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# ON THE ANNIHILATOR IDEALS OF THE RADICAL OF A GROUP ALGEBRA

Dedicated to Professor K. Asano for his 60th birthday

YUKIO TSUSHIMA

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## 1. Introduction

Let $G$ be a finite group, and $k$ a field of characteristic $p$. Let $\mathfrak{n}$ denote the Jacobson radical of the group algebra $kG$, and $r(\mathfrak{n})$ the right annihilator ideal of $\mathfrak{n}$. In this paper we shall show some connections between $r(\mathfrak{n})$ and $p$-elements of $G$. One of them will state that $r(\mathfrak{n})$ contains the sum of all $p$-elements of $G$ (including the identity). This may be regarded in a sense as a refinement of Maschke's theorem. In fact, if $p$ does not divide the order of $G$ then the identity is the only $p$-element, which implies $r(\mathfrak{n})\ni 1$ and hence $\mathfrak{n}=0$. On the other hand, as is easily seen from a theorem of T. Nakayama on Frobenius algebras (see §2), $r(\mathfrak{n})$ is a principal ideal. We shall show that it is generated by an element which is left invariant by every automorphism of $kG$ induced by that of $G$. As an application of this fact, we shall give a lower bound for the first Cartan invariant in terms of the chief composition factors of $G$. The present study owes heavily to some general results on Frobenius algebras and symmetric algebras, which will be summarized in the next section.

NOTATION. If $A$ is a ring, rad($A$) will denote the Jacobson radical of $A$. For a subset $T$ of $A$, $r(T)$ and $l(T)$ will denote respectively the set of right annihilators and the set of left annihilators of $T$ in $A$. If $M$ is a subset of a finite group $G$, then $\Delta_M=\sum_{\sigma\in M}\sigma\in kG$.

## 2. Preliminary results

Let $A(\ni 1)$ be a finite dimensional algebra over a field $k$.

DEFINITION. A linear function $\lambda$ ($\in A^*=\mathrm{Hom}_k(A, k)$) is called *non-singular* if its kernel contains no left or right ideals other than zero. While, $\lambda$ is called *symmetric if* $\lambda(ab)=\lambda(ba)$ for all $a$, $b\in A$.

If $\lambda$ is a linear function and $a\in A$, we denote by $\lambda_a$ the linear function defined by $\lambda_a(x)=\lambda(xa)$, $x\in A$. One may remark here that $f\colon a\to\lambda_a$ is a left $A$–homomorphism from $A$ into $A^*$. It is an (onto) isomorphism if and only if is non-singular. $A$ is a Frobenius [symmetric] algebra if and only if it has (at least) one non-singular [symmetric, non–singular] linear function.

**Theorem A** (T. Nakayama [6], [8], see also [2]). *Let $A$ be a Frobenius [symmetric] algebra, $\lambda$ a non-singular [symmetric, non-singular] linear function on $A$, and $\mathfrak{z}$ a two-sided ideal of $A$. If $A/\mathfrak{z}$ is Frobenius [symmetric], and $\mu$ a non-singular [symmetric, non-singular] linear function on $A/\mathfrak{z}$ then there exists an element [central element] $c\in A$ such that $\mu\psi=\lambda_c$ and $r(\mathfrak{z})=cA$, where $\psi$ is the natural map $A\to A/\mathfrak{z}$. Conversely, if there exists an element [central element] $c\in A$ such that $r(\mathfrak{z})=cA$ then $A/\mathfrak{z}$ is Frobenius [symmetric].*

Proof. As was noted above, there exists an element $c\in A$ such that $\mu\psi=\lambda_c$. We shall show $r(\mathfrak{z})=cA$. Since $\lambda(\mathfrak{z}c)=\lambda_c(\mathfrak{z})=0$ and $\lambda$ is non-singular, it follows at once $cA\subset r(\mathfrak{z})$. On the other hand, if $xc=0$ then $\lambda_c(Ax)=\lambda(Axc)=0$. Since $\lambda_c$ is non-singular as a linear function on $A/\mathfrak{z}$, it follows $x\in\mathfrak{z}$ and hence $l(cA)\subset\mathfrak{z}$. Recalling here that $A$ is Frobenius, we have then $cA=r(l(cA))\supset r(\mathfrak{z})$. Now, suppose further both $\lambda$ and $\mu$ are symmetric. Then $\lambda(xyc)=\lambda_c(xy)=\lambda_c(yx)=\lambda(yxc)=\lambda(xcy)$ for all $x, y\in A$. Therefore, $yc=cy$ and $c$ is central. Next, we shall prove the converse. Suppose $r(\mathfrak{z})=cA$. Then $\lambda_c$ gives rise to a non-singular linear function on $A/\mathfrak{z}$. If $c$ is central, the linear function is evidently symmetric.

