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A NOTE ON SYMMETRIC LINEAR FORMS AND TRACES ON THE RESTRICTED QUANTUM GROUP $\bar{U}_q(\mathfrak{sl}(2))$

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Abstract

In this paper we prove two results about $\mathrm{SLF}(\bar{U}_q)$, the algebra of symmetric linear forms on the restricted quantum group $\bar{U}_q = \bar{U}_q(\mathfrak{sl}(2))$. First, we express any trace on finite dimensional projective $\bar{U}_q$-modules as a linear combination in the basis of $\mathrm{SLF}(\bar{U}_q)$ constructed by Gainutdinov - Tipunin and also by Arike. In particular, this allows us to determine the symmetric linear form corresponding to the modified trace on projective $\bar{U}_q$-modules. Second, we give the explicit multiplication rules between symmetric linear forms in this basis.

1. Introduction

Let $\bar{U}_q = \bar{U}_q(\mathfrak{sl}(2))$ be the restricted quantum group associated to $\mathfrak{sl}(2)$ and $\mathrm{SLF}(\bar{U}_q)$ its space of *symmetric linear forms*, which is naturally endowed with an algebra structure. In [9] and [1], an interesting basis of $\mathrm{SLF}(\bar{U}_q)$ is introduced, that will be called the GTA basis in the sequel, and whose construction is based on the simple and the projective $\bar{U}_q$-modules (see section 3). In this paper, we prove two results about this basis, namely the relation with traces on projectives modules, and the formulas for multiplication of symmetric linear forms.

First, we show in the general setting of a finite dimensional k-algebra A that there is a correspondence between traces on finite dimensional projective A-modules and symmetric linear forms on A (Theorem 4.1). In the case of $A = \bar{U}_q$, the natural question is to express the image of a trace through this correspondence in the GTA basis. We answer this question and show that this basis is relevant with regard to this correspondence in Theorem 4.2. The modified trace computed in [3] is an interesting example of a trace on projective $\bar{U}_q$-modules. We determine the symmetric linear form corresponding to the modified trace, and get that it is $\mu(K^{p+1}\cdot)$, where μ is a suitably normalized right integral of $\bar{U}_q$ (see section 4.3). This last result has been found simultaneously in [2] in a general framework including $\bar{U}_q$.

With regard to the structure of algebra on $\mathrm{SLF}(\bar{U}_q)$, a natural and important problem is to determine the multiplication rules of the elements in the GTA basis. In section 5, we find the decomposition of the product of two basis elements in the GTA basis. The resulting formulas are surprisingly simple (Theorem 5.1). Note that a similar problem (namely the multiplication in the space of q-characters $\mathrm{qCh}(\bar{U}_q)$, which is isomorphic as an algebra to $\mathrm{SLF}(\bar{U}_q)$) has been solved in [9], but I was not aware of the existence of this paper when preparing this work. It turns out that our proofs are different. In [9], they use the fact that the

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multiplication in the canonical basis of $\mathcal{Z}(\bar{U}_q)$ is very simple. They first express the image of their basis of $\mathrm{qCh}(\bar{U}_q)$ through the Radford mapping in the canonical basis of $\mathcal{Z}(\bar{U}_q)$. This gives a basis of $\mathcal{Z}(\bar{U}_q)$ called the Radford basis. Then they use the $\mathcal{S}$-transformation of the $\mathrm{SL}_2(\mathbb{Z})$ representation on $\mathcal{Z}(\bar{U}_q)$ to express the Drinfeld basis (which is the image of their basis of $\mathrm{qCh}(\bar{U}_q)$ by the Drinfeld map) in the Radford basis. This gives the multiplication rules in the Drinfeld basis. Since the Drinfeld map is an isomorphism of algebras between $\mathrm{qCh}(\bar{U}_q)$ and $\mathcal{Z}(\bar{U}_q)$, this gives also the multiplication rules in the GTA basis. Here we directly work in $\mathrm{SLF}(\bar{U}_q)$. We first prove an elementary lemma which shows that there are not many coefficients to determine, and then we compute these coefficients by using the evaluation on suitable elements of $\bar{U}_q$.

To make the paper self-contained and fix notations, we recall some facts about the structure of $\bar{U}_q$ and its representation theory in section 2. In section 3, we introduce $\mathrm{SLF}(\bar{U}_q)$ and the GTA basis. We then state some properties that are needed to prove our results.

In [5], the GTA basis and its multiplication rules are extensively used to describe in detail the projective representation of $\mathrm{SL}_2(\mathbb{Z})$ (the mapping class group of the torus) on $\mathrm{SLF}(\bar{U}_q)$ provided by the graph algebra of the torus with the gauge algebra $\bar{U}_q$ (which is a quantum analogue of the algebra of functions associated to lattice gauge theory on the torus).

NOTATIONS. If A is a k-algebra (with k a field), V is a finite dimensional A-module and $x \in A$, we denote by $\overset{V}{x} \in \mathrm{End}(V)$ the representation of x on the module V. We will work only with finite dimensional modules and mainly with left modules, thus often we simply write "module" instead of "finite dimensional left module". The *socle* of V, denoted by $\mathrm{Soc}(V)$ is the largest semi-simple submodule of V. The *top* of V, denoted by $\mathrm{Top}(V)$, is $V/\mathrm{Rad}(V)$, where $\mathrm{Rad}(V)$ is the Jacobson radical of V. See [4, Chap. IV and VIII] for background material about representation theory.

For $q \in \mathbb{C} \setminus \{-1, 0, 1\}$, we define the q-integer $[n]$ (with $n \in \mathbb{Z}$) and the q-factorial $[m]!$ (with $m \in \mathbb{N}$) by:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [0]! = 1, \ [m]! = [1][2] \ldots [m] \ \text{ for } m \geq 1.$$

In what follows q is a primitive $2p$-root of unity (where p is a fixed integer ≥ 2), say $q = e^{i\pi/p}$. Observe that in this case $[n] = \frac{\sin(n\pi/p)}{\sin(\pi/p)}$, $[p] = 0$ and $[p - n] = [n]$.

As usual, $\delta_{i,j}$ will denote the Kronecker symbol and I_n the identity matrix of size n.

2. Preliminaries

2.1. The restricted quantum group $\bar{U}_q(\mathfrak{sl}(2))$. As mentioned above, q is a primitive root of unity of order $2p$, with $p \geq 2$. Recall that $\bar{U}_q(\mathfrak{sl}(2))$, the *restricted quantum group* associated to $\mathfrak{sl}(2)$, is the $\mathbb{C}$-algebra generated by E, F, K together with the relations

$$E^p = F^p = 0, \quad K^{2p} = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF = FE + \frac{K - K^{-1}}{q - q^{-1}}.$$

It will be simply denoted by $\bar{U}_q$ in the sequel. It is a $2p^3$-dimensional Hopf algebra, with comultiplication Δ, counit ε and antipode S given by the following formulas:

$$\begin{array}{lll} \Delta(E) = 1 \otimes E + E \otimes K, & \Delta(F) = F \otimes 1 + K^{-1} \otimes F, & \Delta(K) = K \otimes K, \\ \varepsilon(E) = 0, & \varepsilon(F) = 0, & \varepsilon(K) = 1, \\ S(E) = -EK^{-1}, & S(F) = -KF, & S(K) = K^{-1}. \end{array}$$

The monomials $E^m F^n K^l$ with $0 \le m, n \le p-1$, $0 \le l \le 2p-1$, form a basis of $\bar{U}_q$, usually referred as the PBW-basis. Recall the formula (see [11, Prop. VII.1.3]):

$$(1) \qquad \Delta(E^m F^n K^l) = \sum_{i=0}^{m} \sum_{j=0}^{n} q^{i(m-i)+j(n-j)-2(m-i)(n-j)} \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} E^{m-i} F^j K^{l+j-n} \otimes E^i F^{n-j} K^{l+m-i}.$$

Recall that the q-binomial coefficients are defined by $\begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!}$ for $a \ge b$.

Since K is annihilated by the polynomial $X^{2p} - 1$, which has simple roots over $\mathbb{C}$, the action of K is diagonalizable on each $\bar{U}_q$-module, and the eigenvalues are $2p$-roots of unity.

Due to the Hopf algebra structure on $\bar{U}_q$, its category of modules is a monoidal category with duals. It is not braided (see [12]).

2.2. Simple and projective $\bar{U}_q$-modules. The finite dimensional representations of $\bar{U}_q$ are classified ([15] and [7]). Two types of modules are important for our purposes: the simple and the projective modules. As in [6] (see also [10]), we denote the simple modules by $\mathcal{X}^\alpha(s)$, with $\alpha \in \{\pm\}$, $1 \le s \le p$. The modules $\mathcal{X}^\pm(p)$ are simple and projective simultaneously. The other indecomposable projective modules are not simple. We denote them by $\mathcal{P}^\alpha(s)$ with $\alpha \in \{\pm\}$, $1 \le s \le p-1$.

The module $\mathcal{X}^\alpha(s)$ admits a *canonical basis* $(v_i)_{0 \le i \le s-1}$ such that

$$(2) \qquad Kv_i = \alpha q^{s-1-2i} v_i,\ Ev_0 = 0,\ Ev_i = \alpha[i][s-i]v_{i-1},\ Fv_i = v_{i+1},\ Fv_{s-1} = 0.$$

The module $\mathcal{P}^\alpha(s)$ admits a *standard basis* $\left(b_i, x_j, y_k, a_l\right)_{\substack{0 \le i,l \le s-1 \\ 0 \le j,k \le p-s-1}}$ such that

$$(3) \qquad \begin{array}{lll} Kb_i = \alpha q^{s-1-2i} b_i, & Eb_i = \alpha[i][s-i]b_{i-1} + a_{i-1}, & Fb_i = b_{i+1}, \\ & Eb_0 = x_{p-s-1}, & Fb_{s-1} = y_0, \\ Kx_j = -\alpha q^{p-s-1-2j} x_j, & Ex_j = -\alpha[j][p-s-j]x_{j-1}, & Fx_j = x_{j+1}, \\ & Ex_0 = 0, & Fx_{p-s-1} = a_0, \\ Ky_k = -\alpha q^{p-s-1-2k} y_k, & Ey_k = -\alpha[k][p-s-k]y_{k-1}, & Fy_k = y_{k+1}, \\ & Ey_0 = a_{s-1}, & Fy_{p-s-1} = 0, \\ Ka_l = \alpha q^{s-1-2l} a_l, & Ea_l = \alpha[l][s-l]a_{l-1}, & Fa_l = a_{l+1}, \\ & Ea_0 = 0, & Fa_{s-1} = 0. \end{array}$$

Note that such a basis is not unique up to scalar since we can replace b_i by $b_i + \lambda a_i$ (with $\lambda \in \mathbb{C}$) without changing the action.

