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CERTAIN INVARIANT SUBRINGS ARE GORENSTEIN II

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Introduction

Let $R=k[X_1, \dots, X_n]$ be a polynomial ring over a field k and G be a finite subgroup of $GL(n, k)$ with $(|G|, ch(k))=1$, if $ch(k)\neq 0$. We want to investigate the problem; “When is the invariant subring R^G Gorenstein?” The main result of this paper is the following theorem.

Theorem 1. *We assume that G contains no pseudo-reflections. Then R^G is Gorenstein if and only if $G \subset SL(n, k)$.*

Recall that $g \in GL(n, k)$ is a pseudo-reflection if $\text{rank}(g-I)=1$ and g has a finite order (where I denotes the unit matrix). It is known that R^G is again a polynomial ring if and only if G is generated by its pseudo-reflections (cf. [7], Théorème 1). So it would be natural to assume that G contains no pseudo-reflections.

The “if” part was treated in [13]. So, in this paper, we consider the “only if” part. To achieve the proof, we need the theory of the canonical module of a Macaulay ring developed in [2]. As R^G is a Macaulay ring, it has the canonical module K_{R^G} , which is unique up to isomorphisms. R^G is Gorenstein if and only if $K_{R^G} \cong R^G$. We want to construct a canonical module of R^G . In this case, as R^G is normal, a canonical module is isomorphic to a divisorial ideal of R^G . Thus the canonical module K_{R^G} determines a well-defined class $c(K_{R^G})$ of the divisor class group $C(R^G)$ of R^G . R^G is Gorenstein if and only if $c(K_{R^G})=0$. But by the “Galois descente” theory of divisor class groups, $C(R^G) \cong \text{Hom}(G, k^*)$ (where k^* denotes the multiplicative group of non-zero elements of k). We show that by this isomorphism, $c(K_{R^G})$ corresponds to \det , the determinant, in $\text{Hom}(G, k^*)$ and conclude the proof of Theorem 1.

We can apply Theorem 1 to the case of regular local rings. If (A, \mathfrak{m}) is a local ring and if $g \in \text{Aut}(A)$, g induces a linear transformation of the tangent space $\mathfrak{m}/\mathfrak{m}^2$ of A . We denote this correspondence by $\lambda: \text{Aut}(A) \rightarrow GL(\mathfrak{m}/\mathfrak{m}^2)$. We call an element g of $\text{Aut}(A)$ a pseudo-reflection if $\lambda(g)$ is a pseudo-reflection. Then, we have the following

Theorem 2. *Let (B, n) be a regular local ring and let G be a finite subgroup of $\text{Aut}(B)$ satisfying the following conditions.*

1. $|G|$ is a unit in B .
2. The automorphisms of $k = B/n$ induced by the elements of G are the identity.
3. G contains no pseudo-reflections.

Then R^G is Gorenstein if and only if $\lambda(G) \subset SL(n/n^2)$.

Examining some examples, it is shown that Theorem 2 fails for non-regular Gorenstein local rings.

1. Preliminaries

(1.1) Canonical modules.

In this paragraph, A denotes a Noetherian local ring with maximal ideal m and residue class field k . All modules are assumed to be unitary and finitely generated.

DEFINITION 1. Let M be a Macaulay A -module of dimension s . The type of M , $r(M)$, is defined by

$$r(M) = \dim_k \text{Ext}_A^s(k, M).$$

The type of a Macaulay ring A is the type of A as an A -module. A is Gorenstein if and only if A is Macaulay and $r(A)=1$.

Proposition A. *If $x \in m$ is an M -regular (resp. A -regular) element, then $r(M/xM) = r(M)$ (resp. $r(A/xA) = r(A)$). (The type of M/xM as an A -module equals to the type of M/xM as an A/xA -module.)*

DEFINITION 2. An A -module K is a canonical module of A if it satisfies the following equivalent conditions.

