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Osaka University
A NOTE ON SYMMETRIC LINEAR FORMS AND TRACES
ON THE RESTRICTED QUANTUM GROUP \( \bar{U}_q(\mathfrak{sl}(2)) \)

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Abstract
In this paper we prove two results about \( \text{SLF}(\bar{U}_q) \), the algebra of symmetric linear forms on the restricted quantum group \( \bar{U}_q = \bar{U}_q(\mathfrak{sl}(2)) \). First, we express any trace on finite dimensional projective \( \bar{U}_q \)-modules as a linear combination in the basis of \( \text{SLF}(\bar{U}_q) \) constructed by Gainutdinov - Tipunin and also by Arike. In particular, this allows us to determine the symmetric linear form corresponding to the modified trace on projective \( \bar{U}_q \)-modules. Second, we give the explicit multiplication rules between symmetric linear forms.

1. Introduction
Let \( \bar{U}_q = \bar{U}_q(\mathfrak{sl}(2)) \) be the restricted quantum group associated to \( \mathfrak{sl}(2) \) and \( \text{SLF}(\bar{U}_q) \) its space of symmetric linear forms, which is naturally endowed with an algebra structure. In [9] and [1], an interesting basis of \( \text{SLF}(\bar{U}_q) \) is introduced, that will be called the GTA basis in the sequel, and whose construction is based on the simple and the projective \( \bar{U}_q \)-modules (see section 3). In this paper, we prove two results about this basis, namely the relation with traces on projectives modules, and the formulas for multiplication of symmetric linear forms.

First, we show in the general setting of a finite dimensional \( k \)-algebra \( A \) that there is a correspondence between traces on finite dimensional projective \( A \)-modules and symmetric linear forms on \( A \) (Theorem 4.1). In the case of \( A = \bar{U}_q \), the natural question is to express the image of a trace through this correspondence in the GTA basis. We answer this question and show that this basis is relevant with regard to this correspondence in Theorem 4.2. The modified trace computed in [3] is an interesting example of a trace on projective \( \bar{U}_q \)-modules. We determine the symmetric linear form corresponding to the modified trace, and get that it is \( \mu(K^{p+1}) \), where \( \mu \) is a suitably normalized right integral of \( \bar{U}_q \) (see section 4.3). This last result has been found simultaneously in [2] in a general framework including \( \bar{U}_q \).

With regard to the structure of algebra on \( \text{SLF}(\bar{U}_q) \), a natural and important problem is to determine the multiplication rules of the elements in the GTA basis. In section 5, we find the decomposition of the product of two basis elements in the GTA basis. The resulting formulas are surprisingly simple (Theorem 5.1). Note that a similar problem (namely the multiplication in the space of \( q \)-characters \( q\text{Ch}(\bar{U}_q) \), which is isomorphic as an algebra to \( \text{SLF}(\bar{U}_q) \)) has been solved in [9], but I was not aware of the existence of this paper when preparing this work. It turns out that our proofs are different. In [9], they use the fact that the

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multiplication in the canonical basis of $Z(\tilde{U}_q)$ is very simple. They first express the image of their basis of $q\text{Ch}(\tilde{U}_q)$ through the Radford mapping in the canonical basis of $Z(\tilde{U}_q)$. This gives a basis of $Z(\tilde{U}_q)$ called the Radford basis. Then they use the $S$-transformation of the $\text{SL}_2(\mathbb{Z})$ representation on $Z(\tilde{U}_q)$ to express the Drinfeld basis (which is the image of their basis of $q\text{Ch}(\tilde{U}_q)$ by the Drinfeld map) in the Radford basis. This gives the multiplication rules in the Drinfeld basis. Since the Drinfeld map is an isomorphism of algebras between $q\text{Ch}(\tilde{U}_q)$ and $Z(\tilde{U}_q)$, this gives also the multiplication rules in the GTA basis. Here we directly work in $\text{SLF}(\tilde{U}_q)$. We first prove an elementary lemma which shows that there are not many coefficients to determine, and then we compute these coefficients by using the evaluation on suitable elements of $\tilde{U}_q$.

To make the paper self-contained and fix notations, we recall some facts about the structure of $\tilde{U}_q$ and its representation theory in section 2. In section 3, we introduce $\text{SLF}(\tilde{U}_q)$ and the GTA basis. We then state some properties that are needed to prove our results.

In [5], the GTA basis and its multiplication rules are extensively used to describe in detail $\text{Top}(\tilde{U}_q)$, which is a quantum analogue of the algebra of functions associated to lattice gauge theory on the torus. Here we use it as a means to describe the relations of $\text{Top}(\tilde{U}_q)$.

Notations. If $A$ is a $k$-algebra (with $k$ a field), $V$ is a finite dimensional $A$-module and $x \in A$, we denote by $\hat{x} \in \text{End}(V)$ the representation of $x$ on the module $V$. We will work only with finite dimensional modules and mainly with left modules, thus often we simply write “module” instead of “finite dimensional left module”. The socle of $V$, denoted by $\text{soc}(V)$, is the largest semi-simple submodule of $V$. The top of $V$, denoted by $\text{Top}(V)$, is $V/\text{Rad}(V)$, where $\text{Rad}(V)$ is the Jacobson radical of $V$. See [4, Chap. IV and VIII] for background material about representation theory.

For $q \in \mathbb{C} \setminus \{-1,0,1\}$, we define the $q$-integer $[n]$ (with $n \in \mathbb{Z}$) and the $q$-factorial $[m]!$ (with $m \in \mathbb{N}$) by:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [0]! = 1, \quad [m]! = [1][2] \ldots [m] \quad \text{for} \ m \geq 1.$$ 

In what follows $q$ is a primitive $2p$-root of unity (where $p$ is a fixed integer $\geq 2$), say $q = e^{\pi i/p}$. Observe that in this case $[n] = \frac{\sin(n \pi/p)}{\sin(\pi/p)}$, $[p] = 0$ and $[p - n] = [n]$.

As usual, $\delta_{i,j}$ will denote the Kronecker symbol and $I_n$ the identity matrix of size $n$.

2. Preliminaries

2.1. The restricted quantum group $\tilde{U}_q(\mathfrak{sl}(2))$. As mentioned above, $q$ is a primitive root of unity of order $2p$, with $p \geq 2$. Recall that $\tilde{U}_q(\mathfrak{sl}(2))$, the restricted quantum group associated to $\mathfrak{sl}(2)$, is the $\mathbb{C}$-algebra generated by $E, F, K$ together with the relations

$$E^p = F^p = 0, \quad K^{2p} = 1, \quad KE = q^2 KE, \quad KF = q^{-2} FK, \quad EF = FE + \frac{K - K^{-1}}{q - q^{-1}}.$$ 

It will be simply denoted by $\tilde{U}_q$ in the sequel. It is a $2p^3$-dimensional Hopf algebra, with comultiplication $\Delta$, counit $e$ and antipode $S$ given by the following formulas:
The monomials $E^mF^nK^l$ with $0 \leq m, n \leq p - 1$, $0 \leq l \leq 2p - 1$, form a basis of $\bar{U}_q$, usually referred as the PBW-basis. Recall the formula (see \cite[Prop. VII.1.3]{11}): \hspace{1cm}

$$
\Delta(E) = 1 \otimes E + E \otimes K, \hspace{0.5cm} \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \hspace{0.5cm} \Delta(K) = K \otimes K, \\
e(\varepsilon(E) = 0, \hspace{0.5cm} e(F) = 0, \hspace{0.5cm} e(K) = 1, \\
S(E) = -EK^{-1}, \hspace{0.5cm} S(F) = -KF, \hspace{0.5cm} S(K) = K^{-1}.
$$

The monomials $E^mF^nK^l$ with $0 \leq m, n \leq p - 1$, $0 \leq l \leq 2p - 1$, form a basis of $\bar{U}_q$, usually referred as the PBW-basis. Recall the formula (see \cite[Prop. VII.1.3]{11}): \hspace{1cm}

1. $\Delta(E^mF^nK^l) = \sum_{i=0}^{m} \sum_{j=0}^{n} (m-i+j(n-j)-2(m-i)(n-j)) q^{2i} F_j K^{j-n} \otimes E^i F^n K^l E^i - F^{i+n} K^{l+i}$. 

Recall that the $q$-binomial coefficients are defined by $\begin{pmatrix} a \选择 b \\ \end{pmatrix} = \frac{[a]_q \times [a-b]_q}{[b]_q}$. For $a \geq b$. Since $K$ is annihilated by the polynomial $X^2p - 1$, which has simple roots over $\mathbb{C}$, the action of $K$ is diagonalizable on each $\bar{U}_q$-module, and the eigenvalues are 2p-roots of unity.

Due to the Hopf algebra structure on $\bar{U}_q$, its category of modules is a monoidal category with duals. It is not braided (see \cite{12}).

2.2. Simple and projective $\bar{U}_q$-modules. The finite dimensional representations of $\bar{U}_q$ are classified (\cite{15} and \cite{7}). Two types of modules are important for our purposes: the simple and the projective modules. As in \cite{6} (see also \cite{10}), we denote the simple modules by $\mathcal{X}^\alpha(s)$, with $\alpha \in \{\pm\}, 1 \leq s \leq p$. The modules $\mathcal{X}^\pm(p)$ are simple and projective simultaneously. The other indecomposable projective modules are not simple. We denote them by $\mathcal{P}^\alpha(s)$ with $\alpha \in \{\pm\}, 1 \leq s \leq p - 1$.