In terms of composition factors, the structure of $\mathcal{P}^\alpha(s)$ can be schematically represented as follows (with the basis vectors corresponding to each factor and the action of E and F):

(4)

$$\begin{tikzcd}
& \mathrm{Top}\,(\mathcal{P}^{\alpha}(s)) \cong \mathcal{X}^{\alpha}(s), (b_i)_{0\leq i\leq s-1} \arrow[dl, "E"'] \arrow[dd, "E"] \arrow[dr, "F"] & \\
(x_j)_{0\leq j\leq p-s-1}, \mathcal{X}^{-\alpha}(p-s) \arrow[dr, "F"'] & & \mathcal{X}^{-\alpha}(p-s), (y_k)_{0\leq k\leq p-s-1} \arrow[dl, "E"] \\
& \mathrm{Soc}\,(\mathcal{P}^{\alpha}(s)) \cong \mathcal{X}^{\alpha}(s), (a_l)_{0\leq l\leq s-1} &
\end{tikzcd}$$

If we need to emphasize the module in which we are working, we will use the following notations: $v_i^{\alpha}(s)$ for the canonical basis of $\mathcal{X}^{\alpha}(s)$ and $b_i^{\alpha}(s)$, $x_j^{\alpha}(s)$, $y_k^{\alpha}(s)$, $a_l^{\alpha}(s)$ for a standard basis of $\mathcal{P}^{\alpha}(s)$ (these are the notations used in [1]).

Let us recall the $\bar{U}_q$-morphisms between these modules. Observe that $\mathcal{X}^{\alpha}(s)$ is $\bar{U}_q$-generated by $v_0^{\alpha}(s)$ and $\mathcal{P}^{\alpha}(s)$ is $\bar{U}_q$-generated by $b_0^{\alpha}(s)$, so the images of these vectors suffice to define $\bar{U}_q$-morphisms. $\mathcal{X}^{\alpha}(s)$ is simple, so by Schur's lemma $\mathrm{End}_{\bar{U}_q}(\mathcal{X}^{\alpha}(s)) = \mathbb{C}\mathrm{Id}$. Since

$$\mathcal{X}^{\alpha}(s) \cong \mathrm{Top}\,(\mathcal{P}^{\alpha}(s)) \cong \mathrm{Soc}\,(\mathcal{P}^{\alpha}(s))$$

there exist injection and projection maps defined by:

$$\begin{array}{ccc} \mathcal{X}^{\alpha}(s) & \hookrightarrow & \mathcal{P}^{\alpha}(s) \\ v_0^{\alpha}(s) & \mapsto & a_0^{\alpha}(s) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{P}^{\alpha}(s) & \twoheadrightarrow & \mathcal{X}^{\alpha}(s) \\ b_0^{\alpha}(s) & \mapsto & v_0^{\alpha}(s). \end{array}$$

We have $\mathrm{End}_{\bar{U}_q}(\mathcal{P}^{\alpha}(s)) = \mathbb{C}\mathrm{Id} \oplus \mathbb{C}p_s^{\alpha}$ and $\mathrm{Hom}_{\bar{U}_q}(\mathcal{P}^{\alpha}(s), \mathcal{P}^{-\alpha}(p-s)) = \mathbb{C}P_s^{\alpha} \oplus \mathbb{C}\overline{P}_s^{\alpha}$, where:

$$p_s^{\alpha}\left(b_0^{\alpha}(s)\right) = a_0^{\alpha}(s), \qquad P_s^{\alpha}\left(b_0^{\alpha}(s)\right) = x_0^{-\alpha}(p-s), \qquad \overline{P}_s^{\alpha}\left(b_0^{\alpha}(s)\right) = y_0^{-\alpha}(p-s). \tag{5}$$

The other Hom-spaces involving only simple modules and indecomposable projective modules are null.

2.3. Structure of the bimodule ${}_{\bar{U}_q}\left(\bar{U}_q\right)_{\bar{U}_q}$ and the center of $\bar{U}_q$. Recall that if M is a left module (over any k-algebra A), then $M^* = \mathrm{Hom}_{\mathbb{C}}(M, k)$ is endowed with a *right* A-module structure, given by:

$$\forall\, a \in A,\ \forall\, \varphi \in M^*,\ \ \varphi a = \varphi(a\cdot)$$

where $\cdot$ is the place of the variable. We denote by $R^*(M)$ the so-defined right module. Note that if we define $R^*(f)$ as the transpose of f, then R^* becomes a contravariant functor. If A is a Hopf algebra, one must be aware not to confuse $R^*(M)$ with the categorical dual M^*, which is a left module on which A acts by:

$$\forall\, a \in A,\ \forall\, \varphi \in M^*,\ \ a\varphi = \varphi(S(a)\cdot).$$

Lemma 2.1. *The right $\bar{U}_q$-module $R^*(\mathcal{X}^{\alpha}(s))$ admits a basis $(\bar{v}_i)_{0\leq i\leq s-1}$ such that*

$$\bar{v}_i K = \alpha q^{1-s+2i}\bar{v}_i, \quad \bar{v}_i E = \alpha[i][s-i]\bar{v}_{i-1}, \quad \bar{v}_0 E = 0, \quad v_i F = \bar{v}_{i+1}, \quad \bar{v}_{s-1} F = 0.$$

The right $\bar{U}_q$-module $R^*(\mathcal{P}^{\alpha}(s))$ admits a basis $\left(\bar{b}_i, \bar{x}_j, \bar{y}_k, \bar{a}_l\right)_{\substack{0\leq i,l\leq s-1 \\ 0\leq j,k\leq p-s-1}}$ such that

$$\begin{array}{lll}
\bar{b}_i K = \alpha q^{1-s+2i}\bar{b}_i, & \bar{b}_i E = \bar{a}_{i-1} + \alpha[i][s-i]\bar{b}_{i-1}, & \bar{b}_i F = \bar{b}_{i+1}, \\
 & \bar{b}_0 E = \bar{x}_{p-s-1}, & \bar{b}_{s-1} F = \bar{y}_0, \\
\bar{x}_j K = -\alpha q^{-p+s+1+2j}\bar{x}_j, & \bar{x}_j E = -\alpha[j][p-s-j]\bar{x}_{j-1}, & \bar{x}_j F = \bar{x}_{j+1}, \\
 & \bar{x}_0 E = 0, & \bar{x}_{p-s-1} F = \bar{a}_0, \\
\bar{y}_k K = -\alpha q^{-p+s+1+2k}\bar{y}_k, & \bar{y}_k E = -\alpha[k][p-s-k]\bar{y}_{k-1}, & \bar{y}_k F = \bar{y}_{k+1}, \\
 & \bar{y}_0 E = \bar{a}_{s-1}, & \bar{y}_{p-s-1} F = 0, \\
\bar{a}_l K = \alpha q^{1-s+2l}\bar{a}_l, & \bar{a}_l E = \alpha[l][s-l]\bar{a}_{l-1}, & \bar{a}_l F = \bar{a}_{l+1}, \\
 & \bar{a}_0 E = 0, & \bar{a}_{s-1} F = 0.
\end{array}$$

Such basis will be termed respectively a *canonical basis* and a *standard basis* in the sequel.

Proof. Let $(v^i)_{0\le i\le s-1}$ be the basis dual to the canonical basis given in (2). Then $\bar{v}_i = v^{s-1-i}$ gives the desired result. Similarly, let $\left(b^i, x^j, y^k, a^l\right)_{\substack{0\le i,l\le s-1\\ 0\le j,k\le p-s-1}}$ be the basis dual to a standard basis given in (3). Then

$$\bar{b}_i = a^{s-1-i},\ \ \bar{x}_j = y^{p-s-1-j},\ \ \bar{y}_k = x^{p-s-1-k},\ \ \bar{a}_l = b^{s-1-l}$$

gives the desired result. $\square$

We denote by ${}_{\bar{U}_q}\!\left(\bar{U}_q\right)_{\bar{U}_q}$ the regular bimodule, where the left and right actions are respectively the left and right multiplication of $\bar{U}_q$ on itself. Recall that a block of ${}_{\bar{U}_q}\!\left(\bar{U}_q\right)_{\bar{U}_q}$ is just an indecomposable two-sided ideal (see [4, Section 55]). The block decomposition of $\bar{U}_q$ is (see [6])

$${}_{\bar{U}_q}\!\left(\bar{U}_q\right)_{\bar{U}_q} = \bigoplus_{s=0}^{p} Q(s)$$

where the structure of each block $Q(s)$ as a left $\bar{U}_q$-module is:

$$\begin{array}{l}
Q(0) \cong p\mathcal{X}^-(p), \quad Q(p) \cong p\mathcal{X}^+(p), \\
Q(s) \cong s\mathcal{P}^+(s) \oplus (p-s)\mathcal{P}^-(p-s) \ \text{ for } 1 \le s \le p-1
\end{array} \tag{6}$$

and the structure of each block as a right $\bar{U}_q$-module is:

$$\begin{array}{l}
Q(0) \cong pR^*(\mathcal{X}^-(p)), \quad Q(p) \cong pR^*(\mathcal{X}^+(p)), \\
Q(s) \cong sR^*(\mathcal{P}^+(s)) \oplus (p-s)R^*(\mathcal{P}^-(p-s)) \ \text{ for } 1 \le s \le p-1.
\end{array}$$

The following proposition is a reformulation of [6, Prop. 4.4.2] (see also [10, Th. II.1.4]). It will be used for the proof of Theorem 4.2.

Proposition 2.1. *For $1 \le s \le p-1$, the block $Q(s)$ admits a basis*

$$\left(B^{++}_{ab}(s), X^{-+}_{cd}(s), Y^{-+}_{ef}(s), A^{++}_{gh}(s), B^{--}_{ij}(s), X^{+-}_{kl}(s), Y^{+-}_{mn}(s), A^{--}_{or}(s)\right)$$

with $0 \le a,b,d,f,g,h,k,m \le s-1,\ \ 0 \le c,e,i,j,l,n,o,r \le p-s-1$, such that

1. $\forall\, 0 \le j \le s-1$, $\left(B^{++}_{ij}(s), X^{-+}_{kj}(s), Y^{-+}_{lj}(s), A^{++}_{mj}(s)\right)_{\substack{0\le i,m\le s-1\\ 0\le k,l\le p-s-1}}$ *is a standard basis of $\mathcal{P}^+(s)$ for the left action.*
2. $\forall\, 0 \le j \le p-s-1$, $\left(B^{--}_{ij}(s), X^{+-}_{kj}(s), Y^{+-}_{lj}(s), A^{--}_{mj}(s)\right)_{\substack{0\le k,l\le s-1\\ 0\le i,m\le p-s-1}}$ *is a standard basis of $\mathcal{P}^-(p-s)$ for the left action.*

3. $\forall\, 0 \le i \le s-1$, $\left(B^{++}_{ij}(s), X^{+-}_{ik}(s), Y^{+-}_{il}(s), A^{++}_{im}(s)\right)_{\substack{0\le j,m\le s-1\\ 0\le k,l\le p-s-1}}$ *is a standard basis of* $R^*\left(\mathcal{P}^+(s)\right)$ *for the right action.*
4. $\forall\, 0 \le i \le p-s-1$, $\left(B^{--}_{ij}(s), X^{-+}_{ik}(s), Y^{-+}_{il}(s), A^{--}_{im}(s)\right)_{\substack{0\le k,l\le s-1\\ 0\le j,m\le p-s-1}}$ *is a standard basis of* $R^*\left(\mathcal{P}^-(p-s)\right)$ *for the right action.*

