Kim, J.-H.
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CONJUGATE CONNECTIONS AND $SU(3)$-INSTANTON INVARIANTS

JIN-HONG KIM

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1. Introduction

In their paper [8], S. Kobayashi and E. Shinozaki introduced the concept of conjugate connection for a reducible principal bundle P. From the point of view of the geometry of 4-manifolds, the importance of this concept is that an automorphism of a Lie group G fixing a Lie subgroup H induces a compatible action on the quotient space $\mathcal{B}(P)$ of connections by the group of gauge transformations. When we fix a Riemannian metric g on the base manifold and G is compact and semi-simple, we can also see that the automorphism group acts on the moduli space $\mathcal{M}_P(g)$ of Yang-Mills connections modulo the group of gauge transformations. In particular, the automorphism group acts on the moduli space of anti-self-dual (ASD) connections, when the dimension of the base manifold is 4.

One advantage of this action on the moduli space of ASD connections is that it is not the action induced from an action on the base manifold. When we attempt to use the induced action on the moduli space coming from the base manifold, typically we have to overcome some serious transversality issues. M. Itoh, however, observed in [5] that the inner automorphism group induces a trivial action on the quotient space $\mathcal{B}(P)$. It is well known [12] that for compact simple Lie groups only Lie groups of type A_r ($r > 1$), D_r ($r \geq 4$), and E_6 have a nontrivial outer automorphism group. As a first application of conjugate connections to the geometry of 4-manifolds, in this paper we exclusively deal with the simplest compact simple Lie group $SU(3)$ other than $SU(2)$. In this case the outer automorphism group is isomorphic to the cyclic group $\mathbb{Z}_2$.

One of the aims of this paper is to prove a fixed point theorem under the group $\mathbb{Z}_2$ action on the irreducible part $\mathcal{B}^*(P)$ of the quotient space $\mathcal{B}(P)$ in a reducible principal $SU(3)$-bundle P along the irreducible part $\mathcal{B}^*(Q)$ of the quotient spaces $\mathcal{B}(Q)$ of $SO(3)$-subbundles Q of P. More precisely, in Section 3 we show the following

Theorem 1.1. *Let X be a closed oriented simply-connected manifold. Fix a Riemannian metric g on X, and let P be a principal $SU(3)$-bundle reducible to an $SO(3)$-subbundle Q. Let $\mathcal{M}^*_P(g)_\sigma$ denote the fixed point set of the irreducible part $\mathcal{M}^*_P(g)$ of the moduli space $\mathcal{M}_P(g)$ of Yang-Mills (or ASD) on P under the action of the outer automorphism group of $SU(3)$ fixing a Lie subgroup $SO(3)$ generated by σ.*

Then we have

$$\mathcal{M}^*_P(g)_\sigma = \coprod_{Q' \in \Gamma} \mathcal{M}^*_{Q'}(g),$$

where Γ *denotes the set of* $SO(3)$*-bundle isomorphism classes in* P.

We then use this fixed point theorem to show a mod 2 vanishing theorem for Donaldson's type instanton invariants for the Lie group $SU(3)$. To the author's knowledge, however, there has been no well-developed theory for instanton invariants using the higher rank Lie groups, or even the Lie group $SU(3)$. Thus the second aim of this paper is to give well-defined $SU(3)$-*instanton invariants* for certain smooth 4-manifolds.

Unlike the $SU(2)$ or $SO(3)$ case whose associated adjoint bundles have rank 3, in general we cannot expect that for a generic metric on a 4-manifold the moduli space of irreducible ASD $SU(3)$-connections is a smooth manifold with the virtual dimension given by the Atiyah-Singer index theorem. This mainly comes from the discrepancy between the dimension of its associated adjoint bundle and the dimension of the space of self-dual 2-forms. For any metric g, it is possible that the moduli space of ASD $SU(3)$-connections on an $SU(3)$-bundle P always contains the moduli space of ASD connections on an $SU(2)$-bundle P' with $c_2(P') = c_2(P)$. In this case it is easy to see through the Atiyah-Singer index calculation that the virtual dimension of the reducible part of the moduli space of the ASD $SU(3)$-connections is larger than that of the irreducible part of the moduli space ASD $SU(3)$-connections.

Hence it seems to be impossible to define $SU(3)$ instanton invariants independent of the metric g with the irreducible part of the moduli space of ASD $SU(3)$-connections. M. Marino and G. Moore also pointed out in [10] that deriving the higher rank Donaldson invariants using the standard mathematical methods in [2] did not work well because of the singularities of instanton moduli space. Donaldson, however, suggested that it might be possible to define the higher rank invariants using the techniques he developed in his paper [1]. It will be very useful to consider the higher rank invariants for many reasons. For example, by considering even dimensional Lie groups such as $SU(3)$ in case the first Betti number is zero, one could obtain results about 4-manifolds with b_2^+ even, while the Donaldson invariants using $SU(2)$ or $SO(3)$ have a restriction to 4-manifolds with b_2^+ odd for a very simple reason.

Following a line suggested by Donaldson, in this paper we consider an even dimensional Lie group $SU(3)$, and as a first step to more general $SU(3)$-instanton invariants on smooth 4-manifolds we define simple $SU(3)$-instanton invariants $q_k(X)$ for certain smooth 4-manifolds. We hope we can come back to the construction of $SU(3)$-instanton polynomial invariants on smooth 4-manifolds in a future paper.

More precisely, in Section 4 we show the following

Theorem 1.2. *Let X be a closed, oriented, simply connected, smooth 4-manifold with a generic metric and $b_2^+(X) = 3k - 1$ $(k \geq 1)$. We assume that the following properties hold*:

(H1) *The signature $\sigma(X)$ of X is not $-2k \mod 8$.*

(H2) *There exists a principal $SU(3)$-bundle P over X with $c_2(P) = 2k$ that is reducible to an $SO(3)$-subbundle Q satisfying $w_2(Q) \equiv w_2(TX) \mod 2$.*

Then there is a well-defined integer $q_k(X)$ depending only on the smooth structure of X, and the smooth invariants $q_k(X)$ are always zero modulo 2.