The module $\mathcal{X}^\alpha(s)$ admits a canonical basis $(v_i)_{0 \leq i \leq s-1}$ such that \hspace{1cm}

$$(2) \hspace{0.5cm} K v_i = \alpha q^{s-i-2} v_i, \hspace{0.5cm} E v_0 = 0, \hspace{0.5cm} E v_i = \alpha[i] s - i v_{i-1}, \hspace{0.5cm} F v_i = v_{i+1}, \hspace{0.5cm} F v_{s-1} = 0.
$$

The module $\mathcal{P}^\alpha(s)$ admits a standard basis $(b_i, x_j, y_k, a_l)_{0 \leq i, j \leq s-1, 0 \leq k \leq s, 0 \leq l \leq s}$ such that \hspace{1cm}

$$(3) \hspace{0.5cm} K b_i = \alpha q^{s-2} b_i, \hspace{0.5cm} E b_i = \alpha[i] s - i b_{i-1} + a_{i-1}, \hspace{0.5cm} F b_i = b_{i+1}, \hspace{0.5cm} F b_{s-1} = y_0, \\
K x_j = -\alpha q^{p-s-2} x_j, \hspace{0.5cm} E x_j = -\alpha[j] p - s - j x_{j-1}, \hspace{0.5cm} F x_j = x_{j+1}, \hspace{0.5cm} F x_{p-s-1} = a_0, \\
K y_k = -\alpha q^{p-s-2} y_k, \hspace{0.5cm} E y_k = -\alpha[k] p - s - k y_{k-1}, \hspace{0.5cm} F y_k = y_{k+1}, \hspace{0.5cm} F y_{p-s-1} = 0, \\
K a_l = \alpha q^{l-2} a_l, \hspace{0.5cm} E a_l = \alpha[l] s - l a_{l-1}, \hspace{0.5cm} F a_l = a_{l+1}, \hspace{0.5cm} F a_{s-1} = 0.
$$

Note that such a basis is not unique up to scalar since we can replace $b_i$ by $b_i + \lambda a_i$ (with $\lambda \in \mathbb{C}$) without changing the action.

In terms of composition factors, the structure of $\mathcal{P}^\alpha(s)$ can be schematically represented as follows (with the basis vectors corresponding to each factor and the action of $E$ and $F$):

\begin{align*}
\Delta(E) &= 1 \otimes E + E \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K, \\
\varepsilon(E) &= 0, \quad \varepsilon(F) = 0, \quad \varepsilon(K) = 1, \\
S(E) &= -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.
\end{align*}
If we need to emphasize the module in which we are working, we will use the following notations: \( v_{0}^{\alpha} (s) \) for the canonical basis of \( \mathcal{X}^{\alpha}(s) \) and \( b_{0}^{\alpha}(s) \), \( \chi_{0}^{\alpha}(s) \), \( a_{0}^{\alpha}(s) \) for a standard basis of \( \mathcal{P}^{\alpha}(s) \) (these are the notations used in [1]).

Let us recall the \( \bar{U}_q \)-morphisms between these modules. Observe that \( \mathcal{X}^{\alpha}(s) \) is \( \bar{U}_q \)-generated by \( v_{0}^{\alpha}(s) \) and \( \mathcal{P}^{\alpha}(s) \) is \( \bar{U}_q \)-generated by \( b_{0}^{\alpha}(s) \), so the images of these vectors suffice to define \( \bar{U}_q \)-morphisms. \( \mathcal{X}^{\alpha}(s) \) is simple, so by Schur’s lemma \( \text{End}_{\bar{U}_q} (\mathcal{X}^{\alpha}(s)) = \mathbb{C} \text{Id} \). Since

\[
\mathcal{X}^{\alpha}(s) \cong \text{Top} (\mathcal{P}^{\alpha}(s)) \equiv \text{Soc} (\mathcal{P}^{\alpha}(s))
\]

there exist injection and projection maps defined by:

\[
\begin{align*}
\mathcal{X}^{\alpha}(s) & \hookrightarrow \mathcal{P}^{\alpha}(s) & \text{and} & \mathcal{P}^{\alpha}(s) & \rightarrow \mathcal{X}^{\alpha}(s) \\
v_{0}^{\alpha}(s) & \mapsto a_{0}^{\alpha}(s) & b_{0}^{\alpha}(s) & \mapsto v_{0}^{\alpha}(s).
\end{align*}
\]

We have \( \text{End}_{\bar{U}_q} (\mathcal{P}^{\alpha}(s)) = \mathbb{C} \text{Id} \oplus \mathbb{C} p^{\alpha}_{s} \) and \( \text{Hom}_{\bar{U}_q} (\mathcal{P}^{\alpha}(s), \mathcal{P}^{-\alpha}(p-s)) = \mathbb{C} p^{\alpha}_{s} \oplus \mathbb{C} \bar{P}^{\alpha}_{s} \), where:

\[
p^{\alpha}_{s} (b_{0}^{\alpha}(s)) = a_{0}^{\alpha}(s), \quad P^{\alpha}_{s} (b_{0}^{\alpha}(s)) = \chi_{0}^{\alpha}(p-s), \quad \bar{P}^{\alpha}_{s} (b_{0}^{\alpha}(s)) = y_{0}^{\alpha}(p-s).
\]

The other Hom-spaces involving only simple modules and indecomposable projective modules are null.

### 2.3. Structure of the bimodule \( \bar{U}_q \) and the center of \( U_q \)

Recall that if \( M \) is a left module (over any \( k \)-algebra \( A \)), then \( M^{\ast} = \text{Hom}_{\mathbb{C}}(M, k) \) is endowed with a right \( A \)-module structure, given by:

\[
\forall a \in A, \forall \varphi \in M^{\ast}, \quad \varphi a = \varphi (a \cdot)
\]

where \( \cdot \) is the place of the variable. We denote by \( R^*(M) \) the so-defined right module. Note that if we define \( R^*(f) \) as the transpose of \( f \), then \( R^* \) becomes a contravariant functor. If \( A \) is a Hopf algebra, one must be aware not to confuse \( R^*(M) \) with the categorical dual \( M^{\ast} \), which is a left module on which \( A \) acts by:

\[
\forall a \in A, \forall \varphi \in M^{\ast}, \quad a \varphi = \varphi (S(a) \cdot).
\]

**Lemma 2.1.** The right \( \bar{U}_q \)-module \( R^*(\mathcal{X}^{\alpha}(s)) \) admits a basis \( (\bar{v}_{i})_{0 \leq i \leq s-1} \) such that

\[
\bar{v}_{i} K = a q^{1-s+2i} \bar{v}_{i}, \quad \bar{v}_{i} E = a [i][s-i] \bar{v}_{i-1}, \quad \bar{v}_{i} F = 0, \quad v_{i} F = \bar{v}_{i+1}, \quad \bar{v}_{s-1} F = 0.
\]

The right \( \bar{U}_q \)-module \( R^*(\mathcal{P}^{\alpha}(s)) \) admits a basis \( (\bar{b}_{i}, \bar{v}_{j}, \bar{y}_{k}, \bar{a}_{i})_{0 \leq i \leq s-1} \) such that
\[ \bar{b}_i K = a q^{1-s+2l} \bar{b}_i, \quad \bar{b}_i E = \bar{a}_{l-1} + a[l][s - l] \bar{b}_i, \quad \bar{b}_i F = \bar{b}_{l+1}, \]
\[ \bar{x}_j K = -a q^{-p+s+2j} \bar{x}_j, \quad \bar{x}_j E = -a[j][p - s - j] \bar{x}_{j-1}, \quad \bar{x}_j F = \bar{x}_{j+1}, \]
\[ \bar{y}_k K = -a q^{-p+s+2k} \bar{y}_k, \quad \bar{y}_k E = -a[k][p - s - k] \bar{y}_{k-1}, \quad \bar{y}_k F = \bar{y}_{k+1}. \]
\[ \bar{a}_l K = a q^{1-s+2l} \bar{a}_l, \quad \bar{a}_l E = a[l][s - l] \bar{a}_{l-1}, \quad \bar{a}_l F = \bar{a}_{l+1}, \]
\[ \bar{a}_0 E = 0, \quad \bar{a}_0 F = 0. \]

Such basis will be termed respectively a canonical basis and a standard basis in the sequel.

Proof. Let \((\psi')_{0 \leq i \leq s-1}\) be the basis dual to the canonical basis given in (2). Then \(\bar{v}_i = v^{s-i}\) gives the desired result. Similarly, let \((\psi^{s-i})_{0 \leq i \leq p-s-1}\) be the basis dual to a standard basis given in (3). Then
\[ \bar{b}_i = a^{s-i}, \quad \bar{x}_j = y^{p-s-j}, \quad \bar{y}_k = x^{p-s-k}, \quad \bar{a}_l = b^{s-l} \]
gives the desired result. \(\square\)

We denote by \(\bar{U}_q (U_q)\) the regular bimodule, where the left and right actions are respectively the left and right multiplications of \(U_q\) on itself. Recall that a block of \(\bar{U}_q (U_q)\) is just an indecomposable two-sided ideal (see [4, Section 55]). The block decomposition of \(\bar{U}_q\) is (see [6])
\[ \bar{U}_q (U_q) = \bigoplus_{s=0}^{p} Q(s) \]
where the structure of each block \(Q(s)\) as a left \(\bar{U}_q\)-module is:
\[ Q(0) \cong p \mathcal{X}^-(p), \quad Q(p) \cong p \mathcal{X}^+(p), \]
\[ Q(s) \cong sp^*(s) \oplus (p - s)p^*(p - s) \quad \text{for} \ 1 \leq s \leq p - 1 \]
and the structure of each block as a right \(\bar{U}_q\)-module is:
\[ Q(0) \cong p R^*(\mathcal{X}^-(p)), \quad Q(p) \cong p R^*(\mathcal{X}^+(p)), \]
\[ Q(s) \cong sp^*(s) \oplus (p - s)p^*(p - s) \quad \text{for} \ 1 \leq s \leq p - 1. \]

The following proposition is a reformulation of [6, Prop. 4.4.2] (see also [10, Th. II.1.4]). It will be used for the proof of Theorem 4.2.

