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FORMALLY SELF ADJOINTNESS FOR THE DIRAC OPERATOR ON HOMOGENEOUS SPACES

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Introduction. In [5], Wolf proved that the Dirac operator is essentially self adjoint over a Riemannian spin manifold M and he used it to give explicit realization of unitary representations of Lie groups.

Let K be a Lie group and α a Lie group homomorphism of K into $SO(n)$ which factors through $\text{Spin}(n)$. He defined the Dirac operator on spinors with values in a certain vector bundle under the assumption that the Riemannian connection on the oriented orthonormal frame bundle P over M can be reduced to some principal K -bundle over M by the homomorphism α .

The purpose of this paper is to give the Dirac operator on a homogeneous space in a more general situation using an invariant connection, and to determine connections that define the formally self adjoint Dirac operator.

Let G be a unimodular Lie group and K a compact subgroup of G . We assume G/K has an invariant spin structure. First, we define the Dirac operator D on spinors using an invariant connection on the oriented orthonormal frame bundle P over G/K . Next, we introduce an invariant connection $\nabla^{\mathcal{C}\mathcal{V}}$ to a homogeneous vector bundle $\mathcal{C}\mathcal{V}$ associated to a unitary representation of K , then we define the Dirac operator $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$ on spinors with values in $\mathcal{C}\mathcal{V}$ according

to [4]. As for a metric on spinors, we use a Lemma given by Parthasarathy in [3]. Using this metric and an invariant measure on G/K , we define a hermitian inner product on the space of spinors with values in $\mathcal{C}\mathcal{V}$. Then we determine connections that define the formally self adjoint Dirac operator with respect to this inner product. In some cases (cf. Remarks in 4), $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$ is

always formally self adjoint if an invariant connection on $\mathcal{C}\mathcal{V}$ is a metric connection. Moreover, in the same way as Wolf [3], we see that if $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$ is formally self adjoint, then $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$ and $(D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1)^2$ are essentially self adjoint.

1. Spin construction

Let \mathfrak{m} be an n -dimensional oriented real vector space with an inner product

\langle, \rangle . We define the Clifford algebra $\text{Cliff}(\mathfrak{m})$ over \mathfrak{m} by $T(\mathfrak{m})/I$, where $T(\mathfrak{m})$ is the tensor algebra over \mathfrak{m} and I is the ideal generated by all elements $v \otimes v + \langle v, v \rangle 1$, $v \in \mathfrak{m}$. The multiplication of $\text{Cliff}(\mathfrak{m})$ will be denoted by $x \cdot y$. Let $p: T(\mathfrak{m}) \rightarrow \text{Cliff}(\mathfrak{m})$ denote the canonical projection. Then $\text{Cliff}(\mathfrak{m})$ is decomposed into the direct sum $\text{Cliff}^+(\mathfrak{m}) \oplus \text{Cliff}^-(\mathfrak{m})$ of the p -images of elements of even and odd degree of $T(\mathfrak{m})$, and \mathfrak{m} is identified with a subspace of $\text{Cliff}(\mathfrak{m})$ through the projection p . Let $\{e_1, e_2, \dots, e_n\}$ be an oriented orthonormal base of \mathfrak{m} . The map, $e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_p} \mapsto (-1)^p e_{i_p} \cdot \dots \cdot e_{i_2} \cdot e_{i_1}$ defines a linear map of $\text{Cliff}(\mathfrak{m})$ and the image of $x \in \text{Cliff}(\mathfrak{m})$ by this linear map is denoted by \bar{x} . The Spin group is defined by

$$\text{Spin}(\mathfrak{m}) = \{x \in \text{Cliff}^+(\mathfrak{m}) : x \text{ is invertible, } x \cdot \mathfrak{m} \cdot x^{-1} \subset \mathfrak{m} \text{ and } x \cdot \bar{x} = 1\}$$

$\text{Spin}(\mathfrak{m})$ is a two fold covering group of $SO(\mathfrak{m})$ through the following map $\pi: \text{Spin}(\mathfrak{m}) \rightarrow SO(\mathfrak{m})$ defined by $\pi(x)v = x \cdot v \cdot x^{-1}$ for $x \in \text{Spin}(\mathfrak{m})$ and $v \in \mathfrak{m}$. When $n \geq 3$, $\text{Spin}(\mathfrak{m})$ is the universal covering group of $SO(\mathfrak{m})$. Moreover, the subspace $\mathfrak{spin}(\mathfrak{m})$ of $\text{Cliff}(\mathfrak{m})$ spanned by $\{e_i \cdot e_j\}_{i < j}$ becomes a Lie algebra by the bracket operation $[x, y] = x \cdot y - y \cdot x$. This is identified with the Lie algebra of $\text{Spin}(\mathfrak{m})$ in such a way that $\exp: \mathfrak{spin}(\mathfrak{m}) \rightarrow \text{Spin}(\mathfrak{m})$ is nothing but the restriction of the exponential map of the algebra $\text{Cliff}(\mathfrak{m})$ into $\text{Cliff}(\mathfrak{m})$. The differential $\dot{\pi}$ of π is given by

$$(1.1) \quad \dot{\pi}(x)v = x \cdot v - v \cdot x \text{ for } x \in \mathfrak{spin}(\mathfrak{m}) \text{ and } v \in \mathfrak{m}.$$

