Title	SMOOTHING OF SEMISTABLE FANO VARIETIES
Author(s)	Shentu, Junchao
Citation	Osaka Journal of Mathematics. 2020, 57(3), p. 617-645
Version Type	VoR
URL	https://doi.org/10.18910/76676
rights	
Note	

Shentu, J.
Osaka J. Math.
57 (2020), 617–645

SMOOTHING OF SEMISTABLE FANO VARIETIES

Junchao SHENTU

(Received June 16, 2017, revised March 22, 2019)

Abstract

In this paper we study the smoothing of a semistable Fano variety over a perfect field k. In characteristic 0, the reduced semistable Fano degenerate fibers of Mori fibrations are classified. In positive characteristic, under a suitable W_2 lifting assumption, we prove that a semistable Fano variety always appears as a degenerate fiber in a semistable family if it has a global log structure (in the sense of Fontaine-Illusie-Kato) of semistable type. A geometric criterion for the existence of a log structure of semistable type is given.

1. Introduction

In this paper we investigate the possible reduced semistable degenerate fibers of a smooth Fano family. The investigation is motivated by the following two problems:

(1) Given a projective algebraic variety X (defined over the field of complex numbers) whose Kodaira dimension is $-\infty$. Then conjecturally there is a modification $X' \to X$ and a fibration $f : X' \to B$ (Mori fibration) such that X' is smooth projective and the general fibers of f are smooth Fano varieties. Analogous to Kodaira's theory of degenerations of elliptic curves, the geometry of the singular fibers (usually non-normal) of f have great influence on the birational geometry of X. The simplest possible non-normal singularities in the fibers of f are the semistable singularities (those which are analytically isomorphic to a product of normal crossing singularities). Conjecturally ([1, Conjecture 0.2]), after a finite base change and a birational modification, the family can always be brought to a semistable family ([1]). Hence it's natural to ask about what kind of Fano varieties with semistable singularities appears in a semistable family with Fano general fibers.

(2) Let X be a projective smooth Fano variety over a local field K of mixed characteristic. One can not hope generally that X has a smooth model even after a base field extension. However, it is conjectured that after a finite extension $K \subseteq L$, X_L would have a semistable model, i.e., there exists a semistable family $\mathscr{X}$ over O_L whose generic fiber is isomorphic to X_L. Therefore it is natural to ask about what kind of Fano varieties with semistable singularities appears to be the central fiber of a semistable model of a Fano manifold defined over K.

The degeneration of Fano manifolds is studied by T. Fujita [6], who gives a complete list of the reducible singular fibers in a (minimal) family of del Pezzo manifolds over an algebraic

2010 Mathematics Subject Classification. 13D10, 14F17.

The author is supported by the National Natural Science Foundation of China for General Program [Grant No. 11771086].

curve (A Fano manifold X is called del Pezzo if $\mathrm{ind}X = \dim X - 1$). Later, Y. Kachi [12] proves that all d-semistable normal crossing del Pezzo surfaces are contained in Fujita's list by showing that each d-semistable del Pezzo surface has a smoothing. This smoothability property is generalized to arbitrary dimensional d-semistable normal crossing Fano varieties defined over an algebraically closed field of characteristic zero by N. Tziolas [25]. Since any family over a curve can always be modified into a family with normal crossing fibers ([21]), Tziolas's work classifies the simplest Fano degenerate fibers of a 1-parameter family of Fano manifolds.

However the Mori fibrations generally have higher dimensional bases, in which case the normal crossing singularities are no longer enough for the degenerate fibers ([13]). The same problem appears in the moduli problem of varieties, while the base space of a family is usually of higher dimension. The simplest example in which the degeneration can not be normal crossing is the 2-parameter family of surfaces defined by $t_1 = x_1y_1$ and $t_2 = x_2y_2$. Conjecturally ([1, Conjecture 0.2]), the best singularities that one can hope for is the semistable singularities (étale locally a product of normal crossing singularities) and any geometric generic integral family can be modified into a semistable family. Given pairs (X, D_X) and (Y, D_Y) such that X, Y are smooth and D_X, D_Y are normal crossing divisors. A log morphism $f : (X, D_X) \to (Y, D_Y)$ (A morphism $f : X \to Y$ such that $f^{-1}(D_Y) \subseteq D_X$) is called semistable if formal locally f is isomorphic to the spectrum of

$$k[[y_1, \dots, y_r]] \to k[[x_1, \dots, x_n]],$$

$$y_i \mapsto x_{l_{i-1}+1} + \cdots + x_{l_i},$$

where $0 = l_0 < l_1 < \cdots < l_k \leq n$. Here $k \leq r$, $D_Y = \{y_1 \cdots y_k = 0\}$ and $D_X = \{x_1 \cdots x_{l_k} = 0\}$.

In the first part of this paper, we study the problem about which kind of semistable Fano varieties may appear in a semistable family whose general fibers are smooth Fano varieties. We work over a perfect field and allow the degenerate fibers to have self intersections. The main results are:

Theorem 1.1 (Corollary 4.4). *Let k be a field of characteristic 0 and X be a Fano semistable variety. Let $r \geq 1$ be an integer. Then the followings are equivalent:*

(1) *there exists a smooth variety with a normal crossing divisor $(\mathscr{X}, D_{\mathscr{X}})$ which is semistable* (Definition 2.1) *over (B, D_B) such that*
   (a) *B is an r-dimensional smooth variety over k, $0 \in B$ is a k-point, and D_B is a simple normal crossing divisor whose number of branches at* 0 *is r;*
   (b) *$\mathscr{X}_0 \simeq X$ as log varieties.*

(2) *X has a log structure (in the sense of Kato-Fontaine-Illusie) of semistable type over* $(k, \mathbb{N}^r \mapsto 0)$ (Definition 2.3).

A variety with semistable singularities is called 'Fano' if its dualizing sheaf is an anti-ample invertible sheaf. Since ampleness is open in family, the fibers near X in Theorem 1.1 are automatically Fano.

For the readers who are not familiar with log geometry in the sense of Kato-Fontaine-Illusie, the condition that a semistable variety has a log structure of semistable type is a global condition on how the components of the variety intersect each other. For example, if X is a union of two smooth components X_1 and X_2 which intersect transversely along a smooth variety D. Then X has a log structure of semistable type if and only if $N_{X_1/D} \otimes N_{X_2/D}$ is a trivial line bundle on D (d-semistable condition in [5]).

Theorem 1.2 (Corollary 4.5)**.** *Let k be a perfect field of characteristic $p > 0$ and X be a log variety which is semistable log smooth over $(k, \mathbb{N}^r \mapsto 0)$ (Definition 2.3) for some $r \geq 0$. Assume that $\dim X < p$. If X is Fano and admits a log smooth lifting over $(W_2(k), \mathbb{N}^r \mapsto 0)$, then there exists a smooth variety with a normal crossing divisor $(\mathscr{X}, D_{\mathscr{X}})$ which is semistable over (B, D_B) such that*

(1) *B is an r-dimensional smooth variety over k, $0 \in B$ is a k-point, and D_B is a simple normal crossing divisor whose number of branches at 0 is r;*
(2) *$\mathscr{X}_0 \simeq X$ as log varieties.*

Moreover if $r = 1$ (i.e., X is d-semistable in the sense of [5]), then X appears in a semistable reduction over the ring of Witt vectors $W(k)$.

The second part of this paper is devoted to a geometric description of the existence of a log structure of semistable type. When the variety admits only normal crossing singularities, this is Friedman's d-semistable condition ([5]) on the first tangent sheaf of X. In fact, a normal crossing variety admits a log structure of semistable type over $(k, \mathbb{N} \mapsto 0)$ if and only if its first tangent sheaf T^1_X is trivial ([20], [14]). When X has general semistable singularities, some new phenomena appear:

- Even when X has only simple normal crossing singularities, the log structure of semistable type may exist for $r \neq 1$. This log structure is induced from an embedding of X into a virtually r-dimensional family and is different from the log structure constructed in [20] and [14] when $r = 1$. As a simple example illustrating this phenomenon, let us consider the case when M is a smooth variety and D_i, $i = 1, 2, 3$ are connected smooth divisors of M such that $D_1 \cup D_2 \cup D_3$ is normal crossing. Let $X = D_3 \cap (D_1 \cup D_2)$, then X is a normal crossing variety with two components $X_i = D_3 \cap D_i$, $i = 1, 2$. X admits a log structure $\mathscr{M}_X$ by restricting the log structure on M induced by $D_1 \cup D_2 \cup D_3$ on X. Such log structure has $\mathbb{N}^2$ on each component, i.e., $\overline{\mathscr{M}_X}|_{X \setminus X_1 \cap X_2} \simeq \mathbb{N}^2$. However, the log structure constructed in [20] and [14] for X has $\mathbb{N}$ on each component. The extra $\mathbb{N}$ comes from the normal direction along D_3. In other words, the log structures of X are sensitive to the **embedding codimension**.
- There is no simple criterion for the existence of log structures of '*embedding type*' for higher codimension r as in [14]. Let us consider the example above. The log structure $\mathscr{M}_X$ naturally decomposes into a direct sum of the log structures induced by D_3 and $D_1 \cup D_2$. These two log structures represent two normal directions of X in M. More generally, if a semistable variety is a complete intersection of normal crossing divisors $D_1, \ldots, D_r$, it is natural to consider the log structures induced from each D_i, instead of the log structure induced from $\sum D_i$. It turns out that there is a simple obstruction theory on the existence of such family of log structures (instead of a single log structure). See Theorem 5.8. The anonymous referee informs us that a similar phenomenon is also observed in [8].
- When X admits general semistable singularities or $r > 1$, T^1_X is not the right obstruction to the existence of a log structure of semistable type. We have a generalized notion of d-semistability (Definition 5.9). A semistable variety admits a log struc-

ture of semistable type if and only if it is d-semistable (Theorem 5.10).

As an application, we show that if X is a product of Fano hypersufaces and one of the product factors is of dimension ≥ 2. Then X does not admit a log structure of semistable type. As a consequence, it has no semistable smoothing (Example 5.12).

The technique that we use in the proof of Theorem 1.1 and Theorem 1.2 is the log deformation, which is first used by Y. Kawamata and Y. Namikawa to smooth certain normal crossing Calabi-Yau varieties in [20] (although they do not use the formal language of log geometry). There are two advantages of using log deformations:

(1) If we endow a normal crossing singularity $x_1 \cdots x_r = 0$ with the canonical log structure, it becomes smooth in the category of log scheme. Although there are many types of deformations of $x_1 \cdots x_r = 0$, the log smooth deformation of $x_1 \cdots x_r = 0$ which respects the canonical log structure is quite simple. There are two types of log smooth deformations of $x_1 \cdots x_r = 0$: one keeps the type of the singularity by $x_1 \cdots x_r = 0$, the other one smooths it by $x_1 \cdots x_r = t$. Therefore, if we choose the log smooth deformation which smooths the singularities, we automatically get a semistable family.

(2) Generally, semistable Fano varieties are obstructed. However, one can show that they are unobstructed under the log smooth deformation. In fact, their log obstruction spaces vanish (Proposition 3.6).

In [25], Tziolas proves the vanishing of the log obstruction space of a d-semistable normal crossing Fano variety by normalizing the singularities and reducing the vanishing theorem to the Akizuki-Nakano-Kodaira vanishing theorem of log pairs. However, in the semistable case, the method of normalizing becomes combinatorially more complicated. In this paper, we use the full power of log geometry which is introduced by Fontaine-Illusie and is developed by K. Kato [16]. By using Kato's decomposition theorem of log de Rham complex (Theorem 3[1]), we are able to prove a general Akizuki-Nakano-Kodaira type vanishing theorem for semistable log varieties (Theorem 3.5 and Corollary 3.6). The vanishing of the log obstruction space of semistable log Fano varieties is an easy consequence.

As long as the log deformation is unobstructed, we are able to lift the semistable log variety over a complete ring (provided that there is a lifting of an ample line bundle). Then we prove the limit preserving property of semistable log smooth morphisms (Proposition 4.2) and use Artin's approximation theorem to extend the family to a variety base.

Y. Zhu informs us that Kawamata also obtains a vanishing theorem of semistable varieties in [18], aiming at generalizing J. Kollar's work in [22] to semistable varieties. Our Akizuki-Nakano-Kodaira type vanishing theorems (Theorem 3.5 and Corollary 3.6) differ from Kawamata's vanishing theorem. We consider the log de Rham complex (which controls the log deformation) instead of the de Rham complex of Du Bois which is considered in [18].