**Theorem B** (T. Nakayama [6]). *If $\mathfrak{z}$ is a two-sided ideal of a symmetric algebra $A$ then $r(\mathfrak{z})=l(\mathfrak{z})$.*

Proof. Let $\lambda$ be a symmetric, non-singular linear function on $A$. Then $r(\mathfrak{z})=\{x\in A\,|\,\lambda(\mathfrak{z}x)=0\}=\{x\in A\,|\,\lambda(x\mathfrak{z})=0\}=l(\mathfrak{z})$.

## 3. The generator of $r(\mathfrak{n})$

From now on, $k$ will denote a field of characteristic $p$, and $G$ a finite group of order $|G|=p^n g_0$, where $(p, g_0)=1$. Let $\nu_p(l)$ denote the exponent of $p$ in the primary decomposition of an integer $l$. Let $\mathfrak{n}$ be the radical of the group algebra $kG$ as before. To be easily seen, $kG$ is a symmetric algebra through the following linear function $\lambda$ which will be fixed throughout the subsequent study: $\lambda(\sum_{\sigma\in G} a_\sigma\sigma)=a_1$, where $a_\sigma\in k$ and 1 denotes the identity of $G$.

REMARK 1. Since there exists a splitting field for $G$ which is finite separable over $k$, $kG/\mathfrak{n}$ is a separable algebra over $k$. In particular, if $K$ is an arbitrary extension field of $k$ and $\mathfrak{n}_K$ denotes the radical of $KG$, then $\mathfrak{n}_K=K\mathfrak{n}$ and $\mathfrak{n}_K\cap kG=\mathfrak{n}$. Similar relations hold for $r(\mathfrak{n})$ and the right annihilator ideal of $\mathfrak{n}_K$ in $KG$.

Let $\bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_r$ be the distinct irreducible characters over a suitable splitting field for $G$ containing $k$, and let $\bar{\phi}=\Sigma_{i=1}^r \phi_i$. Then, it is clear that $\bar{\phi}(\sigma)$ is contained in the prime field for every $\sigma\in G$.

**Proposition 1.** *If* $v=\Sigma_{\sigma\in G} \bar{\phi}(\sigma^{-1})\sigma$ *then* $r(\mathfrak{n})=(kG)v$.

Proof. By the above remark, we may assume that $k$ is a splitting field for $G$. Then, $kG/\mathfrak{n}$ is a direct sum of total matrix algebras over $k$: $kG/\mathfrak{n}=\Sigma_{i=1}^r(k)_{n_i}$. Let $\psi$ and $p_i$ denote respectively the natural map $kG\rightarrow kG/\mathfrak{n}$ and the projection $kG/\mathfrak{n}\rightarrow(k)_{n_i}$. Since the trace map $tr_i$: $(k)_{n_i}\rightarrow k$ is a symmetric, non-singular linear function, so is $\mu=\Sigma_{i=1}^r tr_i p_i$ on $kG/\mathfrak{n}$. Therefore, by Theorem $A$, there exists a central element $v=\Sigma_\sigma a_\sigma\sigma\in kG$ such that $\mu\psi=\lambda_v$ and $r(\mathfrak{n})=v(kG)$. Noting here that $\mu\psi=\bar{\phi}$ on $G$, we obtain $a_\sigma=\lambda(\sigma^{-1}v)=\lambda_v(\sigma^{-1})=\bar{\phi}(\sigma^{-1})$. This completes the proof.

REMARK 2. Let $f$ be an arbitrary automorphism of $G$. Then, it permutes the irreducible characters by $\bar{\phi}_i\rightarrow\bar{\phi}_i^f$, where $\bar{\phi}_i^f(\sigma)=\bar{\phi}_i(\sigma^f)$, $\sigma\in G$. In particular, it follows $\bar{\phi}(\sigma^f)=\bar{\phi}^f(\sigma)=\bar{\phi}(\sigma)$ for all $\sigma\in G$. Hence, regarding $f$ naturally as an automorphism of $kG$, we obtain $v^f=\Sigma_\sigma \bar{\phi}(\sigma^{-1})\sigma^f=\Sigma_\sigma \bar{\phi}((\sigma^f)^{-1})\sigma^f=\Sigma_\tau \phi(\tau^{-1})\tau=v$.

