EXTRINSIC SYMMETRIC SUBSPACES

Jost ESCHENBURG and Makiko Sumi TANAKA

(Received December 4, 2017, revised March 29, 2019)

Abstract
An extrinsic symmetric space is a submanifold $M \subset V = \mathbb{R}^n$ which is kept invariant by the reflection $s_x$ along every normal space $N_xM$. An extrinsic symmetric subspace is a connected component $M'$ of the intersection $M \cap V'$ for some subspace $V' \subset V$ which is $s_x$-invariant for any $x \in M'$. We give an algebraic charactrization of all such subspaces $V'$.

1. Introduction
It is well known that totally geodesic subspaces of a symmetric space $M$ correspond one-to-one to Lie subtriples of the corresponding Lie triple (which is the tangent space of $M$ with the curvature tensor as algebraic structure). In the present note we study the same question for an important subclass of symmetric spaces, those which allow a nice embedding into euclidean space $V = \mathbb{R}^n$. These are the so called extrinsic symmetric spaces or symmetric R-spaces. More precisely, an extrinsic symmetric space is a submanifold $M \subset V$ such that for any point $x \in M$, the reflection $s_x$ along the normal space $N = N_xM$ keeps $M$ invariant. An extrinsic symmetric subspace $M' \subset M$ will be a connected component $M'$ of the intersection $M \cap V'$ with a subspace $V' \subset V$ which is invariant under $s_x$ for all $x \in M'$; in particular, $M' \subset M$ is totally geodesic. We may assume that $M' \subset V'$ is full. Our main result Theorem 2 characterizes these subspaces $V'$ as follows. By a result of Ferus [5, 6], after splitting o off an affine subspace, $V$ is itself a Lie triple (a tangent space of another symmetric space), and our result is:

A connected component $M'$ of $M \cap V'$ which is full in $V'$ is an extrinsic symmetric subspace if and only if $V' \subset V$ is a Lie subtriple.

The main idea for the proof is given by an alternative approach [3] to Ferus’ theorem where the Lie structure is computed in terms of submanifold geometry. At the end we briefly discuss two questions.

1. Which Lie subtriples $V'$ actually do intersect a given extrinsic symmetric space $M'$?
2. Suppose that $V' \subset V$ is a subspace preserved by $s_x$ for all $x \in M \cap V'$. Suppose further that $M \cap V'$ spans $V'$, but no single connected component of $M \cap V'$ is full in $V'$ (e.g. $M \cap V$ could be discrete). Is $V' \subset V$ still a Lie subtriple? The answer to this question seems to be open.

It is our pleasure to thank Peter Quast for several useful hints and discussion during the

2010 Mathematics Subject Classification. 53C35, 53C40, 57S15.

The second author was partly supported by the Grant-in-Aid for Science Research (C) (No.15K04855), Japan Society for the Promotion of Science.
preparation of this work.

2. Extrinsic symmetric spaces and subspaces

Let $M \subset V$ be a closed submanifold (not necessarily connected) of some euclidean vector space $V = \mathbb{R}^n$. For simplicity of notation\(^1\) we assume that $M$ is contained in the unit sphere $S = S^{n-1} \subset V$. Let $O_n$ denote the orthogonal group on $\mathbb{R}^n$.

\[
O_n = \{ A \in \mathbb{R}^{n \times n} : A'A = I \}
\]

where $\mathbb{R}^{n \times n}$ is the space of real $(n \times n)$-matrices, $A'$ the transposed of the matrix $A$ and $I$ the unit matrix. For any $x \in M$ let $s_x \in O_n$ be the reflection along the normal space $N = N_x M$, that is $s_x = I$ on $N$ and $s_x = -I$ on $T = T_x M$. The submanifold $M$ is called extrinsic symmetric if

\[
s_x(M) = M \forall x \in M.
\]

Then $s_x$ is called the (extrinsic) symmetry at $x$ and the subgroup $K \subset O_n$ generated by all $s_x$, $x \in M$ is the symmetry group of $M$. It acts transitively on every connected component $M' \subset M$.

Example: Orthogonal group. Let $M := O_p \subset V = \mathbb{R}^{p \times p}$ be the orthogonal group (1). This is extrinsic symmetric (with two connected components): the symmetry at $x \in M$ is $s_x(v) = xv'x$ for all $x \in M$ and $v \in V$. Clearly, $s_x(v) = det v$, hence $s_x$ preserves the two connected components of $M$.

