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ON A CHARACTERIZATION OF CERTAIN ADDITIVE FUNCTIONALS OF MARKOV PROCESSES

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0. Introduction

The problem of characterizing positive additive functionals of Markov processes by means of their expectation (called characteristic) has been studied by Volkonsky [5], Meyer [Part II of 3], and Motoo and S. Watanabe [1]. In the present paper we shall give a characterization of square-integrable additive functionals with expectation zero.

Our method is different from, and more elementary than, those of Meyer, Motoo and S. Watanabe; it is a version of the method adopted by Meyer [Part I of 3, 4] in the study of absolute continuity of two Markov processes. Our method is also used for characterizing almost additive functionals without assuming the strong Markov property.

Let $\mathbf{X}=(x_t, P_x, x\in S)$ be a Markov process with a Markov transition function $(P_t(x, S)=1)$. Let A be a locally integrable (not necessarily positive) almost additive functional of $\mathbf{X}$, and let us define a system of kernels(1) $(Q_t(x, dy))$ on S by

$$Q_t(x, E) = E_x(A_t\cdot 1_E(x_t))\,, \tag{0.1}$$

where 1_E is the indicator of the set E. $Q_t(x, dy)$ is absolutely continuous with respect to the transition function $P_t(x, dy)=P_x(x_t\in dy)$, and the following equation, called the *characteristic equation*, holds;

$$P_sQ_t+Q_sP_t = Q_{s+t} \qquad \text{for any} \quad s, t\geqq 0\,. \tag{0.2}$$

If a system of kernels $(Q_t(x, dy); t\geqq 0)$ on S satisfies the characteristic equation and if each $Q_t(x, dy)$ is absolutely continuous with respect to $P_t(x, dy)$ we will call it *a system of characteristic kernels*. The density of $Q_t(x, dy)$ with

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(1) A map $k(x, dy)$ from $S\times \boldsymbol{F}(S)$ to $(-\infty, \infty]$ is called a *kernel* if it satisfies the following properties:
(i) For each $E\in\mathbf{F}(S)$, $k(\cdot, E)$ is a $\mathbf{F}(S)$-measurable function on S.
(ii) For each $x\in S$, $k(x, \cdot)$ is a measure on $\mathbf{F}(S)$ with a finite total variation.

respect to $P_t(x, dy)$ is denoted by $q_t(x, y)$. As stated above any locally integrable almost additive functional A defines a system of characteristic kernels by (0.1). In such a case we will say that A *represents* (Q_t). We can verify (section 2) that there exists at most one almost additive functional that represents (Q_t). If (Q_t) is represented by (A_t) then we have (section 2)

$$q_t(x, y) = E_x(A_t \mid x_t = y) . \tag{0.3}$$

Therefore $q_t(x, y)$ is the conditional expectation of the corresponding almost additive functional on the set of all trajectories that start from x and stay at y at the time t. Recall that if (A_t) is a positive additive functional it is characterized by its characteristic

$$m(t, x) = E_x(A_t) = Q_t(x, S) .$$

However if (A_t) has expectation zero it is evident that (A_t) can not be characterized by $m(t, x) = Q_t(x, S)$. Hence it is natural to ask under what conditions a given system of characteristic kernels is represented by a locally integrable almost additive functional. This problem is solved in Theorem 2 of section 3.

In section 4 the system of characteristic kernels which corresponds to a square-integrable almost additive functionals with expectation zero is discussed. In the case of one-dimentional Brownian motion starting from the origin the trajectory itself is a square-integrable additive functional with expectation zero, and it is characterized by the coordinate function $q_t(0, y) = y$. In general the following theorem will be proved.

Theorem 1. *Let $(Q_t(x, dy);\ t \geqq 0)$ be a system of characteristic kernels. Then there exists a square-integrable almost additive functional with mean zero that represents (Q_t) if and only if there exists a characteristic $m(t, x)$*(2) *such that*

$$\int_S q_t(x, y)^2 P_t(x, dy) \leqq m(t, x)$$

for any $x \in S$ and $t \geqq 0$, and $Q_t 1 = 0$ for any $t \geqq 0$.

In section 5 we shall give some examples of (Q_t) and $(q_t(x, y))$.

In section 6 and thereafter we shall confine ourselves to deal with a Hunt process with a reference measure. We shall prove that for each square-integrable almost additive functional with expectation zero there exists an equivalent(3) additive functional, and consequently, in the result of Theorem 1 we can replace the term "almost additive" by the term "additive".

Finally in section 7 some properties of the Laplace transform of charac-

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(2) c.f. Definition 4.1.

(3) (A_t) and (B_t) are equivalent if $A_t = B_t$ (a.s.) for any t.

teristic kernels are studied. It seems important to know what properties of the processes are reflected to the quantities described in section 7.

1. Preliminaries

Let S be a locally compact Hausdorff space with a countable base for its topology. Let Δ be a point adjoined to S as the point at infinity if S is non-compact and as an isolated point if S is compact. Let $\mathbf{B}(S)$ be the topological Borel field of S and let $M(S)$ be the set of all bounded positive measures on S, and $\mathbf{F}(S)$ be the completion of $\mathbf{B}(S)$ by the system of measures in $M(S)$. Let $B(S)$ and $F(S)$ be the sets of all bounded $\mathbf{B}(S)$-measurable functions and bounded $\mathbf{F}(S)$-measurable functions. Let W denote the space of all maps w from $[0, \infty]$ into $S \cup \{\Delta\}$ such that w is right continuous and has left-hand limits on $[0, \infty)$, $w(\infty)=\Delta$, and if $w(t_0)=\Delta$ then $w(t)=\Delta$ for all $t \geqq t_0$. As usual $x_t(w)=w(t)$ denotes the t-th coordinate function. The shift operator ϑ_t is defined on W by $\vartheta_t w(s)=w(t+s)$, $\vartheta_t w$ is also denoted as w_t^+. Let $\mathcal{B}_t$ be the σ-algebra of subsets of W generated by the cylinder sets $\{w; x_s(w) \in A\{ (s \leqq t, A \in \mathbf{B}(S))$, and $\mathcal{B}=\mathcal{B}_\infty$. We assume that for each x in $S \cup \{\Delta\}$ we are given a probability measure on $\mathcal{B}$ satisfying:

(i) $P_x(x_0(w)=x)=1$ for any $x \in S \cup \{\Delta\}$.

(ii) $P_x(\Lambda)$ is $\mathbf{B}(S)$-measurable function of x for any $\Lambda \in \mathcal{B}$.

(iii) (Conservativity) $P_x(x_t \in S)=1$ for each $x \in S$ and $0 \leqq t < \infty$.

(iv) (Markov property) For each $x \in S$, $t>0$, and bounded $\mathcal{B}$-measurable function η on W, we have

$$E_x(\eta(w_t^+) \mid \mathcal{B}_t) = E_{x_t}(\eta) \quad \text{a.s. } P_x .$$

For each $\mu \in M(S)$ we define $P_\mu(\Lambda)=\int P_x(\Lambda)\mu(dx)$ for $\Lambda \in \mathcal{B}$ and then define $\mathcal{F}(\mathcal{F}_t)$ to be the interesection over all $\mu \in M(S)$ of the P_μ-completions of $\mathcal{B}(\mathcal{B}_t)$. Henceforth almost surely (abbreviated *a.s.*) means almost surely with respect to each P_x. Further we assume:

(v) $\mathcal{F}_t=\mathcal{F}_{t+} \equiv \bigcap_{t<s} \mathcal{F}_s$ for each $t \geqq 0$.

In this paper we shall deal with a Markov process $\mathbf{X}=(W, \mathcal{F}_t, x_t(w), \vartheta_t, P_x, x \in S \cup \{\Delta\})$ satisfying (i)—(v). As usual $P_t(x, dy)=P_x(x_t \in dy)$ is the transition probability of $\mathbf{X}$.