Now, let $H$ be a normal subgroup of $G$, and $\mathfrak{m}=\text{rad } kH$. Then $\mathfrak{l}=(kG)\mathfrak{m}=\mathfrak{m}(kG)$ is a nilpotent two-sided ideal of $kG$.

**Corollary 1.** *Under the above notation, there holds the following:*

(1) *$kG/\mathfrak{l}$ is a symmetric algebra over $k$.*

(2) *Let* $(\mathfrak{l}:\mathfrak{n})=\{x\in kG\mid \mathfrak{n}x\subset\mathfrak{l}\}$. *Then, for an arbitrary primitive idempotent $e$ of $kG$ there holds* $(kG)e/\mathfrak{n}e\cong(\mathfrak{l}:\mathfrak{n})e/\mathfrak{l}e$.

Proof. (1) Let $r_H(\mathfrak{m})$ be the right annihilator ideal of $\mathfrak{m}$ in $kH$. Let $v$ be as in Proposition 1 applied to $kH$. Since $G$ induces an automorphism group on $H$, Remark 2 proves that $v$ lies in the center of $kG$. Hence, $r(\mathfrak{l})=kG\cdot r_H(\mathfrak{l})=(kG)v$ is a principal ideal generated by a central element. Theorem $A$ proves therefore $kG/\mathfrak{l}$ is a symmetric algebra.

(2) Evidently, the residue class $\bar{e}$ of $e$ modulo $\mathfrak{l}$ is a primitive idempotent of the symmetric algebra $kG/\mathfrak{l}$. Since $(\mathfrak{l}:\mathfrak{n})/\mathfrak{l}$ is the right annihilator ideal of $\mathfrak{n}/\mathfrak{l}=\text{rad}(kG/\mathfrak{l})$, there holds then $(kG/\mathfrak{l})\bar{e}/(\mathfrak{n}/\mathfrak{l})\bar{e}\cong((\mathfrak{l}:\mathfrak{n})/\mathfrak{l})\bar{e}$, which proves $(kG)e/\mathfrak{n}e\cong(\mathfrak{l}:\mathfrak{n})e/\mathfrak{l}e$.

Under the above notation, if $H$ is a non-trivial $p$-group, then it is well-known that $\mathfrak{m}$ coincides with the augmentation ideal $\mathrm{I}(H)=\{\Sigma_{\sigma\in H} a_\sigma\sigma\mid\Sigma_\sigma a_\sigma=0\}$, so that $r_H(\mathfrak{m})=kH\cdot\Delta_H=k\Delta_H$. We obtain therefore $r(\mathfrak{n})\subset r(\mathfrak{l})=kG\cdot\Delta_H\subset\mathfrak{l}$.

**Lemma 1.** *Let* $P_1\supsetneqq H_1$ *be normal subgroups of* $G$ *such that* $P_1/H_1$ *is a*

*$p$-group. Let $\mathfrak{l}_1=kG\cdot\mathrm{rad}\,kP_1$, and $\mathfrak{h}_1=\ker(kG\to k(G/H_1))$. Then, there holds the following:*

(1) *If $e$ is a primitive idempotent of $kG$ not contained in $\mathfrak{h}_1$ then $(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})e/(\mathfrak{l}_1+\mathfrak{h}_1)e\cong(kG)e/\mathfrak{n}e$ and $(\mathfrak{l}_1+\mathfrak{h}_1)e$ contains a submodule isomorphic to $(kG)e/\mathfrak{n}e$.*

(2) *If $P_2\supsetneqq H_2$ are normal subgroups of $G$ containing $P_1$ such that $P_2/H_2$ is a $p$-group then $\mathfrak{l}_2+\mathfrak{h}_2\supset(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})$, where $\mathfrak{l}_2=kG\cdot\mathrm{rad}\,kP_2$ and $\mathfrak{h}_2=\ker(kG\to k(G/H_2))$.*