The block $Q(0)$ *admits a basis* $\left(A^{--}_{ij}(0)\right)_{0\le i,j\le p-1}$ *such that*

1. $\forall\, 0 \le j \le p-1$, $\left(A^{--}_{ij}(0)\right)_{0\le i\le p-1}$ *is a standard basis of* $\mathcal{X}^-(p)$ *for the left action.*
2. $\forall\, 0 \le i \le p-1$, $\left(A^{--}_{ij}(0)\right)_{0\le j\le p-1}$ *is a standard basis of* $R^*\left(\mathcal{X}^-(p)\right)$ *for the right action.*

The block $Q(p)$ *admits a basis* $\left(A^{++}_{ij}(p)\right)_{0\le i,j\le p-1}$ *such that*

1. $\forall\, 0 \le j \le p-1$, $\left(A^{++}_{ij}(p)\right)_{0\le i\le p-1}$ *is a standard basis of* $\mathcal{X}^+(p)$ *for the left action.*
2. $\forall\, 0 \le i \le p-1$, $\left(A^{++}_{ij}(p)\right)_{0\le j\le p-1}$ *is a standard basis of* $R^*\left(\mathcal{X}^+(p)\right)$ *for the right action.*

As in [6], the structure of $Q(s)$ in terms of composition factors can be schematically represented as follows (each vertex represents a composition factor and is labelled by the basis vectors of this factor):

$$
\begin{tikzcd}
 & \left(B^{++}_{ab}(s)\right) \arrow[dl, "E"'] \arrow[dd, "E"] \arrow[dr, "F"] & \\
\left(X^{-+}_{cd}(s)\right) \arrow[dr, "F"'] & & \left(Y^{-+}_{ef}(s)\right) \arrow[dl, "E"] \\
 & \left(A^{++}_{gh}(s)\right) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \left(B^{--}_{ij}(s)\right) \arrow[dl, "E"'] \arrow[dd, "E"] \arrow[dr, "F"] & \\
\left(X^{+-}_{cd}(s)\right) \arrow[dr, "F"'] & & \left(Y^{+-}_{mn}(s)\right) \arrow[dl, "E"] \\
 & \left(A^{--}_{or}(s)\right) &
\end{tikzcd}
$$

for the left action, and

$$
\begin{tikzcd}
 & \left(B^{++}_{ab}(s)\right) \arrow[dd, "E"] \arrow[drr, "E"] \arrow[drrrrr, "E"] & & & & \left(B^{--}_{ij}(s)\right) \arrow[dlllll, "F"'] \arrow[dlll, "F"'] \arrow[dd, "E"] & \\
\left(X^{-+}_{cd}(s)\right) \arrow[drrrrr, "E"] & & \left(Y^{-+}_{ef}(s)\right) \arrow[drrr, "E"] & \left(X^{+-}_{cd}(s)\right) \arrow[dll, "F"'] & & & \left(Y^{+-}_{mn}(s)\right) \arrow[dlllll, "F"'] \\
 & \left(A^{++}_{gh}(s)\right) & & & & \left(A^{--}_{or}(s)\right) &
\end{tikzcd}
$$

for the right action.

The knowledge of the structure of the bimodule ${}_{\bar{U}_q}\left(\bar{U}_q\right)_{\bar{U}_q}$ allows us to determine the center of $\bar{U}_q$. Indeed, each central element determines a bimodule endomorphism and conversely. Recall from [6] that $\mathcal{Z}(\bar{U}_q)$ is a $(3p-1)$-dimensional algebra with basis elements e_s $(0 \le s \le p)$ and $w^{\pm}_t$ $(1 \le t \le p-1)$. The element e_s is just the unit of the block $Q(s)$, thus by (6) and (4) the action of e_s on the simple and the projective modules is given by

$$(7)\quad \begin{array}{llll} \text{For } s=0, & e_0 v_0^+(t)=0, & e_0 v_0^-(t)=\delta_{t,p} v_0^-(p), & e_0 b_0^{\pm}(t)=0, \\ \text{For } 1\le s\le p-1, & e_s v_0^+(t)=\delta_{s,t} v_0^+(s), & e_s v_0^-(t)=\delta_{p-s,t} v_0^-(p-s), & \\ & e_s b_0^+(t)=\delta_{s,t} b_0^+(s), & e_s b_0^-(t)=\delta_{t,p-s} b_0^-(p-s), & \\ \text{For } s=p, & e_p v_0^+(t)=\delta_{t,p} v_0^+(p), & e_p v_0^-(t)=0, & e_p b_0^{\pm}(t)=0 \end{array}$$

while for the elements $w_s^{\pm}$:

$$(8)\quad \begin{array}{lll} w_s^+ v_0^{\pm}(t)=0, & w_s^+ b_0^+(t)=\delta_{s,t} a_0^+(s), & w_s^+ b_0^-(t)=0, \\ w_s^- v_0^{\pm}(t)=0, & w_s^- b_0^+(t)=0, & w_s^- b_0^-(t)=\delta_{t,p-s} a_0^-(p-s). \end{array}$$

Observe that

$$\overset{\mathcal{P}^+(s)}{w_s^+} = p_s^+, \qquad \overset{\mathcal{P}^-(p-s)}{w_s^-} = p_{p-s}^-.$$

The action of the central elements on $\mathcal{P}^{\alpha}(s)$ is enough to recover their action on every module, using projective covers. From these formulas, we deduce the multiplication rules of these elements:

$$(9)\quad e_s e_t = \delta_{s,t} e_s, \quad e_s w_t^{\pm} = \delta_{s,t} w_s^{\pm}, \quad w_s^{\pm} w_t^{\pm} = 0.$$

Let us mention that the idempotents e_s are not primitive: there exists primitive orthogonal idempotents $e_{s,i}$ such that $e_s = \sum_i e_{s,i}$, see [1].

3. Symmetric linear forms and the GTA basis

Let A be a k-algebra, and let $\mathrm{SLF}(A)$ be the space of *symmetric linear forms* on A:

$$\mathrm{SLF}(A) = \{\varphi \in A^* \mid \forall\, x, y \in A,\ \ \varphi(xy) = \varphi(yx)\}.$$

If A is a bialgebra, then A^* is an algebra whose product is defined by:

$$\varphi\psi(x) = \sum_{(x)} \varphi(x')\psi(x'')$$

with $\Delta(x) = \sum_{(x)} x' \otimes x''$ (Sweedler's notation, see e.g. [11, Chap. 3]). Then $\mathrm{SLF}(A)$ is a subalgebra of A^*. Indeed, if $\varphi, \psi \in \mathrm{SLF}(A)$, we have:

$$\varphi\psi(xy) = \sum_{(x),(y)} \varphi(x'y')\psi(x''y'') = \sum_{(x),(y)} \varphi(y'x')\psi(y''x'') = \varphi\psi(yx)$$

which shows that $\varphi\psi \in \mathrm{SLF}(A)$. If moreover A is finite dimensional, then A^* is a bialgebra whose coproduct is defined by $\Delta(\varphi)(x \otimes y) = \varphi(xy)$, but $\mathrm{SLF}(A)$ is not in general a subcoalgebra of A^*, see Remark 1 below.

Recall (see [6]) that there is a universal R-matrix R belonging to the extension of $\bar{U}_q$ by a square root of K. It satisfies $RR' \in \bar{U}_q^{\otimes 2}$, where $R' = \tau(R)$, with τ the flip map defined by $\tau(x \otimes y) = y \otimes x$. Moreover $\bar{U}_q$ is factorizable (in a generalized sense since it does not contain the R-matrix) and K^{p+1} is a pivotal element, thus it is known from general theory that the Drinfeld morphism which we denote $\mathcal{D}$ provides an isomorphism of algebras

$$(10)\quad \begin{array}{rccl} \mathcal{D}: & \mathrm{SLF}(\bar{U}_q) & \xrightarrow{\sim} & \mathcal{Z}(\bar{U}_q) \\ & \varphi & \mapsto & (\varphi \otimes \mathrm{Id})\Big((K^{p+1} \otimes 1)\cdot RR'\Big) \end{array}$$

Let A be a k-algebra, and V an n-dimensional A-module. If we choose a basis on V, we get a matrix $\overset{V}{T} \in \mathrm{Mat}_n(A^*)$, simply defined by

$$\overset{V}{T}(x) = \overset{V}{x} \tag{11}$$

where $\overset{V}{x}$ is the representation of $x \in A$ in $\mathrm{End}(V)$ expressed in the choosen basis. In our case, we will always choose the canonical bases of the simple modules and standard bases of the projective modules.

An interesting basis of $\mathrm{SLF}(\bar{U}_q)$ was found by Gainutdinov and Tipunin in [9] and by Arike in [1]. To be precise, a basis of the space $\mathrm{qCh}(\bar{U}_q)$ of q-characters is constructed in [9], but the shift by the pivotal element $g = K^{p+1}$ provides an isomorphism

$$\mathrm{qCh}(\bar{U}_q) \xrightarrow{\sim} \mathrm{SLF}(\bar{U}_q), \quad \psi \mapsto \psi(g\,\cdot).$$

This basis is built from the simple and the projective modules. First, define $2p$ linear forms[1] χ_s^α, $\alpha \in \{\pm\}$, $1 \leq s \leq p$, by:

$$\chi_s^\alpha = \mathrm{tr}\big(\overset{\mathcal{X}^\alpha(s)}{T}\big). \tag{12}$$

They are obviously symmetric. Observe that $\chi_1^+ = \varepsilon$ is the unit for the algebra structure on $\mathrm{SLF}(\bar{U}_q)$ described above. To construct the $p-1$ missing linear forms, observe with the help of (4) that the matrix of the action on $\mathcal{P}^\alpha(s)$ has the following block form in a standard basis:

$$\overset{\mathcal{P}^\alpha(s)}{T} = \begin{array}{cc} \begin{array}{cccc} (b_i) & (x_j) & (y_k) & (a_l) \end{array} & \\ \begin{pmatrix} \overset{\mathcal{X}^\alpha(s)}{T} & 0 & 0 & 0 \\ A_s^\alpha & \overset{\mathcal{X}^{-\alpha}(p-s)}{T} & 0 & 0 \\ B_s^\alpha & 0 & \overset{\mathcal{X}^{-\alpha}(p-s)}{T} & 0 \\ H_s^\alpha & D_s^\alpha & C_s^\alpha & \overset{\mathcal{X}^\alpha(s)}{T} \end{pmatrix} & \begin{array}{c} (b_i) \\ (x_j) \\ (y_k) \\ (a_l). \end{array} \end{array}$$

It is not difficult to see that these matrices satisfy the following symmetries:

$$A_{p-s}^- = C_s^+, \quad B_{p-s}^- = D_s^+, \quad D_{p-s}^- = B_s^+, \quad C_{p-s}^- = A_s^+.$$

By computing the matrices $\overset{\mathcal{P}^+(s)}{(xy)} = \overset{\mathcal{P}^+(s)}{x}\overset{\mathcal{P}^+(s)}{y}$ and $\overset{\mathcal{P}^-(p-s)}{(xy)} = \overset{\mathcal{P}^-(p-s)}{x}\overset{\mathcal{P}^-(p-s)}{y}$, these symmetries allow us to see that the linear form G_s ($1 \leq s \leq p-1$) defined by

$$G_s = \mathrm{tr}\,(H_s^+) + \mathrm{tr}\,(H_{p-s}^-) \tag{13}$$

is a symmetric linear form.