This paper is organized as follows:

In section 2 we introduce the notions in log geometry which are necessary for this paper. We also introduce Kato's obstruction theory of log smooth deformations.

In section 3 we prove an Akizuki-Nakano-Kodaira type vanishing theorem for semistable log varieties (Theorem 3.5 and Corollary 3.6) by using Kato's decomposition theorem of log

[1] A detailed proof of this theorem is lacked in [16]. We present a proof in the appendix.

de Rham complex (Theorem 3). As a consequence, we prove that the obstruction space of the log smooth deformation of a semistable log Fano variety vanishes (Proposition 3.7).

In section 4 we prove the limit preserving property of the semistable log smooth morphisms (Proposition 4.2) and the main results of this paper (Theorem 1.1 and Theorem 1.2).

Section 5 is devoted to a geometric criterion for the existence of a log structure of semistable type. These results generalize [20] and [14] in several directions. The main results are Theorem 5.8 and Theorem 5.10. Some examples are also given.

The appendix presents a proof of Kato's decomposition theorem (Theorem 3) for the readers' convenience. This proof is standard but is missing in the literatures.

Notations: We mainly follow the notions and notations in [16] with some exceptions:

- We use the capital letters X, Y, etc. to denote log schemes. If X is a log scheme, we denote $\underline{X}$ to be the underlying scheme and $\alpha_X : \mathscr{M}_X \to \mathscr{O}_X$ to be the log structure. Denote $\overline{\mathscr{M}_X} := \mathscr{M}_X/\mathscr{O}_X^*$.
- the notation $\mathbb{N}^r \mapsto 0$ stands for the log structure associated to the zero map from $\mathbb{N}^r$ to the structure ring.
- We denote a log cotangent sheaf by Ω instead of ω as in [16].
- All log structures are defined on the small étale site.

2. Logarithmic Geometry and Logarithmic Deformation

A log scheme is a triple $X = (\underline{X}, \mathscr{M}_X, \alpha)$ consisting of

- a scheme $\underline{X}$,
- a sheaf $\mathscr{M}_X$ (on the small étale topology on X) of monoids and
- a morphism of sheaves of monoids $\alpha : \mathscr{M}_X \to \mathscr{O}_X$ from $\mathscr{M}_X$ to the multiplication monoid $(\mathscr{O}_X, \times)$ such that

$$\alpha|_{\alpha^{-1}\mathscr{O}_X^*} : \alpha^{-1}\mathscr{O}_X^* \to \mathscr{O}_X^*$$

is an isomorphism.

$(\mathscr{M}_X, \alpha)$ is called the log structure of the log scheme. Usually, α is omitted in the notation if there is no danger of ambiguity.

For any morphism of sheaves of monoids $\alpha : \mathscr{P} \to \mathscr{O}_X$, one can associate to it a log structure $\mathscr{P}^a \to \mathscr{O}_X$ functorially. If $\mathscr{P}$ is a constant sheaf of monoids, then it is called a chart of $\mathscr{P}^a$.

A typical example of log schemes is the log pair. Let (X, D) be a pair consisting of a scheme X and a reduced subscheme D on X of codimension 1. We can construct a log structure $\mathscr{M}_D$ on X by

$$\mathscr{M}_D(U) = \{f \in \mathscr{O}_X(U) | f|_{U\setminus D} \in \mathscr{O}_X^*(U \setminus D)\}.$$

A morphism of log schemes $f : X \to Y$ consists of a morphism of the underlying schemes $\underline{f} : \underline{X} \to \underline{Y}$ and a morphism of sheaves of monoids $f^* : f^{-1}\mathscr{M}_Y \to \mathscr{M}_X$. We say that f is strict if $(f^{-1}\mathscr{M}_Y)^a \simeq \mathscr{M}_X$. f is called log smooth if

(1) the underlying morphism is locally of finite presentation and
(2) for any commutative diagram

$$\begin{array}{ccc} T' & \longrightarrow & X \\ \downarrow i & \overset{g}{\nearrow} & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

of log schemes such that i is a strict closed immersion whose ideal of definition I satisfies $I^2 = 0$, there exists locally on T a dotted arrow g rendering the diagram commutative.

A typical example of log smooth morphisms is the semistable morphism. Our notion generalizes the notion in [1, §0.3].

DEFINITION 2.1. Let X and Y be Noetherian schemes. Suppose that $D_X \subseteq X$ and $D_Y \subseteq Y$ are reduced Cartier divisors. A morphism of pairs $f : (X, D_X) \to (Y, D_Y)$ is called **semistable** if the following two conditions hold.

(1) X, Y are regular and $D_X \subseteq X$, $D_Y \subseteq Y$ are normal crossing divisors,

(2) for each point $x \in X$ such that $f(x) = y$, there are étale morphisms $U \to X$ and $V \to Y$ which send $x' \in U$ and $y' \in V$ to x and y respectively, and a morphism g rendering the diagram

$$\begin{array}{ccc} U & \xrightarrow{x' \mapsto x} & X \\ \downarrow g & & \downarrow f \\ V & \xrightarrow{y' \mapsto y} & Y \end{array}$$

commutative. Here g is formally isomorphic to the formal spectrum of

$$\widehat{\mathscr{O}_{V,x}} \to (\mathscr{O}_{V,x}[[x_1, \dots, x_n]]/I)^\wedge,$$

$$f \mapsto f,$$

where the ideal I is generated by

$$r_i - \prod_{j=l_{i-1}+1}^{l_i} x_j.$$

Here $0 = l_0 < l_1 < \cdots < l_m \leq n$, and

$$D_Y \times_Y V = \{r_1 \cdots r_m = 0\}, \quad D_X \times_X U = \{x_1 \cdots x_{l_m} = 0\}.$$

In particular, $\{r_i\}$ is a subset of a regular systems of parameters in $\widehat{\mathscr{O}_{V,x}}$.

A semistable morphism is log smooth if X and Y are endowed with the log structures induced by the divisors D_X and D_Y. This is a direct consequence of the following criterion of Kato.

Theorem 2.2 ([16, Theorem 3.5]). *Let $f : X \to Y$ be a morphism of fine log schemes. Then f is log smooth if and only if given any étale local chart $Q \to \mathscr{M}_Y$, there exists an étale local chart $P \to \mathscr{M}_X$, and a chart of f, $Q \to P$, such that*

(a): Ker$(Q^{gp} \to P^{gp})$ *and the torsion part of* Coker$(Q^{gp} \to P^{gp})$ *are finite groups of order invertible on X, and*

(b): *the induced morphism* $X \to Y \times_{\mathrm{Spec}\mathbb{Z}[Q]} \mathrm{Spec}\mathbb{Z}[P]$ *is smooth (in the usual sense).*

Let $f : X \to Y$ be a semistable morphism. By restricting the log structures on the fibers of f at $y \in Y$, we get a log variety X_y which is log smooth over $(k, \mathbb{N}^r \mapsto 0)$ where r is the number of formal branches of D_Y at y. This suggests the following definition.

Definition 2.3. A log variety X is **semistable log smooth** over a log field $(k, \mathbb{N}^r \mapsto 0)$ if there is a log smooth morphism $X \to (\mathrm{Spec}(k), \mathbb{N}^r \mapsto 0)$, such that for every point $x \in X$, there exists

- a pointed scheme (U, u),
- an étale morphism $U \to X$ which sends u to x [2] and
- a diagram of log schemes

$$\begin{array}{ccc} U & \xrightarrow{\;g\;} & \mathrm{Spec}(k[x_1, \dots, x_n]/(x_1 \cdots x_{l_1}, x_{l_1+1} \cdots x_{l_2}, \dots, x_{l_{r-1}+1} \cdots x_n), \mathbb{N}^n) \\ \downarrow & \swarrow_{\pi} & \\ (k, \mathbb{N}^r \mapsto 0), & & \end{array}$$

where g is strict, log smooth and the log structure of

$$\mathrm{Spec}(k[x_1, \dots, x_n]/(x_1 \cdots x_{l_1}, x_{l_1+1} \cdots x_{l_2}, \dots, x_{l_{r-1}+1} \cdots x_n)$$

is induced by

$$\alpha : \mathbb{N}^n \to k[x_1, \dots, x_n]/(x_1 \cdots x_{l_1}, x_{l_1+1} \cdots x_{l_2}, \dots, x_{l_{r-1}+1} \cdots x_n), \quad \alpha(e_i) = x_i.$$

Here $e_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^n$ where 1 is placed on the i-th component.

Under a suitable order of $x_1, \dots, x_n$ and $0 = l_0 < l_1 < \cdots < l_r = n$, the map on the log structure of the morphism π is induced by

$$\mathbb{N}^r \to \mathbb{N}^n, \quad e_i \mapsto e_{l_{i-1}+1} + \cdots + e_{l_i}.$$

In this case we say that the log structure $\mathscr{M}_X$ is **of semistable type** over $(k, \mathbb{N}^r \mapsto 0)$.

Given a morphism of log schemes $f : X \to Y$, the relative cotangent sheaf of f is defined by

$$\Omega_{X/Y} := \Omega_{\underline{X}/\underline{Y}} \oplus (\mathscr{M}_X^{gp} \otimes_{\mathbb{Z}} \mathscr{O}_X)/\sim,$$

where $\sim$ is a $\mathscr{O}_X$-submodule generated by

(1) $(d\alpha(a), 0) - (0, a \otimes \alpha(a))$ with $a \in \mathscr{M}_X$, and
(2) $(0, 1 \otimes a)$ with $a \in \mathrm{Image}(f^{-1}(\mathscr{M}_Y) \to \mathscr{M}_X)$.

The canonical morphism $\mathscr{M}_X^{gp} \to \Omega_{X/Y}$ is denoted by dlog. If f is log smooth, then $\Omega_{X/Y}$ is locally free ([16, Proposition 3.10]). The log cotangent sheaf controls the log smooth deformation of a log smooth variety ([16], [15]).

Theorem 2.4 ([16, Proposition 3.14])**.** *Let $f : X \to Y$ be a log smooth morphism between fine log schemes and let $i : Y \to Y'$ be a strict closed immersion such that Y is defined in Y' by a square zero ideal I. Then there is an obstruction class $ob_{f,i} \in H^2(X, (\Omega_{X/Y})^\vee \otimes I)$*

[2] We do not require that $k(u) \simeq k(x)$.

such that $ob_{f,i} = 0$ if and only if there exists a log smooth morphism of fine log schemes $f' : X' \to Y'$ whose restriction on Y is isomorphic to f.

Remark 2.5. Let Y be a fine log scheme and $i : Y \to Y'$ be a strict thickening (i.e., i is a closed immersion defined by a nilpotent ideal sheaf where i is strict as a log morphism). Let X be a fine log scheme which is log smooth over Y. If X is affine, then there exists a unique lifting (up to non-unique isomorphisms) over Y' ([16, Proposition 3.14]). Étale locally a log smooth lifting can be described as follows (loc. cit.).

Choosing a chart P of X and a chart Q of Y as in Theorem 2.2, we get the following diagram

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow f & & \downarrow f' \\ \mathrm{Spec}(\mathbb{Z}[P]) \times_{\mathrm{Spec}(\mathbb{Z}[Q])} Y & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[P]) \times_{\mathrm{Spec}(\mathbb{Z}[Q])} Y' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y', \end{array}$$

where f is strict and étale. We can complete the diagram by the dotted arrows so that f' is strict, étale and the square on the top is a fiber product ([10, Exposé 1, Proposition 8.1]). Here X' is a log smooth lifting of X over Y'.

3. Kodaira-Akizuki-Nakano Vanish for Semistable Log Varieties

In this section we prove that semistable log Fano varieties are unobstructed under log smooth deformations. To prove this we need to establish the Kodaira-Akizuki-Nakano vanishing theorem for semistable log smooth varieties. As corollaries, we prove the main results in this paper.

Definition 3.1. Let k be a field and X be a variety over k. X is said to be semistable if for each point $x \in X$, there exists an étale morphism $U \to X$ which sends $u \in U$ to x (we do not require that $k(u) \simeq k(x)$) such that

$$\widehat{\mathscr{O}_{U,u}} \simeq k(u)[[x_1, \dots, x_n]]/(x_1 \cdots x_{l_1}, x_{l_1+1} \cdots x_{l_2}, \dots, x_{l_{r-1}+1} \cdots x_{l_r}).$$

Here $0 = l_0 < l_1 < \cdots < l_r \leq n$.

This definition allows singularities like $x^2 + y^2 = 0$ defined over $k = \mathbb{R}$.