DEFINITION 1.1. A $(-\infty, \infty]$-valued function $Y(t, w)$ on $[0, \infty) \times W$ is said to be an *additive functional of* $\mathbf{X}$ if the following conditions are satisfied:

(1.1) For fixed t, $Y(t, w)$ is $\mathcal{F}_t$-measurable. There exists $W' \subset W$, $W' \in \mathcal{F}$ such that $\vartheta_t W' \subset W'$ and $P_x(W')=1$ for all $x \in S$, and for any fixed $w \in W'$,

(1.2) $Y(\cdot, w)$ is finite and right continuous and has left limits.

(1.3) $Y(t+s, w) = Y(t, w) + Y(s, \vartheta_t w)$ for any t, s.

(1.4) $Y(0, w) = 0$.

DEFINITION 1.2. $Y(t, w)$ is said to be an *almost additive functional of* $\mathbf{X}$ if it satisfies (1.1) and

(1.2′) $Y(\cdot, w)$ is finite and right continuous and has left limits. (a.s.),

(1.4′) $Y(0, w) = 0$ (a.s.),

(1.5) $P_x(Y_{s+t} = Y_t + Y_s \circ \vartheta_t) = 1$ for any $x \in S$ and t, s.

An almost additive functional Y is called *locally integrable* if $E_x(|Y_t|) < \infty$ for any $t \geqq 0$ and $x \in S$.

DEFINITION 1.3. Let $\mathfrak{M}$ (resp. $\mathfrak{M}^a$) be the class of all additive (resp. almost additive) functionals such that

(1.6) $E_x(Y_t^2) < \infty$ and,

(1.7) $E_x(Y_t) = 0$ for any s, t and $x \in S$.

We now list a fundamental lemma on conditional expectations.

Lemma 1.1. *Let η be a $\mathcal{F}$-measurable function on W and $\mathcal{G}_t$ be a countably generated sub σ-algebra of $\mathcal{F}_t$ that makes $x_0(w)$ measurable. Suppose η is P_μ-integrable for all $\mu \in M(S)$, then we can take a version of conditional expectations of η relative to $\mathcal{G}_t$ which is independent from the initial distribution $\mu \in M(S)$.*

Proof. For any bounded $\mathcal{G}_t$-measurable function ξ on W, $E_x(\xi \cdot E_x(\eta \mid \mathcal{G}_t)) = E_x(\xi \cdot \eta)$ is a $\mathbf{F}(S)$-measurable function on S. Thus for each $x \in S$ $R(x, dw) = E_x(\eta \mid \mathcal{G}_t)(w) \cdot P_x(dw)$ is a measure on $(W, \mathcal{G}_t)$ and for each $\Lambda \in \mathcal{G}_t$ $R(x, \Lambda)$ is a $\mathbf{F}(S)$-measurabel function. Therefore a $\mathbf{F}(S) \times \mathcal{G}_t$-measurable function $e(x, w)$ on $S \times W$ such that $e(x, w) = E_x(\eta \mid \mathcal{G}_t)(w)$ (a.s. P_x) exists [See 2. p.p. 194]. Set $d(w) = e(x_0(w), w)$. It is easy to verify

$$d(w) = E_\mu(\eta \mid \mathcal{G}_t)(w) . \qquad \text{a.s. } P_\mu .$$

for any $\mu \in M(S)$.
We shall write the conditional expectation obtained in the above lemma by $E(\cdot \mid \mathcal{G}_t)$ omitting the subscript x or μ.

2. The densities of characteristic kernels

Let $(Q_t(x, dy); t \geqq 0)$ be a system of kernels(4) on $S \times \mathbf{F}(S)$, and $(Q_t; t \geqq 0)$ be the corresponding linear operators on $F(S)$ defined as follows;

(4) c.f. footnote (1)

$$(2.1) \qquad Q_tf(x) = \int_S Q_t(x, dy)f(y), \quad f \in F(S).$$

Definition 2.1. A system of kernels $(Q_t(x, dy))$ is called *a system of characteristic kernels* if the following properties are satisfied:

1. $Q_t(x, dy)$ is absolutely continuous with respect to $P_t(x, dy)$ for each $t \geqq 0$, and if we denote by $|Q_s|(x, dy)$ the total variation of the measure $Q_s(x, dy)$ then we have $P_t|Q_s|\ 1 < \infty$ for any $s, t \geqq 0$.
2. For each $x \in S$, $s, t \geqq 0$ and $f \in F(S)$.

$$(2.2) \quad Q_sP_tf(x) + P_sQ_tf(x) = Q_{s+t}f(x).$$

(2.2) is called the *characteristic equation.*

Proposition 2.1. *Let (A_t) be an almost additive functional with $E_x(|A_t|) < \infty$ for every $x \in S$ and $t \geqq 0$, and let $Q_t(x, E) = E_x(A_t \cdot 1_E(x_t))$ $(E \in \mathbf{F}(S))$. Then $(Q_t(x, dy);\ t \geqq 0)$ is a system of characteristic kernels.*

Proof. That each $Q_t(x, dy)$ is a kernel follows from $E_x(|A_t|) < \infty$. The absolute continuity of $Q_t(x, dy)$ relative to $P_t(x, dy)$ is obvious and

$$\begin{aligned} P_t|Q_s|1(x) &= E_x(E_{x_t}(|A_s|)) \\ &= E_x(|A_{t+s} - A_s|) \leqq E_x(|A_{t+s}|) + E_x(|A_s|) < \infty . \end{aligned}$$

The characteristic equation follows from the almost additivity of (A_t) by a simple calculation.

Proposition 2.2. *It is necessary and sufficient for $(Q_t(x, dy);\ t \geqq 0)$ to be a system of characteristic kernels that the following properties are satisfied:*

1. *For each $t \geqq 0$ there exists a $\mathbf{F}(S) \times \mathbf{F}(S)$-measurable function $q_t(x, y)$ such that it is $P_t(x, \cdot)$-integrable as a function of y and $Q_t(x, dy) = q_t(x, y) P_t(x, dy)$, and $E_x(|q_t(x_s, x_{s+t})|) < \infty$ for any $s \geqq 0$ and $t \geqq 0$.*
2. *For any s, t and u $(0 \leqq s \leqq t \leqq u)$ we have*

$$(2.3) \qquad E(q_{t-s}(x_s, x_t) + q_{u-t}(x_t, x_u) | x_s, x_u)^{(5)} = q_{u-s}(x_s, x_u) \qquad \text{(a.s.)}$$

Proof. Suppose $Q_t(x, dy)$ is absolutely continuous with respect to $P_t(x, dy)$. Since $Q_t(x, dy)$ is a measure with a finite total variation, the well known method of differentiating one measure with respect to another [2. p.p. 194] is also applicable here, and we can find a $\mathbf{F}(S) \times \mathbf{F}(S)$-measurable version of the densities of $Q_t(x, dy)$ with respect to $P_t(x, dy)$. Once the $\mathbf{F}(S) \times \mathbf{F}(S)$-measurability of

(5) $E(\cdot\,|x_s, x_u)$ means a conditional expectation with respect to the completed Borel field in W generated by x_s and x_u, which does not depend on the initial distribution. (c.f. Lemma 1.1.)

$q_t(x, y)$ is proved, the proof of the equivalence of (2.2) and (2.3) is reduced to a simple calculation on the Markov property, so is omitted.

REMARK. From (2.3) we can prove

$$E\Big(\sum_{j=1}^{n} q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j}) \,|\, x_{t_0}, x_{t_n}\Big) = q_{t_n-t_0}(x_{t_0}, x_{t_n}) \qquad \text{(a.s.)} \tag{2.4}$$

for any sequence $(t_j)_{j=0}^{n}$ such as $t_0 \leqq t_1 \leqq \cdots \leqq t_n$.

The following proposition is an immediate consequence of Proposition 2.1. and 2.2.