It is instructive for our purposes to see a proof that these symmetric linear forms are linearly independent. Let us begin by introducing important elements for $0 \leq n \leq p-1$ (they are discrete Fourier transforms of $(K^l)_{0 \leq l \leq 2p-1}$):

[1]The correspondence of notations with [1] is: $T_s^+ = \chi_s^+$, $T_s^- = \chi_{p-s}^-$. The letter T is here reserved for the matrices $\overset{V}{T}$ described above.

$$\Phi_n^{\alpha} = \frac{1}{2p}\sum_{l=0}^{2p-1}\left(\alpha q^{-n}\right)^l K^l.$$

The following easy lemma shows that these elements allow one to select vectors which have a given weight, and this turns out to be very useful.

Lemma 3.1. 1) *Let M be a left $\bar{U}_q$-module, and let $m_i^+(s)$ be a vector of weight q^{s-1-2i}, $m_i^-(p-s)$ be a vector of weight $-q^{(p-s)-1-2i} = q^{-s-1-2i}$, $m_i^-(s)$ be a vector of weight $-q^{s-1-2i}$, $m_i^+(p-s)$ be a vector of weight $q^{(p-s)-1-2i} = -q^{-s-1-2i}$. Then:*

$$\Phi_{s-1}^+ m_i^+(s) = \delta_{i,0} m_0^+(s), \quad \Phi_{s-1}^+ m_i^-(p-s) = 0,$$
$$\Phi_{s-1}^- m_i^-(s) = \delta_{i,0} m_0^-(s), \quad \Phi_{s-1}^- m_i^+(p-s) = 0.$$

2) *Let N be a right $\bar{U}_q$-module, and let $n_i^+(s)$ be a vector of weight q^{1-s+2i}, $n_i^-(p-s)$ be a vector of weight $-q^{1-(p-s)+2i} = q^{1+s+2i}$, $n_i^-(s)$ be a vector of weight $-q^{1-s+2i}$, $n_i^+(p-s)$ be a vector of weight $q^{1-(p-s)+2i} = -q^{1+s+2i}$. Then:*

$$n_i^+(s)\Phi_{s-1}^+ = \delta_{i,s-1} n_{s-1}^+(s), \quad n_i^-(p-s)\Phi_{s-1}^+ = 0,$$
$$n_i^-(s)\Phi_{s-1}^- = \delta_{i,s-1} n_{s-1}^-(s), \quad n_i^+(p-s)\Phi_{s-1}^- = 0.$$

Proof. It follows from easy computations with sums of roots of unity. □

We can now state the key observation.

Proposition 3.1. *Let*

$$\varphi = \sum_{s=1}^{p}\left(\lambda_s^+\chi_s^+ + \lambda_s^-\chi_s^-\right) + \sum_{s'=1}^{p-1}\mu_{s'}G_{s'} \in \mathrm{SLF}\left(\bar{U}_q\right).$$

Then:

$$\lambda_s^+ = \varphi\left(\Phi_{s-1}^+ e_s\right), \;\; \lambda_s^- = \varphi\left(\Phi_{s-1}^- e_{p-s}\right), \;\; \mu_{s'} = \frac{\varphi\left(w_{s'}^+\right)}{s'} = \frac{\varphi(w_{s'}^-)}{p-s'}.$$

Proof. It is a corollary of (7) and (8). Indeed, we have:

$$\begin{array}{lll}
\overset{\mathcal{X}^+(s)}{T}(e_t) = \delta_{s,t} I_s, & \overset{\mathcal{X}^+(s)}{T}(w_t^{\pm}) = 0, & \overset{\mathcal{X}^-(s)}{T}(e_t) = \delta_{s,p-t} I_s, \\
\overset{\mathcal{X}^-(s)}{T}(w_t^{\pm}) = 0, & H_s^{\pm}(e_t) = 0, & H_s^+(w_s^+) = \delta_{s,t} I_s, \\
H_s^+(w_t^-) = 0, & H_{p-s}^-(w_t^+) = 0, & H_{p-s}^-(w_t^-) = \delta_{s,t} I_{p-s}.
\end{array}$$

This gives the formula for μ_s. The formulas for $\lambda_s^{\pm}$ follow from this and Lemma 3.1. □

If we have $\sum_{s=1}^{p}\left(\lambda_s^+\chi_s^+ + \lambda_s^-\chi_s^-\right) + \sum_{s'=1}^{p-1}\mu_{s'}G_{s'} = 0$, we can evaluate the left-hand side on the elements appearing in Proposition 3.1 to get that all the coefficients are equal to 0. Thus we have a free family of cardinal $3p-1$, hence a basis of $\mathrm{SLF}(\bar{U}_q)$, since $\dim(\mathrm{SLF}(\bar{U}_q)) = 3p-1$ by (10).

Theorem 3.1. *The symmetric linear forms $\chi_s^{\pm}$ $(1 \le s \le p)$ and $G_{s'}$ $(1 \le s' \le p-1)$ form a basis of* $\mathrm{SLF}(\bar{U}_q)$.

Definition 3.1. *The basis of* Theorem 3.1 *will be called the* GTA basis *(for Gainutdinov, Tipunin, Arike).*

REMARK 1. Let $\varphi \in \mathrm{SLF}(\bar{U}_q)$. It is easy to see that $\varphi(K^j E^n F^m) = 0$ if $n \neq m$. From this we deduce that $\mathrm{SLF}(\bar{U}_q)$ is not a sub-coalgebra of $\bar{U}_q^*$. Indeed, write $\Delta(\chi_2^+) = \sum_i \varphi_i \otimes \psi_i$, and assume that $\varphi_i, \psi_i \in \mathrm{SLF}(\bar{U}_q)$. Then $1 = \chi_2^+(EF) = \sum_i \varphi_i(E)\psi_i(F) = 0$, a contradiction.

REMARK 2. If we choose a basis of $\mathcal{Z}(\bar{U}_q)$, then its dual basis can not be entirely contained in $\mathrm{SLF}(\bar{U}_q)$. Indeed, let $\varphi = \sum_{s=0}^{p} \lambda_s^{\pm}\chi_s^{\pm} + \sum_{s=1}^{p-1} \mu_s G_s \in \mathrm{SLF}(\bar{U}_q)$. Then $\varphi(w_s^+) = s\mu_s, \varphi(w_s^-) = (p-s)\mu_s$, and we see that there does not exist $\varphi \in \mathrm{SLF}(\bar{U}_q)$ such that $\varphi(w_s^+) = 1,\ \varphi(w_s^-) = 0$. Hence, $\mathrm{SLF}(\bar{U}_q) \subset \bar{U}_q^*$ is not the dual of $\mathcal{Z}(\bar{U}_q) \subset \bar{U}_q$.

4. Traces on projective $\bar{U}_q$-modules and the GTA basis

4.1. Correspondence between traces and symmetric linear forms. Let A be a finite dimensional k-algebra. We have an anti-isomorphism of algebras:

$$A \to \mathrm{End}_A(A), \quad a \mapsto \rho_a \text{ defined by } \rho_a(x) = xa.$$

Observe that the right action of A naturally appears. Let t be a trace on A, that is, an element of $\mathrm{SLF}(\mathrm{End}_A(A))$. Then:

$$t(\rho_{ab}) = t(\rho_b \circ \rho_a) = t(\rho_a \circ \rho_b) = t(\rho_{ba}).$$

So we get an isomorphism of vector spaces

$$\begin{array}{rcl} \{\text{Traces on } \mathrm{End}_A(A)\} = \mathrm{SLF}\,(\mathrm{End}_A(A)) & \to & \mathrm{SLF}(A) \\ t & \mapsto & \varphi^t \text{ defined by } \varphi^t(a) = t(\rho_a). \end{array}$$

whose inverse is:

$$\begin{array}{rcl} \mathrm{SLF}(A) & \to & \{\text{Traces on } \mathrm{End}_A(A)\} = \mathrm{SLF}\,(\mathrm{End}_A(A)) \\ \varphi & \mapsto & t^\varphi \text{ defined by } t^\varphi(\rho_a) = \varphi(a). \end{array}$$

In the case of $A = \bar{U}_q$, we can express φ^t in the GTA basis, which will be the object of the next section.

Let Proj_A be the full subcategory of the category of finite dimensional A-modules whose objects are the projective A-modules.

Definition 4.1. *A* trace *on* Proj_A *is a family of linear maps* $t = (t_U : \mathrm{End}_A(U) \to k)_{U \in \mathrm{Proj}_A}$ *such that*

$$\forall\, f \in \mathrm{Hom}_A(U, V),\ \forall\, g \in \mathrm{Hom}_A(V, U),\ \ t_V(g \circ f) = t_U(f \circ g).$$

We denote by $\mathcal{T}_{\mathrm{Proj}_A}$ *the vector space of traces on* Proj_A.

This cyclic property of traces on Proj_A is one of the axioms of the so-called modified traces, defined for instance in [8]. Note that this definition could be restated in the following way (and could be generalized to other abelian full subcategories than Proj_A).

Lemma 4.1. *Let* $t = (t_U : \mathrm{End}_A(U) \to k)_{U \in \mathrm{Proj}_A}$ *be a family of linear maps. Then* t *is a trace on* Proj_A *if and only if:*

- $\forall f, g \in \mathrm{End}_A(U)$, $t_U(g \circ f) = t_U(f \circ g)$,
- $t_{U\oplus V}(f) = t_U(p_U \circ f \circ i_U) + t_V(p_V \circ f \circ i_V)$, *where* p_U, p_V *are the canonical projection maps and* i_U, i_V *are the canonical injection maps.*

Proof. If t is a trace and $f \in \mathrm{End}_A(U \oplus V)$, we have:

$$t_{U\oplus V}(f) = t_{U\oplus V}((i_U p_U + i_V p_V) f) = t_U(p_U f i_U) + t_V(p_V f i_V).$$

Conversely, let $f : U \to V$, $g : V \to U$. Define $F = i_V f p_U, G = i_U g p_V$. Then $FG = i_V f g p_V$ and $GF = i_U g f p_U$. We have $p_U GF i_U = gf$, $p_V GF i_V = 0$, $p_U FG i_U = 0$, $p_V FG i_V = fg$, thus:

$$t_V(fg) = t_{U\oplus V}(FG) = t_{U\oplus V}(GF) = t_U(gf).$$

This shows the equivalence. □

Now, consider:

$$\begin{array}{rccccl} \Pi_A \ : & \mathcal{T}_{\mathrm{Proj}_A} & \to & \mathrm{SLF}(\mathrm{End}_A(A)) & \xrightarrow{\sim} & \mathrm{SLF}(A) \\ & t = (t_U)_{U \in \mathrm{Ob}(\mathrm{Proj}_A)} & \mapsto & t_A & \mapsto & \varphi^t \text{ defined by } \varphi^t(a) = t_A(\rho_a). \end{array}$$

Theorem 4.1. *The map* Π_A *is an isomorphism. In other words,* t_A *entirely characterizes* $t = (t_U)$.