Let X be a semistable variety over k. Since it is étale locally a product of normal crossing singularities, X is Gorenstein and the dualizing sheaf $\omega_{X/k}$ is an invertible sheaf.

Definition 3.2. A complete semistable variety X is called Fano if its dualizing sheaf $\omega_{X/k}$ is anti-ample, i.e., $\omega_{X/k}^{-1}$ is ample.

For a semistable log variety, the log canonical sheaf is isomorphic to the dualizing sheaf of the underlying variety.

Lemma 3.3. *Let X be a semistable log smooth variety over $\mathbf{k} = (k, \mathbb{N}^r \mapsto 0)$, then* $\bigwedge^{\dim X} \Omega_{X/\mathbf{k}} \simeq \omega_{X/k}$.

Proof. For simplicity, let us consider the normal crossing singularity. Denote by $Z = \{x_1 \cdots x_r = 0\}$ a closed subspace of $U = \mathrm{Spec}(k[x_1, \ldots, x_n])$. Then $\bigwedge^{n-r} \Omega_{Z/\mathbf{k}}$ is an invertible sheaf generated by

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{x_r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n.$$

The dualizing sheaf $\omega_{Z/k}$ has the same expression by the adjunction formula. Since semistable varieties are locally products of normal crossing singularities, the lemma is proved (We omit the verification of compatibilities here). □

The following theorem is due to Kato [16]. Since the arguments are simple modifications of those given in [3], the author does not provide the detailed argument. For the convenience of the readers, we provide the detailed proof in the appendix.

Theorem 3.4 ([16, Theorem 4.12])**.** *Let k be a perfect field of characteristic $p > 0$ and X be a log variety which is semistable log smooth over $\mathbf{k} = (\mathrm{Spec}(k), \mathbb{N}^r \mapsto 0)$. If X has a log smooth lifting over $(\mathrm{Spec} W_2(k), \mathbb{N}^r \mapsto 0)$, then there is an isomorphism*

$$\tau_{<p} F_* \Omega^\bullet_{X/\mathbf{k}} \simeq \bigoplus_{0 \le i < p} \Omega^i_{X'/\mathbf{k}}[-i]$$

in $D(X')$. Here $F : X \to X'$ is the relative Frobenius morphism over $\mathbf{k}$.

By using the same method as in [3, Lemma 2.9], we have the following

Theorem 3.5. *Let k be a perfect field of characteristic $p > 0$ and X be a semistable log smooth variety over $\mathbf{k} = (\mathrm{Spec}(k), \mathbb{N}^r \mapsto 0)$. Assume that X is proper over k and $\dim X < p$. Suppose that X has a log smooth lifting over $(\mathrm{Spec}(W_2(k)), \mathbb{N}^r \mapsto 0)$. Then for any ample line bundle A on X, we have*

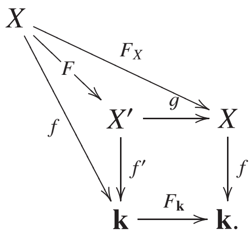
$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A) = 0$$

for $\dim X < i + j$, and

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A^{-1}) = 0$$

for $i + j < \dim X$.

Proof. Consider the Frobenius square of log varieties



First Part. Since A is ample, by Serre's vanishing there is an integer $m \gg 0$ such that

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A^{p^{m+1}}) = 0, \quad i > 0.$$

By the spectral sequence of hypercohomology

$$E_1^{pq} := H^q(X, \Omega^p_{X/\mathbf{k}} \otimes A^{p^{m+1}}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbf{k}} \otimes A^{p^{m+1}}),$$

we obtain

$$\mathbb{H}^i(X, \Omega^\bullet_{X/\mathbf{k}} \otimes A^{p^{m+1}}) = 0, \quad i > \dim X.$$

Due to Theorem 3.4, we obtain

$$F_*(\Omega^\bullet_{X/\mathbf{k}} \otimes A^{p^{m+1}}) \simeq \bigoplus_{0 \le i < \dim X} \Omega^i_{X'/\mathbf{k}} \otimes g^* A^{p^m}[-i].$$

Hence we deduce that

$$H^i(X', \Omega^j_{X'/\mathbf{k}} \otimes g^* A^{p^m}) = 0, \quad \dim X < i + j.$$

Since the underlying morphism of g is an isomorphism,

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A^{p^m}) = 0, \quad \dim X < i + j.$$

Continuing this induction, we obtain

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A) = 0, \quad \dim X < i + j.$$

Second Part. The proof of the second part is almost the same. Since X is Gorenstein, by Serre-Grothendieck duality (cf. [11]), we have

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A^{-p^m}) = 0, \quad m \gg 0, \quad i + j < \dim X.$$

This can be proved as follows:

Since X is Gorenstein, by Serre-Grothendieck duality, we have

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A^{-p^m})^\vee = H^{\dim X - i}(X, \omega_{X/k} \otimes (\Omega^j_{X/\mathbf{k}})^\vee \otimes A^{p^m}) = 0, \quad m \gg 0$$

for $0 < i < \dim X$. Hence

$$\mathbb{H}^i(X, \Omega^\bullet_{X/\mathbf{k}} \otimes A^{-p^m}) = 0, \quad 0 < i < \dim X, \quad m \gg 0.$$

Due to Theorem 3.4, we deduce that

$$F_*(\Omega^\bullet_{X/\mathbf{k}} \otimes A^{-p^m}) \simeq \bigoplus_{0 \le i < \dim X} \Omega^i_{X'/\mathbf{k}} \otimes g^* A^{-p^{m-1}}[-i].$$

We have

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A^{-p^{m-1}}) = 0, \quad i + j < \dim X.$$

By induction we obtain that

$$H^i(X, \Omega^j_{X/\mathbf{k}} \otimes A^{-1}) = 0, \quad i + j < \dim X.$$

□

In this case, $\bigwedge^{\dim X} \Omega_{X/\mathbf{k}}$ is isomorphic to the dualizing sheaf $\omega_{X/k}$. Therefore the two vanishing results are related by Serre-Grothendieck duality. We present here the separated proofs because our argument holds for more general log smooth varieties (e.g. allowing horizontal divisors), in which case the dualizing sheaf may be different from the log canonical sheaf.

By the standard technique of reduction mod p , we have the following corollary.

Corollary 3.6. *Let K be a field of characteristic* 0 *and X be a semistable log smooth variety over* $\mathbf{K} = (\mathrm{Spec}(K), \mathbb{N}^r \mapsto 0)$. *Then for any ample line bundle A on X, we have*

$$H^i(X, \Omega^j_{X/\mathbf{K}} \otimes A) = 0, \quad \dim X < i + j$$

and

$$H^i(X, \Omega^j_{X/\mathbf{K}} \otimes A^{-1}) = 0, \quad i + j < \dim X.$$

Proof. Let R be a subring of K such that R is finitely generated over $\mathbb{Z}$ and X is defined over $\mathrm{Spec}(R)$. Namely there is a log smooth morphism

$$\mathscr{X} \to (\mathrm{Spec}(R), \mathbb{N}^r \mapsto 0)$$

such that $X \simeq \mathscr{X} \times_{(\mathrm{Spec}(R),\mathbb{N}^r\mapsto 0)} \mathbf{K}$ and the closed fibers of $\mathscr{X}$ are semistable log smooth. By shrinking $\mathrm{Spec}(R)$ we may assume that $\mathrm{Spec}(R)$ is smooth over $\mathrm{Spec}(\mathbb{Z})$. Hence for each geometric closed point $\mathrm{Spec}(k(p)) \to \mathrm{Spec}(R)$ where $k(p)$ is a perfect field of characteristic $p > \dim X$, there is a dotted arrow fulfilling the diagram

$$\begin{array}{ccc} \mathrm{Spec}(k(p)) & \longrightarrow & \mathrm{Spec}(R) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(W_2(k(p))) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array} .$$

As a consequence, the closed fiber $\mathscr{X}_{k(p)}$ has a log smooth lifting to $(\mathrm{Spec}(W_2(k(p))), \mathbb{N}^r \to 0)$. By Theorem 3.5,

$$H^i(X, \Omega^j_{\mathscr{X}_{k(p)}} \otimes A) = 0$$

for $\dim X < i + j$, and

$$H^i(X, \Omega^j_{\mathscr{X}_{k(p)}} \otimes A^{-1}) = 0$$

for $i + j < \dim X$. Now the corollary follows from the semi-continuity theorem for cohomologies. □

Proposition 3.7. *Let k be a field and X be a semistable log Fano variety over* $\mathbf{k} = (k, \mathbb{N}^r \mapsto 0)$.

(1) *if* $\mathrm{char}(k) = 0$*, then* $H^2(X, (\Omega_{X/\mathbf{k}})^\vee) = 0$ *and* $H^2(X, \mathscr{O}_X) = 0$.

(2) *Assume that k is perfect and* $\mathrm{char}(k) = p > 0$. *If* $\dim X < p$ *and X admits a log smooth lifting over* $(W_2(k), \mathbb{N}^r \mapsto 0)$*, then* $H^2(X, (\Omega_{X/\mathbf{k}})^\vee) = 0$ *and* $H^2(X, \mathscr{O}_X) = 0$.

Proof. By Theorem 3.5 and Corollary 3.6, we see that

$$H^2(X, (\Omega_{X/\mathbf{k}})^\vee) = H^{n-2}(X, \Omega_{X/\mathbf{k}} \otimes \omega_{X/k}) = 0.$$

By Lemma 3.3, we obtain that

$$H^2(X, \mathscr{O}_X) = H^2(X, \bigwedge^{\dim X} \Omega_{X/\mathbf{k}} \otimes \omega_{X/k}^{-1}) = 0.$$

□

4. Smoothing of Semistable Log Fano Varieties

The vanishing theorems in the last section ensure the lifting of a semistable log Fano variety to a complete DVR. To extend this family to a variety base, we need to study the limit preserving property of semistable log smooth morphisms (Proposition 4.2). First we would like to generalize the notion of semistable log smooth morphisms.

Definition 4.1. Let Y be a log scheme which has a global chart $\mathbb{N}^r$. A morphism $f : X \to Y$ is a semistable log smooth morphism if étale locally there exists a chart $\mathbb{N}^n \to \mathscr{M}_X$ on X and a chart of f,

$$\mathbb{N}^r = \bigoplus_{i=1}^{r} \mathbb{N}e_i \to \mathbb{N}^n = \bigoplus_{i=1}^{n} \mathbb{N}e_i'$$

$$e_i \mapsto e_{l_{i-1}} + \cdots + e_{l_i}$$

where $0 = l_0 < l_1 < \cdots < l_r = n$, such that the induced morphism

$$X \to Y \times_{\mathrm{Spec}\mathbb{Z}[\mathbb{N}^r]} \mathrm{Spec}\mathbb{Z}[\mathbb{N}^n]$$

is strict and log smooth.

This definition generalizes Definition 2.3. If (Y, D_Y) is a smooth variety with the log structure induced by a simple normal crossing divisor D_Y, then the underlying morphism of a semistable log smooth morphism $f : X \to Y$ is semistable in the sense of Definition 2.1.

Proposition 4.2. *Given a $k[\mathbb{N}^r]$-algebra A, denote by $A^{\log}$ the log algebra A whose log structure is given by the composition of $\mathbb{N}^r \to k[\mathbb{N}^r]$ and the structure homomorphism $k[\mathbb{N}^r] \to A$. Then the pseudo-functor from the category of $k[\mathbb{N}^r]$-algebras to the category of groupoids*

$\mathscr{F}(k[\mathbb{N}^r] \to A) =$ *the groupoid of log scheme X which is semistable log smooth over A^{log}*

is limit preserving. That is to say, for any directed system $\{A_i\}_{i\in I}$ of $k[\mathbb{N}^r]$-algebras A_i, the canonical functor

$$\sigma : \mathscr{F}(\varinjlim_{i\in I} A_i) \to \varinjlim_{i\in I} \mathscr{F}(A_i)$$

is an equivalence.

Proof. Denote $A' = \varinjlim_i A_i$. The fully faithfulness of σ is formal. It suffices to prove that σ is essentially surjective, i.e., for any $X \in \mathscr{F}(A')$, there exists an index i_0 and $X_{i_0} \in \mathscr{F}(A_{i_0})$ such that its base change on A' is X.

Step 1: Since $\underline{X}$ is locally of finite presentation over $\mathrm{Spec}(A')$, there exists an index i_1 and a scheme $\underline{X_{i_1}}$ locally of finite presentation over $\mathrm{Spec}(A_{i_1})$ such that its base change on $\mathrm{Spec}(A')$ is $\underline{X}$.