Proof. (1) Since $\mathfrak{l}_1+\mathfrak{h}_1=k(G/H_1)\cdot\mathrm{rad}\,k(P_1/H_1)$ and $P_1/H_1$ is a non-trivial $p$-group, the above remark proves $r(\mathrm{rad}\,k(G/H_1))\subset\mathfrak{l}_1+\mathfrak{h}_1/\mathfrak{h}_1$. Let $\bar{e}$ be the residue class of $e$ modulo $\mathfrak{h}_1$. Then, it is still primitive by assumption, and the former is evident by Corollary 1. Further, noting that $(\mathfrak{l}_1+\mathfrak{h}_1/\mathfrak{h}_1)\bar{e}\neq 0$, it follows at once $0\neq(\mathfrak{l}_1+\mathfrak{h}_1)e\supset r(\mathfrak{n})e\cong(kG)e/\mathfrak{n}e$, proving the latter.

(2) As in (1), we obtain $r(\mathrm{rad}\,k(G/H_2))\subset\mathfrak{l}_2+\mathfrak{h}_2/\mathfrak{h}_2$. If $\mathfrak{p}_1=\ker(kG\to k(G/P_1))$ then $(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})+\mathfrak{p}_1/\mathfrak{p}_1\subset(\mathfrak{l}_1+\mathfrak{p}_1/\mathfrak{p}_1:\mathfrak{n}+\mathfrak{p}_1/\mathfrak{p}_1)=r(\mathrm{rad}\,k(G/P_1))$. Since the natural map $k(G/P_1)\to k(G/H_2)$ is an epimorphism, it sends $r(\mathrm{rad}\,k(G/P_1))$ into $r(\mathrm{rad}\,k(G/H_2))$. Therefore, $(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})+\mathfrak{h}_2/\mathfrak{h}_2\subset r(\mathrm{rad}\,k(G/H_2))\subset\mathfrak{l}_2+\mathfrak{h}_2/\mathfrak{h}_2$.

**Theorem 1.** *Let $m$ be the number of the chief composition factors of $G$ which are (non-trivial) $p$-groups. Then the first Cartan invariant $c_{11}$ is at least $m+1$.*

Proof. We take a primitive idempotent $e$ of $kG$ such that $(kG)e/\mathfrak{n}e$ is isomorphic to the trivial $kG$-module $k\cong k\Delta_G$. Then, $e\Delta_G$ being non-zero, $e$ is not contained in $\ker(kG\to k(G/N))$ ($=$ the ideal generated by $\{1-\eta\mid\eta\in N\}$) for any normal subgroup $N$ of $G$. Hence, by Lemma 1, we can easily see that $(kG)e$ possesses at least $m+1$ composition factors isomorphic to $k$, namely, $c_{11}\geqslant m+1$.

## 4. The sum of all p-elements

First, we shall introduce some notations. Let $\bar{\varepsilon}$ be a primitive $g_0$-th root of unity over the prime field of characteristic $p$. In what follows, whenever we consider Brauer characters, it is assumed that there is defined (and fixed)a homomorphism $Z[\varepsilon]\to k[\bar{\varepsilon}]$ such that $\bar{\varepsilon}$ is the image of a primitive $g_0$-th root of unity $\varepsilon$ in the complex number field. As is well-known, there exists a unique (up to isomorphisms) indecomposable projective module $P_i$ such that $P_i/\mathfrak{n}P_i$ affords the irreducible character $\bar{\phi}_i$. Let $\bar{\eta}_i$ be the character of the representation afforded by $P_i$ and $u_i$ the dimension of $P_i$. As is well-known, $u_i$ is divisible by $p^n$. Let $u_i=p^n h_i$. We may assume, after a suitable change of index if necessary, the first $u_1, u_2, \cdots, u_t$ are all such that $\nu_p(u_i)=n$. Let $\phi_i$ and $\eta_i$ be the Brauer characters of $\phi_i$ and $\eta_i$, respectively.

