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AN INVARIANT DERIVED FROM THE ALEXANDER POLYNOMIAL FOR HANDLEBODY-KNOTS

SHIN’YA OKAZAKI

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Abstract

A handlebody-knot is a handlebody embedded in the 3-sphere. We introduce an invariant for handlebody-knots derived from their Alexander polynomials. The value of the invariant is a vertex-weighted graph. As an application, we describe a sufficient condition for a handlebody-knot to be irreducible and a necessary condition for a link to be a constituent link of a handlebody-knot.

1. Introduction

A genus $g$ handlebody-knot is a genus $g$ handlebody embedded in the 3-sphere $S^3$, denoted by $H$. Any handlebody-knot can be represented by some connected spatial graph. Two handlebody-knots are equivalent if one can be transformed into the other by an isotopy of $S^3$. Suzuki [12] introduced the notion of neighborhood congruence for spatial graphs. The neighborhood congruence class of a connected spatial graph corresponds to a handlebody-knot.

In this paper, we introduce an invariant for handlebody-knots whose value is a vertex-weighted graph. We introduce an equivalence relation $\sim$ on the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. We define the vertex-weighted graph $G_f$ for a Laurent polynomial $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ as an invariant for $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$. The $d$-th Alexander polynomial $\Delta^{(d)}_{(H,M)}(t_1, t_2, \ldots, t_g)$ is an invariant for a pair of a genus $g$ handlebody-knot $H$ and its (oriented and ordered) meridian system $M$. This invariant is in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$. We define an invariant $G_H$ for handlebody-knots as $G^{(g)}_{H(t_1, t_2, \ldots, t_g)}$. The invariant $G_H$ does not depend on the choice of the meridian system of $H$.

In Section 2, we recall the definition of the Alexander polynomial for a pair of a handlebody-knot $H$ and its meridian system $M$, and we define an invariant $G_H$ for handlebody-knots. As applications of the invariant $G_H$, we describe a sufficient condition for a handlebody-knot to be irreducible in Section 3 and a necessary condition for a link to be a constituent link of a handlebody-knot in Section 4. In Section 5, we introduce an equivalence class of handlebody-knots, and demonstrate that $G_H$ is an invariant for this equivalence class of handlebody-knots. The appendix contains a table of $G_H$ and $\hat{G}_H$ for handlebody-knots in the table of genus 2 handlebody-knots with up to six crossings in [4].

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2. An invariant for handlebody-knots

Throughout this paper, we work in the PL category. A genus $g$ handlebody-knot is a genus $g$ handlebody embedded in the 3-sphere $S^3$, denoted by $H$. Any handlebody-knot can be represented by some connected spatial graph. A diagram of a handlebody-knot is a diagram of a connected spatial graph that represents the handlebody-knot. Two handlebody-knots are equivalent if one can be transformed into the other by an isotopy of $S^3$.

We recall the definitions of the universal abelian covering spaces and the Alexander polynomial for handlebody-knots [6, 10]. Let $H$ be a genus $g$ handlebody-knot in $S^3$ and $M = \{m_1, m_2, \ldots, m_g\}$ an (oriented and ordered) meridian system of $H$. Let $E$ be the exterior of $H$, that is, the closure of $S^3 \setminus H$. Let $G = \pi_1(E)$ be the fundamental group of $E$. Let $t_i$ be the homology class in the integral homology group $H_1(E)$ represented by $m_i$ for $i = 1, 2, \ldots, g$. Then, $H_1(E)$ is a free abelian group of rank $g$ generated by $t_1, \ldots, t_g$. Let $\gamma : G \to H_1(E)$ be the Hurewicz epimorphism. The covering space over $E$ corresponding to the subgroup $\text{Ker}(\gamma) = \{[G, G] \mid G\}$ of $G$ is called the universal abelian covering space $E_\gamma$ of $E$. Because $H_1(E)$ acts on $E_\gamma$ as the covering transformation group, $H_1(E_\gamma)$ is regarded as a module over the integral group ring $\mathbb{Z}H_1(E)$ of $H_1(E)$. By regarding $H_1(E)$ as the multiplicative free abelian group $\Pi_{\{m\}}(t_i)$ with basis $t_1, t_2, \ldots, t_g$, we identify $\mathbb{Z}H_1(E)$ with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ in the variables $t_1, \ldots, t_g$. Thus, we can regard $H_1(E_\gamma)$ as a $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$-module. Let $p : E_\gamma \to E$ be the covering projection and $b$ be a point in $E$. Then, $H_1(E_\gamma, p^{-1}(b))$ can also be regarded as a $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$-module.

