is nilpotent.

(2) Since $kT$ is Frobenius, there holds $m=L_T(r_T(m))$. Then, one will easily see that $kG \cdot m=L(r_T(m) \cdot kG)$ and $kG \cdot m^\sigma=L(r_T(m^\sigma) \cdot kG)=L(r_T(m)^\sigma \cdot kG)$. Hence, $\tilde{m}=\bigcap_{\sigma} L(r_T(m)^\sigma \cdot kG)=L(\Sigma_{\sigma} r_T(m)^\sigma \cdot kG)$. Since $kG$ is Frobenius, our assertion is clear by the last.

(3) Using freely the fact that the left annihilator ideal of a two-sided ideal in a symmetric algebra coincides with the right one (Theorem B), we obtain $\tilde{m}=L(\Sigma_{\sigma} r_T(m)^\sigma \cdot kG)=r(\Sigma_{\sigma} L_T(m)^\sigma \cdot kG)=r(\Sigma_{\sigma} kG \cdot L_T(m)^\sigma) = \bigcap_{\sigma} m^\sigma \cdot kG$.

**Theorem 3.** Let $\Omega$ be the set of all Sylow $p$-subgroups of $G$. Then, $r(\Omega) \subseteq \Sigma_{\sigma \in \Omega} \Delta_{S^\sigma} kG$.

Proof. In proposition 2, we set $T=S \in \Omega$ and $m=\text{rad} (kS)=I(S)$. Since every Sylow $p$-subgroup is conjugate to each other, Proposition 2 proves that $\tilde{m}=\bigcap_{\sigma \in \Omega} kG \cdot I(S)$ is contained in $\Omega$. We obtain therefore $r(\Omega) \subseteq r(\tilde{m}) = \Sigma_{\sigma \in \Omega} \Delta_{S^\sigma} kG$.

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