Now, put $a_i = \sqrt{-1} e_{2i-1} \cdot e_{2i}$, $1 \leq i \leq \left[\frac{n}{2} \right]$, then $a_i^2 = 1$ and $a_i \cdot a_j = a_j \cdot a_i$. We consider the right multiplication by a_i 's on $\text{Cliff}(\mathfrak{m}) \otimes C$. For a multi-index $q = (q_1, q_2, \dots, q_{\left[\frac{n}{2} \right]})$, where $q_i = 1$ or -1 , we put

$$L_q^\pm = \left\{ x \in \text{Cliff}^\pm(\mathfrak{m}) \otimes C : x \cdot a_i = q_i x, 1 \leq i \leq \left[\frac{n}{2} \right] \right\}.$$

These spaces give irreducible representations of $\text{Spin}(\mathfrak{m})$ by the left multiplication. When n is odd, these representations are equivalent each other. Any one of these representations is called the spin representation. Choosing a multi-index q , we put $L = L_q^+$ and denote by s the representation of $\text{Spin}(\mathfrak{m})$ on L . When n is even, just two inequivalent irreducible representations appear, according to the sign of $\pm \prod q_i$. Each of these representations is called the positive or negative spin representation according to the sign of $\pm \prod q_i$. Choosing a multi-index q with $\prod q_i = 1$, we put $L^+ = L_q^+$, $L^- = L_q^-$, $L = L^+ + L^-$ and denote by s^+ , s^- and s the representations of $\text{Spin}(\mathfrak{m})$ on L^+ , L^- and L respectively. We identify each element of \mathfrak{m} with an element of $\text{Cliff}(\mathfrak{m}) \otimes C$ by the

natural inclusion. Then if n is even, by the left Clifford multiplication the following symbol maps are induced;

$$(1.2) \quad \begin{cases} \varepsilon^\pm: \mathfrak{m} \otimes L^\pm \rightarrow L^\mp \\ \varepsilon: \mathfrak{m} \otimes L \rightarrow L. \end{cases}$$

If n is odd, by the left multiplication, we have the following map;

$$\varepsilon': \mathfrak{m} \otimes L_q^+ \rightarrow L_q^-.$$

Identifying L_q^+ with L_q^- through the spin module isomorphism induced by right multiplication of e_n , we also have the symbol map;

$$(1.2)' \quad \varepsilon: \mathfrak{m} \otimes L \rightarrow L.$$

The definition yield the properties (i), (ii) in the following lemma.

Lemma 1. *We have*

- (i) *The symbol maps ε commute with the action of $\text{Spin}(\mathfrak{m})$, i.e., it holds $\varepsilon(\pi(x)v \otimes x \cdot l) = x \cdot \varepsilon(v \otimes l)$ for $x \in \text{Spin}(\mathfrak{m})$, $v \in \mathfrak{m}$, $l \in L$.*
- (ii) *If $\varepsilon(v \otimes l) = 0$ (resp. $\varepsilon^\pm(v \otimes l) = 0$) for some $v \in \mathfrak{m}$ and $l \in L$ (resp. for some $v \in \mathfrak{m}$ and $l \in L^\pm$), then $v = 0$ or $l = 0$.*
- (iii) *(Lemma 5.1, §5 in [3]). There exist a hermitian inner product $\langle \cdot, \cdot \rangle$, on L satisfying*

$$\langle \varepsilon(v \otimes l), l' \rangle + \langle l, \varepsilon(v \otimes l') \rangle = 0 \quad \text{for } v \in \mathfrak{m}$$

and $l, l' \in L$.

REMARK. We give explicitly a base of L_q^\pm and an inner product on L satisfying the above condition. When $n = 2m$ (resp. $n = 2m + 1$), let $e_1, e_1', \dots, e_m, e_m'$ (resp. $e_1, e_1', \dots, e_m, e_m', e_n$) be an oriented orthonormal base of \mathfrak{m} . Put $f_i = \frac{e_i - \sqrt{-1}e_i'}{2}, f_i' = -\frac{e_i + \sqrt{-1}e_i'}{2}$ and $a_i = -\sqrt{-1}e_i \cdot e_i'$, then we have

$$\begin{aligned} f_i' \cdot a_i &= -f_i' \\ f_i \cdot a_i &= f_i \\ f_i' \cdot a_j &= a_j \cdot f_i' \quad \text{if } i \neq j \\ f_i \cdot a_j &= a_j \cdot f_i \quad \text{if } i \neq j \\ f_i \cdot f_i &= f_i' \cdot f_i' = 0 \\ f_i \cdot f_i' + f_i' \cdot f_i &= 1 \\ f_i \cdot f_j + f_j \cdot f_i &= 0 \quad \text{if } i \neq j, \end{aligned}$$

For a multi-index $q = (q_1, q_2, \dots, q_m)$ we define

$$g_i = \begin{cases} f_i & \text{if } q_i = 1 \\ f'_i & \text{if } q_i = -1, \end{cases}$$

$$h_i = \begin{cases} f_i & \text{if } q_i = -1 \\ f'_i & \text{if } q_i = 1 \end{cases}$$

and put

$$\tau_q = h_1 \cdot h_2 \cdots h_m.$$

Then

$$\{g_{i_1} \cdots g_{i_p} \cdot \tau_q \in \text{Cliff}^\pm(m) \otimes C : 1 \leq i_1 < \cdots < i_p \leq m$$

(resp.

$$\{g_{i_1} \cdots g_{i_p} \cdot \tau_q, g_{i_1} \cdots g_{i_p} \cdot \tau_q \cdot e_n \in \text{Cliff}^\pm(m) \otimes C : \\ 1 \leq i_1 < \cdots < i_p \leq m\})$$

is a base of L_q^\pm . The inner product which makes the above base into an orthonormal base satisfies the condition of Lemma 1-(iii).