It is worth mentioning that the hypothesis **(H2)** is never strong because we can always get such a principal $SU(3)$-bundle by extending a principal $SO(3)$-bundle Q satisfying $w_2(Q) \equiv w_2(TX) \mod 2$. In view of our proof of this theorem, it seems to be possible to define more general $SU(3)$-instanton invariants without the hypothesis **(H1)**. Thus we conjecture that in that case $SU(3)$-instanton invariants should be equal to $SU(2)$-instanton polynomial invariants modulo 2. But, in order to avoid any complications, in this paper we keep imposing the hypothesis **(H1)**, and we will address this issue somewhere else, later.

The mod 2 vanishing result in the above theorem is an immediate consequence of the fixed point theorem (Theorem 1.1).

Let X be an algebraic surface. On $U(2)$-bundles over algebraic surfaces we have the notion of Hermitian-Einstein connection. A $U(2)$-connection A is called *Hermitian-Einstein* if its curvature F_A satisfies $\Lambda F_A = \lambda$, where Λ is the Kähler trace operator on forms of type (1,1) and λ is constant.

S. Donaldson [2] proved that if E is an $SU(3)$ bundle over an algebraic surface X, the moduli space of irreducible ASD connections is naturally identified as a set with the set of equivalence classes of stable holomorphic $SL(3, \mathbb{C})$ bundles $\mathcal{E}$ which are topologically equivalent to E, i.e., $c_2(\mathcal{E}) = c_2(E)$. Since the moduli spaces of stable bundles are defined as complex spaces, they have a natural orientation. Thus we note that the signs in our simple invariants at different points agree, i.e., there is no cancellation. For algebraic surfaces we have the following

Corollary 1.3. *Let X be a simply connected algebraic surface with $b_2^+(X) = 6k - 1$ $(k \geq 1)$. Assume that the hypotheses* **(H1)** *and* **(H2)** *hold. Let E be a complex rank 3 vector bundle associated to P, and let F be the real rank 3 bundle associated to Q. Then the set of equivalence classes of stable holomorphic $SL(3, \mathbb{C})$ bundles $\mathcal{E}$ which are topologically equivalent to E is finite and the number is even.*

We organize this paper as follows. In Section 2, we recall the definition and basic properties of conjugate connections in reducible principal bundles. Section 3 is devoted to showing a fixed point theorem for the action of outer automorphism group of $SU(3)$ on the quotient space $\mathcal{B}(P)$ in a principal $SU(3)$-bundle P that is reducible to an $SO(3)$-subbundle Q. In Section 4, we define $SU(3)$ instanton invariants for certain smooth 4-manifolds. We also show the mod 2 vanishing theorem for the smooth invariants in that section.

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2. Conjugate Connections in Principal Bundles

The purpose of this section is to review basic definitions and general properties for the conjugate connections introduced by S. Kobayashi and E. Shinozaki. See [8] and [9] for more details.

Let X be a manifold, G a Lie group, and P a principal G-bundle over X with projection π. Let $\{U_i\}$ be an open cover of X with local sections $\xi_i : U_i \longrightarrow \pi^{-1}(U_i)$. Then, we define the transition function $a_{ij} : U_i \cap U_j \longrightarrow G$ by

$$\xi_j(x) = \xi_i(x) a_{ij}(x).$$

A *connection form* A on P is defined as a family of $\mathfrak{g}$-valued 1-forms A_i on U_i which satisfies the following transformation rule from A_i to A_j:

$$A_j = a_{ij}^{-1} A_i a_{ij} + a_{ij}^{-1} da_{ij} \quad \text{on } U_i \cap U_j. \tag{2.1}$$

Since A_i is not defined on all of X, we define a $\mathfrak{g}$-valued 1-form on P from A_i as follows:

$$A = g^{-1} A_i g + g^{-1} dg, \quad g \in G,$$

on $\pi^{-1}(U_i) = U_i \times G$. Then it is easy to see that A is a $\mathfrak{g}$-valued 1-form on P, and it follows that $\xi_i^*(A) = A_i$. The set of all connections on P is denoted by $\mathcal{A}(P)$.

Now we recall the definition of conjugate connection on principal fiber bundles. Let G be a Lie group with its Lie algebra $\mathfrak{g}$, and let H be a closed subgroup with its Lie algebra $\mathfrak{h}$. Let $\mathrm{Aut}(G, H)$ be the group of automorphisms of G leaving all elements of H fixed, $\mathrm{Inn}(G, H)$ be the group of all inner automorphisms in $\mathrm{Aut}(G, H)$, and let $E(G, H) = \mathrm{Aut}(G, H)/\,\mathrm{Inn}(G, H)$. $E(G, H)$ is called the *outer automorphism group*. Let σ be an automorphism in $\mathrm{Aut}(G, H)$. The induced automorphism of $\mathfrak{g}$ is denoted also σ.

Let Q be a principal H-subbundle of P. In general, such a subbundle Q over X does not necessarily exist. However, we can always extend a principal H-bundle Q to the principal G-bundle P so that Q is a principal H-subbundle of P. In order to define the σ-conjugate connection on principal fiber bundles we cover X by open sets U_i with local sections $\xi_i : U_i \longrightarrow Q$. It is important to take local sections of the subbundle Q so that the transition functions a_{ij}'s are H-valued.

