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NOTE ON KRULL-REMAK-SCHMIDT-AZUMAYA'S THEOREM

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The Krull-Remak-Schmidt's theorem is very important in the ring theory and many algebraists tried to generalize it. Especially, Azumaya [1] succeeded to extend this theorem in a case of infinite direct sums.

In this note, we shall further generalize Azumaya's theorem. Let R be a ring with unit, and let all modules in this note be unitary R -modules. We assume that an R -module M has decompositions: $M = \sum_{\alpha \in I} \oplus M_{\alpha} = \sum_{\beta \in J} \oplus N_{\beta}$, where M_{α} and N_{β} are completely indecomposable modules for every $\alpha \in I$ and $\beta \in J$, (namely, $\text{End}_R(M_{\alpha})$, $\text{End}_R(N_{\beta})$ are local rings). In this case we shall consider the following condition.

(*) For any subset I' in I (resp. J' in J), there exists a one-to-one mapping φ of I' into J (resp. J' into I), such that $M_{\alpha} \approx N_{\varphi(\alpha)}$ for all $\alpha \in I'$ (resp. $N_{\beta} \approx M_{\varphi(\beta)}$ for all $\beta \in J'$) and $M = \sum_{\alpha' \in I'} \oplus N_{\varphi(\alpha')} \oplus \sum_{\alpha \in I - I'} \oplus M_{\alpha}$ (resp. $M = \sum_{\beta' \in J'} \oplus N_{\beta'} \oplus \sum_{\alpha \in I - \varphi(J')} \oplus M_{\alpha}$).

Then we have a problem whether the condition (*) is satisfied for any two decompositions of M or not.

Azumaya [1] showed that if I' is finite, (*) is satisfied. In a case of infinite set I' , P. Crawley and B. Jónsson gave an example in which (*) was not satisfied. On the other hand, M. Harada and Y. Sai [3] considered a category related to $\{M_{\alpha}\}_{\alpha \in I}$ and gave a necessary and sufficient condition for (*) being satisfied for any objects in the category

In this note, we shall give a complete answer that problem for any two direct decompositions: $M = \sum_{\alpha \in I} \oplus M_{\alpha} = \sum_{\beta \in J} \oplus N_{\beta}$.

We shall refer the reader to [3] for the notations and definitions.

The author would like to express his thanks to Professor Manabu Harada for suggesting the topic and his guidance.

Let $\{M_{\alpha}\}_{\alpha \in I}$ be a family of completely indecomposable modules and $\{f_i\}_{i=1}^{\infty}$ any sequence of non isomorphic R -homomorphisms of M_{α_i} to $M_{\alpha_{i+1}}$ in $\{M_{\alpha}\}_{\alpha \in I}$. We call $\{M_{\alpha}\}_{\alpha \in I}$ a T -nilpotent system, if there exists n which depends on the sequence and any element m in M such that $f_n f_{n-1} \cdots f_1(m) = 0$

for any set $\{M_{\alpha_i}, f_i\}_i$, and we call $\{M_\alpha\}_{\alpha \in I}$ a *semi-T-nilpotent system*, if the above condition is satisfied only for any sequence $\{f_i\}_{i=1}^\infty$ such that $\alpha_i \neq \alpha_j$ for $i \neq j$ (cf. [3], [4]).

We shall show the following theorem.

Theorem. *The following conditions (i), (ii) are equivalent for $M = \sum_{\alpha \in I} \oplus M_\alpha = \sum_{\beta \in J} \oplus N_\beta$ with infinite sets I and J , where M_α 's and N_β 's are completely indecomposable modules.*

- (i) $\{M_\alpha\}_{\alpha \in I}$ is a semi-T-nilpotent system.
- (ii) M satisfies the condition (*) for any decomposition of M .

In the above case, we have

- (iii) Every direct summand of M is also a directsum of completely indecomposable modules which are isomorphic to some M_α .
- (cf. condition III in [3].)

In order to prove the theorem, we give several lemmas and definitions.

Let I be a well ordered set and $M = \sum_{\alpha \in I} \oplus M_\alpha$, then $S_M = \text{End}_R(M)$ is equal to the ring of column summable matrices, whose entries consist of elements in $\text{Hom}_R(M_\tau, M_\sigma)$, namely for $f \in S_M$ and $x_\tau \in M_\tau$, $f = (b_{\sigma\tau})$ and $b_{\sigma\tau}(x_\tau) = 0$ for almost all $\sigma \in I$. Let \mathfrak{A} be the set of all endomorphisms $(b_{\sigma\tau})$ of M such that $b_{\sigma\tau}$ is a non-isomorphic R -homomorphism of M_τ to M_σ for every σ, τ . Then \mathfrak{A} is a two-sided ideal of S_M and \mathfrak{A} is independent of a decomposition of M . (cf. [3] Corollary to Lemma 5).