**Definition 2.1 (Alexander polynomial for handlebody-knots).** The Alexander matrix $A$ of a pair consisting of a handlebody-knot $H$ and its meridian system $M$ is an $m \times n$ presentation matrix of $H_1(E_\gamma, p^{-1}(b))$. The $d$-th Alexander polynomial $A_{(H,M)}^{(d)}(t_1, t_2, \ldots, t_g)$ of $(H, M)$ is defined to be the greatest common divisor of all $(n-d)$-minors of $A$ for $d = 0, 1, \ldots, n-1$. For $d \geq n$, we define $A_{(H,M)}^{(d)}(t_1, t_2, \ldots, t_g) = 1$.

The Alexander polynomial is an invariant for a basis of $H_1(E)$ up to multiplication by units in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$. We fix a meridian system $M$ of $H$. Then, we can assume that basis of $H_1(E)$ is $M$. Thus, the Alexander polynomial is an invariant for a pair consisting of $H$ and $M$ up to multiplication by units in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$. For simplicity, we denote $A_{(H,M)}^{(g)}(t_1, t_2, \ldots, t_g)$ by $A_{(H,M)}(t_1, t_2, \ldots, t_g)$, because $A_{(H,M)}^{(g)}(t_1, t_2, \ldots, t_g)$ is useful for genus $g$ handlebody-knots. We can obtain $A_{(H,M)}^{(d)}(t_1, t_2, \ldots, t_g)$ using Fox’s free calculus [5] or the C-complex for $H$ [10].

The Alexander polynomial for $(H, M)$ corresponds to the Alexander polynomial for a spatial graph $\Gamma$ that represents $H$. In [7] and [8], Kinoshita introduced the Alexander polynomial for spatial graphs. In [9], Kinoshita introduced a basis of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ as a basis $z$ of the integral first homology group $H_1(\Gamma)$ of $\Gamma$ which is a dual basis of $H_1(E)$, and introduced the elementary ideals $E_d(\Gamma, z)$ associated with $z$ as an invariant for spatial graphs. In [12], Suzuki introduced a representation of $H$ as a $g$-leafed rose $C$. The basis of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ is determined by meridians of the constituent link of $C$ for calculating the one-variable elementary ideal of $H$.

Let $\text{MCG}(\partial H)$ be the mapping class group of $\partial H$. Let $\text{MCG}^*(\partial H)$ be a subgroup of $\text{MCG}(\partial H)$ consisting of those homeomorphisms which can be extended to homeomor-
phisms of $H$ onto itself. Let $\phi \in MCG(\partial H)$. Replacing a meridian system $M$ with $\phi(M)$ of $H$ corresponds to a change of basis for $H_1(E)$, which is represented by a matrix in $GL(g, \mathbb{Z})$.

That is, there exists a matrix

$$\begin{bmatrix}
x_1^1 & x_2^1 & \cdots & x_g^1 \\
x_1^2 & x_2^2 & \cdots & x_g^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^n & x_2^n & \cdots & x_g^n
\end{bmatrix} \in GL(g, \mathbb{Z})$$

to $t_i' = x_1^i, x_2^i, \ldots, x_g^i$ for $i = 1, 2, \ldots, g$. Here $\{t_1, t_2, \ldots, t_g\}$ and $\{t'_1, t'_2, \ldots, t'_g\}$ are the basis of $H_1(E)$ induced from $M$ and $\phi(M)$, respectively. Throughout this paper, we assume that the action of $GL(g, \mathbb{Z})$ on $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$ is as above. Then the following lemma holds.

**Lemma 2.2.** $A_{(H, \phi(M))}(t_1, t_2, \ldots, t_g) = A_{(H,M)}(t'_1, t'_2, \ldots, t'_g)$.

We introduce an equivalence relation on $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$ as follows: For two Laurent polynomials $f_1$ and $f_2$ in $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$, we say that $f_1$ and $f_2$ are equivalent, denoted by $f_1 \sim f_2$, if $f_1$ is equal to $f_2$ up to multiplication by units in $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$ and the action of $GL(n, \mathbb{Z})$ on $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$.