Remark. In this example, the symmetry group does not act transitively on $M = O_p$, since the connected components of $M$ are preserved. However, the full isometry group of all orthogonal maps of $V$ preserving $M$ does act transitively since it contains the left (or right) translations with all elements of $O_p$.

A subset $M' \subset M$ is called an extrinsic symmetric subspace if $M'$ is a connected component of $M \cap V'$ for some linear subspace $V' \subset V$ with

\[
s_x(V') = V' \text{ for all } x \in M'.
\]

Given an extrinsic symmetric subspace $M' \subset M$, there might be several subspaces $V' \subset V$ with (3) such that $M'$ is a connected component of $M \cap V'$; we will always choose $V'$ to be the smallest one (the intersection of all such subspaces), or equivalently, $V'$ is just the linear span of $M'$ or $M'$ is full in $V'$.

Every $s_x$, $x \in M'$, preserves $V'$ and its orthogonal complement $V''$, thus it decomposes these spaces into its $(\pm 1)$-eigenspaces which are the intersections with $T$ and $N$.

\[
V' = T' \oplus N', \quad V'' = T'' \oplus N''
\]

where $T' = T \cap V'$, $N' = N \cap V'$ etc. Let $\pi_T : V \rightarrow T$ and $\pi_N : V \rightarrow N$ be the orthogonal projection onto $T$ and $N$. Then $\pi_T(V') = T'$ by (4). Hence $\pi_T|_{M'} : M' \rightarrow T'$ is a diffeomorphism near $x$. Thus $M'$ is a submanifold of both $M$ and $V'$, and the tangent and normal

\(^1\)It turns out that indecomposable extrinsic symmetric spaces (other than straight lines) lie in euclidean spheres, cf. [1].
spaces of \( M' \subset V' \) at \( x \) are \( T' \) and \( N' \).

Let \( \alpha, \alpha' \) denote the second fundamental forms of \( M \subset V \) and \( M' \subset V' \), respectively. E.g. \( \alpha(v, w) = \pi_x(\partial_v w) = (\partial_v w)^N \) for all \( v, w \in T \), where \( \partial_v \) is the directional derivative, \( \partial_v w = \frac{d}{dt}|_{t=0} w(x + tv) \). Then
\[
\alpha'(v, w) = (\partial_v w)^N = (\partial_v w)^N = \alpha(v, w)
\]
for all \( v, w \in T' \) since \( \partial_v w \in V' \) and \( (\partial_v w)^N \in V' \cap N = N' \). As a consequence we obtain

**Lemma 1.** Every connected component \( M' \) of \( M \cap V' \) is totally geodesic in \( M \) and extrinsic symmetric in \( V' \).

Proof. \( M' \subset M \) is totally geodesic by (5). Further, the group
\[
K' = \{ k \in K : k(V') = V' \}
\]
contains the symmetries \( s_x, x \in M' \), and any \( s_x \) preserves both \( M \) and \( V' \) and thus \( M \cap V' \), and its connected component through \( x \) which is \( M' \). Hence \( M' \subset V' \) is extrinsic symmetric by (4). \( \square \)

**Example: Grassmannians.** Let \( V = \mathbb{R}^{P\times P} \) and \( M = O_p \) as in the previous example. Let \( V' = S_p = \{ x \in V : \langle x, x \rangle = 1 \} \). Then \( O_p \cap S_p \) is the set of involutions in \( O_p \) (“reflections”) since for each \( x \in O_p \), that is \( x^t = x^{-1} \), the condition \( x^t = x \) is the same as \( x^{-1} = x \). Orthogonal reflections are in 1:1 correspondence to their fixed spaces, thus \( O_p \cap S_p \) can be considered as the union of all Grassmannians \( G_k = G_k(\mathbb{R}^p) \) with \( k \in \{0, \ldots, p\} \). These are the connected components of \( M \cap V' \). Hence each Grassmannian \( G_k \) is an extrinsic symmetric subspace of one of the components of \( M \). The map \( x \mapsto -x \) on \( \mathbb{R}^{P\times P} \) is an isometry of \( O_p \) which interchanges \( G_k \) and \( G_{p-k} \) while \( G_k \) and \( G_l \) for \( l \neq k, p - k \) are non-isometric.