- (i) $\dim_k \text{Ext}_A^i(k, K) = \begin{cases} 1 & \text{if } i = \dim A, \\ 0 & \text{otherwise.} \end{cases}$
- (ii) a) K is a Macaulay A -module and $\dim K = \dim A$.
b) K has a finite injective dimension.
c) $r(K) = 1$.

REMARK. In the works of R.Y. Sharp, “canonical module” is called “basic Gorenstein module” or “Gorenstein module of rank 1”.

Proposition B. (i) *If K and K' are canonical modules of A , then $K \cong K'$ as A -modules.*

- (ii) *If there exists a canonical module of A , then A is Macaulay.*
- (iii) *If A is Macaulay and if A has a canonical module K , then A is Gorenstein if and only if $K \cong A$ as A -modules.*

Proposition C. ([9], [2], [5]) *If $A=B/a$, A is Macaulay, B is Gorenstein and $\dim B - \dim A = s$, $\text{Ext}_B^s(A, B)$ is a canonical module of A . Conversely, if A has a canonical module, A is a quotient of a Gorenstein ring.*

Proposition D. ([2], [4]) *If A is a Macaulay domain and if A has a canonical module K , K is isomorphic to an ideal of A which is pure of height 1.*

Proposition E. ([2], Korollar 6.12) *Let A be a Macaulay domain and M be an A -module. M is a canonical module of A if and only if the following conditions are satisfied.*

- (i) M is a Macaulay A -module and $\dim M = \dim A$.
- (ii) $r(M) = 1$.

Proposition F. ([2], Satz 1.24, [10]) *Let (A, m) , (B, n) be Macaulay local rings and $f: A \rightarrow B$ be a flat local homomorphism. If M is a Macaulay A -module, then $r(M \otimes_A B) = r(M) \cdot r(B/mB)$.*

(1.2) Descent theory for divisor class groups.

If B is a Noetherian normal domain, we denote by $C(B)$ the divisor class group of B .

Let A be a U.F.D. (unique factorization domain) and let G be a finite subgroup of $\text{Aut}(A)$.

Proposition G. ([6], Chapter III, Theorem 1.1) *There is a monomorphism $i: C(A^G) \cong H^1(G, U(A))$ ($U(A)$ denotes the multiplicative group of units of A). If A is divisorially unramified over A^G , i is an isomorphism.*

Proposition H. (cf. [11], Proposition 1) *If $R = k[X_1, \dots, X_n]$ is a polynomial ring over a field k , $G \subset GL(n, k)$ and if G does not contain any pseudo-reflections, then $i: C(R^G) \cong \text{Hom}(G, k^*)$ (k^* is the multiplicative group of non-zero elements of k). If $\omega \in \text{Hom}(G, k^*)$, then there exists an element $f \in R$ such that for every $g \in G$, $g(f) = \omega(g)f$.*

Proposition I. ([11], Theorem 2) *Let (A, m) be a local U.F.D. and let G be a finite subgroup of $\text{Aut}(A)$. We assume the following conditions for A and G .*

1. $|G|$ is a unit in A .
2. A contains a primitive $|G|$ -th root of unity.
3. G acts trivially on $k = A/m$.
4. G contains no pseudo-reflections.

Then $i: C(A^G) \cong \text{Hom}(G, k^)$. If $\omega \in \text{Hom}(G, k^*)$, then there exists an element $f \in m$ which satisfies the following conditions.*

1. *The ideal fA is invariant under G .*
2. *For every $g \in G$, $\text{In}(g(f)) = \omega(g) \cdot \text{In}(f)$. (Recall that if $f \in m^n$ and $f \notin m^{n+1}$,*

then $\text{In}(f)=f \bmod m^{n+1}$ in $\text{Gr}_m^n(A)$.)

(1.3) Associated graded rings of local rings.