**Proposition 2.1.** For \(1 \leq s \leq p - 1\), the block \(Q(s)\) admits a basis
\[ \{ B_{ab}^{+}(s), X_{cd}^{+}(s), Y_{ef}^{+}(s), A_{gh}^{+}(s), B_{ij}^{-}(s), X_{kl}^{-}(s), Y_{mn}^{-}(s), A_{or}^{-}(s) \} \]
with \(0 \leq a, b, d, f, g, h, k, m \leq s - 1, \ 0 \leq c, e, i, j, l, n, o, r \leq p - s - 1\), such that
1. \(\forall 0 \leq j \leq s - 1\), \(\{ B_{ij}^{+}(s), X_{kj}^{+}(s), Y_{ij}^{+}(s), A_{mj}^{+}(s) \}_{0 \leq k, l \leq s-1} \) is a standard basis of \(P^+(s)\) for the left action.
2. \(\forall 0 \leq j \leq p - s - 1\), \(\{ B_{ij}^{-}(s), X_{kj}^{-}(s), Y_{ij}^{-}(s), A_{mj}^{-}(s) \}_{0 \leq k, l \leq p-s-1} \) is a standard basis of \(P^-(p - s)\) for the left action.
3. ∀ 0 ≤ i ≤ s − 1, \((B_{ij}^+(s), X_{ij}^-(s), Y_{ij}^+(s), A_{im}^-(s))_{0 ≤ j, m ≤ p−1}\) is a standard basis of \(R^*(P^+(s))\) for the right action.

4. ∀ 0 ≤ i ≤ p − s − 1, \((B_{ij}^-(s), X_{ij}^+(s), Y_{ij}^-(s), A_{im}^+(s))_{0 ≤ j, m ≤ p−1}\) is a standard basis of \(R^*(P^−(p−s))\) for the right action.

The block \(Q(0)\) admits a basis \((A_{ij}^-(0))_{0 ≤ i, j ≤ p−1}\) such that

1. ∀ 0 ≤ j ≤ p − 1, \((A_{ij}^-(0))_{0 ≤ i ≤ p−1}\) is a standard basis of \(X^−(p)\) for the left action.

2. ∀ 0 ≤ i ≤ p − 1, \((A_{ij}^-(0))_{0 ≤ j ≤ p−1}\) is a standard basis of \(R^*(X^−(p))\) for the right action.

The block \(Q(p)\) admits a basis \((A_{ij}^+(p))_{0 ≤ i, j ≤ p−1}\) such that

1. ∀ 0 ≤ j ≤ p − 1, \((A_{ij}^+(p))_{0 ≤ i ≤ p−1}\) is a standard basis of \(X^+(p)\) for the left action.

2. ∀ 0 ≤ i ≤ p − 1, \((A_{ij}^+(p))_{0 ≤ j ≤ p−1}\) is a standard basis of \(R^*(X^+(p))\) for the right action.

As in [6], the structure of \(Q(s)\) in terms of composition factors can be schematically represented as follows (each vertex represents a composition factor and is labelled by the basis vectors of this factor):

for the left action, and

for the right action.

The knowledge of the structure of the bimodule \(U_q(\tilde{U}_q)\) allows us to determine the center of \(\tilde{U}_q\). Indeed, each central element determines a bimodule endomorphism and conversely. Recall from [6] that \(Z(\tilde{U}_q)\) is a \((3p−1)\)-dimensional algebra with basis elements \(e_s\) (0 ≤ s ≤ p) and \(u_t^+\) (1 ≤ t ≤ p − 1). The element \(e_s\) is just the unit of the block \(Q(s)\), thus by (6) and (4) the action of \(e_s\) on the simple and the projective modules is given by
For $s = 0$,  
\[ e_0v_0^+(t) = 0, \quad e_0v_0^-(t) = \delta_{t,p}v_0^-(p), \quad e_0b_0^+(t) = 0, \]
(7)  
For $1 \leq s \leq p - 1$,  
\[ e_s^+v_0^+(t) = \delta_{t,s}v_0^+(s), \quad e_s^+v_0^-(t) = \delta_{t,p-s}v_0^-(p-s), \]
\[ e_s^+b_0^+(t) = \delta_{t,s}b_0^+(s), \quad e_s^+b_0^-(t) = \delta_{t,s}b_0^-(p-s), \]
(8)  
For $s = p$,  
\[ e_p^+v_0^+(t) = \delta_{t,p}v_0^+(p), \quad e_p^+v_0^-(t) = 0, \quad e_p^+b_0^+(t) = 0. \]
while for the elements $w_s^+$:
\[ w_s^+v_0^+(t) = 0, \quad w_s^+b_0^+(t) = \delta_{t,s}a_0^+(s), \quad w_s^+b_0^-(t) = 0, \]
\[ w_s^-v_0^+(t) = 0, \quad w_s^-b_0^+(t) = 0, \quad w_s^-b_0^-(t) = \delta_{t,p-s}a_0^-(p-s). \]

Observe that
\[ p^{\ast}(s) \quad w_s^+ = p_s^+, \quad p^{\ast}(p-s) \quad w_s^- = p_s^- \]

The action of the central elements on $P^n(s)$ is enough to recover their action on every module, using projective covers. From these formulas, we deduce the multiplication rules of these elements:
\[ e_s e_t = \delta_{s,t} e_s, \quad e_s^+ w_t = \delta_{s,t} w_s^+, \quad w_s^+ w_t^+ = 0. \]

Let us mention that the idempotents $e_s$ are not primitive: there exists primitive orthogonal idempotents $e_{s,i}$ such that $e_s = \sum_i e_{s,i}$, see [1].

3. Symmetric linear forms and the GTA basis

Let $A$ be a $k$-algebra, and let $\text{SLF}(A)$ be the space of symmetric linear forms on $A$:
\[ \text{SLF}(A) = \{ \varphi \in A^* \mid \forall x, y \in A, \ \varphi(xy) = \varphi(yx) \}. \]

If $A$ is a bialgebra, then $A^*$ is an algebra whose product is defined by:
\[ \varphi \psi(x) = \sum_{(x)} \varphi(x') \psi(x'') \]
with $\Delta(x) = \sum_{(x)} x' \otimes x''$ (Sweedler’s notation, see e.g. [11, Chap. 3]). Then $\text{SLF}(A)$ is a subalgebra of $A^*$. Indeed, if $\varphi, \psi \in \text{SLF}(A)$, we have:
\[ \varphi \psi(xy) = \sum_{(xy)} \varphi(x'y') \psi(x''y'') = \sum_{(x'y')} \varphi(y') \psi(y''x') = \varphi \psi(yx) \]
which shows that $\varphi \psi \in \text{SLF}(A)$. If moreover $A$ is finite dimensional, then $A^*$ is a bialgebra whose coproduct is defined by $\Delta(\varphi)(x \otimes y) = \varphi(xy)$, but $\text{SLF}(A)$ is not in general a sub-coalgebra of $A^*$, see Remark 1 below.

Recall (see [6]) that there is a universal $R$-matrix $R$ belonging to the extension of $\bar{U}_q$ by a square root of $K$. It satisfies $RR' \in \bar{U}_q^{\otimes 2}$, where $R' = \tau(R)$, with $\tau$ the flip map defined by $\tau(x \otimes y) = y \otimes x$. Moreover $\bar{U}_q$ is factorizable (in a generalized sense since it does not contain the $R$-matrix) and $K^{p+1}$ is a pivotal element, thus it is known from general theory that the Drinfeld morphism which we denote $D$ provides an isomorphism of algebras
\[ D : \text{SLF}(\bar{U}_q) \quad \varphi \mapsto (\varphi \otimes \text{Id})((K^{p+1} \otimes 1) \cdot RR') \]

Let $A$ be a $k$-algebra, and $V$ an $n$-dimensional $A$-module. If we choose a basis on $V$, we
get a matrix $\mathbf{T} \in \text{Mat}_n(A^\ast)$, simply defined by

$$\mathbf{V} \mathbf{T}(x) = \mathbf{V} x$$

where $\mathbf{V} x$ is the representation of $x \in A$ in $\text{End}(\mathbf{V})$ expressed in the chosen basis. In our case, we will always choose the canonical bases of the simple modules and standard bases of the projective modules.