We are ready to define conjugate connections on principal fiber bundles. Let A be a connection form on P. We put $A_i = \xi_i^* A$. Then A_i is a $\mathfrak{g}$-valued 1-form on U_i, but not necessarily $\mathfrak{h}$-valued. We set $A_i^\sigma = \sigma(A_i)$ and apply σ to (2.1). Since a_{ij} is H-valued, we have

$$A_j^\sigma = a_{ij}^{-1} A_i^\sigma a_{ij} + a_{ij}^{-1} da_{ij}.$$

Thus, $\{A_i^\sigma\}$ defines a connection on P. We call it the *σ-conjugate connection* of A relative to Q, denoted A_Q^σ. If there is no danger of confusion, we write A^σ for A_Q^σ. Note that A^σ is not $\sigma(A)$. In fact, $\sigma(A)$ may not be a connection 1-form.

Given $\sigma \in \mathrm{Aut}(G, H)$, we can also define its action on P as follows. This helps us to understand the notion of conjugate connection on principal fiber bundles. Let $p \in P$ lie over $x \in U_i$. Then, since we can let $p = \xi_i(x)g$ for $g \in G$, we define a transformation $h_{\sigma,Q}$ of P by

$$h_{\sigma,Q}(p) = h_{\sigma,Q}(\xi_i(x)g) = \xi_i(x)\sigma(g).$$

If there is no danger of confusion, we write h_σ for $h_{\sigma,Q}$. Note that as a transformation of $U_i \times G$, h_σ can be given by

$$h_\sigma : (x, g) \mapsto (x, \sigma(g)).$$

Note that the σ-conjugate connection A^σ of A can be given by

$$A^\sigma = (h_\sigma^{-1})^* \sigma(A). \tag{2.2}$$

The curvature form of the σ-conjugate connection $\{A_i^\sigma\}$ (resp. A^σ) is given by $\{F_i^\sigma\}$ (resp. F_A^σ).

A transformation φ of a principal fiber G-bundle P which commutes with the right action of G, i.e.

$$\varphi(ug) = \varphi(u)g \quad \text{for all } g \in G,\ u \in P$$

is called a *gauge transformation*. The set of gauge transformations, denoted by $\mathcal{G} = \mathcal{G}(P)$, is called the *group of gauge transformations*. We have another important characterization of gauge transformations which proves to be useful in constructing a gauge transformation.

Lemma 2.1. (a) *A gauge transformation φ of P defines a map $\varphi_i : U_i \longrightarrow G$ by*

$$\varphi(\xi_i(x)) = \xi_i(x)\varphi_i(x), \qquad x \in U_i. \tag{2.3}$$

(b) *These maps φ_i form a family $\{\varphi_i\}$ satisfying*

$$\varphi_j(x) = a_{ij}(x)^{-1}\varphi_i(x)a_{ij}(x), \quad \text{on} \quad x \in U_i \cap U_j. \tag{2.4}$$

(c) *Conversely, a family* $\{\varphi_i\}$ *satisfying* (2.4) *defines a gauge transformation by* (2.3).

Given a gauge transformation $\{\varphi_i\}$ with respect to local sections ξ_i of Q on U_i, applying σ to (2.4), we get

$$\sigma(\varphi_j(x)) = a_{ij}(x)^{-1}\sigma(\varphi_i(x))a_{ij}(x)$$

on $x \in U_i \cap U_j$. By Lemma 2.1, the family $\{\sigma(\varphi_i)\}$ defines a gauge transformation of P, denoted φ^σ, and called *σ-conjugate gauge transformation*.

It is time to state two important theorems of S. Kobayashi and E. Shinozaki in [8], [9]. To do this, we fix a point $u_0 \in Q$ and let $H_{u_0}(A)$ be the holonomy group of the connection A with respect to the reference point u_0. We call a connection A in P *generic* if its holonomy group $H_{u_0}(A)$ coincides with G, and we call a connection A in P *irreducible* if its isotropy group $\mathcal{G}_A$, as a closed Lie subgroup of G, coincides with the center $C(G)$ of G.

Theorem 2.2. *The group* $\mathrm{Aut}(G, H)$ *acts on the quotient space* $\mathcal{B}(P)$ *of connections.*

Note that since $\mathrm{Inn}(G, H)$ acts trivially on the quotient space $\mathcal{B}(P)$, $E(G, H)$ acts on $\mathcal{B}(P)$ (see [9] and [5]).

Theorem 2.3. *Let* $\sigma \in \mathrm{Aut}(G, H)$ *and* A *be a connection in* P. *Assume that* A^σ *is gauge equivalent to* A *under a gauge transformation* φ. *If we define an element* $a \in G$ *by* $\varphi(u_0) = u_0 a$, *then*

$$\sigma(g) = a^{-1}ga$$

for $g \in H_{u_0}(A)$. *In particular, if the holonomy group is* G, *then* σ *is the inner automorphism defined by* a^{-1} *above.*

From the above theorem, it is easy to see that $E(G, H)$ acts freely on the generic part of the quotient space $\mathcal{B}(P)$ and that $\mathrm{Aut}(G, H)$ acts freely on the generic part of the framed quotient space $\mathcal{B}_0(P)$ of connections by the group of framed gauge transformations.

Finally we need one more observation. That is, while the action of $\mathrm{Aut}(G, H)$ on $\mathcal{G}(P)$ and $\mathcal{A}(P)$ depends on the reduction of P to the H-subbundle Q, the action of $\mathrm{Aut}(G, H)$ on the quotient space of connections does not depend on the reduction of P to the H-subbundle Q. In fact, using (2.2) we have the following:

Proposition 2.4.