Proposition 2.3. *Let* (A_t) *and* (Q_t) *be those of Proposition* 2.1., *then we have*

$$E(A_t - A_s \,|\, x_s, x_t) = q_{t-s}(x_s, x_t) \qquad \text{(a.s.)} \tag{2.5}$$

for any $0 \leqq s \leqq t$.

We shall confine ourselves to consider only systems of characteristic kernels in the sequel.

Given any $t \geqq 0$, let $\{T_t^n = (0 = t_0 < t_1 < \cdots < t_n < t);\ n \geqq 1\}$ be a system of partitions of the interval $[0, t]$. Let $\mathcal{B}_t^n$ be the σ-algebra generated by $(x_{t_j}, 0 \leqq j \leqq n, x_t)$, and $\mathcal{F}_t^n$ be its completion. Set

$$A_t^n(w) = \sum_{j=1}^{n} q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j}) + q_{t-t_n}(x_{t_n}, x_t) \tag{2.6}$$

Proposition 2.4. *For each fixed* t $(A_t^n, \mathcal{F}_t^n, P_x)_{n \geqq 1}$ *is a martingale for* $\forall x \in S$, *and*

$$Q_t f(x) = E_x(A_t^n \cdot f(x_t)) . \tag{2.7}$$

Proof. If we let $T_t^{n+1} = (0 = t_0 < t_1 \cdots < t_k < u < t_{k+1} < \cdots < t_n < t)$,

then

$$\begin{aligned} E(A_t^{n+1} \,|\, \mathcal{F}_t^n) &= \sum_{j=1}^{k} q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j}) \\ &+ E(q_{u-t_k}(x_{t_k}, x_u) + q_{t_{k+1}-u}(x_u, x_{t_{k+1}}) \,|\, \mathcal{F}_n^t) \\ &+ \sum_{j=k+2}^{n} q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j}) + q_{t-t_n}(x_{t_n}, x_t) . \end{aligned}$$

The right-hand side is eual to A_t^n by the Markov property and (2.3). We have $Q_t f(x) = E_x(q_t(x, x_t) \cdot f(x_t)) = E_x(A_t^n \cdot f(x_t))$ from (2.4).

In the next section we will construct an almost additive functional from a given system of characteristic kernels. Here we guarantee its uniqueness.

Proposition 2.5. *There exists at most one*[6] *locally integrable almost additive functional which represents* (Q_t)[7].

Proof. If we are given two almost additive functionals (A_t) and (B_t) such that $E.(|A_t|)<\infty$, $E.(|B_t|)<\infty$ and $Q_tf(\cdot)=E.(A_t\cdot f(x_t))=E.(B_t\cdot f(x_t))$ for any $t\geqq 0$ and $f\in F(S)$, applying the Markov property, we have $E.(A_t\cdot\eta)=E.(B_t\cdot\eta)$ for any $t\geqq 0$ and any bounded $\mathcal{F}_t$-measurable function η. Therefore $A_t=B_t$ (a.s.) for any $t\geqq 0$.

3. Construction of an almost additive functional

Throughout this section we assume that we are given a system of characteristic kernels $(Q_t(x, dy); t\geqq 0)$. Let $\mathcal{S}_t$ be the set of all finite partitions of the interval $[0, t]$. For each $T=(0=t_0<t_1<\cdots<t_n<t)\in\mathcal{S}_t$ let $\mathcal{B}_t(T)$ be the σ-algebra generated by $(x_{t_j}, 0\leqq j\leqq n, x_t)$ and $\mathcal{F}_t(T)$ be its completion. As in (2.6) we define the random variable $A_t(T)$ by

$$(3.1)\qquad A_t(T)=\sum_{j=1}^{n} q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j})+q_{t-t_n}(x_{t_n}, x_t)\,.$$

Proposition 2.4 shows that, for any T, $T'\in\mathcal{S}_t$ such that $T\subset T'$,

$$(3.2)\qquad A_t(T)=E_x(A_t(T')\,|\,\mathcal{F}_t(T))$$

holds. This property will be used later.

In this section we shall frequently deal with the following particular type of partitions. Let D be a denumerable dense subset of $[0, \infty]$ and let $\{T_t^n(D)\}_{n\geqq 1}$ be a system of partitions defined as follows;

(3.3) $T_t^n(D)\in\mathcal{S}_t$, and, for each n, $s\in T_t^n(D)$ implies $s\in D$ and $s\in T_t^{n+1}(D)$,

(3.4) $T_t^n(D)$ increases with n so as to cover all points in $D\cap[0, t]$.

We have $\bigvee_n \mathcal{F}_t^n(D)=\mathcal{F}_t$ by virtue of the right continuity of the trajectories, where $\mathcal{F}_t^n(D)=\mathcal{F}_t(T_t^n(D))$. $A_t(T_t^n(D))$ is also denoted as $A_t^n(D)$.

Theorem 2. *Let (Q_t) be a system of characteristic kernels. Then there exists a locally integrable almost additive functional (A_t) with nonnegative expectation that represents (Q_t), if and only if; for any $t\geqq 0$ and $x\in S$,*

(3.5) $Q_t1(x)\geqq 0,\ \lim_{h\downarrow 0} Q_h1(x)=0,$ *and*

(6) Up to equivalence.

(7) On the word 'represent', see the introduction.

(3.6) $\{A_t(T);\ T\in\mathcal{S}_t\}$ *is* P_x*-uniformly integrable, are satisfied.*

Proof. Suppose that there exists a locally integrable almost additive functional (A_t) with nonnegative mean that represents (Q_t), then from (2.5),

$$E.(A_{t_{k+1}}-A_{t_k}\,|\,x_{t_k},\ x_{t_{k+1}}) = q_{t_{k+1}-t_k}(x_{t_k},\ x_{t_{k+1}}) \qquad \text{(a.s.)}$$

holds for any $t_k,\ t_{k+1}\in T\in\mathcal{S}_t$, so

$$E.(A_{t_{k+1}}-A_{t_k}\,|\,\mathcal{F}_t(T)) = q_{t_{k+1}-t_k}(x_{t_k},\ x_{t_{k+1}}) \qquad \text{(a.s.)}$$

Therefore

$$\begin{aligned} A_t(T) &= \sum_{j=1}^{n} q_{t_j-t_{j-1}}(x_{t_{j-1}},\ x_{t_j})+q_{t-t_n}(x_{t_n},\ x_t) \\ &= E.(\sum_{j=1}^{n}(A_{t_j}-A_{t_{j-1}})+A_t-A_{t_n}\,|\,\mathcal{F}_t(T)) \\ &= E.(A_t\,|\,\mathcal{F}_t(T)) \qquad \text{(a.s.)} \end{aligned}$$

This together with the locally integrability of (A_t) yield that $\{A_t(T),\ T\in\mathcal{S}_t\}$ is uniformly integrable. $Q_t1\geqq 0$ follows from $E.(A_t)\geqq 0$. To prove (3.5) we note that $(A_s;\ 0\leqq s\leqq\delta)$ is a submartingale on the closed interval $[0,\ \delta]$ for any $\delta\geqq 0$, and consequently $(A_s;\ 0\leqq s\leqq\delta)$ is uniformly integrable. These and the right continuity of (A_t) prove

$$\lim_{h\downarrow 0}\ \downarrow Q_h1(x) = \lim_{h\downarrow 0} E_x(A_h) = E_x(A_0) = 0\,.$$

The proof of the converse is carried as follows: Let $(T_t^n)_{n\geqq 1}$ be a system of partitions of $[0,\ t]$ consisting of rational points that satisfies (3.3) and (3.4) ($D=$ the set of all rational points), and let $A_t^n=A_t^n(D)$. Since $(A_t^n)_{n\geqq 1}$ is a uniformly integrable martingale, $\lim A_t^n$ exists in the sense of the convergence in L^1. We will show that this limit satisfies (1.1) (1.4) and (1.5) of Definition 1.2. Next we shall modify it so as to satisfy (1.2′) by making use of the fact that it is a submartingale on R_+.