An interesting basis of $\text{SLF}(\overline{U}_q)$ was found by Gainutdinov and Tipunin in [9] and by Arike in [1]. To be precise, a basis of the space $q\text{Ch}(\overline{U}_q)$ of $q$-characters is constructed in [9], but the shift by the pivotal element $g = K^{p+1}$ provides an isomorphism

$$q\text{Ch}(\overline{U}_q) \xrightarrow{\sim} \text{SLF}(\overline{U}_q), \quad \psi \mapsto \psi(g \cdot \cdot) .$$

This basis is built from the simple and the projective modules. First, define $2p$ linear forms $\chi^{\alpha}_s$, $\alpha \in \{\pm\}$, $1 \leq s \leq p$, by:

$$\chi^{\alpha}_s = \text{tr} \left( \mathbf{V}^{\alpha(s)} \right).$$

They are obviously symmetric. Observe that $\chi^{+}_1 = \varepsilon$ is the unit for the algebra structure on $\text{SLF}(\overline{U}_q)$ described above. To construct the $p - 1$ missing linear forms, observe with the help of (4) that the matrix of the action on $\mathbf{P}^{\alpha(s)}$ has the following block form in a standard basis:

$$\mathbf{P}^{\alpha(s)} \mathbf{T} = \begin{pmatrix}
(b_i) & (x_j) & (y_k) & (a_l) \\
\mathbf{V}^{\alpha(s)} & 0 & 0 & 0 \\
\mathbf{A}_s^\alpha & \mathbf{X}^{\alpha(p-s)} & 0 & 0 \\
\mathbf{B}_s^\alpha & 0 & \mathbf{X}^{\alpha(p-s)} & 0 \\
\mathbf{H}_s^\alpha & \mathbf{D}_s^\alpha & \mathbf{C}_s^\alpha & \mathbf{X}^{\alpha(s)} \end{pmatrix} \mathbf{T} \begin{pmatrix}
(b_i) \\
(x_j) \\
y_k \\
(a_l)
\end{pmatrix} .$$

It is not difficult to see that these matrices satisfy the following symmetries:

$$\mathbf{A}_{p-s} = \mathbf{C}_s^+, \quad \mathbf{B}_{p-s} = \mathbf{D}_s^+, \quad \mathbf{D}_{p-s} = \mathbf{B}_s^+ , \quad \mathbf{C}_{p-s} = \mathbf{A}_s^+ ,$$

By computing the matrices $(xy) = \mathbf{P}^{\alpha(s)} \mathbf{P}^{\alpha(s)}$ and $(xy) = \mathbf{P}^{\alpha(p-s)} \mathbf{P}^{\alpha(p-s)}$, these symmetries allow us to see that the linear form $G_s (1 \leq s \leq p - 1)$ defined by

$$G_s = \text{tr} (\mathbf{H}_s^+) + \text{tr} (\mathbf{H}_{p-s}^-)$$

is a symmetric linear form.

It is instructive for our purposes to see a proof that these symmetric linear forms are linearly independent. Let us begin by introducing important elements for $0 \leq n \leq p - 1$ (they are discrete Fourier transforms of $(K^i)_{0 \leq i \leq 2p-1}$):

---

1The correspondence of notations with [1] is: $T^+_s = \chi^{+}_s$, $T^-_s = \chi^{-}_{p-s}$. The letter $T$ is here reserved for the matrices $\mathbf{T}$ described above.
The following easy lemma shows that these elements allow one to select vectors which have a given weight, and this turns out to be very useful.

**Lemma 3.1.** 1) Let \( M \) be a left \( \bar{U}_q \)-module, and let \( m^+_i(s) \) be a vector of weight \( q^{s-1-2i} \), \( m^-_i(p-s) \) be a vector of weight \( -q^{(p-s)-1-2i} \), \( m^-_i(s) \) be a vector of weight \( -q^{s-1-2i} \), \( m^+_i(p-s) \) be a vector of weight \( q^{(p-s)-1-2i} = -q^{s-1-2i} \). Then:

\[
\Phi_{s-1}^+ m^+_i(s) = \delta_{i,0} m^+_0(s), \quad \Phi_{s-1}^- m^-_i(p-s) = 0,
\]

\[
\Phi_{s-1}^- m^-_i(s) = \delta_{i,0} m^-_0(s), \quad \Phi_{s-1}^+ m^+_i(p-s) = 0.
\]

2) Let \( N \) be a right \( U_q \)-module, and let \( n^+_i(s) \) be a vector of weight \( q^{1-s+2i} \), \( n^-_i(p-s) \) be a vector of weight \( -q^{1-(p-s)+2i} = q^{1+s+2i} \), \( n^-_i(s) \) be a vector of weight \( -q^{1+s+2i} \), \( n^+_i(p-s) \) be a vector of weight \( q^{1-(p-s)+2i} = -q^{1+s+2i} \). Then:

\[
n^+_i(s) \Phi_{s-1}^- = \delta_{i,1-n^-_i(s)}, \quad n^-_i(p-s) \Phi_{s-1}^+ = 0,
\]

\[
n^+_i(s) \Phi_{s-1}^- = \delta_{i,1-n^-_i(s)}, \quad n^-_i(p-s) \Phi_{s-1}^+ = 0.
\]

Proof. It follows from easy computations with sums of roots of unity. \( \square \)

We can now state the key observation.

**Proposition 3.1.** Let

\[
\varphi = \sum_{s=1}^p (\lambda^+_s \chi^+_s + \lambda^-_s \chi^-_s) + \sum_{s=1}^{p-1} \mu_s G_s \in \text{SLF}(\bar{U}_q).
\]

Then:

\[
\lambda^+_s = \varphi(e_{s-1} e_s), \quad \lambda^-_s = \varphi(e_{s-1} e_{p-s}), \quad \mu_s = \frac{\varphi(w^+_s)}{s' - s},
\]

Proof. It is a corollary of (7) and (8). Indeed, we have:

\[
\begin{align*}
\chi^+(s) & = \delta_{s,s} I_s, & \chi^+(s) & = \delta_{s,p-s} I_s, & \chi^+(s) & = \delta_{s,p-s} I_s, \\
T(e_s) & = 0, & T(w^+_s) & = 0, & T(e_s) & = 0,
H^+_s(e_s) & = 0, & H^+_s(w^+_s) & = 0, & H^-_{p-s}(w^-_s) & = \delta_{s,s} I_{s,p-s}.
\end{align*}
\]

This gives the formula for \( \mu_s \). The formulas for \( \lambda^\pm_s \) follow from this and Lemma 3.1. \( \square \)

If we have \( \sum_{s=1}^p (\lambda^+_s \chi^+_s + \lambda^-_s \chi^-_s) + \sum_{s=1}^{p-1} \mu_s G_s = 0 \), we can evaluate the left-hand side on the elements appearing in Proposition 3.1 to get that all the coefficients are equal to 0. Thus we have a free family of cardinal \( 3p-1 \), hence a basis of \( \text{SLF}(\bar{U}_q) \), since \( \text{dim}(\text{SLF}(\bar{U}_q)) = 3p-1 \) by (10).

**Theorem 3.1.** The symmetric linear forms \( \chi^+_s \) (1 \( \leq s \leq p \)) and \( G_s \) (1 \( \leq s' \leq p - 1 \)) form a basis of \( \text{SLF}(\bar{U}_q) \).
Definition 3.1. The basis of Theorem 3.1 will be called the GTA basis (for Gainutdinov, Tipunin, Arike).

Remark 1. Let \( \varphi \in \text{SLF}(\bar{U}_q) \). It is easy to see that \( \varphi(K^IF^m) = 0 \) if \( n \neq m \). From this we deduce that \( \text{SLF}(\bar{U}_q) \) is not a sub-coalgebra of \( \bar{U}_q^* \). Indeed, write \( \Delta(\chi_s^\pm_\pm) = \sum_i \varphi_i \otimes \psi_i \), and assume that \( \varphi_i, \psi_j \in \text{SLF}(\bar{U}_q) \). Then \( 1 = \chi_s^\pm_\pm(EF) = \sum_i \varphi_i(E)\psi_i(F) = 0 \), a contradiction.

Remark 2. If we choose a basis of \( \mathcal{Z}(\bar{U}_q) \), then its dual basis can not be entirely contained in \( \text{SLF}(\bar{U}_q) \). Indeed, let \( \varphi = \sum_{s=0}^{p} \lambda_s^+ \chi_s^\pm_\pm + \sum_{i=1}^{n-1} \mu_i G_i \in \text{SLF}(\bar{U}_q) \). Then \( \varphi(w_s^+) = s\mu_s, \varphi(w_s^-) = (p-s)\mu_s \), and we see that there does not exist \( \varphi \in \text{SLF}(\bar{U}_q) \) such that \( \varphi(w_s^+) = 1, \varphi(w_s^-) = 0 \). Hence, \( \text{SLF}(\bar{U}_q) \subset \bar{U}_q^* \) is not the dual of \( \mathcal{Z}(\bar{U}_q) \subset \bar{U}_q \).