We define an invariant for $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]/\sim$ as follows: For a Laurent polynomial $f = \sum_{i=1}^m c_i t_1^i t_2^i \cdots t_g^i \in \mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$, let $T_i = c_i t_1^i t_2^i \cdots t_g^i$ be the $i$-th term of $f$, $C_f$ the set $\{c_i\}$ of coefficients of terms $T_1, T_2, \ldots, T_m$, and $P_f$ the set $\{p^i = (x_1^i, x_2^i, \ldots, x_g^i) \in \mathbb{R}^g\}$ of position vectors determined by the exponents of $T_i$ for $i = 1, 2, \ldots, m$. Note that $x_j^i \in \mathbb{Z}$ for $j = 1, 2, \ldots, n$. The terms $T_1, T_2, \ldots, T_m$ of $f$ are mapped to mutually different terms of $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$ through multiplication by units in $\mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$ and the action of $GL(g, \mathbb{Z})$. Thus, the following lemmas hold.

**Lemma 2.3.** The set $C_f$, up to multiplication by $\pm 1$ to all elements of $C_f$, is an invariant for $f \in \mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]/\sim$.

**Lemma 2.4.** The set $P_f$, up to parallel translation to all elements of $P_f$ and linear transformation by $GL(n, \mathbb{Z})$ on $\mathbb{R}^n$ to all elements of $P_f$, is an invariant for $f \in \mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]/\sim$.

**Definition 2.5** (Vertex-weighted graph $G_f$). The vertex-weighted graph $G_f$ for $f = \sum_{i=1}^m c_i t_1^i t_2^i \cdots t_g^i \in \mathbb{Z}[t_1^1, t_2^1, \ldots, t_g^1]$ is a simple bipartite graph whose vertex set is a disjoint union of a black vertex set and a white vertex set. The black vertex set consists of black vertices $b_1, b_2, \ldots, b_m$ whose labels are $c_1, c_2, \ldots, c_m$, respectively. For each $(n+1)$-tuple of position vectors $p^0, p^1, \ldots, p^g$ in $P_f$ whose convex hull in $\mathbb{R}^n$ contains no vectors of $P_f \setminus \{p^0, p^1, \ldots, p^g\}$, we take a white vertex labeled by the absolute value of the determinant of

$$\begin{bmatrix}
x_1^1 - x_1^0 & x_2^1 - x_2^0 & \cdots & x_g^1 - x_g^0 \\
x_1^2 - x_1^0 & x_2^2 - x_2^0 & \cdots & x_g^2 - x_g^0 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^n - x_1^0 & x_2^n - x_2^0 & \cdots & x_g^n - x_g^0
\end{bmatrix}$$

The white vertex is connected to the $(n+1)$ black vertices $b_{i_0}, b_{i_1}, \ldots, b_{i_n}$ by edges. The simple bipartite graph thus obtained is $G_f$.

According to Lemma 2.3, the set of labels of black vertices up to multiplication by $\pm 1$
are invariants for $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] \sim$. The label of a white vertex is the $n$-volume of the $n$-parallelotope determined by the $n$ vectors $p_i - p_j$, $p_j - p_k$, ..., $p_k - p_i$ in $\mathbb{R}^n$, which is an invariant for $n$-parallelotopes up to parallel translation and the linear transformation given by $GL(n, \mathbb{Z})$ on $\mathbb{R}^n$. Thus, according to Lemma 2.4, the labels of white vertices are invariants for $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] \sim$. Note that if $m < n + 1$, then $G_f$ does not have a white vertex and $G_f$ is not connected.

An isomorphism of the vertex-weighted graphs $G_f$ and $G'_f$ is a bijection between the vertex sets of $G_f$ and $G'_f$ that maps the black vertex set and the white vertex set of $G_f$ to the black vertex set and the white vertex set of $G'_f$, respectively, such that any two vertices of $G_f$ are adjacent if and only if the images of the two vertices are adjacent in $G'_f$. If an isomorphism exists between $G_f$ and $G'_f$, then $G_f$ and $G'_f$ are said to be isomorphic. The following lemma holds.

**Lemma 2.6.** The isomorphism class of the vertex-weighted graph $G_f$ up to multiplication by $\pm 1$ to all labels of the black vertices is an invariant for $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] \sim$.

We define the vertex-weighted graph $G_H$ for $(H, M)$ as $G_{\Delta(H,M)(t_1,t_2,\ldots,t_n)}$. Note that $G_H$ does not depend on the choice of the meridian system of $H$. We have the following theorem. This is the main result of this paper.

**Theorem 2.7.** The isomorphism class of the vertex-weighted graph $G_H$ up to multiplication by $\pm 1$ to all labels of the black vertices is an invariant for handlebody-knots.