Step 2: Consider the pseudo-functor from the category of A_{i_1}-schemes to groupoids:

$$\mathrm{Log}_{A_{i_1}}(X) = \text{ the groupoid of fine log structures on } X \text{ over } A_{i_1}^{\log}.$$

It is proved in [24, Theorem 1.1] that this pseudo-functor is represented by an algebraic stack locally of finite presentation over A_{i_1}. Therefore $\mathrm{Log}_{A_{i_1}}$ is limit preserving. Applying $\mathrm{Log}_{A_{i_1}}$ on the directed inverse system $\{X_{i_1} \times_{\mathrm{Spec}(A_{i_1})} \mathrm{Spec}(A_i)\}_{i \geq i_1}$, we have a categorical equivalence

$$\mathrm{Log}_{A_{i_1}}(X) \simeq \varinjlim_{i \geq i_1} \mathrm{Log}_{A_{i_1}}(X_{i_1} \times_{\mathrm{Spec}(A_{i_1})} \mathrm{Spec}(A_i)).$$

Therefore there exists an index $i_2 \geq i_1$ and a fine log structure over

$$\underline{X_{i_2}} = \underline{X_{i_1}} \times_{\mathrm{Spec}A_{i_1}^{\log}} \mathrm{Spec}A_{i_2}^{\log}$$

such that the base change of the log scheme X_{i_2} over A' is X. So far, X_{i_2} is not necessarily semistable log smooth over $A_{i_2}^{\log}$.

Step 3: We show in this step that there exists an index $i_0 \geq i_2$ such that

$$X_{i_0} = X_{i_2} \times_{\mathrm{Spec}A_{i_2}^{\log}} \mathrm{Spec}A_{i_0}^{\log}$$

is semistable log smooth over $A_{i_0}^{\log}$. Fixing a point $x \in X$, there exists an étale local chart $\mathbb{N}^n \to \mathscr{M}_X$ at x, and a chart of f,

$$\mathbb{N}^r = \bigoplus_{i=1}^{r} \mathbb{N}e_i \to \mathbb{N}^n = \bigoplus_{i=1}^{n} \mathbb{N}e_i'$$

$$e_i \mapsto e_{l_{i-1}+1} + \cdots + e_{l_i}$$

where $0 = l_0 \leq l_1 < \cdots < l_r = n$, such that the induced morphism

$$X \to \mathrm{Spec}A'^{\log} \times_{\mathrm{Spec}\mathbb{Z}[\mathbb{N}^r]} \mathrm{Spec}\mathbb{Z}[\mathbb{N}^n]$$

is smooth in the usual sense. There exists an index $i_3 \geq i_2$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}\mathbb{Z}[\mathbb{N}^n] \\ \downarrow & & \downarrow \\ \mathrm{Spec}A'^{\log} & \longrightarrow & \mathrm{Spec}\mathbb{Z}[\mathbb{N}^r] \end{array}$$

factors through

$$\begin{array}{ccc} X_{i_3} & \longrightarrow & \mathrm{Spec}\mathbb{Z}[\mathbb{N}^n] \\ \downarrow & & \downarrow \\ \mathrm{Spec}A_{i_3}^{\log} & \longrightarrow & \mathrm{Spec}\mathbb{Z}[\mathbb{N}^r] \end{array}$$

and the induced morphism

$$X_{i_3} \to \mathrm{Spec}A_{i_3}^{\log} \times_{\mathrm{Spec}\mathbb{Z}[\mathbb{N}^r]} \mathrm{Spec}\mathbb{Z}[\mathbb{N}^n]$$

is smooth in the usual sense. Hence locally at the preimages of $x \in X$, X_{i_3} is semistable log smooth over $A_{i_3}^{\log}$. Since X is quasi-compact, there exists an index $i_0 \geq i_2$ such that

$X_{i_0} = X_{i_2} \times_{A^{\log}_{i_2}} A^{\log}_{i_0}$ is semistable log smooth over $A^{\log}_{i_0}$. □

Theorem 4.3. *Let k be a field and X be a projective log variety which is semistable log smooth over $\mathbf{k} = (k, \mathbb{N}^r \mapsto 0)$. Assume that $H^2(X, (\Omega_{X/\mathbf{k}})^\vee) = 0$ and $H^2(X, \mathscr{O}_X) = 0$. Then there exists a log variety $\mathscr{X}$ which is semistable log smooth over (B, D_B) such that*

(1) *B is an r-dimensional smooth variety over k. $0 \in B$ is a k-point. D_B is a simple normal crossing divisor whose number of branches at 0 is r;*
(2) *$\mathscr{X}_0 \simeq X$ as log varieties.*

Proof. By Theorem 2.4, X has a log smooth lifting $\widehat{\mathscr{X}}$ over the formal log scheme

$$\widehat{B} = (\mathrm{Spf} k[[x_1, \dots, x_r]], \mathbb{N}^r = \oplus_{i=1}^r \mathbb{N} e_i, e_i \mapsto x_i).$$

In other words, we have a sequence of log smooth morphisms

$$X^{(m)} \to B^{(m)}$$

where

$$B^{(m)} = \Big(\mathrm{Spec}(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^m), \mathbb{N}^r = \oplus_{i=1}^r \mathbb{N} e_i, e_i \mapsto x_i\Big),$$

such that $X^{(1)} \simeq X$ and for every $m_1 < m_2$, we have

$$X^{(m_1)} \simeq X^{(m_2)} \times_{B^{(m_2)}} B^{(m_1)}.$$

Since $H^2(X, \mathscr{O}_X) = 0$, any ample line bundle on X has a lifting over $\widehat{B}$. Denote

$$B_1 = (\mathrm{Spec}(k[[x_1, \dots, x_r]]), \mathbb{N}^r = \oplus_{i=1}^r \mathbb{N} e_i, e_i \mapsto x_i).$$

By Grothendieck's existence theorem, there exists a variety $\mathscr{X}_1$ over $\mathrm{Spec}(k[[x_1, \dots, x_r]])$, whose formal completion is the underlying formal scheme of $\widehat{\mathscr{X}}$. To make $\mathscr{X}_1$ semistable log smooth over B_1, it remains to put a suitable log structure on $\mathscr{X}_1$. This is done as follows.

By Remark 2.5, étale locally there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{X}_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathrm{Spec}(k[\mathbb{N}^n]) \times_{\mathrm{Spec}(k[\mathbb{N}^r])} k & \longrightarrow & \mathrm{Spec}(k[\mathbb{N}^n]) \times_{\mathrm{Spec}(k[\mathbb{N}^r])} \mathrm{Spec}(k[[x_1, \dots, x_r]]) \\ \downarrow & & \downarrow \\ k & \longrightarrow & \mathrm{Spec}(k[[x_1, \dots, x_r]]), \end{array} \tag{4.1}$$

where f is strict and étale, f' is étale and the square on the top square is a fiber product of schemes. The morphism $\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r]) \to \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^n])$ is induced by the homomorphism of monoids

$$\mathbb{N}^r = \bigoplus_{i=1}^r \mathbb{N} e_i \to \mathbb{N}^n = \bigoplus_{i=1}^n \mathbb{N} e'_i$$

$$e_i \mapsto e_{l_{i-1}+1} + \cdots + e_{l_i}.$$

Here $0 = l_0 < l_1 < \cdots < l_r \le n$. This shows that $\mathscr{X}_1 \to B_1$ is semistable. When

$\mathscr{X}_1$ is endowed with the pullback log structure through f', (4.1) becomes a diagram of log schemes. It can be shown that such log structure on $\mathscr{X}_1$ (étale locally) is determined by unique isomorphisms (c.f. the arguments of Theorem 4.2.8 in [23]). Therefore, these local log structures glue to a global one on $\mathscr{X}_1$ so that $\mathscr{X}_1$ is semistable log smooth over B_1.

To extend the family $\mathscr{X}_1$ over a variety base, let us consider the pseudo-functor from the $k[\mathbb{N}^r]^h$-algebras to groupoids:

$$\mathscr{G}(k[\mathbb{N}^r]^h \to A) = \{\text{groupoids of semistable log smooth varieties over } A^{\log}\}.$$

Here $k[\mathbb{N}^r]^h$ is the henselization of $k[\mathbb{N}^r]$ at the maximal ideal generated by $\mathbb{N}^r$. $A^{\log}$ is the log algebra A whose log structure is given by the composition of $\mathbb{N}^r \to k[\mathbb{N}^r]^h$ and the structure homomorphism $k[\mathbb{N}^r]^h \to A$. By Proposition 4.2, this functor is limit preserving. Since $\mathscr{X}_1 \in \mathscr{G}(k[[\mathbb{N}^r]])$, by applying Artin's approximation ([2, Theorem 1.12]) to the functor $\mathscr{G}$, there exists $\mathscr{X}_1^h \in \mathscr{G}(k[\mathbb{N}^r]^h)$ whose base change on k is X. Since $k[\mathbb{N}^r]^h$ is the direct limit of algebras that are étale over $k[\mathbb{N}^r]$. By Proposition 4.2, there is an $k[\mathbb{N}^r]$-algebra R which is étale over $k[\mathbb{N}^r]$, and a semistable log smooth morphism $\mathscr{X} \to B := \mathrm{Spec}R^{\log}$ such that $\mathscr{X}_1^h \sim \mathscr{X} \times_{k[\mathbb{N}^r]^{\log}} (k[\mathbb{N}^r]^h)^{\log}$ as log schemes. By the constructions, $\mathscr{X} \to B$ satisfies the claims in the theorem. □

Corollary 4.4. *Let k be a field of characteristic 0 and X be a Fano semistable variety. Let $r \geq 1$ be an integer. Then the followings are equivalent:*

(1) *there exists a log variety $\mathscr{X}$ which is semistable log smooth over (B, D_B) such that*
   (a) *B is an r-dimensional smooth variety over k, $0 \in B$ is a k-point, and D_B is a simple normal crossing divisor whose number of branches at 0 is r;*
   (b) *$\mathscr{X}_0 \simeq X$ as log varieties.*

(2) *X has a log structure of semistable type over $(k, \mathbb{N}^r \mapsto 0)$.*

Proof. A semistable morphism $f : (\mathscr{X}, D_{\mathscr{X}}) \to (B, D_B)$ is log smooth if we endow $\mathscr{X}$ and B with the log structures induced by the divisors $D_{\mathscr{X}}$ and D_B. If X is isomorphic to a fiber $\mathscr{X}_b$ of a semistable morphism, then the log structure restricted on $\mathscr{X}_b$ gives X a log structure of semistable type over $(k, \mathbb{N}^r \mapsto 0)$. In this case r is the number of formal branches of D_B passing b.

The converse is the combination of Proposition 3.7 and Theorem 4.3. □

By the same arguments, we get the following corollary.

Corollary 4.5. *Let k be a perfect field of characteristic $p > 0$ and X be a log variety X semistable log smooth over $(\mathrm{Spec}(k), \mathbb{N}^r \mapsto 0)$ for some $r \geq 0$. If X is Fano, $\dim X < p$ and X admits a log smooth lifting over $(\mathrm{Spec}W_2(k), \mathbb{N}^r \mapsto 0)$, then there exists a smooth variety with a normal crossing divisor $(\mathscr{X}, D_{\mathscr{X}})$ which is semistable over (B, D_B) such that*

(1) *B is an r-dimensional smooth variety over k, $0 \in B$ is a k-point, and D_B is a simple normal crossing divisor whose number of branches at 0 is r;*
(2) *$\mathscr{X}_0 \simeq X$ as log varieties.*

In particular, if X is semistable log smooth over $(k, \mathbb{N} \mapsto 0)$ (i.e., d-semistable in [5]), then X appears in a semistable reduction over $W(k)$.

5. A Geometric Criterion for the Existence of log structures of semistable type

Given a semistable variety X, in this section we give a geometric criterion for the existence of the log structure of semistable type over $(k, \mathbb{N}^r \mapsto 0)$. Fix an algebraically closed field k and a semistable variety X of pure dimension n. Denote $T^i_X = \mathscr{E}xt^i(\Omega_X, \mathscr{O}_X)$.