2. Invariant connections on homogeneous spaces

Let G be a Lie group and K a closed subgroup of G . We denote by $\mathfrak{g}, \mathfrak{k}$ their Lie algebras. We assume that the pair (G, K) is reductive, i.e., there exists a subspace \mathfrak{m} of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (direct sum) and $Ad(K)\mathfrak{m} \subset \mathfrak{m}$. We fix such decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and identify \mathfrak{m} with the tangent space at the origin o of G/K . Let

$$\rho: K \rightarrow GL(V)$$

be a real or complex representation of K . $1 \in GL(V)$ denotes the identity automorphism of V . The differential of ρ will be denoted by

$$\dot{\rho}: \mathfrak{k} \rightarrow \mathfrak{gl}(V).$$

Now, we consider G -invariant connections on the principal $GL(V)$ -bundle $P = G \times_{\rho} GL(V)$ over G/K , which is the quotient space of $G \times GL(V)$ under the equivalence relation $(g, h) \sim (gk, \rho(k)^{-1}h)$ for $g \in G, k \in K$ and $h \in GL(V)$. The equivalence class in P containing $(g, h) \in G \times GL(V)$ will be denoted by $\{g, h\}$. G acts on P as bundle automorphisms by the left translation

$$L_x: \{g, h\} \rightarrow \{xg, h\} \quad x \in G.$$

Proposition 1. *There exists a one to one correspondence between the set of G -invariant connections in $P = G \times_{\rho} GL(V)$ and the set of R -linear mappings $M_{\mathfrak{m}}: \mathfrak{m} \rightarrow \mathfrak{gl}(V)$ such that*

$$(2.1) \quad M_{\mathfrak{m}}(\alpha(k)X) = \rho(k)M_{\mathfrak{m}}(X)\rho(k)^{-1} \quad \text{for } X \in \mathfrak{m},$$

and $k \in K$.

The connection form ω of the G -invariant connection in P corresponding to M_m is given by

$$(2.2) \quad \begin{cases} M_m(X) = \omega_{u_0}(\tilde{X}) & \text{for } X \in \mathfrak{m}, \\ \dot{\rho}(X) = \omega_{u_0}(\tilde{X}) & \text{for } X \in \mathfrak{k}, \end{cases}$$

where u_0 is the origin $\{e, 1\}$ of $G \times_{\rho} GL(V) = P$ and \tilde{X} is a vector field on P generated by $L_{\text{expt}x}$.

Proof. See [1].

For a linear map M_m satisfying the condition (2.1), the corresponding G -invariant connection will be called the connection induced by M_m .

Let $\mathcal{C}\mathcal{V} = G \times_{\rho} V$ be the vector bundle over G/K associated to (ρ, V) , which is the quotient space of $G \times V$ under the equivalence relation $(g, v) \sim (gk, \rho^{-1}(k)v)$ for $g \in G, k \in K$ and $v \in V$. We denote by $C^{\infty}(\mathcal{C}\mathcal{V})$ the space of all C^{∞} -sections to the bundle $\mathcal{C}\mathcal{V}$. Then $C^{\infty}(\mathcal{C}\mathcal{V})$ is identified as follows with the space $C^{\infty}_0(G, V)$ of all C^{∞} -functions $\tilde{\phi}: G \rightarrow V$ which satisfy $\tilde{\phi}(gk) = \rho(k)^{-1}\tilde{\phi}(g)$ for all $g \in G$ and $k \in K$; Let p be the natural projection of G onto G/K and q the projection of $G \times V$ onto $\mathcal{C}\mathcal{V}$, then the identification $C^{\infty}(\mathcal{C}\mathcal{V}) \ni \phi \mapsto \tilde{\phi} \in C^{\infty}_0(G, V)$ is given by

$$q(g, \tilde{\phi}(g)) = \tilde{\phi}(p(g)) \quad \text{for } g \in G.$$

The principal bundle $P = G \times_{\rho} GL(V)$ is identified with the bundle of frames of $\mathcal{C}\mathcal{V}$ in a natural way, and $\mathcal{C}\mathcal{V}$ is identified with the vector bundle $P \times_{GL(V)} V$ associated to P by the natural action of $GL(V)$ on V . Thus, for a linear map M_m satisfying (2.1), the connection in P induced by M_m defines the covariant derivative

$$\nabla^{\mathcal{C}\mathcal{V}}: C^{\infty}(\mathcal{C}\mathcal{V}) \rightarrow C^{\infty}(\mathcal{I}^* \otimes \mathcal{C}\mathcal{V})$$

on $\mathcal{C}\mathcal{V}$, where \mathcal{I}^* denotes the cotangent bundle of G/K (cf. [1]). We call $\nabla^{\mathcal{C}\mathcal{V}}$ the covariant derivative on $\mathcal{C}\mathcal{V}$ induced by M_m .