Proof. For all the facts concerning PIMs (Principal Indecomposable Modules) and idempotents in finite dimensional k-algebras, we refer to [4, Chap. VIII]. We first show that Π_A is surjective. Let:

$$1 = e_1 + \ldots + e_n$$

be a decomposition of the unit into primitive orthogonal idempotents ($e_i e_j = \delta_{i,j} e_i$). Then the PIMs of A are isomorphic to the left ideals Ae_i (possibly with multiplicity). We have isomorphisms of vector spaces:

$$\mathrm{Hom}_A(Ae_i, Ae_j) \xrightarrow{\sim} e_i A e_j, \quad f \mapsto f(e_i).$$

For every $\varphi \in \mathrm{SLF}(A)$, define $t^{\varphi}_{Ae_i}$ by:

$$t^{\varphi}_{Ae_i}(f) = \varphi(f(e_i)).$$

Let $f : Ae_i \to Ae_j$, $g : Ae_j \to Ae_i$, and put $f(e_i) = e_i a_f e_j$, $g(e_j) = e_j a_g e_i$. Then using the idempotence of the e_i's and the symmetry of φ we get:

$$t^{\varphi}_{Ae_i}(g \circ f) = \varphi(g \circ f(e_i)) = \varphi\Big((e_i a_f e_j)(e_j a_g e_i)\Big) = \varphi\Big((e_j a_g e_i)(e_i a_f e_j)\Big) = \varphi(f \circ g(e_j)) = t^{\varphi}_{Ae_j}(f \circ g).$$

We know that every projective module is isomorphic to a direct sum of PIMs, so we extend t^{φ} to Proj_A by the following formula:

$$t_{\bigoplus_l A_l}(f) = \sum_l t_{Ae_l}(i_l \circ f \circ p_l)$$

where p_l and i_l are the canonical injection and projection maps. By Lemma 4.1, this defines a trace on Proj_A. We then show that $\Pi_A(t^{\varphi}) = \varphi$, proving surjectivity:

$$\Pi_A(t^{\varphi})(a) = t^{\varphi}_A(\rho_a) = \sum_{j=1}^{n} t^{\varphi}_{Ae_j}\left(p_j \circ \rho_a \circ i_j\right) = \sum_{j=1}^{n} \varphi\left(p_j \circ \rho_a(e_j)\right) = \sum_{j=1}^{n} \varphi\left(p_{Ae_j}(e_j a)\right)$$
$$= \sum_{j,k=1}^{n} \varphi\left(p_{Ae_j}(e_j a e_k)\right) = \sum_{j=1}^{n} \varphi\left(e_j a e_j\right) = \sum_{j=1}^{n} \varphi\left(a e_j\right) = \varphi(a).$$

Note that we used that the e_j's are idempotents and that $a = \sum_{j=1}^{n} ae_j$. We now show injectivity. Assume that $\Pi_A(t) = 0$. Then:

$$\forall\, a \in A,\;\; t_A(\rho_a) = \sum_{j=1}^{n} t_{Ae_j}(p_j \circ \rho_a \circ i_j) = 0.$$

Let $f : Ae_j \to Ae_j$, with $f(e_j) = e_j a_f e_j$. Since $\rho_{f(e_j)}(e_l) = \delta_{j,l} e_j a_f e_j$, we have $p_j \circ \rho_{f(e_j)} \circ i_j = f$ and $p_l \circ \rho_{f(e_j)} \circ i_l = 0$ if $l \neq j$. Hence:

$$t_{Ae_j}(f) = t_A(\rho_{f(e_j)}) = 0.$$

Then $t_{Ae_j} = 0$ for each j, so that $t = 0$. □

4.2. Link with the GTA basis. We leave the general case and focus on $A = \bar{U}_q$. The following theorem expresses $\Pi_{\bar{U}_q}$ in the GTA basis.

Theorem 4.2. *Let $t = (t_U)_{U \in \mathrm{Proj}_{\bar{U}_q}}$ be a trace on $\mathrm{Proj}_{\bar{U}_q}$. Then:*

$$\Pi_{\bar{U}_q}(t) = t_{\mathcal{X}^+(p)}(\mathrm{Id})\chi^+_p + t_{\mathcal{X}^-(p)}(\mathrm{Id})\chi^-_p + \sum_{s=1}^{p-1}\left(t_{\mathcal{P}^+(s)}(\mathrm{Id})\chi^+_s + t_{\mathcal{P}^-(s)}(\mathrm{Id})\chi^-_s + t_{\mathcal{P}^+(s)}(p^+_s)G_s\right).$$

Proof. First of all, we write the decomposition of the left regular representation of $\bar{U}_q$, assigning an index to the multiple factors:

$$\bar{U}_q = \bigoplus_{s=1}^{p-1}\left(\bigoplus_{j=0}^{s-1} \mathcal{P}^+_j(s) \oplus \mathcal{P}^-_j(s)\right) \oplus \bigoplus_{j=0}^{p-1} \mathcal{X}^+_j(p) \oplus \mathcal{X}^-_j(p).$$

Thus, since t is a trace:

$$t_{\bar{U}_q}(\rho_a) = \sum_{s=1}^{p-1}\left(\sum_{j=0}^{s-1} t_{\mathcal{P}^+_j(s)}\left(p_{\mathcal{P}^+_j(s)} \circ \rho_a \circ i_{\mathcal{P}^+_j(s)}\right) + t_{\mathcal{P}^-_j(s)}\left(p_{\mathcal{P}^-_j(s)} \circ \rho_a \circ i_{\mathcal{P}^-_j(s)}\right)\right)$$
$$+ \sum_{j=0}^{p-1} t_{\mathcal{X}^+_j(p)}\left(p_{\mathcal{X}^+_j(p)} \circ \rho_a \circ i_{\mathcal{X}^+_j(p)}\right) + t_{\mathcal{X}^-_j(p)}\left(p_{\mathcal{X}^-_j(p)} \circ \rho_a \circ i_{\mathcal{X}^-_j(p)}\right).$$

Consider the following composite maps for $1 \leq s \leq p-1$ (note that the blocks appear because ρ_a is the right multiplication by a):

$$h^+_{s,j,a} : \mathcal{P}^+(s) \xrightarrow{I^+_{s,j}} \mathcal{P}^+_j(s) \xrightarrow{i_{\mathcal{P}^+_j(s)}} Q(s) \xrightarrow{\rho_a} Q(s) \xrightarrow{p_{\mathcal{P}^+_j(s)}} \mathcal{P}^+_j(s) \xrightarrow{\left(I^+_{s,j}\right)^{-1}} \mathcal{P}^+(s),$$
$$h^-_{s,j,a} : \mathcal{P}^-(s) \xrightarrow{I^-_{s,j}} \mathcal{P}^-_j(s) \xrightarrow{i_{\mathcal{P}^-_j(s)}} Q(p-s) \xrightarrow{\rho_a} Q(p-s) \xrightarrow{p_{\mathcal{P}^-_j(s)}} \mathcal{P}^-_j(s) \xrightarrow{\left(I^-_{s,j}\right)^{-1}} \mathcal{P}^-(s),$$

where $I^+_{s,j}$ and $I^-_{s,j}$ are the isomorphisms defined by (see Proposition 2.1):

$I^+_{s,j}(b^+_i(s)) = B^{++}_{ij}(s)$, $I^+_{s,j}(x^+_i(s)) = X^{-+}_{ij}(s)$, $I^+_{s,j}(y^+_i(s)) = Y^{-+}_{ij}(s)$, $I^+_{s,j}(a^+_i(s)) = A^{++}_{ij}(s)$,
$I^-_{s,j}(b^-_i(s)) = B^{--}_{ij}(p-s)$, $I^-_{s,j}(x^-_i(s)) = X^{+-}_{ij}(p-s)$, $I^-_{s,j}(y^-_i(s)) = Y^{+-}_{ij}(p-s)$,
$I^-_{s,j}(a^-_i(s)) = A^{--}_{ij}(p-s)$.

For $s = p$, consider:

$$h^+_{p,j,a} : \mathcal{X}^+(p) \xrightarrow{I^+_{p,j}} \mathcal{X}^+_j(p) \xrightarrow{i_{\mathcal{X}^+_j(p)}} Q(p) \xrightarrow{\rho_a} Q(p) \xrightarrow{p_{\mathcal{X}^+_j(p)}} \mathcal{X}^+_j(p) \xrightarrow{\left(I^+_{p,j}\right)^{-1}} \mathcal{X}^+(p),$$
$$h^-_{p,j,a} : \mathcal{X}^-(p) \xrightarrow{I^-_{p,j}} \mathcal{X}^-_j(p) \xrightarrow{i_{\mathcal{X}^-_j(p)}} Q(0) \xrightarrow{\rho_a} Q(0) \xrightarrow{p_{\mathcal{X}^-_j(p)}} \mathcal{X}^-_j(p) \xrightarrow{\left(I^-_{p,j}\right)^{-1}} \mathcal{X}^-(p)$$

where $I^+_{p,j}$ and $I^-_{p,j}$ are the isomorphisms defined by (see Proposition 2.1):

$$I^+_{p,j}(v^+_i(p)) = A^{++}_{ij}(p) \quad \text{and} \quad I^-_{p,j}(v^-_i(p)) = A^{--}_{ij}(0).$$

Then for $1 \leq s \leq p-1$:

$$t_{\mathcal{P}^\alpha_j(s)}\left(p_{\mathcal{P}^\alpha_j(s)} \circ \rho_a \circ i_{\mathcal{P}^\alpha_j(s)}\right) = t_{\mathcal{P}^\alpha(s)}\left(h^\alpha_{s,j,a}\right)$$

and for $s = p$:

$$t_{\mathcal{X}^\alpha_j(p)}\left(p_{\mathcal{X}^\alpha_j(p)} \circ \rho_a \circ i_{\mathcal{X}^\alpha_j(p)}\right) = t_{\mathcal{X}^\alpha(p)}\left(h^\alpha_{p,j,a}\right).$$

We must determine the endomorphism $h^\alpha_{s,j,a}$ when a is replaced by the elements given in Proposition 3.1. Using (8), we get:

$$\forall s' \neq s, \forall j,\ h^\pm_{s',j,w^+_s} = 0 \quad \text{and} \quad h^-_{s,j,w^+_s} = 0$$

and:

$$\forall j,\ h^+_{s,j,w^+_s} = p^+_s.$$

Since this does not depend on j and since the block $Q(s)$ contains s copies of $\mathcal{P}^+(s)$, we find that $t_{\bar{U}_q}(\rho_{w^+_s}) = s t_{\mathcal{P}^+(s)}(p^+_s)$. So by Proposition 3.1, the coefficient of G_s is $t_{\mathcal{P}^+(s)}(p^+_s)$.