(1) From the last paragraph of section 2, $(A_t^n,\ \mathcal{F}_t^n,\ P_x)_{n\geqq 1}$ is a martingale, where $\mathcal{F}_t^n=\mathcal{F}_t^n(D)$ for D the set of all rational points. Since it is uniformly integrable, by a general theorem on martingale

$$\begin{aligned} &P_x(\lim A_t^n = \tilde{A}_t) = 1 \\ &A_t^n = E(\tilde{A}_t\,|\,\mathcal{F}_t^n) \qquad \text{(a.s.)} \end{aligned}$$

where $\tilde{A}_t=\limsup A_t^n$. Moreover A_t^n converges to $\tilde{A}_t$ in $L^1(\mathcal{F}_t,\ P_x)$. $\tilde{A}_t$ is obviously $\mathcal{F}_t$-measurable.

Now let D be any denumerable subset of $[0,\ \infty)$ that contains all rational points, and $\{S_t^m=(0=s_0<s_1<\cdots<s_m<t)\subset D\}_{m\geqq 1}$ be a system of partitions of $[0,\ t]$ which satisfies (3.3) and (3.4).

Set

$$B_t^m = \sum_{j=1}^{m} q_{s_j - s_{j-1}}(x_{s_{j-1}}, x_{s_j}) + q_{t-s_m}(x_{s_m}, x_t) \quad \text{and}$$
$$B_t^D = \text{limsup } B_t^m .$$

Since D contains all rational points, $(S_t^m)_{m\geq 1} \supset (T_t^n)_{n\geq 1}$, so we can verify, from (3.2),

$$A_t^n = E(B_t^D \mid x_s, s \in T_t^n) \qquad \text{(a.s.)}$$

Tending n to infinity, we have

$$\begin{aligned}\tilde{A}_t &= E(B_t^D \mid x_t; x_s, s \text{ runs through all rational points in } [0, t]) \\ &= E(B_t^D \mid \mathcal{F}_t) = B_t^D \qquad \text{(a.s.)}\end{aligned}$$

If we take the next special D in the above argument,

$$D = \left\{\begin{array}{l}\text{all rational points in } [0, \infty) \\ s \\ (s+r;\ r, \text{ rational points of } [0, t-s))\end{array}\right\},$$

and if we put $B_t^{2n} = A_s^n + A_{t-s}^n \cdot \vartheta_s$, then we have for any $x \in S$,

$$P_x(\tilde{A}_t = \lim_{n\to\infty} B_t^{2n}) = 1 ,$$
$$P_x(\tilde{A}_s = \lim_{n\to\infty} A_s^n) = 1, \quad \text{and}$$
$$\begin{aligned}&P_x(\lim_{n\to\infty} A_{t-s}^n \circ \vartheta_s = \tilde{A}_{t-s} \circ \vartheta_s) \\ &= E_x(P_{x_s}(\lim_{n\to\infty} A_{t-s}^n = \tilde{A}_{t-s})) = 1 .\end{aligned}$$

Therefore $\tilde{A}_t = \tilde{A}_s + \tilde{A}_{t-s} \circ \vartheta_s$ (a.s.). This shows that $(\tilde{A}_s)$ satisfies (1.5).

We have also $E_x(\tilde{A}_t \cdot f(x_t)) = \lim_{n\to\infty} E_x(A_t^n \cdot f(x_t)) = Q_t f(x)$ for any $f \in F(S)$ from (2.7).

(2) The next argument to obtain a right continuous modification of $(\tilde{A}_s)$ is a routine. Since $\tilde{A}_s$ is $\mathcal{F}_s$-measurable and $E_x(|\tilde{A}_s|) < \infty$ for any s, and $(\tilde{A}_s)$ satisfies (1.5), we have

$$\begin{aligned}&E_x(\tilde{A}_t \mid \mathcal{F}_s) = \tilde{A}_s + E_x(\tilde{A}_{t-s} \circ \vartheta_s \mid \mathcal{F}_s) = \tilde{A}_s + E_{x_s}(\tilde{A}_{t-s}) = \tilde{A}_s + Q_{t-s}1(x_s) \\ &\geqq \tilde{A}_s \qquad \text{(a.s. } P_x) \text{ for any } 0 \leqq s \leqq t .\end{aligned}$$

The last inequality follows from (3.5). Thus $(\tilde{A}_s, \mathcal{F}_s, P_x)_{s\geq 0}$ forms a submartingale. From the general theory of submartingale we know;

$$P_x(\lim_{\substack{r \downarrow t \\ r:\text{rational}}} \tilde{A}_r = \tilde{A}_{t+} \text{ and } \lim_{\substack{r \uparrow t \\ r:\text{rational}}} \tilde{A}_r = \tilde{A}_{t-} \quad \text{exist for any } t \geqq 0) = 1$$

and $\tilde{A}_t \leqq E_x(\tilde{A}_{t+} | \mathcal{F}_t) = E_x(\tilde{A}_{t+} | \mathcal{F}_{t+}) = \tilde{A}_{t+}$ (a.s. P_x). Let $(t_n)_{n\geqq 1}$ be a sequence of rational points such as $t_n \downarrow t (n \uparrow \infty)$, since $\tilde{A}_{t_n} \quad {}_x(\tilde{A}_{t_1} | \mathcal{F}_{t_n})$ $(\forall n)$, $(\tilde{A}_{t_n})_{n\geqq 1}$ is uniformly integrable, so we have $E_x(\tilde{A}_{t+}) = \lim_{n\to\infty} \downarrow E_x(\tilde{A}_{t_n})$. But since $Q_{t_n}1(x) = Q_t P_{t_n - t}1(x) + P_t Q_{t_n - t}1(x) = Q_t 1(x) + P_t Q_{t_n - t} 1(x)$ and $\lim_{n\to\infty} Q_{t_n}1(x) = Q_t 1(x) + \lim_{n\to\infty} \downarrow P_t Q_{t_n - t} 1(x) = Q_t 1(x)$ by (3.5), we have $E_x(\tilde{A}_t) = Q_t 1(x) = \lim_{n\to\infty} Q_{t_n} 1(x) = \lim_{n\to\infty} E_x(\tilde{A}_{t_n})$. Therefore $\tilde{A}_t = \tilde{A}_{t+}$ (a.s.). If we set $A_t(w) = \limsup_{\substack{r \downarrow t \\ r:\text{rational}}} \tilde{A}_r(w)$, we have

$$P_x(\tilde{A}_t = A_t) = 1\,,$$

$P_x(A_t$ is right continuous and has left-hand limits in $[0, \infty)) = 1$.

Now using the facts obtained hitherto we get a locally integrable almost additive functional (A_t), and $E.(A_t \cdot f(x_t)) = E.(\tilde{A}_t \cdot f(x_t)) = Q_t f$ for any $t \geqq 0$ and $f \in F(S)$. In particular $E.(A_t) = Q_t 1 \geqq 0$.