**3. Lie triples and submanifold geometry**

A connected extrinsic symmetric space \( M \subset S \subset V \) is extrinsic homogeneous, \( M = Kx \) for some \( x \in S \). In other words, it is an orbit of a representation. By a theorem of D. Ferus [3, 6], both the representation and the point \( x \) are very special. The vector space \( V \) carries the structure of an orthogonal Lie triple \( v \) (cf. [7]) and \( x \) satisfies
\[
(\text{ad}_x)^3 = -\text{ad}_x.
\]
More precisely, \( V = v \) is a linear subspace of a Lie algebra \( \mathfrak{g} \) with an involution \( \sigma \) with (±1)-eigenspace decomposition
\[
\mathfrak{g} = \mathfrak{t} \oplus v,
\]
hence \( [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t} \), \([\mathfrak{t}, v] \subset v \), and \([v, v] \subset \mathfrak{t} \). In other words, \( V = v \) is the tangent space of another symmetric space \( P = G/K \), and the Lie triple structure on \( V = v \subset \mathfrak{g} \) is \( R(u, v)w := -[[u, v], w] \). The Lie bracket can be chosen such that \( M = \text{Ad}(K)x \) where \( x \in v \) satisfies (6).

Recall from [3] that the Lie structure on \( v \) can be derived from the submanifold geometry of \( M \subset v \) as follows. Consider the decomposition \( \mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_- \) where \( \mathfrak{t}_+ \) is the Lie algebra of the stabilizer group of \( x \in M \) and \( \mathfrak{t}_- \) denotes the space of infinitesimal transvections at \( x \) (the Killing fields \( A \) with \( \nabla A = 0 \) at \( x \)) which can be identified with the tangent space \( T = T_x M \).
Then the infinitesimal transvection \( S_v \) corresponding to any \( v \in T \) is essentially\(^2\) the second fundamental form \( \alpha : S(T) \to N \) of \( M \subset V \):

\[
S_v : \begin{cases} 
T & \to N \ : \ w \mapsto (\partial_w^v)^\alpha = \alpha(v,w), \\
N & \to T \ : \ \xi \mapsto (\partial_\xi^v)^T = -A_\xi v.
\end{cases}
\]

Moreover, the Lie brackets on \( \mathfrak{p} \) are also given in terms of \( \alpha \): for all \( v, w \in T \) and \( \xi, \eta \in N \) we have by [3]:

\[
[v,w] = [S_v,S_w] \in \mathfrak{t}_+,
\]

\[
[v,\xi] = S_{Ad^v} \in \mathfrak{t}_-,
\]

\[
[\xi,\eta] = -[A_\xi, A_\eta] \in \mathfrak{t}_+.
\]

On the other hand, when \( M = \text{Ad}(K)x \subset \mathfrak{p} \) and \( (\text{ad}_x)^3 = -\text{ad}_x \), the extrinsic symmetry \( s_x \) can be expressed by the Lie structure of \( \mathfrak{p} \subset \mathfrak{g} \) as follows:

\[
s_x = e^{\tau \text{ad}_x}
\]

since \( \text{ad}_x \) is a complex structure on \( \mathfrak{t}_- \oplus T \) interchanging these two subspaces while it vanishes on \( \mathfrak{t}_+ \oplus N \).

### 4. Extrinsic symmetric subspaces

**Theorem 2.** Let \( M = \text{Ad}(K)x \subset \mathfrak{p} \) with \( (\text{ad}_x)^3 = -\text{ad}_x \) be an extrinsic symmetric space and \( \mathfrak{p}' \) a linear subspace of \( \mathfrak{p} \) intersecting \( M \). Let \( M' \) be a connected component of \( M \cap \mathfrak{p}' \) and suppose that \( \mathfrak{p}' \) is the linear span of \( M' \). Then \( M' \subset \mathfrak{p}' \) is an extrinsic symmetric subspace if and only if \( \mathfrak{p}' \) is a Lie subtriple.

Proof. Let \( \mathfrak{p}' \subset \mathfrak{p} \) be a Lie subtriple intersecting \( M \). Let \( M' \) be a connected component of \( M \cap \mathfrak{p}' \) and \( x \in M' \). We have to show that the symmetry \( s_x \) preserves \( \mathfrak{p}' \). Since \( s_x = e^{\tau \text{ad}_x} \) by (10), it has a natural extension to an automorphism of the full Lie algebra \( \mathfrak{g} \). Now \( x \in \mathfrak{p}' \) lies in the Lie subalgebra \( \mathfrak{g}' = \mathfrak{p}' + [\mathfrak{p}',\mathfrak{p}'] \subset \mathfrak{g} \). Thus \( s_x \) preserves both \( \mathfrak{g}' \) and \( \mathfrak{p}' \) and hence its intersection \( \mathfrak{g}' \cap \mathfrak{p} = \mathfrak{p}' \) is preserved. Therefore \( M' \subset \mathfrak{p}' \) is an extrinsic symmetric subspace. 