Now, we calculate explicitly the covariant derivative $\nabla^{\mathcal{C}\mathcal{V}}$. Note that $\mathcal{I}^* \otimes \mathcal{C}\mathcal{V}$ is identified with the associated bundle $G \times_{\alpha^* \otimes \rho} (\mathfrak{m}^* \otimes V)$, where

$$\alpha^*: K \rightarrow GL(\mathfrak{m}^*)$$

is the representation contragradient to the adjoint representation α of K on \mathfrak{m} , and hence, for each $\phi \in C^{\infty}(\mathcal{C}\mathcal{V})$, $\nabla^{\mathcal{C}\mathcal{V}}\phi$ defines a C^{∞} -function $\widetilde{\nabla^{\mathcal{C}\mathcal{V}}\phi}$ from G into $\mathfrak{m}^* \otimes V$.

Proposition 2. Let $\{X_i\}_{i=1, \dots, n}$ be a base of \mathfrak{m} and $\{\omega^i\}_{i=1, \dots, n}$ its dual base. For the covariant derivative $\nabla^{\mathcal{C}\mathcal{V}}$ on $\mathcal{C}\mathcal{V}$ induced by M_m and $\phi \in C^{\infty}(\mathcal{C}\mathcal{V})$, we have

$$(2.3) \quad \widetilde{\nabla}^{\mathcal{CV}} \phi = \sum_{i=1}^n \omega^i \otimes (X_i \tilde{\phi} + M_m(X_i) \tilde{\phi}),$$

where $X_i \tilde{\phi}$ is the Lie derivative of $\tilde{\phi}$ with respect to the vector field X_i on G , and $M_m(X_i) \tilde{\phi}$ is a C^∞ -function on G defined by $(M_m(X_i) \tilde{\phi})(g) = M_m(X_i) \tilde{\phi}(g)$ for $g \in G$.

Proof. Through the identification $\mathcal{CV} = P \times_{G/K} V$, for $\phi \in C^\infty(\mathcal{CV})$. We define $\tilde{\phi}$ to be a C^∞ -map from P into V in the same way as $\tilde{\phi}$; precisely, let $q: P \times V \rightarrow \mathcal{CV}$ and $p: P \rightarrow G/K$ be the projections, then $\tilde{\phi}$ is defined by the relation

$$q(u, \tilde{\phi}(u)) = \phi(p(u)) \quad \text{for } u \in P.$$

$\tilde{\phi}$ satisfies $\tilde{\phi}(uh) = h^{-1} \tilde{\phi}(u)$ for $h \in GL(V)$, and $\tilde{\phi}(\{g, 1\}) = \tilde{\phi}(g)$ for the class $\{g, 1\} \in P$ represented by $(g, 1) \in G \times GL(V)$. We denote by l_g and L_g the left translations by g on G/K and P respectively. For $X \in \mathfrak{m} = T_0(G/K)$, the horizontal lift to P of $(l_g)_* X$ is $(L_g)_* \tilde{X}_{u_0} - \omega(L_g^* \tilde{X}_{u_0})^*_{L_g u_0}$ at the point $L_g u_0$, where $\omega(L_g^* \tilde{X}_{u_0})^*$ is the fundamental vector field on P generated by $\omega(L_g^* \tilde{X}_{u_0})$. ω is a G -invariant connection, so that $(L_g)_* \tilde{X}_{u_0} - \omega(L_g^* \tilde{X}_{u_0})^*_{L_g u_0} = L_g^* \tilde{X}_{u_0} - \omega_{u_0}(\tilde{X})^*_{L_g u_0}$. Then we have,

$$\begin{aligned} \widetilde{\nabla}_{l_g^* X} \phi(g) &= \widetilde{\nabla}_{L_g^* X} \phi(\{g, 1\}) \\ &= [(L_g^* \tilde{X}_{u_0}) \tilde{\phi} - \omega_{u_0}(\tilde{X})^*_{L_g u_0} \tilde{\phi}](\{g, 1\}) \\ &= \tilde{X}_{u_0}(L_g^* \tilde{\phi}) - \frac{d}{dt} \tilde{\phi}(\{g, 1\} \exp t \omega_{u_0}(\tilde{X}))|_{t=0} \\ &= \frac{d}{dt} ((L_g^* \tilde{\phi})(\{\exp tx, 1\}) - \frac{d}{dt} (\exp t \omega_{u_0}(\tilde{X}))^{-1} \tilde{\phi}(\{g, 1\})|_{t=0} \\ &= \frac{d}{dt} \tilde{\phi}(\{g \exp tx, 1\})|_{t=0} + \omega_{u_0}(\tilde{X}) \tilde{\phi}(g) \\ &= (X_g \tilde{\phi})(g) + M_m(X) \tilde{\phi}(g) \quad \text{for each } g \in G. \end{aligned}$$

This implies (2.3). q.e.d.