Let us consider the semistable morphism $f : (X, D_X) \to (Y, D_Y)$ and a point $y \in D_Y$. Assume for simplicity that D_Y is simple normal crossing. Let $D_1, \ldots, D_r$ be the components of D_Y that pass through y. Then $f^{-1}(y) = \bigcap_{i=1}^r f^{-1}(D_i)$ is the complete intersection of normal crossing divisors $f^{-1}(D_i)$ in X. This suggests the following:

DEFINITION 5.1. A semistable variety X is r-embeddable if there is a smooth variety M and normal crossing divisors $D_1, \ldots, D_r$ of M such that D_i and D_j have no common components for each $i \neq j$, $D = \bigcup_{i=1}^r D_i$ is a normal crossing divisor and $X = \bigcap_{i=1}^r D_i$.

A semistable variety is always étale locally r-embeddable for some $r \geq 0$.

Lemma 5.2. *Let X be a semistable variety and $x \in X$ be a point. Then there is an integer $r \geq 0$, an étale morphism $\phi : U \to X$ and a point $p \in U$ such that*

(1) *$\phi(p) = x$,*
(2) *$U \subset V$ where V is an affine smooth variety,*
(3) *V has the coordinates*

$$(x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}, y_1, \ldots, y_k),$$

i.e., regular functions $x_{ij}, y_i \in \mathscr{O}_V(V)$ so that the induced morphism

$$\psi : V \to \mathbb{A}^{k_1+\cdots+k_r+k}$$

is étale. These coordinates satisfy the following conditions.

- *$\psi^{-1}(0) = \{p\}$.*
- *$D_{ij} := \{x_{ij} = 0\}$ is a connected smooth divisor in V.*
- *$U = \bigcap_{i=1}^r \bigcup_{j=1}^{k_i} D_{ij}$.*

Proof. By the definition of semistable varieties and Artin's approximation ([2, Corollary 2.6]), there is a diagram of pointed varieties

$$\begin{array}{ccc} & (U,p) & \\ {}^{\phi}\swarrow & & \searrow^{\varphi} \\ (X,x) & & (U',0) \end{array}$$

where ϕ, φ are étale morphisms and U' is an open subvariety of

(5.1) $$V' = \operatorname{Spec}(k[x'_{11}, \ldots, x'_{1k_1}, \ldots, x'_{r1}, \ldots, x'_{rk_r}, y'_1, \ldots, y'_k]/(x'_{11} \cdots x'_{1k_1}, \ldots, x'_{r1} \cdots x'_{rk_r})).$$

By [10, Exposé 1, Proposition 8.1], after a shrinking of U there is an affine variety V rendering the diagram

$$\begin{array}{ccc} U & \xrightarrow{i} & V \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ U' & \longrightarrow & V' \end{array}$$

commutative. Here ϕ' is étale and i is a closed immersion. Shrink V if necessary, then V, $x_{ij} := \phi'^* x'_{ij}$ and $y_i := \phi'^* y'_i$ satisfy the conditions in the Lemma. □

DEFINITION 5.3. The data

$$(\phi : (U, p) \to (X, x), U \subset V, (x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}, y_1, \ldots, y_k))$$

in Lemma 5.2 is called a **standard chart** of X at x.

Let X be a semistable variety and

$$(\phi : (U, p) \to (X, x), U \subset V, (x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}, y_1, \ldots, y_k))$$

be a standard chart. Denote $D_i = \{x_{i1} \cdots x_{ik_i} = 0\}$ and $\pi_i : \widetilde{D_i} \to D_i$ be the normalization. Denote

$$\mathscr{N}_{U,i} := \pi_{i*} \mathbb{N}_{\widetilde{D_i}}|_U \tag{5.2}$$

where $\mathbb{N}_{\widetilde{D_i}}$ is the constant étale sheaf of monoids $\mathbb{N}$.

DEFINITION 5.4. A semistable variety X is **virtually r-embeddable** if there are étale sheaves of monoids $\mathscr{N}_1, \ldots, \mathscr{N}_r$ on X such that for every point $x \in X$ there is a standard chart

$$(\phi : U \to X, U \subset V, (x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}, y_1, \ldots, y_k))$$

satisfying

$$\phi^{-1} \mathscr{N}_i \simeq \mathscr{N}_{U,i}, \text{(c.f. (5.2))} \quad i = 1, \ldots, r.$$

Such standard chart is called **admissible**. Denote the subset

$$Z_{\mathscr{N}_i} = \{x \in X | \mathrm{rank}(\mathscr{N}_i^{\mathrm{gp}})_x > 1\}. \tag{5.3}$$

An r-embeddable semistable variety is always virtually r-embeddable. Let $X = \bigcap_{i=1}^r D_i \subset M$ be as the notations in Definition 5.1. Let $\pi_i : \widetilde{D_i} \to D_i$ be the normalization, then $\mathscr{N}_i := \pi_{i*} \mathbb{N}_{\widetilde{D_i}}|_X$, $i = 1, \ldots, r$ make X virtually r-embeddable.

By the local description of $\mathscr{N}_i$, we see that the associated sheaf of groups $\mathscr{N}_i^{\mathrm{gp}}$ is a constructible sheaf of abelian groups on X. As a consequence, $Z_{\mathscr{N}_i}$ is a closed subvariety of X.

Proposition 5.5. *Let $(X, \mathscr{N}_1, \ldots, \mathscr{N}_r)$ be a virtually r-embeddable semistable variety. Then*

$$X_{\mathrm{sing}} = \bigcup_{i=1}^r Z_{\mathscr{N}_i}.$$

For each $1 \le i \le r$ the subsheaf

$$L_{\mathscr{N}_i} := \{\xi \in T_X^1 \mid \mathrm{supp}(\xi) \subset Z_{\mathscr{N}_i}\}$$

of T_X^1 is invertible on $Z_{\mathscr{N}_i}$ and the natural morphism

$$\bigoplus_{i=1}^{r} \tau_{i*} L_{\mathscr{N}_i} \to T^1_X$$

is an isomorphism. Here $\tau_i : Z_{\mathscr{N}_i} \to X$ is the closed immersion.

Proof. Since all statements are étale locally, it suffices to verify the proposition on an admissible standard chart

$$(\phi : U \to X, U \subset V, (x_{11}, \dots, x_{1k_1}, \dots, x_{r1}, \dots, x_{rk_r}, y_1, \dots, y_k))$$

satisfying

$$\phi^{-1}\mathscr{N}_i \simeq \mathscr{N}_{U,i}, \text{(c.f. (5.2))} \quad i = 1, \dots, r.$$

Denote $x_i = x_{i1} \cdots x_{ik_i}$, then $\phi^{-1} Z_{\mathscr{N}_i}$ is defined by

$$x_i/x_{i1} = 0, x_i/x_{i2} = 0, \dots, x_i/x_{ik_i} = 0$$

in U. From these local equations we see that

$$X_{\text{sing}} = \bigcup_{i=1}^{r} Z_{\mathscr{N}_i}. \tag{5.4}$$

Next, let us compute T^1_X on the admissible standard chart. Recall that Ω_U is quasi-isomorphic to

$$I/I^2 \to \Omega_V|_U$$

where $I = (x_1, \dots, x_r)$ is the defining ideal of U in V. So

$$T^1_U \simeq \operatorname{coker}(\tau : \mathscr{H}om(\Omega_V|_U, \mathscr{O}_U) \simeq \mathscr{D}er_k(\mathscr{O}_V, \mathscr{O}_V) \otimes \mathscr{O}_U \to \mathscr{H}om(I/I^2, \mathscr{O}_U)).$$

Note that I/I^2 is a free $\mathscr{O}_U$-module, of which the quotient classes $[x_1], \cdots, [x_r]$ form a basis. τ sends $\frac{\partial}{\partial x_{ij}}$ to the map

$$I/I^2 \to \mathscr{O}_U, \quad [x_k] \mapsto \begin{cases} x_i/x_{ij}, & k = i \\ 0, & k \neq i. \end{cases}$$

and sends $\frac{\partial}{\partial y_i}$ to the zero map $0 : I/I^2 \to \mathscr{O}_X$. This shows that

$$T^1_U \simeq \bigoplus_{i=1}^{r} \mathscr{H}om(I_i/II_i, \mathscr{O}_U) \otimes \mathscr{O}_U/(x_i/x_{i1}, \dots, x_i/x_{ik_i}). \tag{5.5}$$

Here I_i is the sheaf of ideals in $\mathscr{O}_V$ generated by $x_i = x_{i1} \cdots x_{ik_i}$. It is easy to see that $I_i|_{\phi^{-1}Z_{\mathscr{N}_i}}$ is an invertible sheaf on $\phi^{-1}Z_{\mathscr{N}_i}$. As a consequence,

$$L_{\mathscr{N}_i}|_U \simeq \mathscr{H}om(I_i/II_i, \mathscr{O}_U) \otimes \mathscr{O}_U/(x_i/x_{i1}, \dots, x_i/x_{ik_i}) \tag{5.6}$$

is invertible on $\phi^{-1}Z_{\mathscr{N}_i} \subset U$. Again by (5.5), the canonical morphism

$$\bigoplus_{i=1}^{r} \tau_{i*} L_{\mathscr{N}_i} \to T^1_X$$

is an isomorphism on U. □

Lemma 5.6. *Let $(X, \mathscr{N}_1, \dots, \mathscr{N}_r)$ be a virtually r-embeddable semistable variety. Then for each i, there is a distinguished 'diagonal' element $\Delta_i \in H^0(X, \mathscr{N}_i)$ such that for every $x \in X$, $(\Delta_i)_x \in (\mathscr{N}_i)_x \simeq \mathbb{N}^{k_i(x)}$ is the diagonal element $(1, 1, \dots, 1)$.*

Proof. Choose an admissible standard chart

$$(\phi : U \to X, U \subset V, (x_{11}, \dots, x_{1k_1}, \dots, x_{r1}, \dots, x_{rk_r}, y_1, \dots, y_k)).$$

Denote $D_i = \{x_{i1} \cdots x_{ik_i} = 0\}$. Let $\widetilde{D}_i \to D_i$ be the normalization. Denote

$$\mathbf{1}_i = (1, \dots, 1) \in H^0(\widetilde{D}_i, \mathbb{N}_{\widetilde{D}_i})$$

be the diagonal element. Define $\Delta_i \in H^0(U, \mathscr{N}_i)$ to be the image of $\mathbf{1}_i$ through the canonical maps

$$H^0(\widetilde{D}_i, \mathbb{N}_{\widetilde{D}_i}) \simeq H^0(D_i, \mathscr{N}_{D_i}) \to H^0(U, \mathscr{N}_i).$$

Since the diagonal element $(1, \dots, 1) \in \mathbb{N}^r$ is fixed under every automorphism of $\mathbb{N}^r$, such Δ_is are compatible over different admissible standard charts and glue to the distinguished element required in the Lemma. □

Definition 5.7. Let $(X, \mathscr{N}_1, \dots, \mathscr{N}_r)$ be a virtually r-embeddable semistable variety. An admissible family of log structures of r-embedding type on $(X, \mathscr{N}_1, \dots, \mathscr{N}_r)$ is a family of log structures $(\mathscr{M}_1, \dots, \mathscr{M}_r)$ such that for each $0 < i \leq r$, $\overline{\mathscr{M}_i} \simeq \mathscr{N}_i$.

Let $\mathscr{M}$ be a log structure on X and $a \in H^0(X, \overline{\mathscr{M}})$. Denote by $\tau : \mathscr{M} \to \overline{\mathscr{M}}$ the canonical map. Then

$$\tau^{-1}(a) : U \mapsto \left\{ m \in \mathscr{M}(U) \middle| \tau_U(m) = a|_U \right\}$$

is an $\mathscr{O}_X^*$-torsor. We denote by $\mathscr{L}_a$ its associated invertible sheaf.