4. Characterization of functionals in $\mathfrak{M}^a$

Let (Q_t) be a system of characteristic kernels. In this section we shall further assume that the following conditions are satsfied;

$$(4.1) \quad Q_t(x, S) = Q_t 1(x) = \int_S q_t(x, y) P_t(x, dy) = 0 \qquad \text{for any } t \geqq 0 \text{ and } x \in S\,,$$

$$(4.2) \quad \int_S q_t(x, y)^2 P_t(x, dy) < \infty \qquad \text{for any } t \geqq 0 \text{ and } x \in S\,.$$

Note that (4.1), (4.2) and the Makov property yield

$$E_x((\sum_{j=1}^n q_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j}))^2) = \sum_{j=1}^n E_x(q_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j})^2) \qquad \text{for } 0 = t_0 \leqq t_1 \cdots \leqq t_n\,.$$

If

$$(4.3) \quad \sup \left[\sum_{j=1}^m E_x(q_{s_j - s_{j-1}}(x_{s_{j-1}}, x_{s_j})^2) + E_x(q_{t - s_m}(x_{s_m}, x_t)^2) \right] < \infty \qquad \text{for } \forall t \geqq 0$$
$$(D_t^m = (0 = s_0 < s_1 \cdots < s_m < t))\,,$$

where sup is taken over all partitions of the interval $[0, t]$, then

$$(Y_t^m \equiv \sum_{j=1}^m q_{s_j - s_{j-1}}(x_{s_{j-1}}, x_{s_j}) + q_{t - s_m}(x_{s_m}, x_t))_{m \geqq 1}$$

is uniformly integrable for any $t\geqq 0$ and any system $(D_t^m)_{m\geqq 1}$ of partitions of $[0, t]$. By Theorem 2 and Proposition 2.5 there exists a unique almost additive functional which represents (Q_t), and $E_x(Y_t)=Q_t1(x)=0$.

Proposition 4.1. (4.3) *is a necessary and sufficient condition for* (Y_t) *to be an element of* $\mathfrak{M}^a$.

Proof. If (4.3) is satisfied, then $(Y_t^n, \mathcal{F}_t^n, P_x)_{n\geqq 1}$ is a square-integrable martingale and $\sup E_x((Y_t^n)^2)<\infty$, so in the argument of the proof of Theorem 2 we can take Y_t to be square-integrable. Conversely if $(Y_t)\in\mathfrak{M}^a$, then we have

$$\begin{aligned}&E_x(q_{u-s}(x_s, x_u)^2) = E_x(E_x(Y_u-Y_s\,|\,x_s, x_u)^2)\\&\leqq E_x((Y_u-Y_s)^2), \quad \text{and}\\&\sum_{j=1}^{n} E_x(q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j})^2)+E_x(q_{t-t_n}(x_{t_n}, x_t)^2)\\&\leqq \sum_{j=1}^{n} E_x((Y_{t_j}-Y_{t_{j-1}})^2)+E_x((Y_t-Y_{t_n})^2)\,.\end{aligned}$$

Since $E_x(Y_u)=0$ for $\forall u\geqq 0$, the right-hand side is equal to

$$E_x((\sum_{j=1}^{n}(Y_{t_j}-Y_{t_{j-1}})+Y_t-Y_{t_n})^2) = E_x(Y_t^2)\,.$$

$E_x(Y_t^2)$ does not depend on the choice of a system of partitions, and the proposition is proved.

Now we shall give some conditions that verify (4.3). From (2.3) we have

$$E_x(q_{u-s}(x_s, x_u)^2) \leqq E_x(q_{t-s}(x_s, x_t)^2)+E_x(q_{u-t}(x_t, x_u)^2)$$

for any $s\leqq t\leqq u$, so we can prove in the same way as in the proof of Darboux's theorem in the theory of integration that

$$\lim_{\delta\downarrow 0}\Big[\sum_{j=1}^{n} E_x(q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j})^2)+E_x(q_{t-t_n}(x_{t_n}, x_t)^2)\Big] \tag{4.4}$$
$$(\delta = \max_j(\max|t_j-t_{j-1}|,\ |t-t_n|))$$

is equal to the left-hand side of (4.3) if the following condition (4.5) is satisfied.

Proposition 4.2. (4.3) *is satisfied if*:

(4.5) *For any* $s\geqq 0$ *and* x,

$$\lim_{\delta\downarrow 0}\big[\max\{\sup_{h\leqq\delta} E_x(q_h(x_s, x_{s+h})^2)\,,\quad \sup_{h\leqq\delta} E_x(q_h(x_{s-h}, x_s)^2)\}\big] = 0$$

(4.6) *For any* $t\geqq 0$ *and* $\delta>0$ *there exists a constant* $b=b(\delta, t, x)$ *such that*

$$\sup_{h\leqq\delta}\frac{1}{h}\int_0^t E_x(q_h(x_s, x_{s+h})^2)\,ds\leqq b\,.$$

Proof. Choose a $\delta>0$ and fix it. For any $h\in(0, \delta]$ there exists a integer n such that $t\in(nh, (n+1)h]$ and from (4.6) there exists a $s\in(0, h]$ such that

$$\sum_{j=1}^{n-1}E_x(q_h(x_{s+(j-1)h}, x_{s+jh})^2)\leqq b\,.$$

From the fact noted above this proposition, it is enough to prove that

$$\begin{aligned}(4.7)\qquad &\lim_{h\downarrow 0}\,[E_x(q_s(x_0, x_s)^2)+\sum_{j=1}^{n-1}E_x(q_h(x_{s+(j-1)h}, x_{s+jh})^2)\\ &\quad +E_x(q_{t-s-(n-1)h}(x_{s+(n-1)h}, x_t)^2)]\end{aligned}$$

is finite. Since $h\downarrow 0$ implies $s\downarrow 0$ and $t-s-(n-1)h\downarrow 0$, $\lim_{h\downarrow 0}E_x(q_s(x_0, x_s)^2)=0$ and $\lim_{h\downarrow 0}E_x(q_{t-s-(n-1)h}(x_{s+(n-1)h}, x_t)^2)=0$, so (4.7) is smaller than b.

To give another condition equivalent to (4.3), here we give a definition.

DEFINITION 4.1. A finite positive function $m(t, x)$ on $[0, \infty)\times S$ is called a *characteristic*, if the following conditions are satisfied:

(C.1) $\qquad m(t, \cdot)$ is $\mathbf{F}(S)$-measurable

(C.2) $$\lim_{t\downarrow 0}m(t, x)=0$$

(C.3) $$m(t+s, x)=m(t, x)+E_x(m(s, x_t))\,.$$

Proposition 4.3. (4.3) *is satisfied if and only if there exists a characteristic* $m(t, x)$ *such that*

$$(4.8)\qquad \int_S q_s(x, y)^2 P_s(x, dy)\leqq m(s, x)$$

for any x *and* s.

Proof. If there exists a characteristic $m(t, x)$ such that $E_x(q_s(x, x_s)^2)\leqq m(s, x)$, then we have

$$\begin{aligned}&\sum_{j=1}^{n}E_x(q_{t_j-t_{j-1}}(x_{t_{j-1}}, x_{t_j})^2)+E_x(q_{t-t_n}(x_{t_n}, x_t)^2)\\ &\leqq\sum_{j=1}^{n}E_x(m(t_j-t_{j-1}, x_{t_{j-1}}))+E_x(m(t-t_n, x_{t_n}))\\ &=m(t, x)^{(8)}\end{aligned}$$

(8) See Proposition 2.1 of [1].

for any partition $(0=t_0<t_1<\cdots<t_n<t)$. $m(t, x)$ does not depend on the choice of partitions, and (4.3) is proved. The converse is proved if we put $m(s, x)=E_x(Y_s^2)$, where (Y_s) is the functional in $\mathfrak{M}^a$ constructed in Proposition 4.1.

[Proof of Theorem 1] Proposition 4.1 combined with Proposition 4.3 proves Theorem 1.

5. Examples

(1) Let h be an invariant function of the process $\mathbf{X}$ such that $E_x((h(x_t)-h(x_0))^2)<\infty$ for any x and t, then $Y_t=h(x_t)-h(x_0)$ belongs to $\mathfrak{M}^a$. We have

$$q_t(x, y) = E_x(Y_t \mid x_t=y)=h(y)-h(x) \quad \text{for } P(t, x, dy)\text{—almost all } y\,. \tag{5.1}$$

$$Q_t f(x) = P_t(h\cdot f)(x)-h(x)P_t f(x) \tag{5.2}$$

(2) Let $\mathbf{X}=(x_t, P_x)$ be a one-dimentional Brownian motion on $(-\infty, \infty)$, and let Y_t be the stochastic integral $\int_0^t f(x_s)\,dx_s$. We shall determine $q_t(x, y)$.