Lemma 1. *Let M and \mathfrak{A} be as above. If \mathfrak{A} is the radical of S_M , then Condition (ii) for any subset I', J' and any decomposition of M , and (iii) of the theorem are satisfied. (cf. [3] Corollary 2 to Theorem 7.)*

Lemma 2. *If $\{M_\alpha\}_{\alpha \in I}$ is a T-nilpotent system, then \mathfrak{A} is the radical of S_M .*

Proof. We take the induced additive category \mathfrak{B} from $\{M_\alpha\}_{\alpha \in I}$ and \mathfrak{S} the ideal of \mathfrak{B} in [3]. From [3] Lemma 10 and $\mathfrak{A} = \mathfrak{S}(\text{End}_R(M))$, we obtain the lemma.

Lemma 3. *For every element f in \mathfrak{A} , $1-f$ is monomorphic.*

Proof. We obtain the lemma from [3] the proof of Proposition 10.

Lemma 4. *Let $M = \sum_{\alpha \in K} \oplus \sum_{\beta \in I_\alpha} \oplus M_{\alpha\beta}$ be a directsum of completely indecomposable modules, I_α be an infinite set for all $\alpha \in K$ and $M_{\alpha\beta} \approx M_{\alpha\beta'}$, $M_{\alpha\beta} \not\approx M_{\alpha'\beta'}$ for $\alpha \neq \alpha'$. If $\{M_{\alpha\beta}\}_{\substack{\alpha \in K \\ \beta \in I_\alpha}}$ is a semi-T-nilpotent system, then $\{M_{\alpha\beta}\}_{\substack{\alpha \in K \\ \beta \in I_\alpha}}$ is a T-nilpotent system.*

Proof. Let $\{f_\alpha\}_{i=1}^\infty$ be any sequence of non-isomorphic R -homomorphism of $M_{\alpha_i\beta_i}$ to $M_{\alpha_{i+1}\beta_{i+1}}$ ($M_{\alpha_i\beta_i}$ may be equal to $M_{\alpha_j\beta_j}$ for $i \neq j$). From the assumption of infinite set I_α , we may assume that $\beta_i \neq \beta_j$ for all $i \neq j$. Therefore, we obtain the lemma, since $\{M_{\alpha\beta}\}$ is a semi-T-nilpotent system.

Lemma 5. *Let $M, \{M_{\alpha\beta}\}$ be as above, but I_α be a finite set for all $\alpha \in K$. Then \mathfrak{A} is the radical of S_M .*

Proof. Let f be any element in \mathfrak{A} . $1-f$ is monomorphic by Lemma 3. Therefore, we have only to show that $1-f$ is an epimorphism, that is, f is a quasi-regular element in S_M . From the assumption, we obtain

$$M = \sum_{|I_\alpha|=1} \oplus M_{\alpha_1} \oplus \sum_{|I_\beta|=2} \oplus (M_{\beta_1} \oplus M_{\beta_2}) \oplus \dots \oplus \sum_{|I_\gamma|=n} \oplus (M_{\gamma_1} \oplus \dots \oplus M_{\gamma_n}) \oplus \dots$$

where $|I_\alpha|$ is the cardinal number of I_α . Put $N_{\alpha^{(n)}} = M_{\alpha^{(n)1}} \oplus \dots \oplus M_{\alpha^{(n)n}}$, $M = \sum_{n=1}^\infty \oplus \sum_{\alpha^{(n)}} \oplus N_{\alpha^{(n)}}$, and we consider the following sequence

$$N_{\alpha^{(n)}} \xrightarrow{i} M \xrightarrow{1-f} M \xrightarrow{p_{\alpha^{(n)}}} N_{\alpha^{(n)}}$$

where i is an inclusion and $p_{\alpha^{(n)}}$ is a projection of M to $N_{\alpha^{(n)}}$. Then we have $p_{\alpha^{(n)}}(1-f)i = 1_{N_{\alpha^{(n)}}} \pmod{\mathfrak{A}}$, therefore $p_{\alpha^{(n)}}(1-f)i$ is an isomorphism by [3] Corollary 1 to Theorem 7 and we put $\text{Im}((1-f)i) = N'_{\alpha^{(n)}}$, which is a direct summand of M and a submodule of $\text{Im}(1-f)$, then we have a decomposition

$$*(\alpha^{(n)}) : M = N'_{\alpha^{(n)}} \oplus \sum_{\alpha^{(n')} \neq \alpha^{(n)}} \oplus N_{\alpha^{(n)'}}$$