Theorem 5.8. *Let $(X, \mathscr{N}_1, \dots, \mathscr{N}_r)$ be a virtually r-embeddable semistable variety, then it admits an admissible family of log structures $(\mathscr{M}_1, \dots, \mathscr{M}_r)$ of r-embedding type if and only if there are $\mathscr{L}_1, \dots, \mathscr{L}_r \in \mathrm{Pic}(X)$ such that $\mathscr{L}_i|_{Z_{\mathscr{N}_i}} \simeq L_{\mathscr{N}_i}$ for each i. Moreover, for each $i = 1, \dots, r$, the assignment*

$$\mathscr{M}_i \mapsto \mathscr{L}_{\Delta_i} \quad (\text{c.f. Lemma 5.6})$$

gives a one-to-one correspondence between

- *the isomorphic classes of log structures $\mathscr{M}_i$ on X such that $\overline{\mathscr{M}_i} \simeq \mathscr{N}_i$, and*
- *the isomorphic classes of pairs $(\mathscr{L}_i, \epsilon_i)$ consisting of an invertible sheaf $\mathscr{L}_i \in \mathrm{Pic}(X)$ and an isomorphism $\epsilon_i : \mathscr{L}_i|_{Z_{\mathscr{N}_i}} \simeq L_{\mathscr{N}_i}$.*

Proof. Let $\mathscr{K}_i$ be the sheaf of groups defined by

$$0 \to \mathscr{K}_i \to \mathscr{O}_X^* \to \mathscr{O}_{Z_{\mathscr{N}_i}}^* \to 0.$$

This induces an exact sequence of cohomologies

$$H^1(X, \mathscr{O}_X^*) \to H^1(Z_{\mathscr{N}_i}, \mathscr{O}_{Z_{\mathscr{N}_i}}^*) \xrightarrow{\delta} H^2(X, \mathscr{K}_i). \tag{5.7}$$

Since $L_{\mathscr{N}_i}$ is invertible (Proposition 5.5), it defines a cohomological class $[L_{\mathscr{N}_i}] \in$

$H^1(Z_{\mathcal{N}_i}, \mathcal{O}^*_{Z_{\mathcal{N}_i}})$. The first part of the theorem is a consequence of the following two claims:

Claim 1: The pseudo-functor from the small étale site $X_{ét}$

$$ob_{\mathcal{N}_i} : U \in X_{ét} \mapsto \text{ the groupoid of log structures } \mathcal{M}_U \text{ on } U \text{ such that } \overline{\mathcal{M}_U} \simeq \mathcal{N}_i|_U$$

is a $\mathcal{K}_i$-gerbe. Thus the obstruction to the existence of a log structure $\mathcal{M}_i$ on X such that $\overline{\mathcal{M}_i} \simeq \mathcal{N}_i$ is $[ob_{\mathcal{N}_i}] \in H^2(X, \mathcal{K}_i)$.

Claim 2: $\delta([L_{\mathcal{N}_i}]) = [ob_{\mathcal{N}_i}]$.

In fact, by the two claims, there exists a log structure $\mathcal{M}_i$ on X such that $\overline{\mathcal{M}_i} \simeq \mathcal{N}_i$ if and only if $[ob_{\mathcal{N}_i}] = \delta([L_{\mathcal{N}_i}]) = 0$. Since (5.7) is exact, this is equivalent to that $L_{\mathcal{N}_i}$ can be extended to an invertible sheaf on X. The proof of the last part of the theorem is contained in the proof of claim 2.

PROOF OF CLAIM 1:

By [24, Corollary A.2], $ob_{\mathcal{N}_i}$ is a stack over $X_{ét}$. First we show that $ob_{\mathcal{N}_i}(U) \neq \emptyset$ when

$$(\phi : U \to X, U \subset V, (x_{11}, \dots, x_{1k_1}, \dots, x_{r1}, \dots, x_{rk_r}, y_1, \dots, y_k)).$$

is an admissible standard chart. Then

$$U = \bigcap_{i=1}^{r} D_i \subset V, \quad D_i = \{x_{i1} \cdots x_{ik_i} = 0\}.$$

Define

$$\mathcal{M}_{D_i} : V' \in V_{ét} \mapsto \{f \in \mathcal{O}_{V'} | f \text{ is invertible outside } D_i \times_V V'\},$$

then $\mathcal{M}_{D_i}|_U \in ob_{\mathcal{N}_i}(U)$.

It remains to show that the log structures in $ob_{\mathcal{N}_i}$ are locally isomorphic and the sheaf of automorphism groups of $ob_{\mathcal{N}_i}$ is isomorphic to $\mathcal{K}_i$.

Let $U' \in X_{ét}$ be an étale neighborhood of $x \in X$ and $\mathcal{M} \in ob_{\mathcal{N}_i}(U')$, by [24, Proposition 2.1], there is an admissible standard chart $U \to U' \to X$ such that

- $\mathcal{M}|_U$ has a chart $\mathbb{N}^{k_i} \simeq (\mathcal{N}_i)_x \to \mathcal{M}|_U$, and
- the composition

$$(\mathcal{N}_i)_x \to \mathcal{M}_x \to \overline{\mathcal{M}_x} \simeq (\mathcal{N}_i)_x$$

is the identity.

Denote by

$$(\phi : U \to X, U \subset V, (x_{11}, \dots, x_{1k_1}, \dots, x_{r1}, \dots, x_{rk_r}, y_1, \dots, y_k))$$

the standard chart mentioned above. We see that the log structure $\mathcal{M}_i|_U \to \mathcal{O}_U$ is isomorphic to

$$\mathbb{N}^{k_i} \oplus \mathcal{O}^*_U \to \mathcal{O}_U$$

$$(e_j, u) \mapsto uu_{ij}x_{ij}, \quad j = 1, \dots, k_i$$

for some $u_{ij} \in \mathcal{O}^*_U$ after a suitable shrinking. In the formula, e_j is $(0, \dots, 1, \dots, 0) \in \mathbb{N}^{k_i}$ where 1 lies on the j-th component.

As a consequence, the log structure $\mathcal{M}_i|_U$ is determined by $u_{i1}, \dots, u_{ik_i} \in \mathcal{O}^*_U$. Every two such log structures are isomorphic as shown in the following diagram

$$\begin{array}{ccc} \mathbb{N}^{k_i} \oplus \mathscr{O}_U^* & \xrightarrow{\varphi} & \mathbb{N}^{k_i} \oplus \mathscr{O}_U^* \\ & \searrow & \downarrow \\ & & \mathscr{O}_U. \end{array}$$

Here $\varphi(e_j, u) = (e_j, v_{ij}u)$ for some $v_{ij} \in \mathscr{O}_U^*$ which satisfies

$$v_{ij}x_{ij} = x_{ij}, \quad j = 1, \ldots, k_i.$$

Therefore

$$v_{ij} = 1 + a_j x_i / x_{ij}, \quad x_i = x_{i1} \cdots x_{ik_i}$$

for some $a_j \in \mathscr{O}_U$ after a suitable shrinking of U. Since $x_i = 0$ on U, $(v_{i1}, \ldots, v_{ik_i})$ is determined by their product

$$v_{i1} \cdots v_{ik_i} = 1 + \sum_{j=1}^{k_i} a_j x_i / x_{ij}.$$

This shows that there is a natural isomorphism

$$\mathscr{A}ut_U(\mathscr{M}) \simeq \mathrm{Ker}(\mathscr{O}_U^* \to \mathscr{O}^*_{Z_{\mathscr{N}_i} \times_X U}).$$

Omitting the verification of compatibility we show that

$$\mathscr{A}ut_{U'}(\mathscr{M}) \simeq \mathscr{K}_i|_{U'}.$$

This finishes the proof of claim 1.

PROOF OF CLAIM 2:

By [7, IV. 3.1.1], $\delta(L_{\mathscr{N}_i}) \in H^2(X, \mathscr{K}_i)$ is represented by the gerbe

$$Ex(L_{\mathscr{N}_i}) : U \in X_{\acute{e}t} \mapsto \{(\mathscr{L}, \epsilon) | \mathscr{L} \in \mathrm{Pic}(U), \epsilon : \mathscr{L}|_{Z_{\mathscr{N}_i} \times_X U} \simeq L^\vee_{\mathscr{N}_i}|_{Z_{\mathscr{N}_i} \times_X U}\}.$$

Since any fully-faithful functor between gerbes is an equivalence, it suffices to construct a functor

$$F : ob_{\mathscr{N}_i} \to Ex(L_{\mathscr{N}_i})$$

such that

$$\mathrm{Iso}_U(a, b) \simeq \mathrm{Iso}_U(Fa, Fb)$$

for each admissible standard chart U of X and for each $a, b \in ob_{\mathscr{N}_i}(U)$.

Let $U' \in X_{\acute{e}t}$ and $\mathscr{M}_{U'} \in ob_{\mathscr{N}_i}(U')$, then the preimage $\tau_{U'}^{-1}\Delta_i$ of the diagonal element $\Delta_i \in \mathscr{N}_i$ (Lemma 5.6) under the canonical morphism

$$\tau_{U'} : \mathscr{M}_{U'} \to \overline{\mathscr{M}_{U'}} \simeq \mathscr{N}_i|_{U'}$$

is an $\mathscr{O}_{U'}^*$-torsor. Hence $\tau_{U'}^{-1}\Delta_i \otimes_{\mathscr{O}_{U'}^*} \mathscr{O}_{U'}$ is an invertible sheaf on U' which is generated by x_i locally on each standard chart U. Therefore we have an isomorphism

$$\epsilon_{\mathscr{M}_{U'}} : \tau_{U'}^{-1}\Delta_i \otimes_{\mathscr{O}_{U'}^*} \mathscr{O}_{U'}|_{Z_{\mathscr{N}_i} \times_X U'} \simeq L^\vee_{\mathscr{N}_i}.$$

This shows the equivalence of two gerbes and implies the last statement of the theorem. Define

$$F : ob_{\mathscr{N}_i} \to Ex(L_{\mathscr{N}_i}),$$

$$\mathscr{M}_{U'} \mapsto (\tau_{U'}^{-1}\Delta_i \otimes_{\mathscr{O}^*_{U'}} \mathscr{O}_{U'}, \epsilon_{\mathscr{M}_{U'}}).$$

We leave it to the readers to verify that F keeps the isomorphism groups. This proves claim 2. □

Theorem 5.8 suggests the following generalized notion of d-semistability.

Definition 5.9. A semistable variety X is called *d-semistable* if the following conditions hold.

(1) There is an integer $r > 0$ such that X is virtually r-embeddable. Denote $\mathscr{N}_1, \ldots, \mathscr{N}_r$ be the sheaves of monoids in Definition 5.1.
(2) For each $i = 1, \ldots, r$, $L_{\mathscr{N}_i} \simeq \mathscr{O}_{Z_{\mathscr{N}_i}}$.

When $r = 1$, this is exactly the notion of d-semistability in [5].

Theorem 5.10. *Let X be a semistable variety, then the followings are equivalent:*

(1) *X admits a log structure of semistable type over* $(\mathrm{Spec}(k), \mathbb{N}^r \mapsto 0)$.
(2) *For each* $i = 1, \ldots, r$, *there is a log structure* $\mathscr{M}_i$ *on X and a log morphism*

$$\pi_i : (X, \mathscr{M}_i) \to (\mathrm{Spec}(k), \mathbb{N} \mapsto 0)$$

such that

$$(X, \overline{\mathscr{M}_1}, \ldots, \overline{\mathscr{M}_r})$$

is a virtually r-embeddable semistable variety and $(\mathscr{M}_1, \ldots, \mathscr{M}_r)$ *is an admissible family of log structures of r-embedding type. Moreover the canonical homomorphism*

$$\overline{\pi_i^*} : \mathbb{N} \to \mathscr{M}_i \to \overline{\mathscr{M}_i}$$

maps 1 *to the distinguished diagonal element* $\Delta_i \in \overline{\mathscr{M}_i}$ (Lemma 5.6).
(3) *X is d-semistable* (Definition 5.9).

Proof. **(1)⇔(2):** Assume that we have $(\mathscr{M}_1, \ldots, \mathscr{M}_r)$ as in (2). Denote by $\pi_i^* : \mathbb{N} \to \mathscr{M}_i$ the morphism induced by π_i. The diagram

$$\begin{array}{ccc} \mathscr{M} = \mathscr{M}_1 \oplus_{\mathscr{O}_X^*} \mathscr{M}_2 \oplus_{\mathscr{O}_X^*} \cdots \oplus_{\mathscr{O}_X^*} \mathscr{M}_r & \longrightarrow & \mathscr{O}_X \\ \uparrow{\scriptstyle \bigoplus_{i=1}^r \pi_i^*} & & \uparrow \\ \mathbb{N}^r & \xrightarrow{\quad 0 \quad} & k \end{array}$$

gives a log structure of semistable type over $(\mathrm{Spec}(k), \mathbb{N}^r \mapsto 0)$. Conversely, assume that

$$\pi : (X, \mathscr{M}) \to (\mathrm{Spec}(k), \mathbb{N}^r \mapsto 0)$$

is semistable log smooth. Let $\mathscr{M}_i \subset \mathscr{M}$ be the subsheaf of monoids defined by

$$\mathscr{M}_i(U) = \{m \in \mathscr{M}(U) | \exists m' \in \mathscr{M}(U), m + m' \in \pi^*(\mathbb{N}e_i)\}.$$

On a chart as in Definition 2.3, $\mathscr{M}_i$ is generated by $e_{l_{i-1}+1}, \dots, e_{l_i}$. This shows that $(\mathscr{M}_1, \dots, \mathscr{M}_r)$ satisfies the statements in (2).