Let P *be a principal* G*-bundle that is reducible to* H*-subbundles* Q *and* Q'. *For a connection* A *in* $\mathcal{A}(P)$, A^σ_Q *is gauge equivalent to* $A^\sigma_{Q'}$ *under* $h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q} \in$

$\mathcal{G}(P)$. Therefore, the action of $\mathrm{Aut}(G, H)$ *on the quotient space $\mathcal{B}(P)$ is independent of the chosen subbundle Q.*

Proof. We first show that for any $\sigma \in \mathrm{Aut}(G, H)$, $h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q}$ is a gauge transformation in P. To do it, let $\varphi = h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q}$. It suffices to show that $\varphi(ug) = \varphi(u)g$ for all $g \in G$, $u \in P$. We first cover X by an open covering $\{U_i\}$ so that there exists local sections s_i, s_i' on U_i of Q, Q', respectively with the relation $s_i(x) = s_i'(x)a_{QQ'}(x)$ for some $a_{QQ'}(x) \in G$. Then, we have

$$\begin{aligned}\varphi(ug) &= h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q'}(ug) = h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q}(s_i(x)tg) \quad \text{for some} \quad t \in G\\ &= h_{\sigma,Q'}(s_i(x)\sigma^{-1}(tg)) = h_{\sigma,Q'}(s_i'(x)a_{QQ'}(x)\sigma^{-1}(tg))\\ &= s_i'(x)\sigma(a_{QQ'}(x))tg = h_{\sigma,Q'}(s_i'(x)a_{QQ'}(x)\sigma^{-1}(t))g\\ &= h_{\sigma,Q'}(s_i(x)\sigma^{-1}(t))g = h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q}(s_i(x)t)g\\ &= h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q}(u)g = \varphi(u)g,\end{aligned}$$

which completes the proof of the claim.

It is also clear from (2.2) that

$$A^\sigma_Q = (h^{-1}_{\sigma,Q})^*\sigma(A) = (h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q})^*(h^{-1}_{\sigma,Q'})^*\sigma(A) = (h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q})^* A^\sigma_{Q'}.$$

Since $h_{\sigma,Q'} \circ h^{-1}_{\sigma,Q}$ is a gauge transformation in P, $A^\sigma_{Q'}$ is gauge equivalent to A^σ_Q. This completes the proof. □

3. A Fixed Point Theorem

In this section, we primarily consider the following special case: the symmetric pair $(SU(r), SO(r))$ $(r \geq 3)$ which defines a simply connected symmetric space. It is well known that every automorphism of $SU(2)$ fixing a Lie group $SO(2)$ is inner.

We also fix σ which is the automorphism on $SU(r)$ defined by $g \mapsto \overline{g}$ for $g \in SU(r)$. Since $\sigma(g) = g$ if and only if $g \in SO(r)$, the automorphism σ is in $\mathrm{Aut}(SU(r), SO(r))$ and actually this is a generator for $E(SU(r), SO(r)) \cong \mathbb{Z}_2$ (see [12] for more details).

In particular, if $r = 3$, we get the following fixed point theorem:

Theorem 3.1. *Let X be a simply connected manifold, and let P be a principal $SU(3)$-bundle that is reducible to an $SO(3)$-subbundle Q. Let A be an irreducible connection in P. If $[A]^\sigma = [A]$ in the quotient space $\mathcal{B}(P)$, then the holonomy group $H_{u_0}(A) = SO(3)$ up to conjugacy under inner automorphisms.*

As a consequence, A defines a connection in an $SO(3)$-subbundle Q' of P.

Proof. First note that if a connection has $SU(2)$ or S^1 as a holonomy group, then the connection is not irreducible because the centralizer of $SU(2)$ or S^1 in $SU(3)$ is not $\mathbb{Z}_3$, which is the center of $SU(3)$. On the other hand, every connection having $SO(3)$ as a holonomy group is irreducible.

Suppose that A^σ is gauge equivalent to A under $\varphi \in \mathcal{G}(P)$. Since X is a simply-connected manifold, it is shown in [7] and [6] that the holonomy group $H_{u_0}(A)$ is a connected *closed* Lie subgroup of $SU(3)$. Then, by Theorem 2.3 $\overline{Z} = \sigma(Z) = a^{-1}Za$ for Z in the Lie algebra of $H_{u_0}(A)$, where as before a is an element in $SU(3)$ such that $\varphi(u_0) = u_0 a$. Thus, we have $\overline{\det(Z)} = \det(\overline{Z}) = \det(Z)$ for Z in the Lie algebra of $H_{u_0}(A)$. On the other hand, by the property of the Lie algebra $\mathfrak{su}(3)$, we have $\overline{\det(Z)} = \det(\overline{Z}) = \det(\overline{Z}^T) = -\det(Z)$ for Z in the Lie algebra of $H_{u_0}(A)$. Thus, clearly we have $\det(Z) = 0$ for Z in the Lie algebra of $H_{u_0}(A)$.

We next show that the holonomy group $H_{u_0}(A)$ is a compact connected rank-1 Lie subgroup of $SU(3)$. In fact, suppose that $H_{u_0}(A)$ is a rank-2 Lie subgroup of $SU(3)$. The Lie algebra $\mathfrak{su}(3)$ contains a Lie subalgebra of the following form

$$\left\{ \begin{pmatrix} xi & 0 & 0 \\ 0 & yi & 0 \\ 0 & 0 & zi \end{pmatrix} : x + y + z = 0 \right\}. \tag{3.1}$$

Since $H_{u_0}(A)$ is assumed to be a rank-2 Lie subgroup of $SU(3)$, we may assume without loss of generality that its Lie algebra also contains the Lie subalgebra of the form (3.1). However, this Lie subalgebra contains an element $\mathrm{diag}(i, i, -2i)$ whose determinant is $2i \neq 0$, which is a contradiction to $\det(Z) = 0$ for Z in the Lie algebra of $H_{u_0}(A)$. Hence, we can conclude that the Lie subgroup $H_{u_0}(A)$ of $SU(3)$ contains only a 1-dimensional maximal torus. Moreover, using $\det(Z) = 0$ for Z in the Lie algebra of $H_{u_0}(A)$, the Lie algebra of the maximal torus should be of the following form (3.2) without loss of generality:

$$\left\{ \begin{pmatrix} xi & 0 & 0 \\ 0 & -xi & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\}. \tag{3.2}$$