**Example 2.8.** Let $H$ be a handlebody-knot and $M$ its meridian system as depicted in Fig.1. Then, the Alexander polynomial $\Delta_{(H,M)}(t_1, t_2)$ of $(H, M)$ is $t_1^2 - t_1 + t_2 - t_2 + t_1 t_2$. We have $T_1 = t_1^2$, $T_2 = -t_1$, $T_3 = t_2^3$, $T_4 = -t_2$, and $T_5 = t_1 t_2$; $c_1 = 1$, $c_2 = -1$, $c_3 = 1$, $c_4 = -1$, and $c_5 = 1$; and $p_1 = (2, 0)$, $p_2 = (1, 0)$, $p_3 = (0, 2)$, $p_4 = (0, 1)$, and $p_5 = (1, 1)$ in $\mathbb{R}^2$, as depicted in Fig.2. We take the black vertex $b_1$ labeled with $c_1 = 1$. Similarly, we have $b_2$, $b_3$, $b_4$, and $b_5$ as depicted in Fig.3. For three tuple of position vectors $p_1, p_2,$ and $p_3$ in $P_{\Delta_{(H,M)}(t_1,t_2)}$ whose convex hull in $\mathbb{R}^3$, $p_3$ is in the convex hull. Therefore, $G_H$ does not contain a white vertex connected to $b_1, b_2,$ and $b_3$.

For three tuple of position vectors $p_1, p_2,$ and $p_4$ in $P_{\Delta_{(H,M)}(t_1,t_2)}$ whose convex hull in $\mathbb{R}^3$ contains no vectors of $P_{\Delta_{(H,M)}(t_1,t_2)} \setminus \{p_1, p_2, p_3\}$, we take a white vertex labeled by 1 which is the absolute value of the determinant of $\begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & -1 & 0 & -1 \end{bmatrix}$. The white vertex is connected to the three black vertices $p_1, p_2,$ and $p_4$ by edges. Similarly, we have other white vertices, and we have $G_H$ as depicted in Fig.3.
Fig. 2. Position vectors $p^1, p^2, p^3, p^4,$ and $p^5$

The following example shows that there exist infinitely many handlebody-knots whose invariants are mutually different.

**Example 2.9.** Let $H_n$ be a handlebody-knot for $n \neq 0$ and $M$ its meridian system, as depicted in Fig. 4. We have $A_{(H_n,M)}(t_1, t_2) = t_1^n + t_2 - 1$. The invariant $G_{H_n}$ is as depicted in...
detecting the irreducibility using the quandle coloring invariant.

In this section, as an application of Theorem 2.7, we describe a sufficient condition for a handlebody-knot to be irreducible. A handlebody-knot $H$ is reducible if there exists a 2-sphere in $S^3$ such that the intersection of $H$ and the 2-sphere is an essential disk properly embedded in $H$. A handlebody-knot is irreducible if it is not reducible. In [12], Suzuki introduced the irreducibility as the “primeness” of handlebody-knots and demonstrated the uniqueness of the factorization of $H$. In [3], Ishii and Kishimoto provided methods for detecting the irreducibility using the quandle coloring invariant.

Let $B_1$ and $B_2$ be 3-balls in $S^3$ such that $B_1 \cup B_2 = S^3$ and $B_1 \cap B_2 = \partial B_1 = \partial B_2$. Let $H_i$ be a genus $g_i$ handlebody-knot in $B_i$ for $i = 1, 2$. When $H_1 \cap H_2$ is one disk, $H_1 \cup H_2$ is a genus $g_1 + g_2$ handlebody-knot in $S^3$. We denote this by $H_1 \# H_2$, where we remark that the handlebody-knot $H_1 \# H_2$ depends only on the handlebody-knots $H_1$ and $H_2$. If a handlebody-knot $H$ is reducible, then there exist handlebody-knots $H_1$ and $H_2$ such that $H = H_1 \# H_2$. As $S^3 \setminus H$ is the boundary-connected sum of those of $H_1$ and $H_2$, the fundamental group of $S^3 \setminus H$ is the free product of those of $H_1$ and $H_2$. Thus, the following lemma holds [12].

**Lemma 3.1.** For a genus $g_1$ handlebody-knot $H_1$ and genus $g_2$ handlebody-knot $H_2$ and their meridian systems $M_1$ and $M_2$, respectively, The Alexander polynomial $\Delta^{(g_1+g_2)}_{(H_1,H_2,M_1\cup M_2)}(t_1,t_2,\ldots,t_{g_1+g_2})$ of $(H_1\# H_2, M_1 \cup M_2)$ is the product of $\Delta^{(g_1)}_{(H_1,M_1)}(t_1,t_2,\ldots,t_{g_1})$ and $\Delta^{(g_2)}_{(H_2,M_2)}(t_{g_1+1},t_{g_1+2},\ldots,t_{g_1+g_2})$.