(2)⇔(3): By Theorem 5.8, (2) and (3) imply that X has a family of log structures $(\mathscr{M}_1, \dots, \mathscr{M}_r)$ of r-embedding type. Therefore is suffices to show that for each $i = 1, \dots, r$, the diagonal morphism (Lemma 5.6)

$$\mathbb{N} \to \overline{\mathscr{M}_i}, \quad 1 \mapsto \Delta_i$$

lifts to $\mathbb{N} \to \mathscr{M}_i$ if and only if $L_{\overline{\mathscr{M}_i}} \simeq \mathscr{O}_{Z_{\overline{\mathscr{M}_i}}}$.

The short exact sequence

$$0 \to \mathscr{O}_X^* \to \mathscr{M}_i^{\mathrm{gp}} \xrightarrow{\tau} \overline{\mathscr{M}_i}^{\mathrm{gp}} \to 0$$

induces the exact sequence

$$\mathcal{H}om(\mathbb{Z}, \mathscr{M}_i^{\mathrm{gp}}) \to \mathcal{H}om(\mathbb{Z}, \overline{\mathscr{M}_i}^{\mathrm{gp}}) \xrightarrow{\delta} \mathcal{E}xt^1(\mathbb{Z}, \mathscr{O}_X^*) \simeq H^1(X, \mathscr{O}_X^*).$$

Let $\Delta \in \mathcal{H}om(\mathbb{Z}, \overline{\mathscr{M}_i}^{\mathrm{gp}})$ be the diagonal morphism. Then it lifts to $\mathscr{M}_i^{\mathrm{gp}}$ (which is equivalent to that $\mathbb{N} \to \overline{\mathscr{M}_i}$ lifts to $\mathbb{N} \to \mathscr{M}_i$) if and only if $\delta(\Delta) = 0$. On the other hand, $\delta(\Delta)$ is represented by the $\mathscr{O}_X^*$-torsor $\tau^{-1}(\Delta_i)$. By Theorem 5.8, the invertible sheaf $\mathscr{L}_{\Delta_i}$, associated to $\tau^{-1}(\Delta_i)$, satisfies

$$\mathscr{L}_{\Delta_i}|_{Z_{\mathscr{N}_i}} \simeq L_{\mathscr{N}_i}.$$

If $\mathbb{N} \to \overline{\mathscr{M}_i}$ lifts to $\mathbb{N} \to \mathscr{M}_i$, we have a log morphism

$$\pi_i : (X, \mathscr{M}_i) \to (\mathrm{Spec}(k), \mathbb{N} \mapsto 0).$$

$\tau^{-1}(\Delta_i)$ is isomorphic to the pullback of the k^*-torsor $\tau^{-1}(1) \subset k^* \oplus \mathbb{N}$ lifting $1 \in \mathbb{N}$. Hence it is a trivial torsor, so $L_{\mathscr{N}_i} \simeq \mathscr{O}_{Z_{\mathscr{N}_i}}$.

On the other hand, if $L_{\mathscr{N}_i} \simeq \mathscr{O}_{Z_{\mathscr{N}_i}}$, the pair $(\mathscr{O}_X, \epsilon)$ where $\epsilon : \mathscr{O}_X|_{Z_{\mathscr{N}_i}} \simeq \mathscr{O}_{Z_{\mathscr{N}_i}}$ corresponds to a log structure $\mathscr{M}_i$ such that $\overline{\mathscr{M}_i} \simeq \mathscr{N}_i$. Since the obstruction of $\mathbb{N} \to \overline{\mathscr{M}_i}$ lifting to $\mathbb{N} \to \mathscr{M}_i$ is $\delta(\Delta) = [\mathscr{L}_{\Delta_i}] = [\mathscr{O}_X] = 0$, we obtain the theorem. □

By the proof of theorem 5.10, we see that under the correspondence in Theorem 5.8, the isomorphic classes of $\mathscr{M}_i$ having a diagonal element corresponds to the set $\mathrm{Aut}(\mathscr{O}_{Z_{\mathscr{N}_i}}) \simeq H^0(X, \mathscr{O}^*_{Z_{\mathscr{N}_i}})$. Thus we obtain the following uniqueness of the log structure of semistable type.

Corollary 5.11. *Let $(X, \mathscr{N}_1, \dots, \mathscr{N}_r)$ be virtually r-embeddable such that $L_{\mathscr{N}_i} \simeq \mathscr{O}_{Z_{\mathscr{N}_i}}$ for each $1 \le i \le r$. If $Z_{\mathscr{N}_i}$s are connected, there is a unique (up to isomorphisms) family of log structures $(\mathscr{M}_1, \dots, \mathscr{M}_r)$ satisfying* (2) *in* Theorem 5.10.

As an application of this section, we construct some examples of Fano semistable varieties that do not have a semistable smoothing.

Example 5.12. For each $1 \le i \le r$, let $X_i \subset \mathbb{P}^{n_i}$ be a simple normal crossing divisor of degree $d_i \le n_i$. By the adjunction formula,

$$\omega_{X_i} \simeq \omega_{\mathbb{P}^{n_i}}|_{X_i} \otimes \mathscr{O}_{X_i}(d_i) \simeq \mathscr{O}_{X_i}(d_i - n - 1)$$

is anti-ample. Thus $X = \prod_{i=1}^r X_i$ is a semistable Fano variety. We claim that if there is an index i such that $n_i \geq 3$, then X does not admit a log structure of semistable type.

Denote $p_i : X \to X_i$ be the projection. Then

$$T^1_X \simeq \bigoplus_{i=1}^{r} p_i^* T^1_{X_i}.$$

Now we compute $T^1_{X_i}$. Since we have a short exact sequence

$$0 \to \mathscr{O}(-d_i)|_{X_i} \to \Omega_{\mathbb{P}^{n_i}}|_{X_i} \to \Omega_{X_i} \to 0,$$

consequently,

$$T^1_{X_i} \simeq \mathscr{O}(d_i)|_{(X_i)_{\mathrm{sing}}}.$$

Hence we obtain

$$T^1_X \simeq \bigoplus_{i=1}^{r} p_i^* \mathscr{O}(d_i)|_{(X_i)_{\mathrm{sing}}}.$$

If $n_{i_0} \geq 3$ for some i_0, then $\dim(X_{i_0})_{\mathrm{sing}} > 0$. By Proposition 5.5 and Theorem 5.10, X does not admit a log structure of semistable type. By Theorem 1.1, X does not admit semistable smoothing for every r.

Appendix A Proof of Theorem 3

In this appendix we give the detailed proof of Kato's Theorem 3.4 in the (more general) context of Cartier type morphisms ([16, Definition 4.8]).

Theorem A.1 (K. Kato [16])**.** *Let k be a perfect field of characteristic $p > 0$. Let $f : X \to \mathbf{k} = (\mathrm{Spec}(k), P \mapsto 0)$ be a log smooth and integral morphism between fine log schemes. Let X' be the Frobenius base change of X over $\mathbf{k}$ as a log scheme. Denote $F : X \to X'$ be the relative log Frobenius morphism. If f is of Cartier type and X' has a log smooth lifting over $\mathbf{W}_2 = (W_2(k), P \mapsto 0)$, then $\tau_{<p} F_* \Omega_X$ is decomposable.*

Remark: If $f : X \to \mathbf{k}$ is semistable log smooth (Definition 2.3), then f is of Cartier type.

Before proving the theorem we recall some facts about log smooth lifting. Let P be a monoid and let $\pi : X \to \mathbf{k} = (\mathrm{Spec}(k), P \mapsto 0)$ be a log smooth morphism, then a log smooth lifting of π to $\mathbf{W_2} = (\mathrm{Spec} W_2(k), P \mapsto 0)$ is a log smooth morphism $\pi' : X' \to \mathbf{W_2}$ such that $\pi'|_{\mathbf{k}} \simeq \pi$. Étale locally a log smooth lifting can be described as follows (c.f. [16] Proposition 3.14):

Choosing a chart Q of X as in Theorem 2.2, we get the following diagram

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow f & & \downarrow f' \\ \mathrm{Spec}(k[Q]) \times_{\mathrm{Spec}(k[P])} \mathrm{Spec} k & \longrightarrow & \mathrm{Spec}(W_2[Q]) \times_{\mathrm{Spec}(W_2[P])} \mathrm{Spec} W_2 \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} W_2 \end{array},$$

where f is strict and étale. We can complete the diagram by the dotted arrows so that f'

is strict, étale and the square on the top is a fiber product of schemes (SGA I [10], Exposé 1, Proposition 8.1). X' is a log smooth lifting of X to $\mathbf{W_2}$. When X is affine, such a lifting X' is unique up to isomorphisms. As a consequence, if X is integral over $\mathbf{k}$, then X' is integral over $\mathbf{W_2}$. Hence $\underline{X'}$ is flat over $W_2(k)$ ([16, Corollary 4.5]). Therefore there exists a homomorphism '*multiplication by p*' :

$$\times p : \mathscr{O}_X \to \mathscr{O}_{X'},$$

which can induce an isomorphism

$$\mathscr{O}_X \to p\mathscr{O}_{X'}.$$

Moreover, every element $x \in \mathscr{O}_{X'}$ can be written uniquely as $x = a + pb$ where $a, b \in \mathscr{O}_X$. In particular, we can define a section

$$\sigma : \mathscr{O}_X \to \mathscr{O}_{X'}$$

of the canonical projection $\mathscr{O}_{X'} \to \mathscr{O}_X$ by letting $\sigma(a) = a + p \times 0$.

We can get the following lemma by the above construction.

Lemma A.2. *Let X be an integral log smooth variety over* $\mathbf{k}$ *and X' be a log smooth lifting of X over* $\mathbf{W_2}$. *Then for each element $x \in \mathscr{M}_{\mathbf{W_2}}$, its image in $\mathscr{O}_{X'}$ can be written as $\sigma(a)(1 + p\sigma(u))$ where $a, u \in \mathscr{O}_X$.*

Proof of Theorem A.1. The proof is divided into 5 steps:

Step 1 (Cartier isomorphism):

Lemma A.3. *Notations as in Theorem A.1, there is a canonical isomorphism of $\mathscr{O}_{X'}$-modules*

$$C^{-1} : \Omega^q_{X'/\mathbf{k}} \to H^q(\Omega^\bullet_{X/\mathbf{k}}),$$

for any $q \in \mathbb{Z}$ characterized by

$$C^{-1}(a\mathrm{dlog}(b'_1) \wedge \cdots \wedge \mathrm{dlog}(b'_q)) = F^*(a)\mathrm{dlog}(b_1) \wedge \cdots \wedge \mathrm{dlog}(b_q),$$

where $a \in \mathscr{O}_{X'}$, $b_1, \cdots, b_q \in \mathscr{M}_X$ and for each $i = 1, \ldots, q$, b'_i is the pullback of b_i.

Proof. See Theorem 4.12 in [16]. □

Step 2 (Local Decomposition): Assume that F has a global lifting

$$G : X^{(1)} \to X'^{(1)},$$

where $X^{(1)}$ (resp. $X'^{(1)}$) is a log smooth lifting of X (resp. X') over $\mathbf{W}_2$. Then we have the following facts.

(1) Because $X^{(1)}$ is log smooth and integral over $\mathbf{W}_2$, the underlying scheme is flat over $W_2(k)$ ([16, Corollary 4.5]). Thus the multiplication by p induces an isomorphism

$$\mathbf{p} : \Omega_{X/\mathbf{k}} \xrightarrow{\sim} p\Omega_{X^{(1)}/\mathbf{W}_2}.$$

(2) The image of the canonical homomorphism

$$G^* : \Omega_{X'^{(1)}/\mathbf{W}_2} \to G_*\Omega_{X^{(1)}/\mathbf{W}_2}$$

is contained in $pG_*\Omega_{X^{(1)}/\mathbf{W}_2}$. It follows from the diagram

$$\begin{array}{ccc} \Omega_{X'^{(1)}/\mathbf{W}_2} & \xrightarrow{G^*} & G_*\Omega_{X^{(1)}/\mathbf{W}_2} \\ \downarrow{\scriptstyle \bmod p} & & \downarrow{\scriptstyle \bmod p} \\ \Omega_{X'/\mathbf{k}} & \xrightarrow{F^*=0} & F_*\Omega_{X/\mathbf{k}}. \end{array}$$

(3) Since $\mathbf{p}^{-1}G^*(p\Omega_{X'^{(1)}/\mathbf{W}_2}) = 0$, there is a unique homomorphism $\frac{G^*}{p}$ rendering the square commutative

$$\begin{array}{ccc} \Omega_{X'^{(1)}/\mathbf{W}_2} & \xrightarrow{G^*} & pG_*\Omega_{X^{(1)}/\mathbf{W}_2} \\ \downarrow & & \uparrow{\scriptstyle \mathbf{p}} \\ \Omega_{X'/\mathbf{k}} & \xrightarrow{\frac{G^*}{p}} & F_*\Omega_{X/\mathbf{k}}. \end{array}$$

Noticing that $d\mathrm{dlog}(x) = 0$ for any $x \in \mathscr{M}_{X^{(1)}}$, we have

$$d(\mathbf{p}^{-1}G^*(\mathrm{dlog}(x))) = \mathbf{p}^{-1}G^*d\mathrm{dlog}(x) = 0.$$

Hence the image of $\frac{G^*}{p}$ is contained in the kernel of the differential of the log de Rham complex.