This Lie subalgebra is exactly the Lie algebra of a maximal torus of $SO(3)$ or $SU(2)$ in $SU(3)$ up to conjugation under inner automorphisms. But, since any connection having $SU(2)$ or S^1 as a holonomy group is not irreducible, we can conclude that the holonomy group $H_{u_0}(A)$ is $SO(3)$ up to conjugation under inner automorphisms, completing the proof. □

Since the holonomy group $H_{u_0}(A^\sigma)$ of the conjugate connection A^σ is just $\sigma(H_{u_0}(A))$, it is obvious that the action of the automorphism group $\mathrm{Aut}(G, H)$ on the irreducible part of the quotient space of connections is well defined.

In order to state and prove our main theorem in this section, we begin with the following proposition:

Proposition 3.2. *Let X be a connected manifold, and let P be a principal $SU(r)$-bundle that is reducible to an $SO(r)$-subbundle Q. Then there is a natural embedding from $\mathcal{B}^*(Q)$ to $\mathcal{B}^*(P)$.*

To show Proposition 3.2, it suffices to prove the following lemma:

Lemma 3.3. *Let X, P, and Q be the same as in* Proposition 3.2. *Let $A = \varphi^* B$ for some $\varphi \in \mathcal{G}(P)$, where A and B are irreducible connections on P such that they define connections on Q. Then there exists a gauge transformation $\psi \in \mathcal{G}(Q)$ such that $A = \psi^* B$ when A and B are considered as connections on Q.*

Proof. Set $G = SU(r)$, $H = SO(r)$, for our convenience, and recall that $\sigma \in \mathrm{Aut}(G, H)$ is given by $g \mapsto \overline{g}$. Observe first that

$$A = A^\sigma = (\varphi^\sigma)^* B^\sigma = (\varphi^\sigma)^* B = (\varphi^\sigma)^* (\varphi^{-1})^* A = (\varphi^{-1} \circ \varphi^\sigma)^* A.$$

Thus $\varphi^{-1} \circ \varphi^\sigma \in \mathcal{G}_A = C_G(H_{u_0}(A)) = C(G)$, where $C_G(H_{u_0}(A))$ is the centralizer of the holonomy group $H_{u_0}(A)$ of A in G. Thus, we have

$$\varphi^\sigma(u) = \varphi(u)a \tag{3.3}$$

for some constant $a \in C(G)$, because $C(G)$ is discrete and actually in this case is isomorphic to $\mathbb{Z}_r$.

Next we are going to construct a gauge transformation $\psi \in \mathcal{G}(Q)$ such that $A = \psi^* B$. Let $\{U_i\}$ be an open covering of the base manifold X, and take the local sections $s_i \colon U_i \to Q$. Since the given gauge transformation φ in P defines a family of functions $\{\varphi_i\}$ which satisfies $a_{ij}(x)^{-1} \varphi_i(x) a_{ij}(x) = \varphi_j(x)$, where $a_{ij}(x)$ is the transition function, it follows from (3.3) that we have

$$s_i \varphi_i g a = \varphi(s_i) g a = \varphi^\sigma(s_i) g = s_i \sigma(\varphi_i) g,$$

for $g \in G$. Thus, we have

$$\sigma(\varphi_i(x)) = \varphi_i(x) a. \tag{3.4}$$

On the other hand, since $C(G) \cong \mathbb{Z}_r$, we can set $a = \eta \mathbb{I}$, where η is a complex number with $|\eta| = 1$ and $\eta^r = 1$, and $\mathbb{I}$ is the identity element in G.

Hence, from (3.4), we have $\sigma(\varphi_i(x)) = \varphi_i(x)\eta = \eta\varphi_i(x)$. This implies that

$$\eta^{-1/2} \sigma(\varphi_i(x)) = \eta^{1/2} \varphi_i(x).$$

Since $\sigma(g) = \overline{g}$, we have $\eta^{-1/2}\sigma(\varphi_i(x)) = \sigma(\eta^{1/2}\varphi_i(x))$. This implies that $\sigma(\eta^{1/2}\varphi_i(x)) = \eta^{1/2}\varphi_i(x)$. Thus we have $\eta^{1/2}\varphi_i(x) \in H = SO(r)$. Here we used that H is exactly the set of elements of G which are fixed by $\mathrm{Aut}(G, H)$.

Finally we define $\psi_i(x) = \eta^{1/2}\varphi_i(x) \in SO(r) = H$ on U_i. This family $\{\psi_i(x)\}$ satisfies the condition $a_{ij}^{-1}(x)\psi_i(x)a_{ij}(x) = \psi_j(x)$, because the family $\{\varphi_i(x)\}$ satisfies this condition and η is just a complex number. Hence, this family defines a gauge transformation ψ on Q. Moreover, the connections A and B in Q are gauge equivalent under ψ. In fact, we have

$$\begin{aligned}\psi_i^{-1}B_i\psi_i + \psi_i^{-1}d\psi_i &= \eta^{-1/2}\varphi_i^{-1}B_i\eta^{1/2}\varphi_i + \eta^{-1/2}\varphi_i^{-1}d(\eta^{1/2}\varphi_i)\\ &= \varphi_i^{-1}B_i\varphi_i + \varphi_i^{-1}d\varphi_i = A_i,\end{aligned}$$

which completes the proof. □

Lemma 3.4. *Let $P \longrightarrow X$ be a principal G-bundle that is reducible to two principal H-subbundles P_1, P_2. Assume that P_1 and P_2 are bundle isomorphic. Then, every connection A which defines one in P_1 is gauge equivalent to a connection B which defines one in P_2.*