Next, assume that $1 \leq s \leq p-1$, and let us compute $h^\alpha_{s',j,\Phi^+_{s-1}e_s}$. By (7), we see that

$$\forall s' \notin \{s, p-s\}, \forall j,\ h^\pm_{s',j,\Phi^+_{s-1}e_s} = 0 \quad \text{and} \quad \forall j,\ h^-_{s,j,\Phi^+_{s-1}e_s} = 0,\ h^+_{p-s,j,\Phi^+_{s-1}e_s} = 0.$$

Then, Proposition 2.1 together with Lemma 3.1 gives:

$$\forall j,\ h^-_{p-s,j,\Phi^+_{s-1}e_s} = 0 \quad \text{and} \quad \forall 0 \leq j \leq s-2,\ h^+_{s,j,\Phi^+_{s-1}e_s} = 0 \quad \text{and} \quad h^+_{s,s-1,\Phi^+_{s-1}e_s} = \mathrm{Id}.$$

It follows that $t_{\bar{U}_q}\left(\rho_{\Phi^+_{s-1}e_s}\right) = t_{\mathcal{P}^+(s)}(\mathrm{Id})$. So by Proposition 3.1, the coefficient of χ^+_s is $t_{\mathcal{P}^+(s)}(\mathrm{Id})$.

We now consider $h^\pm_{s',j,\Phi^-_{s-1}e_{p-s}}$. This time, (7) shows that

$$\forall s' \notin \{s, p-s\}, \forall j,\ h^\pm_{s',j,\Phi^-_{s-1}e_{p-s}} = 0 \quad \text{and} \quad \forall j,\ h^-_{p-s,j,\Phi^-_{s-1}e_{p-s}} = 0,\ h^+_{s,j,\Phi^-_{s-1}e_{p-s}} = 0.$$

Then, Proposition 2.1 together with Lemma 3.1 gives:

$$\forall j,\ h^+_{p-s,j,\Phi^-_{s-1}e_{p-s}} = 0 \quad \text{and} \quad \forall 0 \leq j \leq s-2,\ h^-_{s,j,\Phi^-_{s-1}e_{p-s}} = 0 \quad \text{and} \quad h^-_{s,s-1,\Phi^-_{s-1}e_{p-s}} = \mathrm{Id}.$$

It follows that $t_{\bar{U}_q}\left(\rho_{\Phi^-_{s-1}e_{p-s}}\right) = t_{\mathcal{P}^-(s)}(\mathrm{Id})$. So by Proposition 3.1, the coefficient of χ^-_s is

$t_{\mathcal{P}^-(s)}(\mathrm{Id})$.
Finally, in the case where $s = p$:

$$\forall\, s' \neq p, \forall\, j,\;\; h^{\pm}_{s',j,\Phi^+_{p-1}e_p} = 0 \quad \text{and} \quad h^{-}_{p,j,\Phi^+_{p-1}e_p} = 0.$$

Then, Proposition 2.1 together with Lemma 3.1 gives:

$$\forall\, 0 \leq j \leq p-2,\;\; h^{+}_{p,j,\Phi^+_{p-1}e_p} = 0 \quad \text{and} \quad h^{+}_{p,p-1,\Phi^+_{p-1}e_p} = \mathrm{Id}.$$

It follows that $t_{\bar{U}_q}\left(\rho_{\Phi^+_{p-1}e_p}\right) = t_{\mathcal{X}^+(p)}(\mathrm{Id})$. So by Proposition 3.1, the coefficient of χ^+_p is $t_{\mathcal{X}^+(p)}(\mathrm{Id})$ One similarly gets the coefficient of χ^-_p. □

By Proposition 3.1, the coefficient of G_s is also given by: $\frac{1}{p-s} t_{\bar{U}_q}(\rho_{w^-_s})$. Taking back the notations of the proof above, we see using (8) that

$$\forall\, s' \neq p-s, \forall\, j,\;\; h^{\pm}_{s',j,w^-_s} = 0 \quad \text{and} \quad h^{+}_{p-s,j,w^-_s} = 0$$

and:

$$\forall\, j,\;\; h^{-}_{p-s,j,w^-_s} = p^-_{p-s}.$$

Since this does not depend on j and since the block $Q(s)$ contains $p-s$ copies of $\mathcal{P}^-(p-s)$, we find that $t_{\bar{U}_q}(\rho_{w^-_s}) = (p-s)t_{\mathcal{P}^-(p-s)}(p^-_{p-s})$. So by Proposition 3.1, the coefficient of G_s is $t_{\mathcal{P}^-(p-s)}(p^-_{p-s})$. We thus have:

$$t_{\mathcal{P}^-(p-s)}(p^-_{p-s}) = t_{\mathcal{P}^+(s)}(p^+_s). \tag{14}$$

Note that there is an elementary way to see this. Indeed, the morphisms P^+_s and $\bar{P}^-_{p-s}$ defined in (5) satisfy:

$$\bar{P}^-_{p-s} \circ P^+_s = p^+_s, \quad P^+_s \circ \bar{P}^-_{p-s} = p^-_{p-s}.$$

Hence, we recover (14) by property of the traces. From this, we deduce the following corollary.

Corollary 4.1. *Let*

$$\varphi = \sum_{s=1}^{p} (\lambda^+_s \chi^+_s + \lambda^-_s \chi^-_s) + \sum_{s'=1}^{p-1} \mu_{s'} G_{s'} \in \mathrm{SLF}\left(\bar{U}_q\right).$$

Then the trace $t^{\varphi} = \Pi^{-1}_{\bar{U}_q}(\varphi)$ *associated to* φ *is given by:*

$$t^{\varphi}_{\mathcal{X}^{\pm}(p)}(\mathrm{Id}) = \lambda^{\pm}_p, \quad t^{\varphi}_{\mathcal{P}^{\pm}(s)}(\mathrm{Id}) = \lambda^{\pm}_s, \quad t^{\varphi}_{\mathcal{P}^+(s')}(p^+_{s'}) = t^{\varphi}_{\mathcal{P}^-(p-s')}(p^-_{p-s'}) = \mu_{s'}.$$

4.3. Symmetric linear form corresponding to the modified trace on $\mathrm{Proj}_{\bar{U}_q}$**.** Let H be a finite dimensional Hopf algebra. Let us recall that a *modified trace* t on Proj_H is a trace which satisfies the additional property that for $U \in \mathrm{Proj}_H$, for each H-module V and for $f \in \mathrm{End}_H(U \otimes V)$ we have:

$$\mathrm{t}_{U\otimes V}(f) = \mathrm{t}_U(\mathrm{tr}_R(f))$$

where $\mathrm{tr}_R = \mathrm{Id} \otimes \mathrm{tr}_q$ is the right partial quantum trace (see [8, (3.2.2)]). These modified traces are actively studied, having for motivation the construction of invariants in low dimensional topology. We refer to [8] for the general theory in a categorical framework which encapsulates the case of Proj_H.

In [3], it is shown that there exists a unique up to scalar modified trace $\mathsf{t} = (\mathsf{t}_U)$ on $\mathrm{Proj}_{\bar{U}_q}$. Uniqueness comes from the fact that $\mathcal{X}^+(p)$ is both a simple and a projective module. The values of this trace are given by:

$$\begin{array}{lll} \mathsf{t}_{\mathcal{X}^+(p)}(\mathrm{Id}) = (-1)^{p-1}, & \mathsf{t}_{\mathcal{X}^-(p)}(\mathrm{Id}) = 1, & \mathsf{t}_{\mathcal{P}^+(s)}(\mathrm{Id}) = (-1)^s(q^s + q^{-s}), \\ \mathsf{t}_{\mathcal{P}^-(s)}(\mathrm{Id}) = (-1)^{p-s-1}(q^s + q^{-s}), & \mathsf{t}_{\mathcal{P}^+(s)}(p_s^+) = (-1)^s[s]^2 & \mathsf{t}_{\mathcal{P}^-(s)}(p_s^-) = \mathsf{t}_{\mathcal{P}^+(p-s)}(p_{p-s}^+). \end{array}$$

Let H be a finite dimensional unimodular pivotal Hopf algebra with pivotal element g and let $\mu \in H^*$ be a right co-integral on H, which means that

$$\forall\, x \in H, \ \ (\mu \otimes \mathrm{Id})(\Delta(x)) = \mu(x)1.$$

From [14], we know that $\mu(g\cdot)$ is a symmetric linear form. In the recent paper [2], it is shown that modified traces on Proj_H are unique up to scalar, and that the corresponding symmetric linear forms are scalar multiples of $\mu(g\cdot)$. Here, we show how Theorem 4.2 and computations made in [9] (see also [1]) and [6] quickly allow us to recover this result in the case of $H = \bar{U}_q$. First, recall that right integrals μ_ζ of $\bar{U}_q$ are given by:

$$\mu_\zeta(F^m E^n K^j) = \zeta \delta_{m,p-1}\delta_{n,p-1}\delta_{j,p+1},$$

where ζ is an arbitrary scalar. Hence:

$$\mu_\zeta(K^{p+1} F^m E^n K^j) = \zeta \delta_{m,p-1}\delta_{n,p-1}\delta_{j,0}.$$

Using formulas given in [9] (see also [1][2]), we have ($1 \le s \le p-1$):

$$e_0 = \frac{(-1)^{p-1}}{2p[p-1]!^2} \sum_{t=0}^{p-1}\sum_{l=0}^{2p-1} q^{-(-2t-1)l} F^{p-1}E^{p-1}K^l + (\text{terms of lower degree in } E \text{ and } F),$$

$$e_s = \alpha_s \sum_{t=0}^{p-1}\sum_{l=0}^{2p-1} q^{-(s-2t-1)l} F^{p-1}E^{p-1}K^l + (\text{terms of lower degree in } E \text{ and } F),$$

$$e_p = \frac{1}{2p[p-1]!^2} \sum_{t=0}^{p-1}\sum_{l=0}^{2p-1} q^{-(p-2t-1)l} F^{p-1}E^{p-1}K^l + (\text{terms of lower degree in } E \text{ and } F),$$

where α_s is given in the last page of [1] as:

$$\alpha_s = -\frac{(-1)^{p-s-1}}{2p[p-s-1]!^2[s-1]!^2}\left(\sum_{l=1}^{s-1}\frac{1}{[l][s-l]} - \sum_{l=1}^{p-s-1}\frac{1}{[l][p-s-l]}\right).$$

In order to simplify this, it is observed in [13, Proof of Proposition 2], that

$$\sum_{l=1}^{s-1}\frac{1}{[l][s-l]} - \sum_{l=1}^{p-s-1}\frac{1}{[l][p-s-l]} = \frac{-(q^s + q^{-s})}{[s]^2}.$$