Since $E_0(f(x_s)(x_u-x_s) \mid x_t=x)$

$$=\left(\frac{t}{2\pi s(t-s)}\right)^{1/2}\cdot\frac{u-s}{t-s}\cdot\int_{-\infty}^{\infty}(x-y)f(y)e^{-(ty-sx)^2/2st(t-s)}dy \quad (0\leqq s<u<t),$$

we have

$$\begin{aligned} q_t(0, x) &= E_0\left(\int_0^t f(x_s)\,dx_s \mid x_t = x\right) \\ &= \lim_{N\to\infty}\sum_{j=1}^{N} E_0(f(x_{t_{j-1}})(x_{t_j}-x_{t_{j-1}}) \mid x_t = x) \\ &= \lim_{N\to\infty}\sum_{j=1}^{N} (t_j-t_{j-1})F(t_{j-1}) \\ &= \int_0^t F(S)\,ds \qquad (\text{a.s. } P_t(x, dy))\,. \end{aligned}$$

where $F(s)=\left(\frac{t}{2\pi s(t-s)}\right)^{1/2}\frac{1}{t-s}\int_{-\infty}^{\infty}(x-y)f(y)\exp\left(-\frac{(ty-sx)^2}{2st(t-s)}\right)dy\,.$

Since

$$\begin{aligned} &\int_0^t\left(\frac{t}{2\pi s(t-s)}\right)^{1/2}\frac{1}{t-s}\int_{-\infty}^{\infty}|x-y|\cdot|f|(y)\cdot\exp\left(-\frac{(ty-sx)^2}{2st(t-s)}\right)dy\,ds \\ &\leqq \|f\|\cdot\left[(2\pi)^{1/2}|x|+\int_0^t\left(\frac{s}{t(t-s)}\right)^{1/2}ds\cdot\int_{-\infty}^{\infty}|z|\exp\left(-\frac{z^2}{2}\right)dz\right]<\infty\,, \end{aligned} \tag{5.3}$$

by Fubini's theorem we have

$$q_t(0, x) = \int_{-\infty}^{\infty} (x-y) f(y) G(y, x, t) dy , \qquad \text{where} \tag{5.4}$$

$$G(y, x, t) = \int_0^t \left(\frac{t}{2\pi s(t-s)}\right)^{1/2} \frac{1}{t-s} \exp\left(-\frac{(ty-sx)^2}{2st(t-s)}\right) ds$$
$$= \frac{1}{(2\pi t)^{1/2}} \int_0^1 \frac{1}{a^{1/2}(1-a)^{3/2}} \exp\left(-\frac{(y-ax)^2}{2ta(1-a)}\right) da$$

We can verify that $G(y, x, t) < \infty$ for $x \neq y$, and $G(x, x, t) = \infty$.

$$Q_t g(0) = \int_{-\infty}^{\infty} g(x) q_t(0, x) \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{x^2}{2t}\right) dx .$$

By (5.3) we can exchange the order of integration and have

$$Q_t g(0) = \int_0^t \frac{1}{t-s} (P_s f P_{t-s} G - P_s F P_{t-s} g)(0) ds \tag{5.5}$$
$$= \int_0^t \frac{P_{t-s}}{s} [f(P_s \cdot - \cdot P_s) g](0) ds ,$$

where $F(x) = xf(x)$, $G(x) = xg(x)$ and

$$(P_s \cdot)(x, dy) = P_s(x, dy) y ,$$
$$(\cdot P_s)(x, dy) = x P_s(x, dy) .$$

(3) Let $\mathbf{X} = (n_t, P_n, n \in N)$ be a Poisson process with parameter 1 and let $P_t(n, m) = \frac{1}{(m-n)!} t^{m-n} e^{-t}$ be its transition function. Then $Y_t = n_t - n_0 - t$ belongs to $\mathfrak{M}$, and

$$q_t(n_0, n_t) = E(Y_t \mid n_0, n_t) = n_t - n_0 - t \qquad \text{(a.s.)}$$

so we have

$$q_t(n, m) = m - n - t , \quad \text{and} \tag{5.6}$$

$$Q_t f(n) = \sum_{k=0} \frac{t^k}{k!} e^{-t} (k-t) f(k+n) . \tag{5.7}$$

6. $\mathfrak{M}^a = \mathfrak{M}$

In and after this section we shall assume that the underlying Markov process $\mathbf{X} = (W, \mathcal{F}_t, x_t(w), \vartheta_t, P_x, x \in S \cup \{\Delta\})$ is a conservative Hunt process with a reference measure, and shall freely use the standard properties of additive functionals in $\mathfrak{M}$ as well as those of characteristics. (See [1].)

In [4], it was shown that every finite almost additive functional satisfies the strong Markov property;

$$A_\sigma + A_t \circ \vartheta_\sigma = A_{t+\sigma} \qquad \text{a.s.}$$

for any bounded Markov time σ and $t \geqq 0$. So the same arguments as those of Lemma 8.1–8.3 of [1] prove that $m(t, x) = E_x(X_t^2)$ is a regular characteristic for any $(X_t) \in \mathfrak{M}^a$. By Theorem 6.8 in [1] we know that a characteristic $m(t, x)$ is regular if and only if there exists a (unique up to equivalence) $A \in \mathcal{C}_1^+$ (9) such that $m(t, x) = E_x(A_t)$.

Theorem 3. *For every functional* (X_t) *in* $\mathfrak{M}^a$*, there exists a unique equivalent functional in* $\mathfrak{M}$.

Proof. Choose any additive functional (Z_t) in $\mathfrak{M}$ and set $m_0(t, x) = E_x(X_t \cdot Z_t)$. Since $m_0(t, x) = \frac{1}{4} E_x(X_t + Z_t)^2 - \frac{1}{4} E_x(X_t - Z_t)^2$ and $E_x(X_t^2) = E_x(A_t)$ for some $(A_t) \in \mathcal{C}_1^+$, it can be verified that $m_0(t, x)$ is a regular characteristic, so $m_0(t, x) = E_x(B_t)$ for some $(B_t) \in \mathcal{C}_1$ (10). We can prove as in [1] that (6.1) $E_x(|f| \cdot |B|_t^{(11)}) \leqq (E_x(X_t^2) \cdot E_x(f^2 \cdot \langle Z \rangle_t))^{1/2}$ for any $f \in \ell_2(Z)$. From this we can also prove $B_t = g \cdot \langle Z \rangle_t$ for some $g \in \ell_1(\langle Z \rangle)$. If we let

$$g_N(x) = \begin{cases} N & g(x) \geqq N \\ g(x) & -N \leqq g(x) \leqq N \\ -N & g(x) \leqq -N \end{cases},$$

then from (6.1) we have

$$\begin{aligned} &E_x(|g_N| \cdot |g| \cdot \langle Z \rangle_t) = E_x(|g_N| \cdot |B|_t) \\ &\leqq (E_x(X_t^2) \cdot E_x(g_N^2 \cdot \langle Z \rangle_t))^{1/2}, \end{aligned}$$

and

$$\begin{aligned} &(E_x(|g_N| \cdot |g| \cdot \langle Z \rangle_t))^{1/2} \\ &\leqq (E_x(X_t^2))^{1/2} \cdot \frac{(E_x(g_N^2 \cdot \langle Z \rangle_t))^{1/2}}{(E_x(|g_N| \cdot |g| \cdot \langle Z \rangle_t))^{1/2}} \leqq (E_x(X_t^2))^{1/2}. \end{aligned}$$

Tending N to infinity we have $g^2 \in \ell_1(\langle Z \rangle)$, so we can define the stochastic

(9) $\mathcal{C}_1^+$ denotes the set of all continuous, positive, increasing, additive functionals such that $E_x(A_t) < +\infty$ for all $x \in S$ and $t \geqq 0$.