4. Traces on projective \( \bar{U}_q \)-modules and the GTA basis

4.1. Correspondence between traces and symmetric linear forms. Let \( A \) be a finite dimensional \( k \)-algebra. We have an anti-isomorphism of algebras:

\[
A \rightarrow \text{End}_A(A), \quad a \mapsto \rho_a \text{ defined by } \rho_a(x) = xa.
\]

Observe that the right action of \( A \) naturally appears. Let \( t \) be a trace on \( A \), that is, an element of \( \text{SLF}(\text{End}_A(A)) \). Then:

\[
t(\rho_{ab}) = t(\rho_b \circ \rho_a) = t(\rho_a \circ \rho_b) = t(\rho_{ba}).
\]

So we get an isomorphism of vector spaces

\[
\{\text{Traces on } \text{End}_A(A)\} = \text{SLF}(\text{End}_A(A)) \rightarrow \text{SLF}(A)
\]

\[
t \mapsto \varphi' \text{ defined by } \varphi'(a) = t(\rho_a).
\]

whose inverse is:

\[
\text{SLF}(A) \rightarrow \{\text{Traces on } \text{End}_A(A)\} = \text{SLF}(\text{End}_A(A))
\]

\[
\varphi \mapsto t^\varphi \text{ defined by } t^\varphi(\rho_a) = \varphi(a).
\]

In the case of \( A = \bar{U}_q \), we can express \( \varphi' \) in the GTA basis, which will be the object of the next section.

Let \( \text{Proj}_A \) be the full subcategory of the category of finite dimensional \( A \)-modules whose objects are the projective \( A \)-modules.

Definition 4.1. A trace on \( \text{Proj}_A \) is a family of linear maps \( t = (t_U : \text{End}_A(U) \rightarrow k)_{U \in \text{Proj}_A} \) such that

\[
\forall f \in \text{Hom}_A(U, V), \forall g \in \text{Hom}_A(V, U), \ t_V(g \circ f) = t_U(f \circ g).
\]

We denote by \( T_{\text{Proj}_A} \) the vector space of traces on \( \text{Proj}_A \).

This cyclic property of traces on \( \text{Proj}_A \) is one of the axioms of the so-called modified traces, defined for instance in [8]. Note that this definition could be restated in the following way (and could be generalized to other abelian full subcategories than \( \text{Proj}_A \)).

Lemma 4.1. Let \( t = (t_U : \text{End}_A(U) \rightarrow k)_{U \in \text{Proj}_A} \) be a family of linear maps. Then \( t \) is a trace on \( \text{Proj}_A \) if and only if:
For every $\varphi \in \text{End}_A(U)$, $t_U(g \circ f) = t_U(f \circ g)$.

- $t_{U \oplus V}(f) = t_U(p_U \circ f \circ i_U) + t_V(p_V \circ f \circ i_V)$, where $p_U, p_V$ are the canonical projection maps and $i_U, i_V$ are the canonical injection maps.

**Proof.** If $t$ is a trace and $f \in \text{End}_A(U \oplus V)$, we have:

$$t_{U \oplus V}(f) = t_U(i_U p_U + i_V p_V)f = t_U(p_U fi_U) + t_V(p_V fi_V).$$

Conversely, let $f : U \to V$, $g : V \to U$. Define $F = i_V f p_U$, $G = i_U g p_V$. Then $FG = i_V f g p_V$ and $GF = i_U g f p_U$. We have $p_U GFi_U = gf$, $p_V GFi_V = 0$, $p_U FGi_U = 0$, $p_V FGi_V = fg$, thus:

$$t_V(fg) = t_{U \oplus V}(FG) = t_U(gf).$$

This shows the equivalence. \qed

Now, consider:

$$\Pi_A : \mathcal{T}_{\text{Proj}_A} \to \text{SLF}(\text{End}_A(A)) \xrightarrow{\sim} \text{SLF}(A)$$

with $t = (t_U)_{U \in \text{Ob}(\text{Proj}_A)} \mapsto t_A$ defined by $\varphi' = \varphi \circ t_A$. We prove:

**Theorem 4.1.** The map $\Pi_A$ is an isomorphism. In other words, $t_A$ entirely characterizes $t = (t_U)$.

**Proof.** For all the facts concerning PIMs (Principal Indecomposable Modules) and idempotents in finite dimensional $k$-algebras, we refer to [4, Chap. VIII]. We first show that $\Pi_A$ is surjective. Let:

$$1 = e_1 + \ldots + e_n$$

be a decomposition of the unit into primitive orthogonal idempotents ($e_ie_j = \delta_{ij}e_i$). Then the PIMs of $A$ are isomorphic to the left ideals $Ae_i$ (possibly with multiplicity). We have isomorphisms of vector spaces:

$$\text{Hom}_A(Ae_i, Ae_j) \xrightarrow{\sim} e_i Ae_j, \ f \mapsto f(e_i).$$

For every $\varphi \in \text{SLF}(A)$, define $t^\varphi_{Ae_i}$ by:

$$t^\varphi_{Ae_i}(f) = \varphi(f(e_i)).$$

Let $f : Ae_i \to Ae_j$, $g : Ae_j \to Ae_i$, and put $f(e_i) = e_i a_f e_j$, $g(e_j) = e_j a_g e_i$. Then using the idempotence of the $e_i$’s and the symmetry of $\varphi$ we get:

$$t^\varphi_{Ae_i}(g \circ f) = \varphi(g \circ f(e_i)) = \varphi((e_i a_f e_j)(e_j a_g e_i)) = \varphi((e_i a_g e_i)(e_j a_f e_j)) = \varphi(f \circ g(e_j)) = t^\varphi_{Ae_i}(f \circ g).$$

We know that every projective module is isomorphic to a direct sum of PIMs, so we extend $t^\varphi$ to $\text{Proj}_A$ by the following formula:

$$t^\varphi_{\oplus A}(f) = \sum_i t^\varphi_{Ae_i}(i_i \circ f \circ p_i)$$

where $p_i$ and $i_i$ are the canonical injection and projection maps. By Lemma 4.1, this defines a trace on $\text{Proj}_A$. We then show that $\Pi_A(t^\varphi) = \varphi$, proving surjectivity:
where I assign an index to the multiple factors: because Consider the following composite maps for $1 \leq j \leq n$ 

$$\Pi_A(t_{\alpha}(a)) = t_{\alpha}^{\pi}(\rho_{a}) = \sum_{j=1}^{n} t_{A_{ej}}\left(p_{j} \circ \rho_{a} \circ i_{j}\right) = \sum_{j=1}^{n} \varphi\left(p_{j} \circ \rho_{a}(e_{j})\right) = \sum_{j=1}^{n} \varphi\left(p_{ae_{j}}(e_{j},a)\right)$$

$$= \sum_{j,k=1}^{n} \varphi\left(p_{ae_{j}}(e_{j}ae_{k})\right) = \sum_{j=1}^{n} \varphi\left(e_{j}ae_{j}\right) = \sum_{j=1}^{n} \varphi\left(ae_{j}\right) = \varphi(a).$$

Note that we used that the $e_{j}$'s are idempotents and that $a = \sum_{j=1}^{n} ae_{j}$. We now show injectivity. Assume that $\Pi_A(t) = 0$. Then:

$$\forall a \in A, \quad t_{A}(\rho_{a}) = \sum_{j=1}^{n} t_{A_{ej}}(p_{j} \circ \rho_{a} \circ i_{j}) = 0.$$ 

Let $f : Ae_{j} \to Ae_{j}$, with $f(e_{j}) = e_{j}ae_{j}$. Since $\rho_{j(f(e_{j})}(e_{i}) = \delta_{i,j}e_{j}ae_{j}$, we have $p_{j} \circ \rho_{j(f(e_{j})} \circ i_{j} = f$ and $p_{i} \circ \rho_{j(f(e_{j})} \circ i_{i} = 0$ if $i \neq j$. Hence:

$$t_{A_{ej}}(f) = t_{A}(\rho_{j(f(e_{j})}) = 0.$$ 

Then $t_{A_{ej}} = 0$ for each $j$, so that $t = 0$. 

\[\Box\]

**4.2. Link with the GTA basis.** We leave the general case and focus on $A = \bar{U}_{q}$. The following theorem expresses $\Pi_{\bar{U}_{q}}$ in the GTA basis.

**Theorem 4.2.** Let $t = (t_{U_{q}})_{U_{q} \in \text{Proj}_{U_{q}}}$ be a trace on $\text{Proj}_{U_{q}}$. Then:

$$\Pi_{\bar{U}_{q}}(t) = t_{X^{+}(p)}(\text{Id})X_{e_{j}}^{+} + \sum_{j=1}^{p-1} \left(t_{p^{-}(s)}(\text{Id})X_{e_{j}}^{+} + t_{p^{-}(s)}(\text{Id})X_{e_{j}}^{-} + t_{p^{-}(s)}(p_{e_{j}}^{+})G_{s}\right).$$

Proof. First of all, we write the decomposition of the left regular representation of $\bar{U}_{q}$, assigning an index to the multiple factors:

$$\bar{U}_{q} = \bigoplus_{s=1}^{p-1} \left( \bigoplus_{j=0}^{r-1} P_{j}^{+}(s) \oplus P_{j}^{-}(s) \right) \oplus \bigoplus_{j=0}^{p-1} X_{e_{j}}^{+}(p) \oplus X_{e_{j}}^{-}(p).$$

Thus, since $t$ is a trace:

$$t_{\bar{U}_{q}}(\rho_{a}) = \sum_{s=1}^{p-1} \left( \sum_{j=1}^{r-1} t_{p^{-}(s)}(p_{p_{e_{j}}^{+}}(s) \circ \rho_{a} \circ i_{p_{e_{j}}^{+}}(s)) + t_{p^{-}(s)}(p_{p_{e_{j}}^{-}}(s) \circ \rho_{a} \circ i_{p_{e_{j}}^{-}}(s)) \right)$$

$$+ \sum_{j=0}^{p-1} t_{X_{e_{j}}^{+}(p)}(p_{X_{e_{j}}^{+}(p)} \circ \rho_{a} \circ i_{X_{e_{j}}^{+}(p)}) + t_{X_{e_{j}}^{-}(p)}(p_{X_{e_{j}}^{-}(p)} \circ \rho_{a} \circ i_{X_{e_{j}}^{-}(p)}).$$