Let (A, m) be a Noetherian local ring and N be an A -module. Let $(F_n)_{n \geq 0}$ be a filtration on A with $F_0=A$ and $F_1=m$. We assume that $(F_n)_{n \geq 0}$ defines the same topology as the m -adic topology. We put $R=G^*(A)=\bigoplus_{n \geq 0} F_n/F_{n+1}$, $M=R_+=\bigoplus_{n \geq 1} F_n/F_{n+1}$ and $\tilde{N}=G^*(N)=\bigoplus_{n \geq 0} F_n N/F_{n+1} N$. If $a \in F_n$ and $a \notin F_{n+1}$, we write $\text{In}(a)=a \bmod F_{n+1} \in R_n=F_n/F_{n+1}$. We define $\text{In}(x)$ for $x \in N$ in the same manner.

Proposition J. *Let $f_i=\text{In}(a_i)$ ($i=1, \dots, s$) be homogenous elements of M which make an \tilde{N} -regular sequence. Then a_i ($i=1, \dots, s$) make an N -regular sequence and we have a canonical isomorphism $G^*(N/(a_1, \dots, a_s)N) \cong \tilde{N}/(f_1, \dots, f_s)\tilde{N}$.*

Proof. The proof of Lemma 10 of [13] works by the change of notations.

Proposition K. (i) *If R_M is Macaulay, then A is Macaulay.*

(ii) *If R_M is Gorenstein, then A is Gorenstein.*

(iii) *If \tilde{N}_M is a Macaulay R_M -module, then N is a Macaulay A -module.*

(iv) *If \tilde{N}_M is Macaulay and if $r(\tilde{N}_M)=1$, then $r(N)=1$.*

Proof. (i) and (ii) are Theorem 3 of [13]. (iii) is clear by Proposition J. The proof of Lemma 11 of [13] works for the proof of (iv) if we use the following lemma instead of Lemma 3 of [13].

Lemma 1. *If (A, m) is a Noetherian local ring and if N is an Artinian A -module, the following conditions are equivalent.*

(a) $r(N)=1$.

(b) $\text{length}_A(O:m)_N=1$.

(c) *There exists an element $z \neq 0$ in N such that for every $y \neq 0$ in N there exists an element a in A satisfying $ay=z$.*

Proposition L. *Let $A=\bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring with $A_0=k$ a field.*

Let $M=\bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded A -module and $f \in A_d$ be a homogenous M -regular element. Then $\dim_k(M/fM)_n$ depends only on M , n and d .

Proof. $\dim_k(M/fM)_n=\dim_k M_n-\dim_k M_{n-d}$.

(1.4) Invariant subrings and the Reynolds operator.

In this paragraph, R is a Noetherian ring and G is a finite subgroup of $\text{Aut}(R)$. We assume that $|G|$ is a unit in R . We define the Reynolds operator $\rho: R \rightarrow R^G$ by $\rho(r)=\frac{1}{|G|} \sum_{g \in G} g(r)$ for $r \in R$.

Lemma 2. *Let M be an R -module. We assume that G acts on M satisfying the condition $g(ax)=g(a)g(x)$ for $g \in G$, $a \in R$ and $x \in M$. If a_1, \dots, a_s be elements of R^G which make an M -regular sequence, then they make an M^G -regular sequence and there is a canonical isomorphism $M^G/(a_1, \dots, a_s)M^G \cong (M/(a_1, \dots, a_s)M)^G$.*

Proof. It suffices to prove the last equality for $s=1$. We put $a=a_1$. There exists a natural inclusion $M^G/aM^G \hookrightarrow (M/aM)^G$. If $x \in M$ and $g(x)-x \in aM$ for every $g \in G$, then $\rho(x)-x \in aM$ and $\rho(x) \in M^G$. So this inclusion is an isomorphism.

Lemma 3. *Let M be as in Lemma 2. If M is a Macaulay R -module, then M^G is a Macaulay R^G -module.*

Proof. We can take a parameter system a_1, \dots, a_s of M from R^G . Then a_1, \dots, a_s make an M -regular sequence. By Lemma 2, they make an M^G -regular sequence.