(10) $\mathcal{C}_1 = \mathcal{C}_1^+ - \mathcal{C}_1^+$

(11) $f \cdot B_t = \int_0^t f(x_s)\, dB_s$.

$g \cdot |B|_t = \int_0^t g(x_s)\, dB_s^+ + \int_0^t g(x_s)\, dB_s^-$, for $B = B^+ - B^-$, $B^+, B^- \in \mathcal{C}_1^+$.

integral $g\cdot Z_t=\int_0^t g(x_s)\,dZ_s$.

In Proposition 12.3 of [1] it was proved that there exists a sequence $(Y^1, Y^2,\cdots, Y^n,\cdots)$ in $\mathfrak{M}$ such that Y^j's are mutually orthogonal and $\mathfrak{M}=L(Y^1, Y^2,\cdots, Y^n,\cdots)$. For each j, let $m_j(t, x)=E_x(Y_t^j\cdot X_t)$ and $g_j\in \ell_2(Y^j)$ be the function constructed as the above so as to $m_j(t, x)=E_x(g_j\cdot\langle Y^j\rangle_t)$. $Y^{[N]}=\sum^N g_j\cdot Y^j$ obviously belongs to $\mathfrak{M}$. Now we shall show $E_x((Y_t^{[N]}-Y_t^{[M]})^2)\to 0$ $(M, N\to\infty)$ for each x and t. Then by the completeness of $\mathfrak{M}$ $Y\equiv\lim_{N\to\infty} Y^{[N]}$ must belong to $\mathfrak{M}$. It is enough to show the next inequality

$$\sum_{j=1}^{\infty} E_x(g_j^2\cdot\langle Y^j\rangle_t)\leqq E_x(X_t^2)\,. \tag{6.2}$$

But this follows from

$$\begin{aligned}0\leqq E_x((X_t-Y_t^{[N]})^2) &= E_x(X_t^2)\\ &-2\sum_{j=1}^{N} E_x(X_t\cdot(g_j\cdot Y_t^j))+\sum_{j=1}^{N} E_x(g_j^2\cdot\langle Y^j\rangle_t)\\ &=E_x(X_t^2)-\sum_{j=1}^{N} E_x(g_j^2\cdot\langle Y^j\rangle_t)\,.\end{aligned} \tag{6.3}$$

Since $E_x(Y_t\cdot Y_t^k)=E_x(g_k\cdot\langle Y^k\rangle_t)=E_x(X_t\cdot Y_t^k)$ for each k, and since $\mathfrak{M}=L(Y^1, Y^2,\cdots, Y^n,\cdots)$, we have $E_x((Y_t-X_t)\cdot U_t)=0$ for any $(U_t)\in\mathfrak{M}$, and have also $E_x((Y_t-X_t).\cdot\int_0^t e^{-\alpha s}dU_s)=0$. In particular if we let $U_t=f(x_t)-f(x_0)+\int_0^t(g-\alpha f)(x_s)\,ds$, where $f=G^\alpha g$[12] and $g\in C(S)$, we have $\int_0^t e^{-\alpha s}dU_s=e^{-\alpha t}f(x_t)-f(x_0)+\int_0^t e^{-\alpha s}g(x_s)\,ds$ and,

$$\begin{aligned}&e^{-\alpha t}E_x(f(x_t)\cdot(Y_t-X_t))+E_x\Big(\int_0^t e^{-\alpha s}g(x_s)\,ds\cdot(Y_t-X_t)\Big)\\ &=0\,.\end{aligned} \tag{6.4}$$

The first term of (6.4) is equal to $\int_t^\infty e^{-\alpha s}E_x((Y_t-X_t).\,(P_{s-t}g)(x_t)\,ds$. Therefore if we set

$$\begin{aligned}\varphi(s) &= E_x(g(x_s)\cdot(Y_s-X_s)) && s\leqq t\\ &= E_x((P_{s-t}g)(x_t)\cdot(Y_t-X_t)) && s>t\end{aligned}$$

we have $\int_0^\infty e^{-\alpha s}\varphi(s)\,ds=0$. $\varphi(s)$ is a right continuous function in $s\in R_+-t$, so we have $\varphi(s)=0$ for any $s\leqq t$ from the uniqueness of the Laplace transform. Thus we have $E_x(g(x_t)\cdot(Y_t-X_t))=0$ for each $g\in C(S)$, and therefore for each

(12) $G^\alpha g=\int_0^\infty e^{-\alpha s}P_s g\,ds$

$g\in F(S)$. By the same argument as in the proof of Proposition 2.5., we conclude $Y_t=X_t$ (a.s. P_x) for any x and t.

Theorem 4. *In the conclusion of Theorem* 1 *we can replace* $\mathfrak{M}^a$ *by* $\mathfrak{M}$.

7. Some properties of the Laplace transform of characteristic kernels corresponding to functionals in $\mathfrak{M}$.

We shall denote by $(Q_t^{[Y]})$ the characteristic kernels represented by $(Y_t)\in\mathfrak{M}$. By Lemma 8.2. of [1] we can verify that $Q_t^{[Y]}g(x)$ is continuous in t for each $x\in S$ and $g\in F(S)$. We will assume that $\int_0^\infty e^{-\alpha_0 t}E_x(\langle Y\rangle_t)\,dt<\infty$ for all x and some $\alpha_0>0$. Set

$$(7.1)\qquad K^\alpha{}_{[Y]}g(x)=\int_0^\infty e^{-\alpha s}Q_s g(x)\,ds \qquad (\alpha\geqq\alpha_0)\,.$$

Proposition 7.1. *Let* $g\in F(S), f=G^\alpha g(\alpha\geqq\alpha_0)$ *and* $X^\alpha[g]_t=f(x_t)-f(x_0)+\int_0^t(g-\alpha f)(x_s)\,ds.$

Then

1. $E_x(\int_0^t e^{-\alpha s}d\varphi_s{}^\alpha)=e^{-\alpha t}Q_t{}^{[Y]}f(x)+\int_0^t e^{-\alpha s}Q_s{}^{[Y]}g(x)\,ds\,,$
   where $\varphi^\alpha=\langle Y,X^\alpha[g]\rangle\,.$
2. (7.2) $\quad K^\alpha{}_{[Y]}g(x)=E_x(\int_0^\infty e^{-\alpha s}d\varphi_s{}^\alpha)\,.$
   Therefore $K^\alpha{}_{[Y]}g$ *is the difference of regular* α*-excessive functions.*
3. $\lim_{\alpha\to\infty}\alpha K^\alpha{}_{[Y]}g(x)=0\,.$
4. $K^\alpha 1(x)=0$

Proof. Since

$$\int_0^t e^{-\alpha s}d(X^\alpha[g])_s=e^{-\alpha t}f(x_t)-f(x_0)+\int_0^t e^{-\alpha s}g(x_s)\,ds\,,$$

we have

$$\begin{aligned}E_x(\int_0^t e^{-\alpha s}d\varphi_s{}^\alpha)&=E_x(Y_t\cdot\int_0^t e^{-\alpha s}d(X^\alpha[g])_s)\\&=E_x(Y_te^{-\alpha t}f(x_t))+E_x(Y_t\cdot\int_0^t e^{-\alpha s}g(x_s)\,ds)\\&=e^{-\alpha t}Q_t{}^{[Y]}f(x)+\int_0^t e^{-\alpha s}Q_s{}^{[Y]}g(x)\,ds\,.\end{aligned}$$

Tending t to infinity in the above, (7.2) follows from

$$|e^{-\alpha t}Q_t f(x)| \leqq e^{-\alpha t}\|f\|(E_x(\langle Y\rangle_t))^{1/2} \to 0\,.$$

As for 3, $\quad |\alpha K^{\alpha}g(x)| \leqq \alpha\|g\|\int_0^{\infty} e^{-\alpha s}(E_x\langle Y\rangle_s)^{1/2}ds$

$$= \|g\|\int_0^{\infty} e^{-\alpha s}(E_x(\langle Y\rangle_{s/\alpha}))^{1/2}ds \to 0 \qquad (\alpha\uparrow\infty)\,.$$

4 is obvious.