Because $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a unique factorization domain, a Laurent polynomial $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ can be uniquely expressed as $c f_1 f_2 \cdots f_m$, where $f_i \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ is irreducible for $i = 1, 2, \ldots, m$ and $c \in \mathbb{Z}$. For a Laurent polynomial $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, we define the set $\hat{G}_f$ as follows:

$$\hat{G}_f = \begin{cases} \{ G_f | f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] \} & \text{if } f = 0 \\ \{ G_f | 1 \leq i \leq m \} & \text{otherwise}. \end{cases}$$

If $f = 0$, then $\hat{G}_f$ is an infinite set. If $f = 1$, then $\hat{G}_f$ is an empty set. From the definition of $\hat{G}_f$, we have the following lemma. We use this to prove Theorem 4.2 in Section 4.

**Lemma 3.2.** For Laurent polynomials $f$ and $f'$ in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, if $f | f'$, then $\hat{G}_f \subset \hat{G}_{f'}$. 

![Fig.5. The vertex-weighted graph $G_{H_n}$](image)
We define the set $\hat{G}^{(d)}_H$ by $\hat{G}^{(d)}_{(H,M)}(t_1, t_2, \ldots, t_g)$ for the $d$-th Alexander polynomial $A^{(d)}_{(H,M)}(t_1, t_2, \ldots, t_g)$ of $(H,M)$. By Theorem 2.7, $\hat{G}^{(d)}_H$ is an invariant for handlebody-knots.

For simplicity, we denote $\hat{G}^{(g)}_H$ by $\hat{G}_H$ for genus $g$ handlebody-knots. The following theorem gives a sufficient condition for a handlebody-knot to be irreducible.

**Theorem 3.3.** For a handlebody-knot $H$, if there exists $G_f \in \hat{G}_H$ that has a white vertex with a nonzero label, then $H$ is irreducible.

Proof. Let $H_i$ be a genus $g_i$ handlebody-knot and $M_i$ its meridian system for $i = 1, 2$. Set $H = H_1 \# H_2$ and $M = M_1 \cup M_2$. Note that $H$ is a genus $g_1 + g_2$ handlebody-knot and $M$ is its meridian system. We show that, for any $G_f \in \hat{G}^{(g_1+g_2)}_H$, all labels of white vertices of $G_f$ are equal to zero. By Lemma 3.1, $A^{(g_1+g_2)}_{(H,M)}(t_1, t_2, \ldots, t_{g_1+g_2})$ is equal to the product of $A^{(g_1)}_{(H_1,M_1)}(t_1, t_2, \ldots, t_{g_1})$ and $A^{(g_2)}_{(H_2,M_2)}(t_{g_1+1}, t_{g_1+2}, \ldots, t_{g_1+g_2})$. Let $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{g_1+g_2}^{\pm 1}]$ be an irreducible polynomial that is a factor of $A^{(g_1+g_2)}_{(H,M)}(t_1, t_2, \ldots, t_{g_1+g_2})$. Then, $f$ is a factor of $A^{(g_1)}_{(H_1,M_1)}(t_1, t_2, \ldots, t_{g_1})$ or $A^{(g_2)}_{(H_2,M_2)}(t_{g_1+1}, t_{g_1+2}, \ldots, t_{g_1+g_2})$, because $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{g_1+g_2}^{\pm 1}]$ is a unique factorization domain.

If $f$ is a factor of $A^{(g_1)}_{(H_1,M_1)}(t_1, t_2, \ldots, t_{g_1})$, then all labels of white vertices of $G_f$ are zero, because the label of the white vertex of $G_f$ is the absolute value of the $(g_1 + g_2)$-volume of the degenerated $(g_1 + g_2)$-parallelepiped determined by the $g_1 + g_2$ vectors in the $g_1$-dimensional vector space $\mathbb{R}^{g_1} \subset \mathbb{R}^{g_1+g_2}$. Similarly, if $f$ is a factor of $A^{(g_2)}_{(H_2,M_2)}(t_{g_1+1}, t_{g_1+2}, \ldots, t_{g_1+g_2})$, then all labels of white vertices of $G_f$ are zero. Thus, Theorem 3.3 holds. \qed

**Example 3.4.** The handlebody-knot depicted in Fig.6 is $4_1$ in the table of genus 2 handlebody-knots with up to six crossings in [4]. Let $M$ be a meridian system of $4_1$, as depicted in Fig.6. We have $A^{(4_1,M)}(t_1, t_2) = t_1 + t_2 - 1$, and $\hat{G}^{(4_1)}_4$ is as depicted in Fig.7. As $t_1 + t_2 - 1$ is irreducible, we have $\hat{G}^{(4_1)}_4 = \{4_1\}$. Because $4_1$ has a white vertex whose label is 1, $4_1$ is irreducible by Theorem 3.3.