Lemma 4. *Let M be as in Lemma 2. We assume further that R is local with maximal ideal m and M is Artinian with $r(M)=1$. Let z be an element of N which satisfies the condition in (c) of Lemma 1. If $z \in M^G$, then $r(M^G)=1$.*

Proof. The same as the proof of Lemma 4 of [13].

2. The main theorem

Theorem 1. *Let $R=k[X_1, \dots, X_n]$ be a polynomial ring over a field k and G be a finite subgroup of $GL(n, k)$ with $(|G|, ch(k))=1$ if $ch(k) \neq 0$. We also assume that G contains no pseudo-reflections. Then R^G is Gorenstein if and only if $G \subset SL(n, k)$.*

The “if” part was proved in [13]. So it suffices to prove the “only if” part. First we fix our notations. We put $n=(X_1, \dots, X_n)R$, $m=n \cap R^G$, $B=R_n$ and $A=(R^G)_m$.

As A is Macaulay (Lemma 3) and is a quotient of a regular local ring, it has a canonical module K_A . By Proposition D, K_A is isomorphic to a divisorial ideal of A . As isomorphic ideals determine the same element of the divisor class group, K_A determines a well-defined element $c(K_A) \in C(A)$. A is Gorenstein if and only if $c(K_A)=0$. But as R^G is a graded ring, $C(A) \cong C(R^G)$ by Proposition 7.4 of [6] and by Proposition H, $C(R^G) \cong \text{Hom}(G, k^*)$. To prove the theorem, it suffices to show that $c(K_A) \in C(A)$ corresponds to $\det \in \text{Hom}(G, k^*)$, the determinant map, by these isomorphisms. The realization of $\det \in \text{Hom}(G, k^*)$ as a divisorial ideal is done by the following way. Take $f \in R$ as in Proposition H. That is to say, f satisfies the condition $g(f)=\det(g)f$ for $g \in G$. We can assume that f is homogenous. We put $\deg(f)=d$. Then $K=(fR \cap R^G)A$ is a divisorial

ideal of A whose class in $C(A)$ corresponds to \det in $\text{Hom}(G, k^*)$. Thus Theorem 1 reduces to the following

Theorem 1'. *Let f be as above. Then $K=(fR \cap R^G)A$ is a canonical module of A .*

Proof. By Proposition E , it suffices to prove that K is a Macaulay A -module and $r(K)=1$. (It is clear that $\dim(K)=\dim(A)$ as K is an ideal of A .) We divide the proof into several steps.

Lemma 5. *K is a Macaulay A -module.*

Proof. Let (a_1, \dots, a_n) be elements of R^G which make a parameter system for A . Then (a_1, \dots, a_n) make an R -regular sequence. As fR is a free R -module, they also make an fR -regular sequence. As $fR \cap R^G=(fR)^G$, they also make an $fR \cap R^G$ -regular sequence by Lemma 2 and thus make a K -regular sequence. Thus K is a Macaulay A -module.

Lemma 6. $r(K)=1$.

Proof. We divide the proof into two steps.

Case 1. G is cyclic.

If k' is an extension field of k and if we put $R'=R \otimes_k k'$, then G acts naturally on R' and $(R')^G \cong R^G \otimes_k k'$ by Lemma 3 of [13]. If we put $n'=(X_1, \dots, X_n)R'$ and $m'=n' \cap (R')^G$, then $A'=((R')^G)_{m'}$ is faithfully flat over A and $K \otimes_A A' \cong (fR' \cap (R')^G)A'$. By Proposition F , it suffices to show that $r(K \otimes_A A')=1$. Thus we may assume that k is algebraically closed.