Noting that $h_j=0$ in $k$ for $t<j\leq r$ and $\phi_i(\sigma)=\phi_i(\sigma')$ for the $p$-regular part $\sigma'$ or $\sigma$, the orthogonality relation ([3] p. 561)

$$\Sigma_{i=1}^{r}\eta_i(\sigma^{-1})\phi_i(\tau)=\begin{cases}|C_G(\sigma)| & \text{if } \sigma \text{ is conjugate to } \tau,\\ 0 & \text{otherwise}\end{cases}$$

implies

$$(*)\qquad \Sigma_{i=1}^{t} h_i\bar{\phi}_i(\tau)=\begin{cases}g_0 & \text{if } \tau \text{ is a } p\text{-element},\\ 0 & \text{otherwise.}\end{cases}$$

**Lemma 2.** *Assume that $k$ is a splitting field for $G$. Let $\mathfrak{z}=\Sigma_e(kG)e+\mathfrak{n}$, where $e$ runs over the primitive idempotents such that $\nu_p(\dim_k(kG)e)>n$. Let $c$ be the sum of all $p$-elements of $G$. Then there holds $r(\mathfrak{z})=(kG)c$.*

Proof. We use the same notation as in the proof of Proposition 1. Then, $kG/\mathfrak{n}=\Sigma_{i=1}^{t}(k)_{n_i}+\Sigma_{j=t+1}^{r}(k)_{n_j}$ (direct sum), where $n_i=\dim_k P_i/\mathfrak{n}P_i$. It is clear that $\mathfrak{z}$ is the inverse image of $\Sigma_{j=t+1}^{r}(k)_{n_j}$ by the natural map $\psi$. Hence, it is a two-sided ideal and there holds $kG/\mathfrak{z}\cong\Sigma_{i=1}^{t}(k)_{n_i}$. We set here $\mu=\Sigma_{i=1}^{t}h_i tr_i p_i$. Since $h_i\neq 0$ in $k$, $\mu$ is a symmetric, non-singular linear function on $kG/\mathfrak{z}$. Then, by Theorem $A$, there exists a central element $c'\in kG$ such that $\mu\psi'=\lambda_{c'}$ and $r(\mathfrak{z})=(kG)c'$, where $\psi'$ is the natural map $kG\to kG/\mathfrak{z}$. Now, by making use of (*), we can prove $c'=g_0c$ in the same way as in the last part of the proof of Proposition 1. This completes the proof.

Since $\mathfrak{z}\supset\mathfrak{n}$, we obtain in particular $r(\mathfrak{n})\ni c$. However, in virtue of Remark 1, this holds without assuming that $k$ is a splitting field. Thus, we have shown

**Theorem 2.** *$r(\mathfrak{n})$ contains the sum of all $p$-elements of $G$.*

**Corollary 2.** *Let $x=\Sigma_\sigma a_\sigma\sigma$ be an element of $kG$, and $x(p)$ the sum of the coefficient $a_\sigma$ of $p$-elements $\sigma$. If $x$ is in $\mathfrak{n}$ then $x(p)=0$.*

Proof. Note that $x(p)$ is equal to the coefficient of 1 in $xc$. If $x$ is in $\mathfrak{n}$ then $xc=0$, and hence $x(p)=0$.

If $e$ is a primitive idempotent of $kG$, then $r(\mathfrak{n})e$ is a minimal left ideal of $kG$ and contains $(kG)ce$.

**Corollary 3.** *Let $e$ be a primitive idempotent of $kG$. If $\nu_p(\dim_k(kG)e)=n$ then $r(\mathfrak{n})e=(kG)ce$. The converse holds, provided $k$ is a splitting field for $G$.*

Proof. First, we assume that $k$ is a splitting field. By Lemma 2, if $ce=0$, or what is the same, if $e\in l(c)=\mathfrak{z}$, then $\nu_p(\dim_k(kG)\mathrm{e})>n$, and coversely. We have seen therefore that $ce\neq 0$ and $\nu_p(\dim_k(kG)e)=n$ are equivalent.

Secondly, we assume that $\nu_p(\dim_k(kG)e)=n$. Let $K$ be a splitting filed for $G$ containing $k$, and $e=\Sigma_j e_j$ a decomposition of $e$ into orthogonal primitive idempotents of $KG$. Then, by assumption there exists at least one $e_i$ such that $\nu_p(\dim_k(KGe_i)=n$. Since $ce_i\neq 0$ by the first step, we have $ce\neq 0$, completing the proof.

Now, if $G$ is $p$-solvable then $\nu_p(\dim_k (kG)e)=n$ for every primitive idempotent $e$ (P. Fong [5]), whence it follows $\mathfrak{z}=\mathfrak{n}$ and therefore $r(\mathfrak{n})=r(\mathfrak{z})=(kG)c$.