We shall show that the submodule $M_0 = \sum_{n=1}^\infty \sum_{\alpha^{(n)}} N'_{\alpha^{(n)}}$ is a directsum $M = \sum_{n=1}^\infty \oplus \sum_{\alpha^{(n)}} \oplus N'_{\alpha^{(n)}}$. Let x be an element in M_0 and $x = \sum_{i=1}^t x_{\alpha_i^{(n_i)}}$, where $x_{\alpha_i^{(n_i)}} \in N'_{\alpha_i^{(n_i)}}$. From $*(\alpha_1^{(n_1)})$ and $*(\alpha_2^{(n_2)})$, we have $M = N'_{\alpha_1^{(n_1)}} \oplus \sum_{\alpha^{(n')} \neq \alpha^{(n_1)}} \oplus N_{\alpha^{(n')}} = N'_{\alpha_2^{(n_2)}} \oplus \sum_{\alpha^{(n')} \neq \alpha_2^{(n_2)}} \oplus N_{\alpha^{(n)'}}$. We apply [3] Corollary 1 to Theorem 7 to these decomposition, by $M_{\alpha_1^{(n_1)}j_1} \approx M_{\alpha_2^{(n_2)}j_2}$ we have

$$*(\alpha_1^{(n_1)}, \alpha_2^{(n_2)}) : M = N'_{\alpha_1^{(n_1)}} \oplus N'_{\alpha_2^{(n_2)}} \oplus \sum_{\alpha^{(n')} \neq \alpha_1^{(n_1)}, \alpha_2^{(n_2)}} \oplus N_{\alpha^{(n)'}}$$

Using $*(\alpha_3^{(n_3)})$ again, we have

$$*(\alpha_1^{(n_1)}, \alpha_2^{(n_2)}, \alpha_3^{(n_3)}) : M = N'_{\alpha_1^{(n_1)}} \oplus N'_{\alpha_2^{(n_2)}} \oplus N'_{\alpha_3^{(n_3)}} \oplus \sum_{\alpha^{(n')} \neq \alpha_i^{(n_i)}, i=1,2,3} \oplus N_{\alpha^{(n)'}}$$

Repeating this argument, we have $M_0 = \sum_{n=1}^\infty \oplus \sum_{\alpha^{(n)}} \oplus N'_{\alpha^{(n)}}$.

We assume $M \neq M_0$, that is, there exists some $\alpha_1^{(n_1)}$ such that $N_{\alpha_1^{(n_1)}}$ is not contained in M_0 . Let x be an element in $M_{\alpha_1^{(n_1)}j_1}$ which is not contained in M_0 .

By $*(\alpha_1(n_1))$ we have $x = x_1 + \dots + x_t$, where $x_1 \in N'_{\alpha_1(n_1)}$, $x_k \in M_{\alpha_k(n_k)j_k}$ for $\alpha_k(n_k) \neq \alpha_1(n_1)$, $k \geq 2$. Then there exists $k \geq 2$, such that x_k is not contained in M_0 , and x_k is a homomorphic image of some homomorphism f_1 of $M_{\alpha_1(n_1)j_1}$ to $M_{\alpha_k(n_k)j_k}$ and we may assume $\alpha_k(n_k) = \alpha_2(n_2)$. By $*(\alpha_1(n_1), \alpha_2(n_2))$ and the same argument, there is $x_3 \in M_{\alpha_3(n_3)j_3}$, $f_2 \in \text{Hom}_R(M_{\alpha_2(n_2)j_2}, M_{\alpha_3(n_3)j_3})$ such that $x_3 = f_2(x_2) = f_2 f_1(x) \notin M_0$ and $\alpha_3(n_3) \neq \alpha_1(n_1), \alpha_2(n_2)$. Repeating this process, we have a sequence $\{f_i\}$ of non-isomorphisms f_i such that $f_n f_{n-1} \dots f_1(x) \neq 0$ for any n , and $f_i \in \text{Hom}_R(M_{\alpha_i(n_i)j_i}, M_{\alpha_{i+1}(n_{i+1})j_{i+1}})$, $\alpha_k(n_k) \neq \alpha_j(n_j)$ for $k \neq j$, which contradicts the assumption of semi-T-nilpotent system.