The following example shows that there exists an irreducible genus $g$ handlebody-knot for each genus $g$.

**Example 3.5.** Let $H_g$ be a genus $g$ handlebody-knot and $M$ a meridian system of $H_g$, as depicted in Fig.8. Note that $H_2$ is $4_1$. Taniyama showed that $H_g$ is irreducible as a spatial graph [13]. We show that $H_g$ is irreducible as a handlebody-knot using Theorem 3.3.

We have
Fig. 7. The vertex-weighted graph $G_{41}$

$G_{41}:

\begin{center}
\begin{tikzpicture}
\node[vertex] at (0,0) (1) {$1$};
\node[vertex] at (1,1) (2) {$1$};
\node[vertex] at (1,-1) (3) {$-1$};
\draw (1) -- (2);
\end{tikzpicture}
\end{center}

Fig. 8. A handlebody-knot $H_g$

\begin{center}
\begin{tikzpicture}
\node[vertex] at (0,0) (1) {$m_1$};
\node[vertex] at (1,1) (2) {$m_2$};
\node[vertex] at (1,-1) (3) {$m_g$};
\node at (2.5,0) {$\ldots$};
\draw (1) -- (2);
\draw (1) -- (3);
\end{tikzpicture}
\end{center}

\[ A_{(H_g,M^g)}(t_1,t_2,\ldots,t_g) = \prod_{i=1}^{g} (t_i - 1) - \prod_{i=1}^{g} t_i = \sum_{i=1}^{g-1} (-1)^{g+i} \sum_{j_1,j_2,\ldots,j_i \in S} t_{j_1}t_{j_2}\cdots t_{j_i} + (-1)^g, \]

where $S = \{1,2,\ldots,g\}$ and $j_1,j_2,\ldots,j_i$ are mutually different elements in $S$. By induction on $g$, we can check $A_{(H_g,M^g)}(t_1,t_2,\ldots,t_g)$ is irreducible. Hence, we have $\hat{G}_{H_g} = \{G_{H_g}\}$. The set $P_{(H_g,M^g)}(t_1,t_2,\ldots,t_g)$ has unit vectors $e_1,e_2,\ldots,e_g$ and the zero vector 0 in $\mathbb{R}^g$. For each $(g+1)$-tuple of position vectors $e_1,e_2,\ldots,e_g$ and 0 in $P_{(H_g,M^g)}(t_1,t_2,\ldots,t_g)$ whose convex hull in $\mathbb{R}^g$ contains no vectors of $P_f \setminus \{e_1,e_2,\ldots,e_g,0\}$, we take a white vertex labeled by 1. Thus, $G_{H_g}$ has a white vertex whose label is 1, and $H_g$ is irreducible by Theorem 3.3.

4. Constituent links of a handlebody-knot

In this section, as an application of Theorem 2.7, we describe a necessary condition for a link to be a constituent link of a handlebody-knot. In [12], Suzuki introduced a $g$-leafed rose that is a connected spatial graph as follows: A $g$-leafed rose $C = K_1 \cup K_2 \cup \cdots \cup K_g \cup T$ consists of a $g$-component link $L = K_1 \cup K_2 \cup \cdots \cup K_g$ and a star graph $T$, as depicted in Fig. 9.
We call $L$ the constituent link of $C$. For a genus $g$ handlebody-knot, there exist infinitely many $g$-leafed roses representing the handlebody-knot. We define a constituent link of $H$ as the constituent link of a $g$-leafed rose that represents $H$. Therefore, there exist infinitely many constituent links of $H$.

Let $M$ be the meridian system of the constituent link $L$ of $C$. Let $E_d(C,M)$ and $E_d(L)$ be the $d$-th elementary ideals of $(C,M)$ and $L$, respectively; that is, the ideal of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ generated by all $(n-d)$-minors of the Alexander matrix of $C$ and $L$, respectively. The following theorem was proved by Suzuki in [11].