Proof. Let φ be a bundle isomorphism between P_1 and P_2. Note that φ extends to the gauge transformation of P, denoted also φ. Since A is a connection which defines one in P_1, as a connection in P_1 we have $A = \varphi^*B$ for some connection B in P_2. Then, it is easy to see that by extending B as a connection in P, A is gauge equivalent to B under φ, completing the proof. □

Let $\sigma \in \mathrm{Aut}(G, H)$ and let G' be the subgroup of G consisting of elements that are fixed by σ and $\mathfrak{g}'$ be its Lie algebra. Let P' be the G'-bundle extending Q. Then,

Lemma 3.5. *A connection A is invariant by $\sigma \in \mathrm{Aut}(G, H)$, i.e., $A^\sigma = A$ if and only if it defines a connection in P'.*

Proof. Since $H \subset G' \subset G$, we have $Q \subset P' \subset P$. If a connection A is already a connection in P', then A_i is $\mathfrak{g}'$-valued and thus $A^\sigma = A$. Conversely, if A does not define a connection in P', then A_i takes its values outside $\mathfrak{g}'$, and thus $\sigma(A_i) \neq A_i$. Hence, the connection A is not invariant by σ. □

Now we are ready to state our main theorem in this section which is an immediate consequence of Proposition 2.4, Theorem 3.1, Proposition 3.2, Lemma 3.4, and Lemma 3.5.

Theorem 3.6. *Let X be a simply connected oriented manifold, and P be a principal $SU(3)$-bundle that is reducible to an $SO(3)$-subbundle Q. Let $\mathcal{B}^*(P)_\sigma$ denote the fixed point set of the action of $E(SU(3), SO(3)) \cong \langle\sigma\rangle \cong \mathbb{Z}_2$ on $\mathcal{B}^*(P)$. Then* *we have*

$$\mathcal{B}^*(P)_\sigma = \coprod_{Q' \in \Gamma} \mathcal{B}^*(Q'),$$

where Γ *denotes the set of* $SO(3)$*-bundle isomorphism classes in* P.

Proof. It suffices to show that if A is a connection in P which defines one in an $SO(3)$-subbundle Q' then it is invariant under the action of the σ on the quotient space of irreducible connections in P. In fact, since A defines a connection in Q' $A^\sigma_{Q'} = A$ by Lemma 3.5, and $A^\sigma_Q = \varphi^* A^\sigma_{Q'}$ for a gauge transformation $\varphi \in \mathcal{G}(P)$ by Proposition 2.4. Thus, $A^\sigma_Q = \varphi^* A$, which completes the proof. □

Assume that G is compact and semisimple. Then the inner product

$$\langle Z, W \rangle = -\operatorname{tr}((\operatorname{ad} Z)(\operatorname{ad} W)), \quad Z,\ W \in \mathfrak{g}$$

is invariant not only by $\operatorname{Ad} G$ but also by all automorphisms of G. Fix also a Riemannian metric on X. Then, $\operatorname{Aut}(G, H)$ acts on the moduli space of Yang-Mills connections [8]. It is also easy to see that $\operatorname{Aut}(G, H)$ acts on the moduli space of anti-self-dual (ASD) connections, when $\dim X = 4$. Using Theorem 3.6 and the above remark, it is easy to prove Theorem 1.1.

4. *SU*(3)-Instanton Invariants for certain 4-Manifolds

The purpose of this section is to give well-defined $SU(3)$-instanton invariants for certain smooth 4-manifolds, and to prove Theorem 1.2.

Let X be a closed, oriented, smooth 4-manifold with a Riemannian metric g. Let $P \longrightarrow X$ be a principal G-bundle, and let $\mathcal{M}_P(g)$ be the moduli space of ASD connections on P modulo the group of gauge transformations. The Atiyah-Singer index theorem gives the formula for the virtual dimension $s(P)$ of $\mathcal{M}_P(g)$

$$s(P) = -2p_1(\mathfrak{g}_P) - \dim G(1 - b_1 + b_2^+),$$

where b_1 is the first Betti number, b_2^+ is the dimension of maximal positive part of $H^2(X, \mathbb{R})$, and $\mathfrak{g}_P$ is the adjoint bundle of P. In particular, for $SU(r)$-bundles P and the associated vector bundle E of P via the standard representation, we have $2p_1(\mathfrak{g}_P) = -4rc_2(E)$. Thus for $r = 3$ we have $s(P) = 12c_2(E) - 8(1 - b_1 + b_2^+)$. On the orther hand, for $SO(3)$-bundles Q and the associated vector bundle F of Q via the standard representation of $SO(3)$, we have $s(Q) = -2p_1(F) - 3(1 - b_1 + b_2^+)$. Note also that for a generic metric g on X, the moduli space of ASD $SO(3)$-connections is smooth and has the virtual dimension, if it is non-empty ([3], [2]).

For the rest of this section, we fix a principal $SU(3)$-bundle P with positive $c_2(E)$ over a simply connected Riemannian 4-manifold X.

4.1. Reducible connections on *P*. When we work with $SU(2)$ or $SO(3)$ connections over simply connected 4-manifolds, all possible reductions are very simple. For example, for the $SU(2)$ case the only reductions are to a copy of $S^1 \subset SU(2)$ or to the trivial subgroup corresponding to the product connection. In case of an $SU(2)$-bundle E the reductions correspond to splittings $E \cong L \oplus L^{-1}$, and such an isomorphism exists if and only if $c_2(E) = -c_1(L)^2$. In the $SO(3)$ case, we have $p_1(\underline{\mathbb{R}} \oplus L) = -c_2(L_{\mathbb{R}} \otimes \mathbb{C}) = -c_2(L \oplus \bar{L}) = c_1(L)^2$. For the $SU(2)$ or $SO(3)$ case, a line bundle L over the Riemannian 4-manifold X admits an ASD connection if and only if $c_1(L)$ is represented by an anti-self-dual 2-form. Furthermore, when X is simply connected, this connection is unique up to gauge equivalence. Thus we have the following Proposition 4.2.15 in [2] we do not use in this paper.