Assume that V has an inner product or a hermitian inner product \langle, \rangle according to V is a real or complex vector space, such that ρ is an orthogonal or unitary representation with respect to \langle, \rangle . Then \langle, \rangle defines a metric \langle, \rangle on the associated vector bundle \mathcal{CV} . The connection in P induced by M_m is called a metric connection if $M_m(\mathfrak{m})$ is contained in the Lie algebra $\mathfrak{o}(V)$ of the orthogonal group $O(V)$ or in the Lie algebra $\mathfrak{u}(V)$ of the unitary group $U(V)$. This condition is equivalent to that the metric \langle, \rangle on \mathcal{CV} is parallel with respect to the covariant derivative $\nabla^{\mathcal{CV}}$ on \mathcal{CV} induced by M_m .

3. Dirac operators on homogeneous spaces

In what follows, we assume that G is a connected unimodular Lie group

and K is a compact subgroup of G . Then the pair (G, K) is reductive, and so we retain the notation in the previous section.

We choose a G -invariant Riemannian metric \langle , \rangle on G/K . This defines an inner product \langle , \rangle on \mathfrak{m} . We assume that G/K is orientable and the isotropy representation α of K on \mathfrak{m} has a lifting $\tilde{\alpha}$ to $\text{Spin}(\mathfrak{m})$, i.e., there exists a homomorphism $\tilde{\alpha}$ of K into $\text{Spin}(\mathfrak{m})$ such that the following diagram is commutative;

$$\begin{array}{ccc} & & \text{Spin}(\mathfrak{m}) \\ & \nearrow \tilde{\alpha} & \downarrow \pi \\ K & \xrightarrow{\alpha} & SO(\mathfrak{m}) \end{array}$$

Take the representation $(s, L), (s^\pm, L^\pm)$ of $\text{Spin}(\mathfrak{m})$ defined in 1 and define the representations $(\sigma, L), (\sigma^\pm, L^\pm)$ of K by

$$\sigma = s \circ \tilde{\alpha}, \quad \sigma^\pm = s^\pm \circ \tilde{\alpha}.$$

The vector bundles over G/K associated to these representations are denoted by $\mathcal{L}, \mathcal{L}^\pm$ respectively.

Let $\Lambda_{\mathfrak{m}}$ be a linear map of \mathfrak{m} into $\mathfrak{o}(\mathfrak{m})$ satisfying the condition;

$$\Lambda_{\mathfrak{m}}(\alpha(k)X) = \alpha(k)\Lambda_{\mathfrak{m}}(X)\alpha(k)^{-1}$$

for $k \in K$ and $X \in \mathfrak{m}$.

We define a linear map

$$\tilde{\Lambda}_{\mathfrak{m}}: \mathfrak{m} \rightarrow \mathfrak{spin}(\mathfrak{m})$$

by

$$\tilde{\Lambda}_{\mathfrak{m}} = \tilde{\Pi}^{-1} \circ \Lambda_{\mathfrak{m}}.$$

Then $\tilde{\Lambda}_{\mathfrak{m}}$ satisfies the condition

$$\tilde{\Lambda}_{\mathfrak{m}}(\alpha(k)X) = \tilde{\alpha}(k) \cdot \tilde{\Lambda}_{\mathfrak{m}}(X) \cdot \tilde{\alpha}(k)^{-1} \quad \text{for } k \in K, X \in \mathfrak{m}.$$

We imbed $\mathfrak{spin}(\mathfrak{m})$ into $\mathfrak{gl}(L)$ (resp. $\mathfrak{gl}(L^\pm)$) through Clifford left multiplication (the differential of the spin representations). Then the above condition implies

$$\tilde{\Lambda}_{\mathfrak{m}}(\alpha(k)X) = \sigma(k)\tilde{\Lambda}_{\mathfrak{m}}(X)\sigma(k)^{-1}$$

(resp.

$$\tilde{\Lambda}_{\mathfrak{m}}(\alpha(k)X) = \sigma^\pm(k)\tilde{\Lambda}_{\mathfrak{m}}(X)\sigma^\pm(k)^{-1})$$

for $k \in K$ and $X \in \mathfrak{m}$.

We denote by $\mathcal{Q}, \mathcal{Q}^*$ the tangent and cotangent bundles over G/K . The isomorphism of \mathcal{Q}^* onto \mathcal{Q} through the Riemannian metric on G/K is denoted by h . And we denote by μ (resp. μ^\pm) the map from $C^\infty(\mathcal{Q} \otimes \mathcal{L})$ (resp. $C^\infty(\mathcal{Q} \otimes \mathcal{L}^\pm)$) to

$C^\infty(\mathcal{L})$ (resp. $C^\infty(\mathcal{L}^\pm)$) induced by the bundle map defined by the symbol map (1.2), (1.2)'. This can be well defined by Lemma 1-(i). We denote by ∇ (resp. ∇^\pm) the covariant derivative induced by $\tilde{\Lambda}_m$ on \mathcal{L} (resp. \mathcal{L}^\pm). We define the Dirac operators D, D^\pm as follows;

$$D = \mu \circ (h \otimes 1) \circ \nabla: C^\infty(\mathcal{L}) \xrightarrow{\nabla} C^\infty(\mathcal{I}^* \otimes \mathcal{L}) \xrightarrow{h \otimes 1} C^\infty(\mathcal{I} \otimes \mathcal{L}) \xrightarrow{\mu} C^\infty(\mathcal{L}),$$

$$D^\pm = \mu^\pm \circ (h \otimes 1) \circ \nabla^\pm: C^\infty(\mathcal{L}^\pm) \xrightarrow{\nabla^\pm} C^\infty(\mathcal{I}^* \otimes \mathcal{L}^\pm) \xrightarrow{h \otimes 1} C^\infty(\mathcal{I} \otimes \mathcal{L}^\pm) \xrightarrow{\mu^\pm} C^\infty(\mathcal{L}^\pm).$$

Lemma 2. Let $\{X_i\}_{i=1, \dots, n}$ be an orthonormal base of \mathfrak{m} with respect to \langle, \rangle , and $\{\omega^i\}_{i=1, \dots, n}$ its dual base.