**Corollary 4.** *If $G$ is $p$-solvable then $r(\mathfrak{n})=(kG)c$.*

## 5. Some nilpotent ideal of $kG$

The present section is independent of the preceding ones. Let $T$ be a subgroup of $G$, and $\mathfrak{m}$ a left nilpotnet ideal of $kT$. Let $\mathfrak{m}^\sigma=\{\sigma^{-1}x\sigma \mid x\in\mathfrak{m}\}$ for $\sigma\in G$, and $r_T(\mathfrak{m})$ the right annihilator ideal of $\mathfrak{m}$ in $kT$.

**Proposition 2.** *Let $\tilde{\mathfrak{m}}=\bigcap_{\sigma\in G} kG\cdot\mathfrak{m}^\sigma$. Then there hold the following:*
(1) *$\tilde{\mathfrak{m}}$ is a nilpotent two-sided ideal of $kG$.*
(2) *$r(\tilde{\mathfrak{m}})=\Sigma_{\sigma\in G}\, r_T(\mathfrak{m})^\sigma kG$.*
(3) *If $\mathfrak{m}$ is a two-sided ideal of $KT$, then $\tilde{\mathfrak{m}}=\bigcap_{\sigma\in G}\mathfrak{m}^\sigma\cdot kG$.*

Proof. (1) For every $\tau\in G$, there holds $\tilde{\mathfrak{m}}\tau\subset\bigcap_\sigma kG\cdot\mathfrak{m}^{\sigma\tau}=\tilde{\mathfrak{m}}$, and hence $\tilde{\mathfrak{m}}$ is a two-sided ideal. Accordingly, $\mathfrak{m}^2\subset\mathfrak{m}(kG\cdot\mathfrak{m})=\mathfrak{m}\cdot\mathfrak{m}\subset kG\cdot\mathfrak{m}^2$, so that $\tilde{\mathfrak{m}}^t\subset kG\cdot\mathfrak{m}^t$ for every positive integer $t$. We see therefore $\tilde{\mathfrak{m}}$ is nilpotent.

(2) Since $kT$ is Frobenius, there holds $\mathfrak{m}=l_T(r_T(\mathfrak{m}))$. Then, one will easily see that $kG\cdot\mathfrak{m}=l(r_T(\mathfrak{m})\cdot kG)$ and $kG\cdot\mathfrak{m}^\sigma=l(r_{T\sigma}(\mathfrak{m}^\sigma)\cdot kG)=l(r_T(\mathfrak{m})^\sigma\cdot kG)$. Hence, $\tilde{\mathfrak{m}}=\bigcap_\sigma l(r_T(\mathfrak{m})^\sigma\cdot kG)=l(\Sigma_\sigma r_T(\mathfrak{m})^\sigma\cdot kG)$. Since $kG$ is Frobenius, our assertion is clear by the last.

(3) Using freely the fact that the left annihilator ideal of a two-sided ideal in a symmetric algebra coincides with the right one (Theorem B), we obtain $\tilde{\mathfrak{m}}=l(\Sigma_\sigma r_T(\mathfrak{m})^\sigma\cdot kG)=r(\Sigma_\sigma l_T(\mathfrak{m})^\sigma\cdot kG)=r(\Sigma_\sigma kG\cdot l_T(\mathfrak{m})^\sigma)=\bigcap_\sigma\mathfrak{m}^\sigma\cdot kG$.

**Theorem 3.** *Let $\Omega$ be the set of all Sylow $p$-subgroups of $G$. Then, $r(\mathfrak{n})\subset\Sigma_{S\in\Omega}\Delta_S\cdot kG$.*

Proof. In proposition 2, we set $T=S\in\Omega$ and $\mathfrak{m}=\mathrm{rad}\,(kS)=I(S)$. Since every Sylow $p$-subgroup is conjugate to each other, Proposition 2 proves that $\tilde{\mathfrak{m}}=\bigcap_{S\in\Omega} kG\cdot I(S)$ is contained in $\mathfrak{n}$. We obtain therefore $r(\mathfrak{n})\subset r(\tilde{\mathfrak{m}})$ $=\Sigma_{S\in\Omega}\Delta_S\cdot kG$.

OSAKA CITY UNIVERSITY

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