Proposition 4.1. *Let X be a simply connected oriented Riemannian 4-manifold and let E be an $SU(2)$ or $SO(3)$ bundle over X. Then the gauge equivalence classes of reducible ASD connections on E whose holonomy group is S^1 are in one-to-one correspondence with the pairs $\pm c$, where c is a non-zero anti-self-dual class in $H^2(X, \mathbb{Z})$ with $c^2 = -c_2(E)$ or $c^2 = p_1(E)$, respectively.*

On the other hand, in the $SU(3)$ case the possible reductions are much more complicated, compared to $SU(2)$ or $SO(3)$ case. However, it turns out that it suffices to consider only the largest possible reductions. In order to state and prove an analogous statement of Proposition 4.1 for the largest possible reductions in the $SU(3)$ case, we need to set up some notations.

Let $\mathcal{M}(P)_{\text{red}}$ denote the set of the gauge equivalence classes of reducible ASD connections on P whose isotropy group is S^1. Let $\tilde{\mathcal{R}}$ denote the set of pairs (c_1, c_2) in $H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z})$, where c_1 is an anti-self-dual class satisfying $c_2(E) + (1/2)c_1^2 \geq 0$ and $c_2 = c_2(E) + c_1^2$.

Let A be a representative for $[A] \in \mathcal{M}(P)_{\text{red}}$. Since A has an isotropy group S^1, the complex rank 3 vector bundle E associated to P has a reduction to an $S(U(2) \times U(1))$-bundle S formed by a $U(2)$-bundle E' and a line bundle L. Thus, we have a decomposition of A into a diagonal form (A_1, A_2), where A_1 (resp. A_2) denotes an ASD connection on E' (resp. L).

Since $\operatorname{tr}(F_{A_1}) + \operatorname{tr}(F_{A_2}) = 0$, it is easy to see that $c_1(L) = -c_1(E')$. Furthermore, we have

$$\begin{aligned} c_2(E) &= \left[\operatorname{tr}\left(\frac{1}{8\pi^2} F_{A_1}^2\right)\right] + \left[\operatorname{tr}\left(\frac{1}{8\pi^2} F_{A_2}^2\right)\right] \\ &= c_2(E') - \frac{1}{2}c_1(E')^2 - \frac{1}{2}c_1(L)^2 \\ &= c_2(E') - c_1(L)^2. \end{aligned}$$

Note also from Proposition 2.1.42 in [2] that since E' admits an ASD connection we have

$$0 \leq \kappa(E') := c_2(E') - \frac{1}{2}c_1(E')^2 = c_2(E) + \frac{1}{2}c_1(L)^2.$$

If we choose a different representative B for $[A] \in \mathcal{M}(P)_{\rm red}$, we have a different line bundle L' for B. But, since A is gauge equivalent to B, L must be isomorphic to L'. This implies that we have a well-defined map

$$\phi : \mathcal{M}(P)_{\rm red} \to \tilde{\mathcal{R}}, \quad [A] \mapsto (c_1(L), c_2(E')).$$

Let $\mathcal{R}$ denote the image of ϕ. Note that $(0, c_2(E))$ is contained in the set $\mathcal{R}$.

Lemma 4.2. *The preimage of $(c_1, c_2) \in \mathcal{R}$ under ϕ is the set of reducible connections whose isotropy group is S^1, corresponding to each $(c_1, c_2) \in \mathcal{R}$, and is exactly same as the set $\{[A] = [A_1, A_2] \in \mathcal{M}(P)_{\rm red} | A_1 \in \mathcal{M}(E')\}$ for a fixed ASD connection A_2 on the line bundle L satisfying $c_1(L) = c_1$*

Proof. Let L be the line bundle whose first Chern class is c_1. Since X is simply connected, there is a unique gauge equivalence class $[A_2]$ of ASD connections on L. Thus every element $[B_1, B_2] \in \phi^{-1}(c_1, c_2)$ is gauge equivalent to $[B_1, A_2]$. This completes the proof. □

4.2. Transversality. In this subsection, we explain in more detail why the Freed and Uhlenbeck's transversality result for the ASD equations in an principal $SU(3)$-bundle does not hold. This suggests a way to overcome such difficulties in our case.

With a given metric, an irreducible ASD connection A is called *regular* if $H^2_A = \operatorname{coker} d^+_A = 0$ and we call a moduli space *regular* if all its irreducible points in the moduli space are regular points. A regular moduli space of irreducible connections is a smooth manifold of dimension given by the Atiyah-Singer index theorem. In particular, if the moduli space $\mathcal{M}^*_P(g)$ contains only generic connections on P, then the zeros of F^+_A in $\mathcal{B}^*(P)$ are transverse, forming a smooth manifold of dimension $12c_2(P) - 8(1 + b^+_2(X))$.

After we identify $H^2(X, \mathbb{R})$ with the space of harmonic 2-forms, we have a decomposition $H^2(X, \mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^+$ and $\mathcal{H}^-$ are the spaces of harmonic self-dual and anti-self-dual 2-forms, respectively. If a connection A in an $SU(3)$-bundle E is reduced to one on an $S(U(2) \times U(1))$-bundle formed by a $U(2)$-bundle E' and a $U(1)$-bundle L, then for this connection A the line bundle L admits an ASD connection. Thus $c_1(L)$ can be represented by an anti-self-dual harmonic form and lies in $\mathcal{H}^-$. If b^+_2 is non-zero, the space $\mathcal{H}^-$ is a *proper* subspace of $H^2(X, \mathbb{R})$. Thus we can see that generically $\mathcal{H}^-$ meets the integer lattice $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$ only at zero. Hence the reducible connections in an $SU(3)$-bundle are not just the product connection. In fact, it is clear from Lemma 4.2 that for any metric on X the moduli space of ASD $SU(3)$-connections could contain the moduli space of ASD connections in an $SU(2)$-bundle E' with $c_2(E') = c_2(E)$, unless we impose additional hypotheses on X or P.