Proposition 7.2.

$$(7.3)\quad K^{\alpha}g(x)-K^{\beta}g(x)+(\alpha-\beta)(K^{\alpha}G^{\beta}g(x)+G^{\alpha}K^{\beta}g(x)) = 0 \qquad (\alpha,\ \beta\geqq\alpha_0,\ x\in S)$$

, *where* $K^{\alpha}\equiv K^{\alpha}_{[Y]}$.

Proof. Let $h(x)=G^{\beta}g(x)$, and $\psi^{\alpha}=\langle Y, X^{\alpha}[h]\rangle$. By Proposition 7.1, $E_x(\int_0^{\infty} e^{-\alpha s}d\psi_s^{\alpha})=K^{\alpha}h(x)=K^{\alpha}G^{\beta}g(x)$, and $E_x(\int_0^{t} e^{-\alpha s}d\psi_s^{\alpha})=e^{-\alpha t}Q_tG^{\alpha}h(x)$ $+E_x(Y_t\cdot\int_0^t e^{-\alpha s}h(x_s)\,ds)$.

The second term of the right-hand side is equal to

$$E_x(\int_0^t e^{-\alpha s}Y_s h(x_s)\,ds) = E_x(\int_0^t e^{-\alpha s}Y_s E_{x_s}(\int_0^{\infty} e^{-\beta u}g(x_u))\,du\,ds)$$

$$= E_x(\int_0^t e^{(\beta-\alpha)s}\cdot\int_0^{\infty} e^{-\beta(u+s)}Y_{u+s}g(x_{u+s})\,du\,ds)$$

$$-E_x(\int_0^t e^{-\alpha s}\int_0^{\infty} e^{-\beta u}Y_u(w_s^{+})g(x_u(w_s^{+}))\,du\,ds)$$

$$= \int_0^t e^{(\beta-\alpha)s}\int_s^{\infty} e^{-\beta u}Q_u g(x)\,du\,ds - E_x(\int_0^t e^{-\alpha s}\int_0^{\infty} e^{-\beta u}E_{x_s}(Y_u g(x_u))\,du\,ds)$$

$$= \int_0^t e^{-\beta u}Q_u g(x)\cdot\int_0^u e^{(\beta-\alpha)s}ds + \int_t^{\infty} e^{-\beta u}Q_u g(x)\cdot\int_0^t e^{(\beta-\alpha)s}ds$$

$$-E_x(\int_0^t e^{-\alpha s}\cdot\int_0^{\infty} e^{-\beta u}Q_u g(x_s)\,du\,ds)$$

$$= \frac{1}{\beta-\alpha}(\int_0^t e^{-\alpha u}Q_u g(x)\,du - \int_0^t e^{-\beta u}Q_u g(x)\,du)$$

$$+\int_t^{\infty} e^{-\beta u}Q_u g(x)\,du\Big(\frac{1-e^{(\beta-\alpha)t}}{\beta-\alpha}\Big) - E_x(\int_0^t e^{-\alpha s}K^{\beta}g(x_s)\,ds)\,.$$

But since

$$\Big|\int_t^{\infty} e^{-\beta u}Q_u g(x)\,du\Big(\frac{1-e^{(\beta-\alpha)t}}{\beta-\alpha}\Big)\Big| \leqq \frac{\|g\|}{\beta(\beta-\alpha)}(e^{-\alpha t}+e^{-\beta t})(E_x\langle Y\rangle_t)^{1/2} \to 0 \quad (t\uparrow\infty),$$

and

$$|e^{-\alpha t}Q_tG^{\alpha}h(x)| \to 0 \qquad (t\uparrow\infty)\,,$$

we have

$$K^{\alpha}G^{\beta}g(x)=\frac{1}{\beta-\alpha}(K^{\alpha}g(x)-K^{\beta}g(x))-G^{\alpha}K^{\beta}g(x),$$

by tending t to infinity in the above equalities.

The next corollary shows that $(K^{\alpha}{}_{[Y]})$ posesses a composite entrance and exit property.

Corollary 7.3.

$$K^{\alpha}=(I-(\alpha-\beta)G^{\alpha})K^{\beta}(I-(\alpha-\beta)G^{\alpha}). \tag{7.4}$$

Next we shall give a relation between $K^{\alpha}{}_{[Y]}$ and $K^{\alpha}{}_{[Z]}$ for $Z=h\cdot Y=\int_0 h(x_s)\,dY_s$.

Proposition 7.4. *Let* $Z=\int_0 h(x_s)\,dY_s$ $(h\in\ell_2(Y))$, *and* $U^{\alpha}(x)=E_x(\int_0^{\infty}e^{-\alpha s}d(\langle Y, X^{\alpha}[g]\rangle)_s=K^{\alpha}{}_{[Y]}g(x)$, *then we have*

$$K^{\alpha}{}_{[Z]}g(x)=U^{\alpha}h(x)\equiv E_x(\int_0^{\infty}e^{-\alpha s}h(x_s)\,d(\langle Y, X^{\alpha}[g]\rangle)_s). \tag{7.5}$$

Proof.
$$\begin{aligned}K^{\alpha}{}_{[Z]}g(x)&=E_x(\int_0^{\infty}e^{-\alpha s}d(\langle Z, X^{\alpha}[g]\rangle)_s\\&=E_x(\int_0^{\infty}e^{-\alpha s}h(x_s)\,d(\langle Y, X^{\alpha}[g]\rangle)_s.\end{aligned}$$

EXAMPLES

1. When $Y_t=h(x_t)-h(x_0)$ for some invariant function h of the process $\mathbf{X}$ with $E_x(h(x_t)-h(x_0))^2<\infty$, we have $K^{\alpha}f(x)=G^{\alpha}(h\cdot f)(x)-h(x)G^{\alpha}f(x)$.

2. When $Y_t=\int_0^t f(x_s)\,dx_s$ as in example (2) of section 5, we have

$$\begin{aligned}\langle Y, X^{\alpha}[g]\rangle&=\langle\int_0 f(x_s)\,dx_s, \int_0 grad\, G^{\alpha}g(x_s)\,dx_s\rangle\\&=\int_0 f(x_s)\, grad\, G^{\alpha}g(x_s)\,ds, \text{ so } K^{\alpha}g(x)=G^{\alpha}(f\,\mathrm{grad}\,G^{\alpha}g)(x).\end{aligned}$$

3. As for example (3) of section 5, we have

$$\begin{aligned}K^{\alpha}{}_{[Y]}f(n)&=\int_0^{\infty}e^{-\alpha t}Q_tf(n)\,dt\\&=\int_0^{\infty}e^{-\alpha t}\sum_{k=0}\frac{t^k}{k!}e^{-t}(k-t)f(k+n)\,dt.\end{aligned}$$

If we set $a_n=\frac{k}{(\alpha+1)^k}-\frac{k+1}{(\alpha+1)^{k+1}}$, we have $\lim\frac{|a_{n+1}|}{|a_n|}=\frac{1}{\alpha+1}<1$, so $\frac{1}{\alpha+1}$

$\sum_{k=1} a_k f(k+n)$ is absolutely convergent. Since we can verify

$$\sum_{k=0} \frac{1}{k!} \int_0^\infty t^k e^{-(\alpha+1)t}(k-t)\, dt f(k+n)$$
$$= \frac{1}{\alpha+1} \sum_{k=1} a_k f(k+n) < \infty ,$$

we have

$$K^{\alpha}_{[Y]} f(n) = \frac{1}{\alpha+1} \sum_{k=1} a_k f(k+n)$$
$$= \frac{1}{\alpha+1} \sum_{k=1} \frac{k}{(\alpha+1)^k} (f(k+n) - f(k+n-1))$$

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