Consider the following composite maps for $1 \leq s \leq p - 1$ (note that the blocks appear because $\rho_{a}$ is the right multiplication by $a$):

$$h_{s,ja}^{+} : P^{+}(s) \xrightarrow{I_{s}^{+}} P^{+}_{j}(s) \xrightarrow{p_{e_{j}}^{+}} Q(s) \xrightarrow{p_{a}} Q(s) \xrightarrow{p_{p_{e_{j}}^{+}}(s)} P^{+}_{j}(s) \xrightarrow{(I_{s}^{+})^{-1}} P^{+}(s),$$

$$h_{s,ja}^{-} : P^{-}(s) \xrightarrow{I_{s}^{-}} P^{-}_{j}(s) \xrightarrow{p_{e_{j}}^{-}} Q(p-s) \xrightarrow{p_{a}} Q(p-s) \xrightarrow{p_{p_{e_{j}}^{-}}(s)} P^{-}_{j}(s) \xrightarrow{(I_{s}^{-})^{-1}} P^{-}(s),$$

where $I_{s}^{+}$ and $I_{s}^{-}$ are the isomorphisms defined by (see Proposition 2.1):
For $s = p$, consider:

$$h_{p,j}^+(x_+(s)) = \lambda_+(p) \xrightarrow{\iota_{X_+}^+(p)} X_+^+(p) \xrightarrow{\rho_a} Q(p) \xrightarrow{p_{\lambda_+}^+(p)} X_+^+(p) \xrightarrow{(t_{\rho_+})^{-1}} \lambda_+(p),$$

$$h_{p,j}^-(x_-(s)) = \lambda_-^+(p) \xrightarrow{\iota_{X_-}^+(p)} X_-^+(p) \xrightarrow{\rho_a} Q(p) \xrightarrow{p_{\lambda_-}^+(p)} X_-^+(p) \xrightarrow{(t_{\rho_-})^{-1}} \lambda_-^+(p)$$

where $I_{p,j}^+$ and $I_{p,j}^-$ are the isomorphisms defined by (see Proposition 2.1):

$$I_{p,j}^+(v_i^+(p)) = A_{ij}^+(p)$$ and $$I_{p,j}^-(v_i^-(p)) = A_{ij}^-(p).$$

Then for $1 \leq s \leq p - 1$:

$$t_{p_j}^+(p_{\lambda_+}^+(p) \circ \rho_a \circ i_{X_+}^+(p)) = t_{p_j}^+(p_{\lambda_+}^+(p))$$

and for $s = p$:

$$t_{X_-}^+(p_{\lambda_-}^+(p) \circ \rho_a \circ i_{X_-}^+(p)) = t_{X_-}^+(p_{\lambda_-}^+(p)).$$

We must determine the endomorphism $h_{s,j,a}^+$ when $a$ is replaced by the elements given in Proposition 3.1. Using (8), we get:

$$\forall s' \neq s, \forall j, h_{s',j,a}^+ = 0 \quad \text{and} \quad h_{s',j,a}^- = 0$$

and:

$$\forall j, h_{s,j,a}^+ = p_s^+.$$
Finally, in the case where \( s = p \):
\[
\forall s' \neq p, \forall j, \quad h^+_{s', j, \Phi_{p-1}} = 0 \quad \text{and} \quad h^-_{p, j, \Phi_{p-1}} = 0.
\]

Then, Proposition 2.1 together with Lemma 3.1 gives:
\[
\forall 0 \leq j \leq p - 2, \quad h^+_{p, j, \Phi_{p-1}} = 0 \quad \text{and} \quad h^+_{p, p-1, \Phi_{p-1}} = \text{Id}.
\]

It follows that \( t_{U_q}(\rho_{\Phi_{p-1}}) = t_{\chi^+(p)}(\text{Id}) \). So by Proposition 3.1, the coefficient of \( \chi^+_p \) is \( t_{\chi^+(p)}(\text{Id}) \). One similarly gets the coefficient of \( \chi^-_p \).

By Proposition 3.1, the coefficient of \( G_s \) is also given by:
\[
\frac{1}{p-s} t_{U_q}(\rho_{w_i}).
\]

Taking back the notations of the proof above, we see using (8) that
\[
\forall s' \neq p - s, \forall j, \quad h^+_{s', j, w_i} = 0 \quad \text{and} \quad h^-_{p-s, j, w_i} = 0
\]

and:
\[
\forall j, \quad h^-_{p-s, j, w_i} = p_{-s}.
\]

Since this does not depend on \( j \) and since the block \( Q(s) \) contains \( p - s \) copies of \( P^-(p-s) \), we find that \( t_{U_q}(\rho_{w_i}) = (p-s)t_{p-(p-s)}(p^-_{p-s}) \). So by Proposition 3.1, the coefficient of \( G_s \) is \( t_{p-(p-s)}(p^-_{p-s}) \). We thus have:
\[
(14) \quad t_{p-(p-s)}(p^-_{p-s}) = t_{\chi^+(s)}(p^+_s).
\]

Note that there is an elementary way to see this. Indeed, the morphisms \( P^+_s \) and \( P^-_{p-s} \) defined in (5) satisfy:
\[
P^-_{p-s} \circ P^+_s = p^+_s, \quad P^+_s \circ P^-_{p-s} = p^-_{p-s}.
\]

Hence, we recover (14) by property of the traces. From this, we deduce the following corollary.

**Corollary 4.1.** Let
\[
\varphi = \sum_{s=1}^{p} (\chi^+_s \chi^-_s + \chi^-_s \chi^+_s) + \sum_{j=1}^{p-1} \mu_j G_s \in \text{SLF}(\bar{U}_q).
\]

Then the trace \( t^\varphi = \Pi_{U_q}^1(\varphi) \) associated to \( \varphi \) is given by:
\[
t^\varphi_{\chi^+(p)}(\text{Id}) = \lambda^+_p, \quad t^\varphi_{\chi^-(p)}(\text{Id}) = \lambda^-_p, \quad t^\varphi_{\chi^+(s)}(p^+_s) = t^\varphi_{\chi^-(s)}(p^-_{p-s}) = \mu_s.
\]

### 4.3. Symmetric linear form corresponding to the modified trace on \( \text{Proj}_{U_q} \)

Let \( H \) be a finite dimensional Hopf algebra. Let us recall that a *modified trace* \( t \) on \( \text{Proj}_H \) is a trace which satisfies the additional property that for \( U \in \text{Proj}_H \), for each \( H \)-module \( V \) and for \( f \in \text{End}_H(U \otimes V) \) we have:
\[
t_{U \otimes V}(f) = t_U(\text{tr}_R(f))
\]
where \( \text{tr}_R = \text{Id} \otimes \text{tr}_q \) is the right partial quantum trace (see [8, (3.2.2)]). These modified traces are actively studied, having for motivation the construction of invariants in low dimensional
topology. We refer to [8] for the general theory in a categorical framework which encapsulates the case of \(\text{Proj}_H\).

In [3], it is shown that there exists a unique up to scalar modified trace \(t = (t_u)\) on \(\text{Proj}_{\hat{U}}\). Uniqueness comes from the fact that \(\mathcal{X}(p)\) is both a simple and a projective module. The values of this trace are given by:

\[
\begin{align*}
\mu_{(p)}(\text{Id}) &= (-1)^{p-1}, \\
\mu_{(-p)}(\text{Id}) &= 1, \\
\mu_{-p^\ast}(\text{Id}) &= (-1)^s(q^r + q^{-r}), \\
\mu_{p^\ast}(\text{Id}) &= (-1)^s(q^r + q^{-r}).
\end{align*}
\]

Using formulas given in [9] (see also [1] 2), we have (1

\[
\sum_{s=0}^{p-1} q^{(s,2r-1)} F^{p-1} E^{p-1} K^l + (\text{terms of lower degree in } E \text{ and } F),
\]

\[
\sum_{s=0}^{p-1} q^{(s,2r-1)} F^{p-1} E^{p-1} K^l + (\text{terms of lower degree in } E \text{ and } F),
\]

\[
e_p = \frac{1}{2p[p-1]^2} \sum_{t=0}^{p-2} \sum_{l=0}^{p-1} q^{-(s-2r-1)} F^{p-1} E^{p-1} K^l + (\text{terms of lower degree in } E \text{ and } F),
\]

where \(\alpha_s\) is given in the last page of [1] as:

\[
\alpha_s = \frac{(-1)^{p-s-1}}{2p[p-s-1]^2[s-1]^2} \left( \sum_{l=1}^{s-1} \frac{1}{[l][s-l]} - \sum_{l=1}^{p-s-1} \frac{1}{[l][p-s-l]} \right).
\]

In order to simplify this, it is observed in [13, Proof of Proposition 2], that

\[
\sum_{l=1}^{s-1} \frac{1}{[l][s-l]} - \sum_{l=1}^{p-s-1} \frac{1}{[l][p-s-l]} = \frac{-(q^r + q^{-r})}{[s]^2}.
\]

So, since:

\[
[p-s-1]^2[s-1]^2 = \frac{(p-1)_{2}}{[s]^2},
\]