Vice versa, let \( M' \subset \mathfrak{p}' \) be an extrinsic symmetric subspace. Choose \( x \in M' \). Let \( T' = T_xM' \) and \( N' = \mathfrak{p}' \ominus T' \) be the tangent and normal spaces of \( M' \subset \mathfrak{p}' \). We want to show that \( \mathfrak{p}' \) is a Lie subtriple. We know already that \( M' \subset M \) is totally geodesic (see Lemma 1), thus the second fundamental form \( \alpha' \) of \( M' \subset \mathfrak{p}' \) satisfies \( \alpha' = \alpha|_{S(T')} \). Hence by (9), the restriction of the Lie bracket of \( \mathfrak{p} \) to \( \mathfrak{p}' = T' \ominus N' \) takes values in \( T' \). Thus \( [\mathfrak{p}',\mathfrak{p}'] \subset T' \) and \( \mathfrak{p}' \subset \mathfrak{p} \) is a Lie subtriple. \( \square \)

### 5. Lie subtriples \( \mathfrak{p}' \subset \mathfrak{p} \) intersecting \( M \subset \mathfrak{p} \)

It remains to determine those Lie subtriples \( \mathfrak{p}' \) which have non-empty intersection with \( M \). This can be seen from \( M \) and the Dynkin diagrams of \( \mathfrak{p} \) and \( \mathfrak{p}' \).

Let \( x \in \mathfrak{p} \) be an extrinsic symmetric vector, that is \( x \) satisfies (6) or in other words, \( i,0,-i \) are the eigenvalues of \( \text{ad}_x \). We choose a maximal abelian subalgebra \( \mathfrak{a} \subset \mathfrak{p} \) containing \( x \) and a simple root system \( \alpha_1,\ldots,\alpha_r \) with \( \alpha_i(x) \geq 0 \) for \( i = 1,\ldots,r \). Let \( \alpha \) be any positive root.

\(^2\)Note that \( (\xi \mapsto A_\xi) : N \to S(T) \) is the adjoint of \( \alpha : S(T) \to N \).
On the corresponding root space $g_α \subset g \otimes \mathbb{C}$ we have $\text{ad}_x = i\alpha(x) \cdot \text{id}$. Thus $\alpha(x) \in \{0, \pm 1\}$. In particular this holds for the highest root, $\alpha = \delta = \sum n_i \alpha_i$, hence $\delta(x) = \sum n_i \alpha_i(x) = 1$. Since all $n_i \geq 1$, the element $x$ must be a dual root $x = \xi_j$ for some $j \in \{1, \ldots, r\}$, that is $\alpha_j(x) = 1$ for some $j$ with $n_j = 1$ and $\alpha_i(x) = 0$ for all $i \neq j$. Below we display the Dynkin diagrams of the simple root systems with Dynkin diagrams.

When we have a Lie subtriple $\mathfrak{p}' \subset \mathfrak{p}$, we may choose maximal abelian subalgebras $\mathfrak{a}', \mathfrak{a}$ of $\mathfrak{p}'$, $\mathfrak{a}$ with $\mathfrak{a}' \subset \mathfrak{a}$. Since $M$ is an $\text{Ad}(K)$-orbit, it intersects $\mathfrak{a}$ at some point $x$ in a closed Weyl chamber $C \subset \mathfrak{a}$, and $x$ is a dual root of weight 1 for the simple root system corresponding to the Weyl chamber $C$. Thus:

**Theorem 3.** Let $M = \text{Ad}(K)x$ with $x \in \mathfrak{a}$. Then $M \cap \mathfrak{p}'$ is non-empty if and only if $x \in \mathfrak{a}'$ up to transformations of the Weyl group $W_P$ of $P = G/K$ (with $g = 1 + \mathfrak{p}$), more precisely, if (up to Weyl transformations) $x$ is a dual root of weight one with respect to a simple root system of $\mathfrak{p}'$.

An obvious necessary condition is that $\mathfrak{p}'$ contains dual roots of weight one at all. In particular we see:

**Corollary 4.** If $\mathfrak{p}' \subset \mathfrak{p}$ is a Lie subtriple of the same rank as $\mathfrak{p}$, then $M \cap \mathfrak{p}'$ is nonempty and its connected components are extrinsic symmetric subspaces.