Let g be a generator of G . We may assume that g is in a diagonal form. If m is a multiple of $|G|$, we can take X_1^m, \dots, X_n^m as a parameter system for A . As $\bar{B}=R/(X_1^m, \dots, X_n^m)$ is an Artinian Gorenstein local ring and $\bar{M}=fR/(X_1^m, \dots, X_n^m)fR$ is a free \bar{B} -module, $r(\bar{M})=1$ and $z=(X_1 \cdots X_n)^{m-1}f \bmod (X_1^m, \dots, X_n^m)fR$ satisfies the condition in (c) of Lemma 1. But as $g((X_1 \cdots X_n)^{m-1}f)=\det(g)^{-1}(X_1 \cdots X_n)^{m-1}\det(g)f=(X_1 \cdots X_n)^{m-1}f$, $z \in \bar{M}^G$. As $\bar{M}^G=K/(X_1^m, \dots, X_n^m)K$ by Lemma 2, we have $r(K)=1$ by Lemma 4 and Proposition A.

Case 2. General case.

We take a parameter system (a_1, \dots, a_n) of R^G satisfying the following conditions.

1. a_i are homogenous of the same degree m .
2. m is a multiple of $|G|$.

We put $\bar{B}=R/(a_1, \dots, a_n)R$ and $\bar{M}=fR/(a_1, \dots, a_n)fR$. The $\bar{B}^G \cong A/(a_1, \dots, a_n)A$ and $\bar{M}^G \cong K/(a_1, \dots, a_n)K$. As $\bar{B} \cong \bar{M}$ as \bar{B} -modules and as \bar{B} is Gorenstein, $r(\bar{M})=1$. As f is a homogenous element of R , fR is a graded ideal and \bar{B} and \bar{M} have induced structures of a graded ring and a graded \bar{B} -module respectively.

As in the proof of Theorem 1a of [13], $\dim_k \bar{B}_{n(m-1)} = 1$ and $\bar{B}_t = 0$ for $t > n(m-1)$. Similarly, $\dim_k \bar{M}_{n(m-1)+d} = 1$ and $\bar{M}_t = 0$ for $t > n(m-1)+d$. Let H be a cyclic subgroup of G and g be a generator of H . We can assume that g is in a diagonal form. Then we can take (X_1^m, \dots, X_n^m) for a parameter system of R^H . If z is a generator of $(fR/(X_1^m, \dots, X_n^m)fR)_{n(m-1)+d}$, we have seen in Case 1 that z is invariant under H . If we use Proposition L to R^H and $(fR)^H$, we have $\dim_k ((fR)^H/(X_1^m, \dots, X_n^m)(fR)^H)_{n(m-1)+d} = \dim_k ((fR)^H/(a_1, \dots, a_n)(fR)^H)_{n(m-1)+d} = \dim_k (\bar{M}^H)_{n(m-1)+d} = 1$. Thus we see that each element of $\bar{M}_{n(m-1)+d}$ is invariant under the action of H . As H is arbitrary cyclic subgroup of G , we can say that each element of $\bar{M}_{n(m-1)+d}$ is invariant under the action of G . By Lemma 4 and Proposition A , $r(K) = r(K/(a_1, \dots, a_n)K) = r(\bar{M}^G) = 1$.

This completes the proof of Theorem 1'.

3. The case of regular local rings

In this section, let (B, n) be a regular local ring of dimension n and G be a finite subgroup of $\text{Aut}(B)$ with $|G|$ a unit in B . If $g \in G$, g induces a linear transformation of the tangent space n/n^2 of B . We denote this transformation by $\lambda(g)$. Thus we get a group homomorphism $\lambda: G \rightarrow GL(n/n^2)$. We call an element g of G a pseudo-reflection if $\lambda(g)$ is a pseudo-reflection.

Theorem 2. *Let (B, n) be a regular local ring and G be a finite subgroup of $\text{Aut}(B)$ which satisfies the following conditions.*

1. $|G|$ is a unit in B .
2. G acts trivially on $k = B/n$.
3. G contains no pseudo-reflections.