So, since:

$$[p-s-1]!^2[s-1]!^2 = \frac{[p-1]!^2}{[s]^2},$$

[2] In notations of [1], we have $e_s = \sum_{t=1}^{s} e^+(s,t) + \sum_{u=1}^{p-s} e^-(p-s,u)$.

we get:

$$\alpha_s = \frac{(-1)^{p-s-1}}{2p[p-1]!^2}(q^s + q^{-s}).$$

Using formulas given in [6] (see also [10, Prop. II.3.19]), we have:

$$w_s^+ = \frac{(-1)^{p-s-1}}{2p[p-1]!^2}[s]^2 s F^{p-1}E^{p-1} + \text{(other monomials)},$$
$$w_s^- = \frac{(-1)^{p-s-1}}{2p[p-1]!^2}[s]^2 (p-s) F^{p-1}E^{p-1} + \text{(other monomials)}.$$

We now use Proposition 3.1 to get the coefficients of $\mu_\zeta(K^{p+1}\cdot)$ in the GTA basis. For instance:

$$\frac{\mu_\zeta(K^{p+1}w_s^+)}{s} = \zeta\frac{(-1)^{p-s-1}}{2p[p-1]!^2}[s]^2,$$
$$\mu_\zeta(K^{p+1}\Phi_{s-1}^+ e_s) = \frac{\alpha_s}{2p}\mu_\zeta\left(K^{p+1}F^{p-1}E^{p-1}\sum_{t=0}^{p-1}\sum_{l,j=0}^{2p-1} q^{-(s-1)(l+j)+2tl}K^{l+j}\right)$$
$$= \zeta\frac{\alpha_s}{2p}\sum_{t=0}^{p-1}\sum_{l=0}^{2p-1} q^{2tl} = \zeta\alpha_s.$$

Choose the normalization factor to be $\zeta = (-1)^{p-1}2p[p-1]!^2$, and let μ be the so-normalized integral. Then:

$$\mu(K^{p+1}\cdot) = (-1)^{p-1}\chi_p^+ + \chi_p^- + \sum_{s=1}^{p-1}\Big((-1)^s(q^s+q^{-s})\chi_s^+ + (-1)^{p-s-1}(q^s+q^{-s})\chi_s^- + (-1)^s[s]^2 G_s\Big).$$

By Theorem 4.2, we recover $\Pi_{\bar{U}_q}(\mathfrak{t}) = \mu(K^{p+1}\cdot)$.

5. Multiplication rules in the GTA basis

We mentioned in section 3 that $\mathrm{SLF}(\bar{U}_q)$ is a commutative algebra. In this section, we address the problem of the decomposition in the GTA basis of the product of two elements in this basis. The resulting formulas are surprisingly simple.

Let us start by recalling some facts. For every $\bar{U}_q$-module V, we define the character of V as (see (11) for the definition of T):

$$\chi^V = \mathrm{tr}(\overset{V}{T}).$$

This splits on extensions:

$$0 \to V \to M \to W \to 0 \implies \chi^M = \chi^V + \chi^W.$$

Due to the fact that $\bar{U}_q$ is finite dimensional, every finite dimensional $\bar{U}_q$-module has a composition series (*i.e.* is constructed by successive extensions by simple modules). It follows that every χ^V can be written as a linear combination of the $\chi_s^\alpha = \chi^{\mathcal{X}^\alpha(s)}$. Moreover, we see by definition of the product on $\bar{U}_q^*$ that

$$\overset{V\otimes W}{T} = \overset{V}{T}_1\overset{W}{T}_2 \tag{15}$$

where $\overset{V}{T}_1 = \overset{V}{T} \otimes I_{\dim(W)}$ and $\overset{W}{T}_2 = I_{\dim(V)} \otimes \overset{W}{T}$. Thus $\chi^{V\otimes W} = \chi^V\chi^W$. Hence multiplying two χ's is equivalent to tensoring two simples modules and finding the decomposition into simple factors. This means that

$$\mathrm{vect}(\chi_s^\alpha)_{\alpha\in\{\pm\},1\leq s\leq p} \xrightarrow{\sim} \mathfrak{G}(\bar{U}_q)\otimes_{\mathbb{Z}}\mathbb{C}, \quad \chi^I \mapsto [I]$$

where $\mathfrak{G}(\bar{U}_q)$ is the Grothendieck ring of $\bar{U}_q$. By [6], we know the structure of $\mathfrak{G}(\bar{U}_q)$. Recall the decomposition formulas (with $2 \leq s \leq p-1$):

$$\mathcal{X}^-(1)\otimes\mathcal{X}^\alpha(s) \cong \mathcal{X}^{-\alpha}(s),\ \mathcal{X}^+(2)\otimes\mathcal{X}^\alpha(s) \cong \mathcal{X}^\alpha(s-1)\oplus\mathcal{X}^\alpha(s+1),\ \mathcal{X}^+(2)\otimes\mathcal{X}^\alpha(p) \cong \mathcal{P}^\alpha(p-1)$$

so that

$$\chi_1^-\chi_s^\alpha = \chi_s^{-\alpha},\quad \chi_2^+\chi_s^\alpha = \chi_{s-1}^\alpha + \chi_{s+1}^\alpha,\quad \chi_2^+\chi_p^\alpha = 2\chi_{p-1}^\alpha + 2\chi_1^{-\alpha}. \tag{16}$$

We see in particular that χ_2^+ generates the subalgebra $\mathrm{vect}(\chi_s^\alpha)_{\alpha\in\{\pm\},1\leq s\leq p}$. The χ_s^α are expressed as Chebyschev polynomials of χ_2^+, see [6, section 3.3] for details.

Theorem 5.1. *The multiplication rules in the GTA basis are entirely determined by* (16) *and by the following formulas:*

$$\chi_2^+G_1 = [2]G_2, \tag{17}$$

$$\chi_2^+G_s = \frac{[s-1]}{[s]}G_{s-1} + \frac{[s+1]}{[s]}G_{s+1} \ \textit{for}\ 2 \leq s \leq p-2, \tag{18}$$

$$\chi_2^+G_{p-1} = [2]G_{p-2}, \tag{19}$$

$$\chi_1^-G_s = -G_{p-s} \ \textit{for all}\ s, \tag{20}$$

$$G_sG_t = 0 \ \textit{for all}\ s,t. \tag{21}$$

Before giving the proof, let us deduce a few consequences.

Corollary 5.1. *For all* $1 \leq s \leq p-1$ *we have:*

$$G_s = \frac{1}{[s]}\chi_s^+G_1, \quad \chi_p^+G_1 = 0.$$

It follows that $(\chi_s^+ + \chi_{p-s}^-)G_t = 0$*, and that* $\mathcal{V} = \mathrm{vect}(\chi_s^+ + \chi_{p-s}^-, \chi_p^+, \chi_p^-)_{1\leq s\leq p-1}$ *is an ideal of* $\mathrm{SLF}(\bar{U}_q)$.

Proof of Corollary 5.1. The formulas for $\chi_s^+G_1$ are proved by induction using $\chi_{s+1}^+ = \chi_s^+\chi_2^+ - \chi_{s-1}^+$ together with formula (18). We deduce:

$$(\chi_s^+ + \chi_{p-s}^-)G_t = \frac{\chi_t^+}{[t]}(\chi_s^+G_1 + \chi_{p-s}^-G_1) = \frac{\chi_t^+}{[t]}([s]G_s + [s]\chi_1^-G_{p-s}) = 0.$$

It is straightforward that $\mathcal{V}$ is stable by multiplication by χ_2^+, so it is an ideal. □

Remark 3. We have $\chi^{\mathcal{P}^\alpha(s)} = 2\left(\chi_s^\alpha + \chi_{p-s}^{-\alpha}\right)$ for $1 \leq s \leq p-1$. Thus $\mathcal{V}$ is generated by characters of the projective modules. It is well-known that if H is a finite dimensional Hopf algebra, then the full subcategory of finite dimensional projective H-modules is a tensor ideal. Thus we can deduce without any computation that $\mathcal{V}$ is stable under the multiplication by every χ^I.

We now proceed with the proof of the theorem. Observe that we cannot apply Proposition 3.1 to show it since we do not know expressions of $\Delta(e_s)$ and $\Delta(w_s^{\pm})$ which are easy to evaluate in the GTA basis. Recall ([12], see also [10]) the following fusion rules:

$$\mathcal{X}^-(1) \otimes \mathcal{P}^{\alpha}(s) \cong \mathcal{P}^{-\alpha}(s) \quad \text{for all } s, \tag{22}$$

$$\mathcal{X}^+(2) \otimes \mathcal{P}^{\alpha}(1) \cong 2\mathcal{X}^{-\alpha}(p) \oplus \mathcal{P}^{\alpha}(2), \tag{23}$$

$$\mathcal{X}^+(2) \otimes \mathcal{P}^{\alpha}(s) \cong \mathcal{P}^{\alpha}(s-1) \oplus \mathcal{P}^{\alpha}(s+1) \quad \text{for } 2 \leq s \leq p-1, \tag{24}$$

$$\mathcal{X}^+(2) \otimes \mathcal{P}^{\alpha}(p-1) \cong 2\mathcal{X}^{\alpha}(p) \oplus \mathcal{P}^{\alpha}(p-2). \tag{25}$$

They imply the following key lemma.

Lemma 5.1. *There exist scalars $\gamma_s, \beta_s, \lambda_s, \eta_s, \delta_s$ such that*

$$\chi_2^+ G_s = \beta_s G_{s-1} + \gamma_s G_{s+1} + \lambda_s\left(\chi_{s-1}^+ + \chi_{p-s+1}^- - \chi_{s+1}^+ - \chi_{p-s-1}^-\right) \quad (\textit{for } 2 \leq s \leq p-2),$$

$$\chi_2^+ G_1 = \gamma_1 G_2 + \lambda_1\left(\chi_p^- - \chi_2^+ - \chi_{p-2}^-\right), \quad \chi_2^+ G_{p-1} = \beta_{p-1} G_{p-2} + \lambda_{p-1}\left(\chi_{p-2}^+ + \chi_2^- - \chi_p^+\right),$$

$$\chi_1^- G_s = \eta_s G_{p-s} + \delta_s\left(\chi_{p-s}^+ + \chi_s^-\right).$$