**Examples.** 1. Let $\mathfrak{p} = \mathbb{R}^{p \times p}$ and $M_+ \subset \mathfrak{p}$ be the connected components of $O_p$ (with $M_+ = SO_p$). Further, let $\mathfrak{p}' = S_p \subset \mathfrak{p}$ be the space of symmetric $(p \times p)$-matrices. This is the example of the real Grassmannians (see end of section 2). Then $\mathfrak{p}'$ is of type $AI$ [7, p. 532] with Dynkin diagram $A_{p-1}$. The maximal abelian subalgebra of $\mathfrak{p}$ is the space of diagonal matrices $\mathfrak{a}$. Since $\mathfrak{a} \subset \mathfrak{p}'$, the Lie triples $\mathfrak{p}'$ and $\mathfrak{p}$ have the same rank and hence $\mathfrak{p}'$ intersects $M_+$. The positive dimensional connected components of $M_+ \cap \mathfrak{p}'$ are the real Grassmannians $G_k(\mathbb{R}^p)$, $k = 1, \ldots, p - 1$, which correspond to the $p - 1$ dual roots with weight one in the table above.

2. Let $\mathfrak{p} = \mathfrak{u}_n$ with $n = 2m$ be the Lie algebra of $U_n$ with maximal abelian subspace $\mathfrak{a} = \{i \text{diag}(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$. We identify $\mathfrak{a}$ with $\mathbb{R}^n$ and let $\{e_k : k = 1, \ldots, n\}$ denote the standard orthonormal basis of $\mathbb{R}^n$. The root system is of type $A_{n-1}$. The fundamental roots are $\alpha_k = e_k - e_{k+1}$ with $k = 1, \ldots, n - 1$; all of them have weight one. The dual root vector for $\alpha_k$ is $\xi_k = \frac{1}{2}(\sum_{i=1}^k e_i - \sum_{j=k+1}^n e_j)$. The corresponding extrinsic symmetric space

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$^3$Remind that $BC_n$ and $B_n$ have the same simple root system.
$M_k = \text{Ad}(U_p)\mathfrak{k}_k$ is isomorphic to the complex Grassmannian of $k$-planes in $\mathbb{C}^n$.

Now consider $\mathfrak{p}' = \mathfrak{s}_0 \subset \mathfrak{p}$. Passing to a conjugate $\tilde{\mathfrak{p}}' = g \mathfrak{p}' g^{-1}$ for some suitable $g \in U_n$, the maximal abelian subspace of $\tilde{\mathfrak{p}}'$ becomes

$\tilde{\mathfrak{a}}' = \{ x \in \mathfrak{a} : x_{j+m} = -x_j \text{ for all } j = 1, \ldots, m \}.$

This contains $\xi_k \in \mathfrak{a}$ precisely for $k = m$, and $\xi_m$ is a complex structure in $\mathfrak{s}_0$. Hence $M_k \cap \mathfrak{p}' = \emptyset$ for $k \neq m$, and $M_m \cap \mathfrak{p}'$ is the space $SO_n/U_m$ of complex structures in $\mathfrak{s}_0$. This has two connected components which are conjugate in $O_n$ and hence in $U_n$; these correspond to the two fundamental roots with weight 1 at the bifurcation of the Dynkin diagram $D_m$ of $SO_{2m}$.

3. An extrinsic symmetric space $M \subset \mathfrak{p}$ is hermitian symmetric if and only if it is a Lie algebra, $\mathfrak{p} = \mathfrak{g}$, and all other extrinsic symmetric spaces are the real forms of hermitian symmetric spaces, see [1, p. 310f] or [2]. The real forms are obtained as extrinsic symmetric subspaces from a hermitian extrinsic symmetric space $M = \text{Ad}(G)x \subset \mathfrak{g}$ as follows. Let $\sigma$ be an involution on $\mathfrak{g}$ with eigenspace decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ and $x \in \mathfrak{p}$. Then $M' := M \cap \mathfrak{p}$ is a real form of $M$, and every real form arises that way [8].