For each $x \in \mathscr{M}_{X'^{(1)}}$, denote a' be the image of x in $\mathscr{O}_{X'^{(1)}}$. Then, by Lemma A.2, $G^*(a') = F(\overline{a'})(1 + pb) = a^p(1 + pb)$ where $\overline{a'}$ is the image of a' in $\mathscr{O}_{X'}$ and a is the preimage of $\overline{a'}$ in $\mathscr{O}_X$. Applying $\mathbf{p}^{-1}G^*$ to

$$a'\mathrm{dlog}(x) = da',$$

we get

$$a^p\mathbf{p}^{-1}G^*\mathrm{dlog}(x) = a^{p-1}(da + adb).$$

Then by taking mod p, we can get

$$\frac{G^*}{p}(\mathrm{dlog}(x)) = \mathrm{dlog}(a) + db,$$

which induces the Cartier isomorphism C^{-1} in degree one.

Define the homomorphism of complexes

$$\phi_G : \bigoplus \Omega^i_{X'/\mathbf{k}}[-i] \to F_*\Omega^\bullet_{X/\mathbf{k}}$$

as follows.

Let

$$\phi^0_G = F^* : \mathscr{O}_X \to F_*\mathscr{O}_{X'}; \quad \phi^1_G = \frac{G^*}{p} : \Omega_{X'/\mathbf{k}} \to F_*\Omega_{X/\mathbf{k}}.$$

For $i > 1$, ϕ^i_G is the composition of $\bigwedge^i \phi^1_G$ and the product $\bigwedge^i F_*\Omega_{X/\mathbf{k}} \to F_*\Omega^i_{X/\mathbf{k}}$. Then ϕ_G is a quasi-isomorphism, which induces the Cartier isomorphism C^{-1}.

Step 3 (Compatibility):

Lemma A.4. *To a pair* $(G_1 : X^{(1)}_1 \to X'^{(1)}, G_2 : X^{(1)}_2 \to X'^{(1)})$ *of the liftings of* F*, we can associate canonically a homomorphism*

$$h(G_1, G_2) : \Omega_{X'/\mathbf{k}} \to F_*\mathscr{O}_X$$

such that

$$\frac{G_2^*}{p} - \frac{G_1^*}{p} = dh(G_1, G_2).$$

If $G_3 : X_3^{(1)} \to X'^{(1)}$ *is the third lifting of* F, *one has*

$$h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3).$$

Proof. We may assume initially that $X_1^{(1)}$ and $X_2^{(1)}$ are isomorphic over $\mathbf{W}_2$. Choose an isomorphism $u : X_1^{(1)} \simeq X_2^{(1)}$. Then G_2u and G_1 lift F. Since the set of the liftings of F is a torsor under the action of $Hom(F^*\Omega_{X'/\mathbf{k}}, \mathscr{O}_X) \simeq Hom(\Omega_{X'/\mathbf{k}}, F_*\mathscr{O}_X)$ ([16, Proposition 3.9]), G_2u and G_1 differ by a unique $h_u \in Hom(\Omega_{X'/\mathbf{k}}, F_*\mathscr{O}_X)$ and we have

$$\frac{(G_2u)^*}{p} - \frac{G_1^*}{p} = dh_u.$$

If $v : X_1^{(1)} \simeq X_2^{(1)}$ is another isomorphism, then u and v differ by a homomorphism $\delta : \Omega_{X/\mathbf{k}} \to \mathscr{O}_X$. Therefore G_2u and G_2v differ by the composition of δ and the homomorphism $F^*\Omega_{X'/\mathbf{k}} \to \Omega_{X/\mathbf{k}}$, which is zero. Therefore, $G_2u = G_2v$ and h_u do not depend on the choice of u.

Now return to the general case. Since $X_1^{(1)}$ and $X_2^{(1)}$ are locally isomorphic, there exists h_u locally (which does not depend on the choice of u). We can glue them together to obtain $h(G_1, G_2)$ which satisfies the properties in the lemma. □

Step 4 (Global Decomposition): Fix the following data:

- a lifting $X'^{(1)}$ of X' over $\mathbf{W}_2$,
- an open affine covering $\mathscr{U} = (U_i)_{i\in I}$ of X,
- a lifting $G_i : U_i^{(1)} \to X'^{(1)}$ of $F|_{U_i}$ over $\mathbf{W}_2$ for each $i \in I$, where $U_i^{(1)}$ is a log smooth lifting of U_i over $\mathbf{W}_2$.

According to the arguments above, we have for each $i \in I$ a homomorphism of complexes

$$\phi^1_{G_i} : \Omega_{X'/\mathbf{k}}|_{U_i}[-1] \to F_*\Omega^\bullet_{X/\mathbf{k}}|_{U_i},$$

and for each pair (i, j) a homomorphism

$$h_{ij} = h(G_i|_{U_{ij}}, G_j|_{U_{ij}}) : \Omega_{X'/\mathbf{k}}|_{U_{ij}} \to F_*\Omega^\bullet_{X/\mathbf{k}}|_{U_{ij}},$$

where $U_{ij} = U_i \cap U_j$. These data are connected by

$$\phi^1_{G_j} - \phi^1_{G_i} = dh_{ij} \quad (\text{on } U_{ij}),$$

$$h_{ij} + h_{jk} = h_{ik} \quad (\text{on } U_{ijk} = U_i \cap U_j \cap U_k).$$

Hence they define a homomorphism of complexes of $\mathscr{O}_{X'}$-modules

$$\phi^1_{X'^{(1)},(\mathscr{U},(G_i))} : \Omega_{X'/\mathbf{k}} \to \check{\mathscr{C}}(\mathscr{U}, F_*\Omega^\bullet_{X/\mathbf{k}}),$$

where $\check{\mathscr{C}}(\mathscr{U}, F_*\Omega^\bullet_{X/\mathbf{k}})$ is the total complex of the Čech bicomplex of the covering $\mathscr{U}$ with values in $F_*\Omega^\bullet_{X/\mathbf{k}}$.

Notice that there is a canonical quasi-isomorphism of complexes

$$\epsilon : F_*\Omega^\bullet_{X/\mathbf{k}} \to \check{\mathscr{C}}(\mathscr{U}, F_*\Omega^\bullet_{X/\mathbf{k}}).$$

Then we have a homomorphism

$$\phi^1_{X'^{(1)}} = \epsilon^{-1}\phi^1_{X^{(1)},(\mathscr{U},(G_i))} : \Omega_{X'/k}[-1] \to F_*\Omega^\bullet_{X/\mathbf{k}}$$

in $D(X')$, which induces C^{-1}. Therefore, $\tau_{\leq 1}F_*\Omega_{X/\mathbf{k}}$ is decomposable.

Step 5 (Using multiplication structure): $\phi^1_{X'^{(1)}}$ induces a morphism in $D(X')$

$$(\phi^1_{X'^{(1)}})^{\otimes^L i} : (\Omega_{X'/k})^{\otimes i}[-i] \to (F_*\Omega^\bullet_{X/\mathbf{k}})^{\otimes^i}.$$

For $i < p$, by composing $(\phi^1_{X'^{(1)}})^{\otimes^L i}$ with

$$\Omega^i_{X'/k} \to (\Omega_{X'/k})^{\otimes i}, \quad \omega_1 \wedge \cdots \wedge \omega_i \mapsto \frac{1}{i!}\sum_{\sigma\in S_i} \mathrm{sgn}(\sigma)\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(i)}$$

and the canonical multiplication

$$(F_*\Omega^\bullet_{X/\mathbf{k}})^{\otimes^i} \to F_*\Omega^\bullet_{X/\mathbf{k}},$$

we obtain a morphism in $D(X')$

$$\phi^i_{X'^{(1)}} : \Omega^i_{X'/k}[-i] \to F_*\Omega^\bullet_{X/\mathbf{k}}$$

that induces C^{-1} on the cohomological level for each $i < p$. This completes the proof of the theorem.

□

Acknowledgements. The author is grateful to the anonymous referee whose suggestions are of great help.

References

[1] D. Abramovich and K. Karu: *Weak semistable reduction in characteristic 0*, Invent. Math. **139** (2000), 241–27.
[2] M. Artin: *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 23–58.
[3] P. Deligne and L. Illusie: *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), 247–270.
[4] J. Bertin, J.-P. Demailly, L. Illusie and C. Peters: Introduction to Hodge theory, SMF/AMS Texts and Monographs, **8**, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002.
[5] R. Friedman: *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) **118** (1983), 75–114.
[6] T. Fujita: *On del Pezzo fibrations over curves*, Osaka J. Math. **27** (1990), 229–245.
[7] J. Giraud: Cohomologie non abélienne, Springer-Verlag, Berlin-New York, 1971 (French).

[8] M. Green and P. Griffiths: *Deformation theory and limiting mixed Hodge structures*; in Recent advances in Hodge theory, London Math. Soc. Lecture Note Ser. **427**, Cambridge Univ. Press, Cambridge, 2016, 88–133.
[9] A. Grothendieck: *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I.* (French), Publ. Math. Inst. Hautes Études Sci. **20** (1964), 5–259pp.
[10] A. Grothendieck: Revêtements étales et groupe fondamental, Séminaire de Géométrie Algébrique, Springer Lecture Notes **224**, Springer-Verlag, Berlin, 1971.
[11] R. Hartshorne: Residues and duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966.
[12] Y. Kachi: *Global smoothings of degenerate Del Pezzo surfaces with normal crossings*, J. Algebra **307** (2007), 249–253.
[13] K. Karu: *Semistable reduction in characteristic zero*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–Boston University.
[14] F. Kato: *Logarithmic Embeddings and Logarithmic Semistable Reductions*, arXiv:alg-geom/9411006v2.
[15] F. Kato: *Log smooth deformation theory*, Tohoku Math. J. (2) **48** (1996), 317–354.
[16] K. Kato: *Logarithmic structures of Fontaine-Illusie*; in Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, 191–224.
[17] Y. Kawamata: *Semipositivity theorem for reducible algebraic fiber spaces*, Pure Appl. Math. Q. **7** (2011), Special Issue: In memory of Eckart Viehweg, 1427–1447.
[18] Y. Kawamata: *Hodge theory on generalized normal crossing varieties*, Proc. Edinb. Math. Soc. (2) **57** (2014), 175–189.
[19] Y. Kawamata: *Variation of mixed Hodge structures and the positivity for algebraic fiber spaces*, Algebraic geometry in east Asia—Taipei 2011, Adv. Stud. Pure Math. **65**, Math. Soc. Japan, Tokyo, 2015, 27–57.
[20] Y. Kawamata and Y. Namikawa: *Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties*, Invent. Math. **118** (1994), 395–409.
[21] G. Kempf, F.F. Knudsen, D. Mumford and B. Saint-Donat: Toroidal embeddings. I, Lecture Notes in Mathematics **339**, Springer-Verlag, Berlin-New York, 1973.
[22] J. Kollár: *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), 11–42.
[23] M.C. Olsson: *Log algebraic stacks and moduli of log schemes*, ProQuest LLC, Ann Arbor, MI, 2001. Thesis (Ph.D.)–University of California, Berkeley.
[24] M.C. Olsson: *Logarithmic geometry and algebraic stacks*, Ann. Sci. École Norm. Sup. (4) **36** (2003), 747–791.
[25] N. Tziolas: *Smoothings of Fano varieties with normal crossing singularities*, Proc. Edinb. Math. Soc. (2) **58** (2015), 787–806.

School of Mathematical Sciences
University of Science and Technology of China
Hefei, 230026
China
e-mail: stjc@ustc.edu.cn