Proof. Let us fix $2 \leq s \leq p-2$; by (12), (13), (15) and (24) we have:

$$\chi_2^+ G_s \in \mathrm{vect}\left(\overset{\mathcal{X}^+(2)}{T_{ij}} \cdot \overset{\mathcal{P}^+(s)}{T_{kl}}, \overset{\mathcal{X}^+(2)}{T_{ij}} \cdot \overset{\mathcal{P}^-(p-s)}{T_{kl}}\right)_{ijkl} = \mathrm{vect}\left(\overset{\mathcal{X}^+(2)\otimes\mathcal{P}^+(s)}{T_{ijkl}}, \overset{\mathcal{X}^+(2)\otimes\mathcal{P}^-(p-s)}{T_{ijkl}}\right)_{ijkl}$$

$$= \mathrm{vect}\left(\overset{\mathcal{P}^+(s-1)}{T_{ij}}, \overset{\mathcal{P}^+(s+1)}{T_{ij}}, \overset{\mathcal{P}^-(p-s+1)}{T_{ij}}, \overset{\mathcal{P}^-(p-s-1)}{T_{ij}}\right)_{ij}$$

where $\overset{V}{T_{ij}}$ is the matrix element at the i-th row and j-th column of the representation matrix $\overset{V}{T}$ and $\overset{V\otimes W}{T_{ijkl}}$ is the matrix element at the (i, j)-th row and (k, l)-th column of the representation matrix $\overset{V\otimes W}{T}$. Hence, since $\chi_2^+ G_s$ is symmetric, it is necessarily of the form

$$\chi_2^+ G_s = \beta_s G_{s-1} + \gamma_s G_{s+1} + z_1\chi_{s-1}^+ + z_2\chi_{s+1}^+ + z_3\chi_{p-s+1}^- + z_4\chi_{p-s-1}^-.$$

Evaluating this equality on K and K^2, we find (since $G_t(K^l) = 0$ for all t and l):

$$[s-1](z_1 - z_3) + [s+1](z_2 - z_4) = 0, \quad [s-1]_{q^2}(z_1 - z_3) + [s+1]_{q^2}(z_2 - z_4) = 0,$$

with $[n]_{q^2} = \frac{q^{2n}-q^{-2n}}{q^2-q^{-2}}$. The determinant of this linear system with unknowns $z_1 - z_3, z_2 - z_4$ is $\frac{2\sin((s-1)\pi/p)\sin((s+1)\pi/p)}{\sin(\pi/p)\sin(2\pi/p)}(\cos((s+1)\pi/p) - \cos((s-1)\pi/p)) \neq 0$. Hence $z_1 = z_3$, $z_2 = z_4$. Moreover, evaluating the above equality on 1, we find $p(z_1 + z_2) = 0$. Letting $\lambda_s = z_1$, the result follows. The other formulas are obtained in a similar way using (22), (23) and (25). □

We will use the Casimir element C of $\bar{U}_q$ to make computations easier. It is defined by:

$$C = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} = \sum_{j=0}^{p} c_j e_j + \sum_{k=1}^{p-1}(w_k^+ + w_k^-) \in \mathcal{Z}(\bar{U}_q)$$

where $c_j = \frac{q^j+q^{-j}}{(q-q^{-1})^2}$. The second equality is obtained by considering the action of C on the PIMs $\mathcal{P}^\alpha(s)$. Observe that

$$\forall\, x \in \bar{U}_q,\quad \chi_s^\alpha(Cx) = \alpha c_s \chi_s^\alpha(x),\quad G_s(Cx) = c_s G_s(x) + (\chi_s^+ + \chi_{p-s}^-)(x). \tag{26}$$

Then by induction we get $G_s(C^n) = npc_s^{n-1}$ for $n \geq 1$. We will also denote $c_K = \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}$.

Proof of Theorem 5.1. • *Formula* (18). We first evaluate the corresponding formula of Lemma 5.1 on FE. It holds $G_t(FE) = G_t(C) = p$, $(\chi_t^+ + \chi_{p-t}^-)(FE) = (\chi_t^+ + \chi_{p-t}^-)(C) = pc_t$ for all t and $\chi_2^+ G_s(FE) = \chi_2^+(K^{-1})G_s(FE) = [2]p$. Thus we get:

$$\beta_s + \gamma_s + (c_{s-1} - c_{s+1})\lambda_s = \beta_s + \gamma_s - [s]\lambda_s = [2]. \tag{27}$$

Next, we evaluate the formula of Lemma 5.1 on $(FE)^2$. On the one hand,

$$\begin{aligned}(\chi_2^+ G_s)\big((FE)^2\big) &= \chi_2^+(K^{-2})G_s\big((FE)^2\big) = \chi_2^+(K^{-2})G_s\big(C^2 - 2Cc_K + c_K^2\big)\\ &= \chi_2^+(K^{-2})G_s(C^2) = 2p(q^2+q^{-2})c_s.\end{aligned}$$

For the first equality, we used that $\varphi(E^iF^jK^l) = \delta_{i,j}\varphi(E^iF^iK^l)$ for all $\varphi \in \mathrm{SLF}(\bar{U}_q)$, that $G_s(K^l) = 0$ and that $G_s(FEK^l) = 0$ for $1 \leq l \leq p-1$. The third equality is due to (26) and to the fact that $(\chi_s^+ + \chi_{p-s}^-)(K^l) = 0$ for $1 \leq l \leq p-1$. On the other hand, using again the Casimir element,

$$\begin{aligned}&\beta_s G_{s-1}\big((FE)^2\big) + \gamma_s G_{s+1}\big((FE)^2\big) + \lambda_s\big(\chi_{s-1}^+ + \chi_{p-s+1}^- - \chi_{s+1}^+ - \chi_{p-s-1}^-\big)\big((FE)^2\big)\\ &= \beta_s G_{s-1}\big(C^2\big) + \gamma_s G_{s+1}\big(C^2\big) + \lambda_s\big(\chi_{s-1}^+ + \chi_{p-s+1}^- - \chi_{s+1}^+ - \chi_{p-s-1}^-\big)\big(C^2\big)\\ &= 2pc_{s-1}\beta_s + 2pc_{s+1}\gamma_s + p(c_{s-1}^2 - c_{s+1}^2)\lambda_s.\end{aligned}$$

Since $c_{s-1}^2 - c_{s+1}^2 = -(q+q^{-1})c_s[s]$, we get

$$2c_{s-1}\beta_s + 2c_{s+1}\gamma_s - (q+q^{-1})c_s[s]\lambda_s = 2(q^2+q^{-2})c_s. \tag{28}$$

In order to get a third linear equation between β_s, γ_s and λ_s, we use evaluation on $E^{p-1}F^{p-1}$. This has the advantage to annihilate all the χ_t^α appearing in the formula of Lemma 5.1. First:

$$\begin{aligned}E^{p-1}F^{p-1}b_0^\alpha(s) &= E^{p-1}y_{p-s-1}^\alpha(s) = (-\alpha)^{p-s-1}[p-s-1]!^2E^sy_0^\alpha(s)\\ &= (-\alpha)^{p-s-1}\alpha^{s-1}[p-s-1]!^2[s-1]!^2a_0^\alpha(s)\\ &= (-\alpha)^{p-s-1}\alpha^{s-1}\frac{[p-1]!^2}{[s]^2}a_0^\alpha(s)\end{aligned} \tag{29}$$

and $E^{p-1}F^{p-1}$ annihilates all the other basis vectors. Hence:

$$G_s(E^{p-1}F^{p-1}) = 2(-1)^{p-s-1}\frac{[p-1]!^2}{[s]^2}.$$

Next by (1), we have:

$$\chi_2^+ \otimes \mathrm{Id}\left(\Delta(E^{p-1}F^{p-1})\right) = -[2]E^{p-1}F^{p-1} - q^2E^{p-2}F^{p-2}K.$$

As in (29), we find:

$$E^{p-2}F^{p-2}Kb_0^\alpha(s) = (-\alpha)^{p-s}\alpha^sq^{s-1}\frac{[p-1]!^2}{[s+1][s]^2}a_0^\alpha(s),$$

$$E^{p-2}F^{p-2}Kb_1^{\alpha}(s) = (-\alpha)^{p-s-1}\alpha^{s-1}q^{s-3}\frac{[p-1]!^2}{[s-1][s]^2}a_1^{\alpha}(s)$$

and all the others basis vectors are annihilated. Hence:

$$G_s(E^{p-2}F^{p-2}K) = 2(-1)^{p-s-1}\frac{[p-1]!^2}{[s]^2}\frac{q^{-2}[2]}{[s-1][s+1]}.$$

We obtain:

$$\chi_2^+ \otimes G_s\left(\Delta(E^{p-1}F^{p-1})\right) = 2(-1)^{p-s}[p-1]!^2\frac{[2]}{[s-1][s+1]}$$

and thus:

$$\frac{\beta_s}{[s-1]^2} + \frac{\gamma_s}{[s+1]^2} = \frac{[2]}{[s-1][s+1]}. \tag{30}$$

As a result, we have a linear system (27)–(28)–(30) between β_s, γ_s and λ_s. It is easy to check that $\beta_s = \frac{[s-1]}{[s]}, \gamma_s = \frac{[s+1]}{[s]}, \lambda_s = 0$ is a solution. Moreover this solution is unique. Indeed, a straightforward computation reveals that

$$\det\begin{pmatrix} 1 & 1 & -[s] \\ 2c_{s-1} & 2c_{s+1} & -(q+q^{-1})c_s[s] \\ \frac{1}{[s-1]^2} & \frac{1}{[s+1]^2} & 0 \end{pmatrix} = \frac{[s]^2}{[s-1]^2} + \frac{[s]^2}{[s+1]^2} > 0.$$

• *Formulas* (17) *and* (20). Evaluating as above the corresponding formulas of Lemma 5.1 on FE and $(FE)^2$, one gets linear systems with non-zero determinants. It is then easy to see that $\beta_1 = [2], \lambda_1 = 0$ and $\eta_s = -1, \delta_s = 0$ are the unique solutions of each of these two systems.

• *Formula* (19). It can be deduced from the formulas already shown:

$$\chi_2^+ G_{p-1} = -\chi_2^+\chi_1^- G_1 = -\chi_1^-[2]G_2 = [2]G_{p-2}.$$

• *Formula* (21). Recall the isomorphism of algebras $\mathcal{D}$ defined in (10). Taking into account that $\varphi(K^iF^mE^n) = 0$ if $n \neq m$ for any $\varphi \in \mathrm{SLF}(\bar{U}_q)$ and that $G_s(K^i) = 0$ for all i, and making use of the expression of RR' given in [6], we get:

$$\begin{aligned}\mathcal{D}(G_s) &= \sum_{n=0}^{p-1}\sum_{j=0}^{2p-1}\left(\sum_{i=0}^{2p-1}\frac{(q-q^{-1})^n}{[n]!^2}q^{n(j-i-1)-ij}G_s(K^{p+i+1}E^nF^n)\right)K^jF^nE^n \\ &= \sum_{n=1}^{p-1}\sum_{j=0}^{2p-1}\lambda_{j,n}K^jF^nE^n\end{aligned}$$

for some coefficients $\lambda_{j,n}$ (observe that $n \geq 1$). From this it follows that for all $\alpha \in \{\pm\}$ and $1 \leq r \leq p-1$: $\mathcal{D}(G_s)b_0^{\alpha}(r) \in \mathbb{C}a_0^{\alpha}(r)$. By (7), we deduce that $\mathcal{D}(G_s) \in \mathrm{vect}(w_r^{\pm})_{1 \leq r \leq p-1}$ for all s. Thus $\mathcal{D}(G_sG_t) = 0$, thanks to (9). □

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5.1.

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