(i) For $\phi \in C^\infty(\mathcal{L})$, we have

$$(3.1) \quad \tilde{D}\phi = \sum_{i=1}^n \varepsilon \{X_i \otimes (X_i \tilde{\phi} + \tilde{\Lambda}_m(X_i) \tilde{\phi})\}.$$

(ii) The same formulas hold for an element of $C^\infty(\mathcal{L}^\pm)$.

Proof. Immediate consequence of Proposition 2 and the definition of the Dirac operator D .

Let (ρ, V) be a finite dimensional unitary representation of K and V the vector bundle associated to (ρ, V) . Then V carries the invariant metric induced from the hermitian inner product on V . Let M_m be a linear map of \mathfrak{m} into $\mathfrak{gl}(V)$ satisfying the condition (2.1), and $\nabla^{\mathcal{C}\mathcal{V}}$ the covariant derivative on $\mathcal{C}\mathcal{V}$ induced by M_m . In order to define our Dirac operators from $C^\infty(\mathcal{L} \otimes \mathcal{C}\mathcal{V})$ (resp. $C^\infty(\mathcal{L}^\pm \otimes \mathcal{C}\mathcal{V})$) to $C^\infty(\mathcal{L} \otimes \mathcal{C}\mathcal{V})$ (resp. $C^\infty(\mathcal{L}^\pm \otimes \mathcal{C}\mathcal{V})$) We use the following theorem.

Theorem P (Theorem 3, §9, Chapter IV in [4]). Let $\mathcal{U}, \mathcal{C}\mathcal{V}, \mathcal{W}$ be vector bundles over a C^∞ -manifold M , $D: C^\infty(\mathcal{U}) \rightarrow C^\infty(\mathcal{W})$ a first order linear differential operator on M and $\nabla^{\mathcal{C}\mathcal{V}}$ a covariant derivative for $\mathcal{C}\mathcal{V}$. Then there is a unique first order differential operator

$$T: C^\infty(\mathcal{U} \otimes \mathcal{C}\mathcal{V}) \rightarrow C^\infty(\mathcal{W} \otimes \mathcal{C}\mathcal{V})$$

such that

$$T(f \otimes h)(x) = (Df \otimes h)(x) \quad \text{whenever } (\nabla^{\mathcal{C}\mathcal{V}} h)(x) = 0.$$

We denote this operator by $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$. From (2.3), (3.1) and using the above theorem, we can define a differential operator $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$ (resp. $D^\pm \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$), which we call the Dirac operators.

Proposition 3. For $\phi \in C^\infty(\mathcal{L} \otimes \mathcal{C}\mathcal{V})$, we have

$$(3.2) \quad \widetilde{D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1} \phi = \sum_{i=1}^n \varepsilon_V [X_i \otimes \{X_i \tilde{\phi} + (\tilde{\Lambda}_m(X_i) \otimes 1) \tilde{\phi} + (1 \otimes M_m(X_i)) \tilde{\phi}\}]$$

where $\varepsilon_V = \varepsilon \otimes 1: m \otimes L \otimes V \rightarrow L \otimes V$ and $\{X_i\}_{i=1, \dots, n}$ is an orthonormal base of m .

Proof. It suffices to prove that the right hand side of (3.2) is a first order differential operator from $C^\infty(\mathcal{L} \otimes \mathcal{C}V)$ to $C^\infty(\mathcal{L} \otimes \mathcal{C}V)$ and satisfies the condition of theorem *P*. But it is easy to see these conditions by making use of Proposition 2. q.e.d.

4. Formal self adjointness of Dirac operators

We denote by dx the invariant measure on G/K induced by the G -invariant Riemannian metric. Since G is a unimodular Lie group, there exists a bi-invariant measure dg on G such that for any C^∞ -function f with compact support we have

$$\int_G p^* f dg = \int_{G/K} f dx$$

where p is the projection $G \rightarrow G/K$. Then we have in virtue of the invariance of dg

$$(4.1) \quad \int_G Xf dg = 0 \quad \text{for all } X \in \mathfrak{g}.$$

We fix inner products on L and L^\pm satisfying the Lemma 1-(iii). Then \mathcal{L} and \mathcal{L}^\pm carry the metrics induced from the above inner products and also $\mathcal{L} \otimes \mathcal{C}V$, $\mathcal{L}^\pm \otimes \mathcal{C}V$ carry the metrics induced from the metrics of \mathcal{L} , \mathcal{L}^\pm and $\mathcal{C}V$. The inner product $(,)$ on the space $C_c^\infty(\mathcal{L} \otimes \mathcal{C}V)$ of all C^∞ -sections with compact support is defined by

$$(4.2) \quad (\phi, \psi) = \int_{G/K} \langle \phi, \psi \rangle dx = \int_G \langle \tilde{\phi}, \tilde{\psi} \rangle_0 dg$$

where \langle , \rangle is the metric on the vector bundle, and \langle , \rangle_0 is the inner product of $L \otimes V$ or $L^\pm \otimes V$.