**Theorem 4.1.** [11] $E_{d+g-1}(C,M) \supset E_d(L)$.

Proof. Let $L$ be a $g$-leafed rose that represents $H$ and $M$ be the meridian system of the constituent link $L$ of $C$. Then, we have the $(d+g-1)$-th Alexander polynomial $A_{d+g-1}^{(d+g-1)}(t_1, t_2, \ldots, t_g)$ of the pair $(C,M)$ and the $d$-th Alexander polynomial $A_d^{(d)}(t_1, t_2, \ldots, t_g)$ of $L$.

The Alexander polynomials $A_{d+g-1}^{(d+g-1)}(t_1, t_2, \ldots, t_g)$ and $A_d^{(d)}(t_1, t_2, \ldots, t_g)$ can be uniquely expressed as $uf_1f_2 \cdots f_n$ and $u'f'_1f'_2 \cdots f'_m$, respectively, because $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ is a unique factorization domain. Here, $u$ and $u'$ are units of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$, and $f_i, f'_j$ are irreducible in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. We have $\hat{G}_{H}^{(d+g-1)} = \hat{G}_{L}^{(d+g-1)}(t_1, t_2, \ldots, t_g)$.

By Theorem 4.1, $E_{d+g-1}(C,M) \supset E_d(L)$. If $A_{d+g-1}^{(d+g-1)}(t_1, t_2, \ldots, t_g) \neq 0$, then $A_{d+g-1}^{(d+g-1)}(t_1, t_2, \ldots, t_g) \nmid A_d^{(d)}(t_1, t_2, \ldots, t_g)$. By Lemma 3.2, $\hat{G}_{H}^{(d+g-1)} \subset \hat{G}_{L}^{(d)}$. If $A_{d+g-1}^{(d+g-1)}(t_1, t_2, \ldots, t_g) = 0$, then $A_d^{(d)}(t_1, t_2, \ldots, t_g) = 0$ by Theorem 4.1. Thus, we have $\hat{G}_{H}^{(d+g-1)} \subset \hat{G}_{L}^{(d)}$.

For simplicity, we denote $\hat{G}_{L}^{(1)}$ as $\hat{G}_{L}$. By Theorem 4.2, for a constituent link $L$ of a genus $g$ handlebody-knot $H$, $\hat{G}_{H} \subset \hat{G}_{L}$.
Fig. 10. The Hopf link $L$ and the vertex-weighted graph $G_L$.

**Example 4.3.** In Example 3.4, we used $G_{4_1}$ for $4_1$. The 1st Alexander polynomial of the Hopf link $L$ is 1, and $G_L$ is as depicted in Fig.10. We have $\hat{G}_L = \emptyset$. As $\hat{G}_{4_1} \subset \hat{G}_L$, the Hopf link is not a constituent link of $4_1$ by Theorem 4.2.

5. An equivalence class of handlebody-knots

In this section, we introduce an equivalence class of handlebody-knots. A handlebody-knot is represented by a connected spatial graph. A *crossing change* of a handlebody-knot $H$ is a crossing change of a connected spatial graph that represents $H$. For a handlebody-knot $H$, a crossing change between two edges of $H$ whose meridians are null-homologous in $S^3 \setminus H$ is called an $N$-*crossing change*. We say that handlebody-knots $H_1$ and $H_2$ are $N$-*equivalent* if they are transformed into each other by a finite sequence of $N$-crossing changes and an isotopy of $S^3$.

The following proposition shows that the Alexander polynomial is an invariant for $N$-equivalence classes of handlebody-knots up to multiplication by units in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$. This proposition is thought to be mathematical folklore.

**Proposition 5.1.** The Alexander polynomial of a spatial graph $\Gamma$ does not change under the $N$-crossing change on $\Gamma$.

Proof. Spatial graphs $\Gamma_+$ and $\Gamma_-$, which are as depicted in Fig.11, are identical outside a small 3-ball. Let $G_+$ and $G_-$ be the fundamental groups of $S^3 \setminus \Gamma_+$ and $S^3 \setminus \Gamma_-$, respectively. We take the generators $a_i, a_{i+1}, a_j,$ and $a_{j+1}$ of the Wirtinger presentation of $G_+$ and $G_-$ around the crossing point, as depicted in Fig.11.