In notations of [1], we have \(e_s = \sum_{u=1}^{s} e^u(s,t) + \sum_{u=s+1}^{p} e^u(p-s,u).\)
we get:
\[ \alpha_s = \frac{(-1)^{p-s-1}}{2p[p-1]!^2}(q^s + q^{-s}). \]

Using formulas given in [6] (see also [10, Prop. II.3.19]), we have:
\[ w_+^s = \frac{(-1)^{p-s-1}}{2p[p-1]!^2}[s]^2F^{p-1}E^{p-1} + \text{(other monomials)}, \]
\[ w_-^s = \frac{(-1)^{p-s-1}}{2p[p-1]!^2}(p-s)F^{p-1}E^{p-1} + \text{(other monomials)}. \]

We now use Proposition 3.1 to get the coefficients of \( \mu_{\zeta}(K^p\wedge \cdot) \) in the GTA basis. For instance:
\[ \mu_{\zeta}(K^p\wedge \cdot) = \frac{\alpha_s}{2p} \mu_{\zeta} \left( [s]^2 \sum_{t=0}^{p-1} \sum_{l,j=0} q^{-(s-1)(l+j)+2tl} k^{l+j} \right) \]
\[ = \zeta \frac{\alpha_s}{2p} \sum_{t=0}^{p-1} \sum_{l=0} q^{2tl} = \zeta \alpha_s. \]

Choose the normalization factor to be \( \zeta = (-1)^{p-1}2p[p-1]^2 \), and let \( \mu \) be the so-normalized integral. Then:
\[ \mu(K^p\wedge \cdot) = (-1)^{p-1}\chi_p^+ + \chi_p^- + \sum_{s=1}^{p-1} \left( (-1)^s(q^s + q^{-s})\chi_s^+ + (-1)^{p-s-1}(q^s + q^{-s})\chi_s^- \right) \]
\[ + (-1)^s[s]^2G_s. \]

By Theorem 4.2, we recover \( \Pi_{U_q}(t) = \mu(K^p\wedge \cdot) \).

5. Multiplication rules in the GTA basis

We mentioned in section 3 that SLF(\( \tilde{U}_q \)) is a commutative algebra. In this section, we address the problem of the decomposition in the GTA basis of the product of two elements in this basis. The resulting formulas are surprisingly simple.

Let us start by recalling some facts. For every \( \tilde{U}_q \)-module \( V \), we define the character of \( V \) as (see (11) for the definition of \( T \)):
\[ \chi^V = \text{tr}(T). \]

This splits on extensions:
\[ 0 \to V \to M \to W \to 0 \implies \chi^M = \chi^V + \chi^W. \]

Due to the fact that \( \tilde{U}_q \) is finite dimensional, every finite dimensional \( \tilde{U}_q \)-module has a composition series (i.e. is constructed by successive extensions by simple modules). It follows that every \( \chi^V \) can be written as a linear combination of the \( \chi^V_{\alpha} = \chi^{V_{\alpha}(\cdot)} \). Moreover, we see by definition of the product on \( \tilde{U}_q^* \) that
Thus we can deduce without any computation that $\chi^W = \chi^V \chi^W$. Hence multiplying two $\chi$'s is equivalent to tensoring two simple modules and finding the decomposition into simple factors. This means that

$$\text{vect}(\chi_s^a)_{a \in \{1\}, 1 \leq s \leq p} \rightarrow \mathfrak{g}(\bar{U}_q) \otimes \mathbb{C}, \chi^I \mapsto [I]$$

where $\mathfrak{g}(\bar{U}_q)$ is the Grothendieck ring of $\bar{U}_q$. By [6], we know the structure of $\mathfrak{g}(\bar{U}_q)$. Recall the decomposition formulas (with $2 \leq s \leq p - 1$):

$$\mathcal{X}(1) \otimes \mathcal{X}(s) \cong \mathcal{X}(s-1) \oplus \mathcal{X}(s+1), \mathcal{X}(2) \otimes \mathcal{X}(p) \cong \mathcal{P}(p-1)$$

so that

$$\chi_s^a \chi_t^b = \chi_s^{a+b}, \chi_s^a \chi_{s+1}^a = \chi_s^{a+1}, \chi_s^a \chi_{p-1}^a = 2 \chi_{p-1}^a + 2 \chi_1^{-a}. \tag{16}$$

We see in particular that $\chi_s^2$ generates the subalgebra $\text{vect}(\chi_s^a)_{a \in \{1\}, 1 \leq s \leq p}$. The $\chi_s^a$ are expressed as Chebyschev polynomials of $\chi_s^2$, see [6, section 3.3] for details.

**Theorem 5.1.** The multiplication rules in the GTA basis are entirely determined by (16) and by the following formulas:

$$\chi_s^2 G_1 = [2]G_2, \tag{17}$$

$$\chi_s^2 G_s = \frac{[s-1]}{[s]} G_{s-1} + \frac{[s+1]}{[s]} G_{s+1} \text{ for } 2 \leq s \leq p - 2, \tag{18}$$

$$\chi_s^2 G_{p-1} = [2]G_{p-2}, \tag{19}$$

$$\chi_s^2 G_s = -G_{p-s} \text{ for all } s, \tag{20}$$

$$G_s G_t = 0 \text{ for all } s, t. \tag{21}$$

Before giving the proof, let us deduce a few consequences.

**Corollary 5.1.** For all $1 \leq s \leq p - 1$ we have:

$$G_s = \frac{1}{[s]} \chi_s^2 G_1, \quad \chi_s^2 G_1 = 0.$$  

It follows that $\chi_s^2 + \chi_{p-s}^2 G_1 = 0$, and that $\mathcal{V} = \text{vect}(\chi_s^2 + \chi_{p-s}^2, \chi_s^2, \chi_{p-s}^2)_{1 \leq s \leq p-1}$ is an ideal of $\text{SLF}(U_q)$.

Proof of Corollary 5.1. The formulas for $\chi_s^2 G_1$ are proved by induction using $\chi_{s+1}^2 = \chi_s^2 \chi_1^2 - \chi_{s+1}^2$ together with formula (18). We deduce:

$$(\chi_s^2 + \chi_{p-s}^2) G_t = \frac{\chi_t^2}{[t]} (\chi_1^2 G_1 + \chi_{p-s}^2 G_1) = \frac{\chi_t^2}{[t]} ([s] G_s + [s] \chi_1 G_{p-s}) = 0.$$  

It is straightforward that $\mathcal{V}$ is stable by multiplication by $\chi_s^2$, so it is an ideal. $\square$

**Remark 3.** We have $\chi_s^{2n(s)} = 2(\chi_s^a + \chi_{p-s}^a)$ for $1 \leq s \leq p - 1$. Thus $\mathcal{V}$ is generated by characters of the projective modules. It is well-known that if $H$ is a finite dimensional Hopf algebra, then the full subcategory of finite dimensional projective $H$-modules is a tensor ideal. Thus we can deduce without any computation that $\mathcal{V}$ is stable under the multiplication.
by every \( \chi^l \).

We now proceed with the proof of the theorem. Observe that we cannot apply Proposition 3.1 to show it since we do not know expressions of \( \Delta(e_s) \) and \( \Delta(w^+_s) \) which are easy to evaluate in the GTA basis. Recall ([12], see also [10]) the following fusion rules:

\[
\begin{align*}
(22) & \quad \chi^+(1) \otimes P^a(s) \equiv P^{-a}(s) \quad \text{for all } s, \\
(23) & \quad \chi^+(2) \otimes P^a(1) \equiv 2\chi^{-a}(p) \oplus P^a(2), \\
(24) & \quad \chi^+(2) \otimes P^a(s) \equiv P^a(s-1) \oplus P^a(s+1) \quad \text{for } 2 \leq s \leq p-1, \\
(25) & \quad \chi^+(2) \otimes P^a(p-1) \equiv 2\chi^a(p) \oplus P^a(p-2).
\end{align*}
\]

They imply the following key lemma.

**Lemma 5.1.** There exist scalars \( \gamma_s, \beta_s, \lambda_s, \eta_s, \delta_s \) such that

\[
\chi^+_s G_s = \beta_s G_{s-1} + \gamma_s G_{s+1} + \lambda_s \left( \chi^+_s - \chi^+_{s+1} - \chi^+_{s-1} \right) \quad \text{(for } 2 \leq s \leq p-2),
\]

\[
\chi^+_s G_1 = \gamma_1 G_2 + \lambda_1 \left( \chi^+_0 - \chi^+_2 - \chi^+_1 \right), \quad \chi^+_s G_{p-1} = \beta_{p-1} G_{p-2} + \lambda_{p-1} \left( \chi^+_2 - \chi^+_1 \right),
\]

\[
\chi^-_1 G_s = \eta_s G_{p-s} + \delta_s \left( \chi^+_s - \chi^-_s \right).
\]

Proof. Let us fix \( 2 \leq s \leq p-2 \); by (12), (13), (15) and (24) we have:

\[
\chi^+_s G_s \in \text{vec} \left( \chi^+_s \otimes P^a(s) \chi^+_s \otimes P^a(s) \chi^+_s \otimes P^a(s) \chi^+_s \otimes P^a(s) \right)_{ij} = \text{vec} \left( \chi^+_s \otimes P^a(s) \chi^+_s \otimes P^a(s) \chi^+_s \otimes P^a(s) \chi^+_s \otimes P^a(s) \right)_{ij}
\]

where \( T_{ij} \) is the matrix element at the \( i \)-th row and \( j \)-th column of the representation matrix \( V \) and \( T_{ijkl} \) is the matrix element at the \( (i, j) \)-th row and \( (k, l) \)-th column of the representation matrix \( V \). Hence, since \( \chi^+_s G_s \) is symmetric, it is necessarily of the form

\[
\chi^+_s G_s = \beta_s G_{s-1} + \gamma_s G_{s+1} + z_1 \chi^+_s + z_2 \chi^+_{s+1} + z_3 \chi^+_{s-1} + z_4 \chi^+_{s-1}.
\]