Proposition 4. *We have the formula for the formal adjoint operator $(D \hat{\otimes} 1)^*$ of $(D \hat{\otimes} 1)$ as follows;*

$$(4.3) \quad \widetilde{(D \hat{\otimes} 1)^* \phi} = \sum_{i=1}^n [\varepsilon_V \{X_i \otimes (X_i \phi)\} + (\tilde{\Delta}_m(X_i) \otimes 1) \varepsilon_V(X_i \otimes \tilde{\phi}) - \varepsilon_V \{X_i \otimes (1 \otimes M_m^*(X_i)) \tilde{\phi}\}]$$

where M_m^* is the adjoint operator of M_m with respect to the inner product of V .

Proof. For $\phi, \psi \in C_c^\infty(\mathcal{L} \otimes \mathcal{C}V)$,

$$\begin{aligned}
 ((D \hat{\otimes} 1) \phi, \psi) &= \sum_{i=1}^n \int_G \langle \varepsilon_V \{X_i \otimes (X_i \tilde{\phi})\}, \tilde{\psi} \rangle_0 dg \\
 &\quad + \sum_{i=1}^n \int_G \langle \varepsilon_V \{X_i \otimes (\tilde{\Lambda}_m(X_i) \otimes 1) \tilde{\phi}\}, \tilde{\psi} \rangle_0 dg \\
 &\quad + \sum_{i=1}^n \int_G \langle \varepsilon_V \{X_i \otimes (1 \otimes M_m(X_i)) \tilde{\phi}\}, \tilde{\psi} \rangle_0 dg \\
 &= - \sum_{i=1}^n \int_G \langle X_i \tilde{\phi}, \varepsilon_V(X_i \otimes \tilde{\psi}) \rangle_0 dg \\
 &\quad - \sum_{i=1}^n \int_G \langle (\tilde{\Lambda}_m(X_i) \otimes 1) \tilde{\phi}, \varepsilon_V(X_i \otimes \tilde{\psi}) \rangle_0 dg \\
 &\quad - \sum_{i=1}^n \int_G \langle (1 \otimes M_m(X_i)) \tilde{\phi}, \varepsilon_V(X_i \otimes \tilde{\psi}) \rangle_0 dg \\
 &= - \sum_{i=1}^n \int_G X_i \langle \tilde{\phi}, \varepsilon_V(X_i \otimes \tilde{\psi}) \rangle_0 dg \\
 &\quad + \sum_{i=1}^n \int_G \langle \tilde{\phi}, X_i \varepsilon_V(X_i \otimes \tilde{\psi}) \rangle_0 dg \\
 &\quad + \sum_{i=1}^n \int_G \langle \tilde{\phi}, (\tilde{\Lambda}_m(X_i) \otimes 1) \varepsilon_V(X_i \otimes \tilde{\psi}) \rangle_0 dg \\
 &\quad - \sum_{i=1}^n \int_G \langle \tilde{\phi}, (1 \otimes M_m^*(X_i)) \{ \varepsilon_V(X_i \otimes \tilde{\psi}) \} \rangle_0 dg \\
 &= \sum_{i=1}^n \int_G \langle \tilde{\phi}, \varepsilon_V(X_i \otimes X_i \tilde{\psi}) \rangle_0 dg \\
 &\quad + \sum_{i=1}^n \int_G \langle \tilde{\phi}, (\tilde{\Lambda}_m(X_i) \otimes 1) \varepsilon_V(X_i \otimes \tilde{\psi}) \rangle_0 dg \\
 &\quad - \sum_{i=1}^n \int_G \langle \tilde{\phi}, \varepsilon_V \{X_i \otimes (1 \otimes M_m^*(X_i)) \tilde{\psi}\} \rangle_0 dg .
 \end{aligned}$$

Thus we have the proposition 4.

q.e.d.

Theorem. *Suppose the connection induced by M_m is a metric connection. Then a necessary and sufficient condition that $D \hat{\otimes} 1$ is a formal self adjoint operator (resp. $D^\pm \hat{\otimes} 1$ is the formal adjoint operator of $D^\mp \hat{\otimes} 1$) if and only if the following condition;*

$$(4.4) \quad \sum_{i=1}^n \Lambda_m(X_i) X_i = 0$$

holds, where $\{X_i\}_{i=1, \dots, n}$ is an orthonormal base of \mathfrak{m} .