Then B^G is Gorenstein if and only if $\lambda(G) \subset SL(n/n^2)$.

Proof. We put $A = B^G$ and $m = n \cap A$. First, we need a lemma.

Lemma 7. *We may assume that B contains a primitive $|G|$ -th root of unity.*

Proof. We know that A is a Noetherian local ring (cf. the proof of Theorem 4 of [13]). G acts on \hat{B} and we have $(\hat{B})^G \cong \hat{A}$ ([14], Chapter II, Lemma 1, Corollary). (\hat{B} denotes the completion of B .) G induces the same linear transformations on the tangent spaces of B and \hat{B} and A is Gorenstein if and only if $\hat{A} \cong (\hat{B})^G$ is Gorenstein. Thus we may assume that B is complete. If $k = B/n$ contains a primitive $|G|$ -th root of unity, then B contains a primitive $|G|$ -th root of unity by Hensel's lemma. If k does not contain primitive $|G|$ -th roots of unity, let F be a monic polynomial in the polynomial ring $B[T]$ whose image in $k[T]$ is an irreducible polynomial for a primitive $|G|$ -th root of unity. We put $B' = B[T]/(F)$. Then B' is free over B and B' contains a primitive $|G|$ -th root of unity. As B'/nB' is a field, B' is a regular local ring and we can extend

the action of G to B' . As B' is free over B , it is easily seen that $(B')^G$ is also free over A and $(B')^G/m(B')^G \cong B'/nB'$. Thus A is Gorenstein if and only if $(B')^G$ is Gorenstein ([12], Theorem 1). This completes the proof of Lemma 7.

By Lemma 7, we can use Proposition I. By the similar reasoning as in the case of Theorem 1, we reduce Theorem 2 to the following

Theorem 2'. *Let (B, n) , G be as in Theorem 2 and let f be an element of B which satisfies the following conditions.*

1. *The ideal fB is invariant under the action of G .*
2. *For every $g \in G$, $In(g(f)) = \lambda(g)In(f) = \det(\lambda(g))In(f)$.*

Then $(fB) \cap A = K$ is a canonical module of A .

Proof of Theorem 2'. We put $R = Gr_n^*(B)$ the associated graded algebra of B . We know that $R \cong k[X_1, \dots, X_n]$, where $n = \dim(B)$. G acts on R by the action of $\lambda(G)$ on linear forms of R . We define a filtration $(F_i)_{i \geq 0}$ on A by putting $F_i = A \cap n^i = (n^i)^G$. It was shown in [13], §7, that the filtration $(F_i)_{i \geq 0}$ defines the same topology as the m -adic topology on A . The filtration (F_i) induces a filtration on K . We define $F_i \cdot K = (f \cdot n^i)^G = F_{i+d}(K)$ if $\deg(In(f)) = d$. We put $G^*(A) = \bigoplus_{i \geq 0} F_i / F_{i+1}$ and $G^*(K) = \bigoplus_{i \geq d} F_i(K) / F_{i+1}(K)$.

Lemma 8. *K is a Macaulay A -module.*

Proof. This is a direct consequence of Lemma 3.

Lemma 9. *$G^*(A) \cong R^G$ and $G^*(K) = In(f)R \cap G^*(A)$.*

Proof. The first assertion was proved in [13], in the proof of Theorem 4. As K is an ideal of A , $G^*(K)$ is the ideal of $G^*(A)$ generated by $\{In(a) \mid a \in K\}$. But if we consider $G^*(A)$ as a subring of R and if $a \in A$, then $In(a)$ in R is equal to $In(a)$ in $G^*(A)$. As $G_n^*(fB) = In(f) \cdot R$ in R , the second assertion follows.

Lemma 10. *$r(K) = 1$.*

Proof. This is a consequence of Lemma 6, Lemma 5 and Proposition K, (iv).

By Lemma 8 and Lemma 10, the proof of Theorem 2' is complete.