E.g. let $G = U_n$. Then $M \subset \mathfrak{g}$ is the complex Grassmannian $G_p(\mathbb{C}^n)$ for $p \in \{1, \ldots, n-1\}$. There are three types of real forms: real Grassmannians, quaternionic Grassmannians if both $p, n$ are even, and the unitary group $U_p$ if $n = 2p$. Let us consider the latter case, $M' = U_p$. The embedding of $U_p$ into the Grassmannian $G_p(\mathbb{C}^2p)$ is by assigning to each $A \in U_p$ its graph $E_A = \{(x, Ax) : x \in \mathbb{C}^p \} \subset \mathbb{C}^p \times \mathbb{C}^p$. The subgroup $U_p \times U_p \subset U_{2p}$ acts transitively on it since for all $(B, C) \in U_p \times U_p$, 

$$(B \ C) E_A = \{(Bx, CAx) : x \in \mathbb{C}^p \} = \{ \tilde{x}, CAB^{-1}\tilde{x} : \tilde{x} \in \mathbb{C}^p \} = E_{CAB^{-1}}.$$  

The embedding of $M = G_p(\mathbb{C}^{2p})$ into $\mathfrak{g} = \mathfrak{u}_{2p}$ is obtained by assigning to a $p$-dimensional subspace $E \subset \mathbb{C}^{2p}$ the matrix $r_E$ with eigenvalues $i$ on $E$ and $-i$ on $E^\perp$. This matrix is not only in $\mathfrak{u}_{2p}$ but also in $U_{2p}$. In particular, for the subspace $E_I = \{(x, x) : x \in \mathbb{C}^p \}$ we have $r_{E_I} = i(I^I)$. The group $U_{2p}$ acts by conjugation on these matrices, hence for $E' = (B \ C) E_I$ we have

$r_{E'} = (B \ C)i(I^I)(B^* \ C^*) = i(CB^*BC^*) = (-A^* \ A)$

with $A = iBC^*$. Thus $M' \subset \mathfrak{p}' := \left\{ \left( -X, X \right) : X \in \mathbb{C}^{p \times p} \right\}$. Vice versa, if $\left( -X, X \right) \in M \subset U_{2p}$, then $X \in U_p$, thus $M' = M \cap \mathfrak{p}'$. The subtriple $\mathfrak{p}'$ belongs to the Grassmannian $G_p(\mathbb{C}^{2p})$ and has Dynkin diagram $C_p$, see [7, pp. 517, 532], which has just one weight 1.

6. Open problems

In some sense, $\hat{M}' := M \cap \mathfrak{p}'$ should be considered as one single object with several connected components, like in the case of the Grassmannians. However, given $\hat{M}'$, we are not able to show that in general the smallest linear subspace $\mathfrak{p}'$ containing $\hat{M}'$ is a Lie triple. The question is easy when $\hat{M}'$ is the fixed set of a group of isometries: any isometry of $M \subset \mathfrak{p}$ extends as a linear isometry to the ambient space $\mathfrak{p}$, see [4], and $\mathfrak{p}'$ is the common
fixed space of these extensions which is a Lie subtriple. In general, if $M'_i$ are the connected components of $\hat{M}'$, then $\mathfrak{p}' = \sum \mathfrak{p}'_i$ where $\mathfrak{p}'_i$ is the linear span of $M'_i$, and all $\mathfrak{p}'_i$ are Lie triples acted on by the same group $K' = \{ k \in K : k(\mathfrak{p}') = \mathfrak{p}' \}$ containing the symmetries $s_x$ for all $x \in \hat{M}'$. But is $\mathfrak{p}'$ itself a Lie triple? So far, we have no information on the Lie brackets $[\mathfrak{p}'_i, \mathfrak{p}'_j]$ for $i \neq j$.

Maybe the worst case is when $\hat{M}'$ is discrete. This happens when $\mathfrak{p}'$ is abelian. In particular, when $\mathfrak{p}' = \mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, then $\hat{M}' = M \cap \mathfrak{a}$ is a Weyl orbit: It is the intersection of the $\text{Ad}(K)$-orbit $M$ on $\mathfrak{p}$ with the section $\mathfrak{a}$ of this polar representation. We conjecture that the converse is also true:

**Conjecture 5.** Let $M \subset \mathfrak{p}$ be extrinsic symmetric and $\mathfrak{p}' \subset \mathfrak{p}$ a Lie subtriple intersecting $M$. Then $M' = M \cap \mathfrak{p}'$ is discrete (finite) if and only if $\mathfrak{p}'$ is abelian.

References


Jost Eschenburg
Institut für Mathematik
Universität Augsburg
D-86135 Augsburg
Germany
e-mail: eschenburg@math.uni-augsburg.de

Makiko Sumi Tanaka
Faculty of Science and Technology
Tokyo University of Science
Noda, Chiba, 278–8510
Japan
e-mail: tanaka_makiko@ma.noda.tus.ac.jp