Proof. From our assumption, $M_m(X) = -M_m^*(X)$ for each $X \in \mathfrak{m}$. For $\phi \in C_c^\infty(\mathcal{L} \otimes \mathcal{V})$, from Proposition 4 we have

$$\widetilde{(D \hat{\otimes} 1) \phi} - \widetilde{(D \hat{\otimes} 1)^* \phi} = \sum_{i=1}^n [(\tilde{\Lambda}(X_i) \otimes 1) \varepsilon_V(X_i \otimes \tilde{\phi}) - \varepsilon_V \{X_i \otimes (\tilde{\Lambda}_m(X_i) \otimes 1) \tilde{\phi}\}].$$

Thus, we see that the condition,

$$(4.5) \quad \sum_{i=1}^n \{ \tilde{\Lambda}_m(X_i) \varepsilon(X_i \otimes l) - \varepsilon(X_i \otimes \tilde{\Lambda}_m(X_i) l) \} = 0 \quad \text{for } l \in L,$$

is necessary and sufficient in order that $D \hat{\otimes}_{\nabla^{CV}} 1$ becomes the formal self adjoint operator. From Lemma 1-(ii) and (1.1), the condition (4.5) is equivalent to

$$\sum_{i=1}^n \Lambda_m(X_i) X_i = 0. \quad \text{q.e.d.}$$

Corollary. *In the following two cases, the condition of theorem is satisfied;*

- (i) α has no fixed point except 0.
- (ii) The connection induced by Λ_m is Riemannian, i.e., it coincides with the Riemannian connection defined by a G -invariant Riemannian metric g on G/K .

Proof. (i) for any $k \in K$ we have,

$$\begin{aligned} \alpha(k) \left(\sum_{i=1}^n \Lambda_m(X_i) X_i \right) &= \sum_{i=1}^n \alpha(k) \Lambda_m(X_i) \alpha(k)^{-1} \alpha(k) X_i \\ &= \sum_{i=1}^n \Lambda_m(\alpha(k) X_i) \alpha(k) X_i \\ &= \sum_{i=1}^n \Lambda_m(X_i) X_i, \end{aligned}$$

where the last equality follows from the fact that $\alpha(k)$ is an orthogonal transformation on \mathfrak{m} . Hence from our assumption, we have

$$\sum_{i=1}^n \Lambda_m(X_i) X_i = 0.$$

- (ii) We denote by B the inner product on \mathfrak{m} induced by g . Then the Riemannian connection defined by g is given by

$$\Lambda_m(X) Y = \frac{1}{2} [X, Y]_{\mathfrak{m}} + U(X, Y)$$

where $[X, Y]$ is the \mathfrak{m} -component of $[X, Y]$ and $U(X, Y)$ is the symmetric bilinear mapping of $\mathfrak{m} \times \mathfrak{m}$ into \mathfrak{m} defined by

$$2B(U(X, Y), Z) = B(X, [Z, Y]_{\mathfrak{m}}) + B([Z, X]_{\mathfrak{m}}, Y)$$

for all $X, Y, Z \in \mathfrak{m}$. (cf. Theorem 3.3 Chapter X in [1]-(b)). From the above formula, we have

$$\begin{aligned} \sum_{i=1}^n \Lambda_m(X_i) X_i &= \sum_{i=1}^n U(X_i, X_i) \\ &= \sum_{i,i} B(U(X_i, X_i), X_j) X_j \end{aligned}$$

$$= \sum_{i,j}^n B(X_i, [X_j, X_i])X_j,$$

here we extend B to \mathfrak{g} such that (1) B is an $Ad(k)$ -invariant metric on \mathfrak{g} , and (2) \mathfrak{m} and \mathfrak{k} are mutually orthogonal with respect to B . We choose an orthonormal base $\{Y_1, \dots, Y_p\}$ of \mathfrak{k} . Then using $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, we have

$$\begin{aligned} \sum_{i=1}^n B(X_i, [X_j, X_i])\mathfrak{m} &= \sum_{i=1}^n \{B(X_i, [X_j, X_i]) + B(Y_i, [X_j, X_i])\} \\ &= Tr \, ad_{\mathfrak{g}} X_j \\ &= 0, \end{aligned}$$

where the last equality holds since G is unimodular. Thus we have

$$\sum_{i=1}^n \Lambda_{\mathfrak{m}}(X_i)X_i = 0. \quad \text{q.e.d.}$$

REMARKS (i) Suppose G is compact and $\text{rank } G = \text{rank } K$, then the condition (i) of Corollary is always satisfied. In fact, let T be a maximal torus of G contained in K . Adjoint representation of T on \mathfrak{g} is decomposed as follows;

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_{n/2},$$

where each \mathfrak{p}_i is two dimensional subspace of \mathfrak{g} . On each \mathfrak{p}_i , T act as nontrivial rotational elements. Thus the isotropy representation has no fixed point except 0.

(ii) Suppose G is semi-simple and G/K is a symmetric space. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. Let H be a non-zero element of \mathfrak{p} and \mathfrak{a} a maximal abelian subspace of \mathfrak{p} containing H . Then there exist an element w of the Weyl group of G/K and an element k of K such that $Ad(k)H = wH \neq H$. Thus the condition of Corollary (i) is satisfied.

(iii) If $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$ is formally self adjoint, then in the same way as Wolf [5], We see that $D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1$ and $(D \hat{\otimes}_{\nabla^{\mathcal{C}\mathcal{V}}} 1)^2$ are essentially self adjoint operators. Because, his proof (Theorem 5.1 and Theorem 6.1 in [5]) has no use that the connection is Riemannian.

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