4. Base extensions

In this section, let A be a Noetherian ring and G be a finite subgroup of $GL(n, A)$ with $|G|$ a unit in A . G acts naturally on $R = A[X_1, \dots, X_n]$, the polynomial ring over A . When $p \in \text{Spec}(A)$ and $g \in G$, we say that g is a pseudo-reflection at p if the canonical image of g in $GL(n, k(p))$ is a pseudo-reflection. ($k(p) = A_p/pA_p$) Under these terminologies, we have the following

Theorem 3. *If G does not contain any pseudo-reflections at each point of $\text{Spec}(A)$, then R^G is Gorenstein if and only if A is Gorenstein and $\det(g) - 1$ is nilpotent for all $g \in G$.*

Proof. By Lemma 9 of [13], R^G is faithfully flat over A . Thus R^G is Gorenstein if and only if A is Gorenstein and $R^G \otimes_A k(p)$ is Gorenstein for all $p \in \text{Spec}(A)$ ([12], Theorem 1'). But as $R^G \otimes_A k(p) \cong (k(p)[X_1, \dots, X_n])^G$, $R^G \otimes_A k(p)$ is Gorenstein if and only if the canonical image of G in $GL(n, k(p))$ is contained in $SL(n, k(p))$ by Theorem 1. From these facts, Theorem 3 follows easily.

5. Examples

In this section, k always denotes a field and e_m denotes a primitive m -th root of unity in k . We assume always that $|G|$ is not a multiple of $\text{ch}(k)$. We denote $G = \langle g \rangle$ if G is a cyclic group generated by g .

(5.1) “ R^G is Gorenstein” does not imply “ R is Gorenstein”.

EXAMPLE 1. Let $R = k[T^3, T^4, T^5]$ and $G = \langle g \rangle$. If g acts on R by $g(T) = e_3 T$, then $R^G = k[T^3]$ is regular but R is not Gorenstein.

EXAMPLE 2. Let $R = k[S^4, S^3T, S^2T^2, ST^3, T^4]$ and $G = \langle g \rangle$ acting on R by $g(S) = e_8 S$ and $g(T) = e_8^{-1} T$. Then $R^G = k[S^2T^2, S^8, T^8]$ is Gorenstein but R is not Gorenstein.

(5.2) “ (A, m) is Gorenstein local and $\lambda(G) \supset SL(m/m^2)$ ” does not imply “ A^G is Gorenstein”.

EXAMPLE 3. Let $R = k[S^2, ST, T^2]$ and $G = \langle g \rangle$ acting on R by $g(S) = e_3 S$ and $g(T) = e_3 T$. If A is the local ring of R at the maximal ideal (S^2, ST, T^2) , then A is Gorenstein, $\lambda(g) \in SL(m/m^2)$ and A^G is not Gorenstein because $R^G = k[S^6, S^5T, S^4T^2, S^3T^3, S^2T^4, ST^5, T^6]$.

(5.3) “ (A, m) is Gorenstein local and A^G is Gorenstein” does not imply “ $\lambda(G) \subset SL(m/m^2)$ ”.

EXAMPLE 4. Let $R = k[T^2, T^3]$ and $G = \langle g \rangle$ acting on R by $g(T) = e_4 T$. Then R is Gorenstein and $R^G = k[T^4]$ is regular. But if A is the local ring of R at the maximal ideal (T^2, T^3) , $\lambda(g) \notin SL(m/m^2)$.

EXAMPLE 5. Let $R = k[X^2, XY, Y^2, Z]$ and $G = \langle g \rangle$ acting on R by $g(X) = X, g(Y) = e_3 Y, g(Z) = e_3^{-1} Z$. Then R is Gorenstein and by Theorem 1, R^G is Gorenstein. But if A is the local ring of R at the maximal ideal (X^2, XY, Y^2, Z) , $\lambda(g) \notin SL(m/m^2)$.

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