Evaluating this equality on \( K \) and \( K^2 \), we find (since \( G_i(K^2) = 0 \) for all \( i \) and \( j \))

\[
[s-1](z_1 - z_3) + [s+1](z_2 - z_4) = 0, \quad [s-1]q^2(z_1 - z_3) + [s+1]q^2(z_2 - z_4) = 0,
\]

with \( [n]_q = 2 \sin((s+1)\pi/p) \sin((s+1)\pi/p) - \cos((s+1)\pi/p) \cos((s-1)\pi/p) \neq 0 \). Hence \( z_1 = z_3, z_2 = z_4 \). Moreover, evaluating the above equality on 1, we find \( p(z_1 + z_2) = 0 \). Letting \( \lambda_s = z_1 \), the result follows. The other formulas are obtained in a similar way using (22), (23) and (25).

\[ \square \]

We will use the Casimir element \( C \) of \( U_q \) to make computations easier. It is defined by:

\[
C = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} = \sum_{j=0}^p c_j e_j + \sum_{k=1}^{p-1} (w_k^+ + w_k^-) \in \mathcal{Z}(U_q)
\]
where \( c_j = \frac{q^j - q^{-j}}{(q - q^{-1})^j} \). The second equality is obtained by considering the action of \( C \) on the \( \text{PIMs} \) \( P^0(s) \). Observe that

\[
\forall x \in \bar{U}_q, \quad \chi_a^t(Cx) = ac_x \chi_a^t(x), \quad G_s(Cx) = c_s G_s(x) + (\chi_s^+ + \chi_{p-s}^-)(x).
\]

Then by induction we get \( G_s(C^n) = n p c_{s-1} \) for \( n \geq 1 \). We will also denote \( c_K = \frac{q^{K+1} - q^{-K}}{q - q^{-1}} \).

**Proof of Theorem 5.1.** Formula (18). We first evaluate the corresponding formula of Lemma 5.1 on \( FE \). It holds \( G_s(FE) = G_s(C) = p, (\chi_s^+ + \chi_{p-s}^-)(FE) = (\chi_s^+ + \chi_{p-s}^-)(C) = p c_t \) for all \( t \) and \( \chi_s^2 G_s(FE) = \chi_s^2(K^{-1})G_s(FE) = [2]p \). Thus we get:

\[
\beta_s + \gamma_s + (c_{s-1} - c_{s+1}) \lambda_s = \beta_s + \gamma_s - [s] \lambda_s = [2].
\]

Next, we evaluate the formula of Lemma 5.1 on \((FE)^2\). On the one hand,

\[
\chi_s(G_s)((FE)^2) = \chi_s^2(K^{-2})G_s((FE)^2) = \chi_s^2(K^{-2})G_s(C^2 - 2 C c_K + c_K^2) = \chi_s^2(K^{-2})G_s(C^2) = 2 p (q^2 + q^{-2}) c_s.
\]

For the first equality, we used the fact that \( \varphi(E^l F^l K^l) = \delta_{l,s} \varphi(E^l F^l K^l) \) for all \( \varphi \in \text{SLF}(\bar{U}_q) \), that \( G_s(K^l) = 0 \) and that \( G_s(FEK^l) = 0 \) for \( 1 \leq l \leq p - 1 \). The third equality is due to (26) and to the fact that \( (\chi_s^+ + \chi_{p-s}^-)(K^l) = 0 \) for \( 1 \leq l \leq p - 1 \). On the other hand, using again the Casimir element,

\[
\beta_s G_{s-1}((FE)^2) + \gamma_s G_{s+1}((FE)^2) + \lambda_s (\chi_{s-1}^+ + \chi_{p-s+1}^- - \chi_{s+1}^+ - \chi_{p-s-1}^-)(FE)^2
\]

\[
= \beta_s G_{s-1}(C^2) + \gamma_s G_{s+1}(C^2) + \lambda_s (\chi_{s-1}^+ + \chi_{p-s+1}^- - \chi_{s+1}^+ - \chi_{p-s-1}^-)(C^2)
\]

\[
= 2 p c_{s-1} \beta_s + 2 p c_{s+1} \gamma_s + p (c_{s-1}^2 - c_{s+1}^2) \lambda_s.
\]

Since \( c_{s-1}^2 - c_{s+1}^2 = -(q + q^{-1}) c_s [s] \), we get

\[
(28) \quad 2 c_{s-1} \beta_s + 2 c_{s+1} \gamma_s - (q + q^{-1}) c_s [s] \lambda_s = 2 (q^2 + q^{-2}) c_s.
\]

In order to get a third linear equation between \( \beta_s, \gamma_s \), and \( \lambda_s \), we use evaluation on \( E^{p-1} F^{p-1} \). This has the advantage to annihilate all the \( \chi^{t,a}_s \) appearing in the formula of Lemma 5.1. First:

\[
E^{p-1} F^{p-1} b_0^a(s) = E^{p-1} y_{p-s+1}^s(s) = (-\alpha)^{p-s+1} [p - s - 1]^2 E^{p-1} y_{p-s+1}^s(s)
\]

\[
= (-\alpha)^{p-s+1} \alpha^{p-s} [p - s - 1]^2 [s - 1]^2 \alpha^a_0(s)
\]

\[
= (-\alpha)^{p-s+1} \alpha^{p-s} \frac{(p - 1)!}{[s]!^2} d_0^a(s)
\]

and \( E^{p-1} F^{p-1} \) annihilates all the other basis vectors. Hence:

\[
G_s(E^{p-1} F^{p-1}) = 2(-1)^{p-s-1} \frac{(p - 1)!}{[s]!^2}.
\]

Next by (1), we have:

\[
\chi_s^2 \otimes \text{Id} \left( \Delta(E^{p-1} F^{p-1}) \right) = [2] E^{p-1} F^{p-1} - q^2 E^{p-2} F^{p-2} K.
\]

As in (29), we find:

\[
E^{p-2} F^{p-2} K b_0^a(s) = (-\alpha)^{p-s} \alpha^{p-s} \frac{(p - 1)!}{[s + 1]! [s]!} d_0^a(s),
\]
5.1 on $FE$ \(\leq (30)\) and thus:

\[
\text{and thus:} \\
\beta_s = \frac{\gamma_s}{[s + 1]^2} = \frac{[2]}{[s - 1][s + 1]}.
\]

As a result, we have a linear system (27)–(28)–(30) between \(\beta_s, \gamma_s, \text{ and } \lambda_s\). It is easy to check that \(\beta_1 = \frac{[s - 1]}{[s + 1]}, \gamma_1 = \frac{[s + 1]}{[s - 1]}, \lambda_0 = 0\) is a solution. Moreover this solution is unique. Indeed, a straightforward computation reveals that

\[
\det \begin{pmatrix} 1 & 1 & -[s] \\ 2c_{s-1} & 2c_{s+1} & -(q + q^{-1})c_s[s] \\ \frac{1}{[s - 1]^2} & \frac{1}{[s + 1]^2} & 0 \end{pmatrix} = \frac{[s]^2}{[s - 1]^2} + \frac{[s]^2}{[s + 1]^2} > 0.
\]

- **Formulas (17) and (20).** Evaluating as above the corresponding formulas of Lemma 5.1 on $FE$ and $(FE)^2$, one gets linear systems with non-zero determinants. It is then easy to see that $\beta_1 = [2], \lambda_1 = 0$ and $\eta_s = -1, \delta_s = 0$ are the unique solutions of each of these two systems.

- **Formula (19).** It can be deduced from the formulas already shown:

\[
\]

- **Formula (21).** Recall the isomorphism of algebras $D$ defined in (10). Taking into account that \(\varphi(K^n F^m E^n) = 0\) if \(n \neq m\) for any \(\varphi \in \text{SL}(\bar{U}_n)\) and that \(G_s(K^n) = 0\) for all \(i\), and making use of the expression of $RR'$ given in [6], we get:

\[
D(G_s) = \sum_{n=0}^{p-1} \sum_{j=0}^{2p-1} \left( \sum_{i=0}^{2p-1} \frac{(q - q^{-1})^n}{[n]^2} q^{n(j-i-1)-i} G_s(K^{p+i+1} F^n E^n) \right) K^j F^n E^n
\]

\[
= \sum_{n=1}^{p-1} \sum_{j=0}^{2p-1} \lambda_{jn} K^j F^n E^n
\]

for some coefficients \(\lambda_{jn}\) (observe that \(n \geq 1\)). From this it follows that for all \(\alpha \in \{\pm\} \text{ and } 1 \leq r \leq p - 1\): $D(G_s) b^n_r \in \text{C}(\alpha_0^r (r))$. By (7), we deduce that $D(G_s) \in \text{vect}(w^n_r)_{1 \leq r \leq p-1}$ for all \(s\). Thus $D(G_s, G_t) = 0$, thanks to (9).

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5.1.

References


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