We have $r_1 : a_i = a_{i+1}$ and $r_2 : a_j a_i = a_{i+1} a_{j+1}$ as relators of $G_+$, and $r'_1 : a_j = a_{j+1}$ and $r_2' : a_j a_i = a_{i+1} a_{j+1}$ as relators of $G_-$. Therefore, we have the following presentations of $G_+$ and $G_-:

$$G_+ = \langle a_1, a_2, \ldots, a_m | r_1, r_2, \ldots, r_n \rangle, \quad G_- = \langle a_1, a_2, \ldots, a_m | r'_1, r_2, \ldots, r_n \rangle.$$
We assume that the generators \( a_i, a_{i+1}, a_j, \) and \( a_{j+1} \) are mapped to 1 by the abelianizer \( \alpha_+ \) of \( G_+ \) and \( \alpha_- \) of \( G_- \). Let \( A_+ \) and \( A_- \) be the Alexander matrices of \( G_+ \) and \( G_- \), respectively.

\[
A_+ \sim \begin{bmatrix}
\cdots & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & -1 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots \\
\cdots & * & c_i & c_{i+1} & * & \cdots & * & c_j & c_{j+1} & * & \cdots
\end{bmatrix}
\]

\[
A_- \sim \begin{bmatrix}
\cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots \\
\cdots & * & c_i & c_i + c_{i+1} & * & \cdots & * & c_j & c_{j+1} & * & \cdots
\end{bmatrix}
\]

\[
A_+ \sim \begin{bmatrix}
\cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots \\
\cdots & * & c_i + c_{i+1} & * & \cdots & * & c_j & c_{j+1} & * & \cdots
\end{bmatrix}
\]

\[
A_- \sim \begin{bmatrix}
\cdots & * & c_i + c_{i+1} & * & \cdots & * & c_j + c_{j+1} & * & \cdots
\end{bmatrix}
\]

Similarly, \( A_- \sim \begin{bmatrix}
\cdots & * & c_i + c_{i+1} & * & \cdots & * & c_j + c_{j+1} & * & \cdots
\end{bmatrix} \). Thus, we have \( A_+ \sim A_- \). □

By Proposition 5.1, we have the following corollary of Theorem 2.7.

**Corollary 5.2.** The isomorphism class of the vertex-weighted graph \( G_H \), up to multiplication by \( \pm 1 \) to all labels of the black vertices, is an invariant for the \( N \)-equivalence classes of handlebody-knots.

**Example 5.3.** The handlebody-knot depicted in Fig.12 is \( 5_4 \) in the table of genus 2 handlebody-knots with up to six crossings in [4]. It is clear that \( 5_4 \) is \( N \)-equivalent to the trivial handlebody-knot \( 0_1 \). Thus, \( \Delta_{(5_4,M)}(t_1,t_2) = \Delta_{(0_1,M)}(t_1,t_2) = 1 \).

\[
5_4:
\]

Fig.12. The handlebody-knot \( 5_4 \)

**Appendix A Table of \( G_H \) and \( \hat{G}_H \)**

In this appendix, we present the table of \( \Delta^{(2)}_{(H,M)}(t_1,t_2), G_H, \) and \( \hat{G}_H \) for handlebody-knots in the table of genus 2 handlebody-knots with up to six crossings in [4]. Here, \( M \) is a
meridian system of $H$. Let $G_1$, $G_2$, $G_3$, $G_4$, and $G_5$ be the vertex-weighted graphs depicted in Fig.13. Then, we have Table 1.

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Fig.13. The vertex-weighted graph $G_1$, $G_2$, $G_3$, $G_4$, and $G_5$.
Invariant Derived from Alexander Polynomial

Table 1. Table of $\Delta^{(2)}_{(H,M)}(t_1, t_2)$, $G_H$, and $\hat{G}_H$.

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\Delta^{(2)}_{(H,M)}(t_1, t_2)$</th>
<th>$G_H$</th>
<th>$\hat{G}_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0_1$</td>
<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$4_1$</td>
<td>$t_1 + t_2 - 1$</td>
<td>$G_3$</td>
<td>$[G_3]$</td>
</tr>
<tr>
<td>$5_1$</td>
<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$5_2$</td>
<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$5_3$</td>
<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$5_4$</td>
<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$6_1$</td>
<td>$t_1^2 + t_2 - 1$</td>
<td>$G_4$</td>
<td>$[G_4]$</td>
</tr>
<tr>
<td>$6_2$</td>
<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
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<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
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<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
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<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
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<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$6_7$</td>
<td>$t_1t_2 - t_1 - t_2 + 2$</td>
<td>$G_5$</td>
<td>$[G_5]$</td>
</tr>
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</tr>
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<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
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<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
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<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
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<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
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<td>$1$</td>
<td>$G_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
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<td>$t_1^2 - t_1 + 1$</td>
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<tr>
<td>$6_{15}$</td>
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<td>$[G_2]$</td>
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<tr>
<td>$6_{16}$</td>
<td>$1$</td>
<td>$G_1$</td>
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References


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