Proof of Theorem (ii) \Rightarrow (i) is proved in [3] Lemma 9. (i) \Rightarrow (ii) Let $M = \sum_{\alpha \in K} \bigoplus_{\beta \in I_\alpha} \bigoplus M_{\alpha\beta} \oplus \sum_{\sigma \in L} \bigoplus_{\rho \in J_\sigma} \bigoplus M_{\sigma\rho}$, where $M_{\alpha\beta}$ and $M_{\sigma\rho}$ are completely indecomposable modules for all $\alpha, \beta, \sigma, \rho$, I_α is a finite set for all $\alpha \in K$, J_σ is an infinite set for all $\sigma \in L$, and $M_{\nu\mu} \approx M_{\nu'\mu'}$, $M_{\nu\mu} \approx M_{\nu'\mu'}$ for $\nu \neq \nu'$. Put $M_1 = \sum_{\alpha \in K} \bigoplus_{\beta \in I_\alpha} \bigoplus M_{\alpha\beta}$, $M_2 = \sum_{\sigma \in L} \bigoplus_{\rho \in J_\sigma} \bigoplus M_{\sigma\rho}$, $M = M_1 \oplus M_2$ and

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathfrak{A} \subset \begin{pmatrix} \text{Hom}_R(M_1, M_1) & \text{Hom}_R(M_2, M_1) \\ \text{Hom}_R(M_1, M_2) & \text{Hom}_R(M_2, M_2) \end{pmatrix}$$

Then $1-f$ is a unit in S_M , since $1-f_{ii}$ is a unit in $\text{Hom}_R(M_i, M_i)$ by lemmas 4 and 5, (cf. [3] the proof of Lemma 8). (ii) is clear from Lemma 1.

In the following, we shall consider the several results given by M. Harada in [4], which are related to semi-T-nilpotent system. We use definitions in [4].

Let $\{M_\alpha\}_{\alpha \in I}$ be a family of completely indecomposable right R -modules. We define an additive category \mathfrak{B} in the same way as in [3] and [4]. The objects of \mathfrak{B} consist of some directsum of M_α 's and the morphisms of \mathfrak{B} consist of all R -homomorphisms, Furthermore, we consider the ideal \mathfrak{F} of \mathfrak{B} (see [3], [4]). Then we know from Theorem 7 in [3] that $\mathfrak{B}/\mathfrak{F}$ is a C_3 -completely reducible abelian category. Let M, N be objects in \mathfrak{B} and i be an inclusion of N to M . If i is isomorphic modulo \mathfrak{F} , then we call $\text{Im } i$ a dense submodule in M (see [4]).

Corollary 1. *Let $M = \sum_{\alpha \in I} \bigoplus M_\alpha$, then the following conditions are equivalent.*

1. $\{M_\alpha\}_{\alpha \in I}$ is a semi-T-nilpotent system.
2. Condition (ii) of Theorem.
3. \mathfrak{A} is the radical of S_M .
4. Every dense submodule of M coincides with M .

It is clear from Theorem and [4], Theorem 2.

Corollary 2. ([4], Proposition 2) *Let M and N be in \mathfrak{B} and $\bar{M} \supseteq \bar{N}$, then there exists a submodule N in M satisfying the following conditions.*

1. N_0 is an object in $\mathfrak{B} : N_0 = \sum_{\alpha \in I'} \oplus M_\alpha$.
2. N_{0J} is a direct summand of M for any finite subset J of I' , (if $\{M_\alpha\}_{\alpha \in I'}$ is a semi- T -nilpotent system, J needs not be finite).
3. $\bar{N}_0 = \bar{N}$

Furthermore, if $N = \text{Im } e$ and e is an idempotent in S_M , then we can choose N_0 in N .

Corollary 3. ([4], Lemma 2) Assume that $M = \sum_{\alpha \in I} \oplus M_\alpha = N_1 \oplus N_2$. If either N_i is finitely generated or a dense submodule of N is semi- T -nilpotent system, then N_i is in \mathfrak{B} .

Corollary 4. $M = \sum_{i=1}^{\infty} \oplus M_i$, where M_i is a comcompletely indecomposable module such that $[M_i, M_j] = 0$ for $i < j$. Then M satisfies the condition (*).

Proof. It is clear that the assumption implies that $\{M_i\}$ is a semi- T -nilpotent system.

Proposition 13 in [3] is a special case of Corollary 4.

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