Now we are in a position to state and prove the statement which is analogous to Corollary 4.3.15 in [2]. We first need some notations. Let $\mathcal{C}$ denote the space of conformal structures of C^r metrics on X for some fixed suitable non-negative integer r. The space of $\mathcal{C}$ is just the quotient of C^r metrics by the C^r conformal changes.

Lemma 4.3. *If $b_2^+(X) > 0$ then for any $l > 0$, there is an open dense subset $\mathcal{C}(l) \subset \mathcal{C}$ such that for $[g] \in \mathcal{C}(l)$ the only possible reducible g-ASD connections on an $SU(3)$-bundle E over X with $c_2(E) \leq l$ are either the connections consisting of an irreducible g-ASD connection on the $SU(2)$-bundle E' satisfying $c_2(E') = c_2(E)$ and the trivial product connection on the trivial $U(1)$-bundle $\underline{\mathbb{C}}$ or the trivial product connection on E.*

Moreover, if $d < b_2^+(X)$ and $h\colon R \to \mathcal{C}$ is a smooth family of metrics parametrized by a d-dimensional manifold R, then there is an arbitrarily small perturbation of h whose image lies in $\mathcal{C}(l)$.

Proof. We prove the first assertion. The proof of the second assertion is similar to Corollary 4.3.14 in [2].

The proof of the first assertion is essentially same as that of Corollary 4.3.15 in [2]. From the above discussion, we see that generically the space $\mathcal{H}^-$ meets the integer lattice $H^2(X, \mathbb{Z})$ only at zero. Thus generically we have $c_1 = 0$. Since $(0, c_2(E))$ is contained in the set $\mathcal{R}$, by Lemma 4.2 and above argument, there is an open dense subset $\mathcal{C}_1(l) \subset \mathcal{C}$ such that, for $[g] \in \mathcal{C}_1(l)$, the reducible g-ASD connections on E are connections consisting of a g-ASD connection on E' satisfying $c_2(E) = c_2(E')$ and the trivial product connection on $\underline{\mathbb{C}}$.

Applying once again, if necessary, the same argument as in the proof of Corollary 4.3.15 in [2] to the g-ASD connections on E', we get an open dense subset $\mathcal{C}_2(l) \subset \mathcal{C}$ such that, for $[g] \in \mathcal{C}_2(l)$, the only reducible g-ASD connection on E' is the trivial product connection on E'. Thus we have an open dense subset $\mathcal{C}(l) = \mathcal{C}_1(l) \cap \mathcal{C}_2(l) \subset \mathcal{C}$ satisfying the statements of the theorem. □

For the orientation of the moduli space consisting of zeros F_A^+ in $\mathcal{B}^*(P)$, it can be shown as in Proposition 5.4.1 and Section 5.4.3 of [2] that a choice of orientations of $H^0(X; \mathbb{R})$ and $H_+^2(X; \mathbb{R})$ defines a natural orientation of the moduli space.

4.3. Proof of Theorem 1.2. Since we want to get differential-topological invariants of X from $\mathcal{M}_P(g)$, we have to study its properties under the change of metric. As we can see in Lemma 4.3, an one-parameter family of metrics could contain non-generic metrics for which the largest possible reducible connections occur. This implies that a singular moduli space and the homology class of $\mathcal{M}_P(g)$ changes inside the quotient space of connections on P. However, by making a good choice of an $SO(3)$-subbundle Q of P, we can obtain the homology class independent of the metric.

We first claim that the moduli space $\mathcal{M}_P(g)$ contains only irreducible connections under the hypotheses **(H1)** and **(H2)**. To do this, we show that there are no reductions of E, and it suffices to consider only $\underline{\mathbb{C}} \oplus E'$ reductions for a $SU(2)$ vector bundle E' in the moduli space of ASD $SU(3)$-connections for a generic metric, since they are the largest possible reductions by Lemma 4.3.

Suppose that the complex rank 3 vector bundle E has a reduction of $\underline{\mathbb{C}} \oplus E'$. Then clearly the real rank 3 subbundle F has also a reduction of $\underline{\mathbb{R}} \oplus L$. Thus we have $p_1(F) = c_1(L)^2$. Since $c_1(L)$ is a lift of $w_2(F)$ and any lift of $w_2(TX) = w_2(F)$ to the integers is characteristic, we must have modulo 8

$$-2k = -c_2(E) = -c_2(F \otimes \mathbb{C}) = p_1(F) = c_1(L)^2 = \sigma(X),$$

where we used in the last equality the standard fact (e.g., see Subsection 1.1.3 in [2] or Lemma 1.2.20 in [4]) that for any characteristic element $c \in H^2(X, \mathbb{Z})$ and the intersection form of X we have

$$(c \cup c)[X] = \sigma(X) \mod 8.$$

Hence the signature $\sigma(X)$ must be $-2k$ mod 8, which is a contradiction.

Note also that the dimension of the moduli space of ASD $SO(3)$-connections on any $SO(3)$-subbundle Q' of P is $s(Q') = -2p_1(Q') - 3(1 + b_2^+(X)) = 4k - 9k = -5k < 0$. Thus these moduli spaces are generically empty and so the moduli space of ASD $SU(3)$-connections on P, in fact, consists of only generic connections. Thus the moduli space $\mathcal{M}_P(g)$ is a submanifold cut out transversely in $\mathcal{B}^*(P)$. Moreover, since $c_2(E) = 2k$ and $b_2^+(X) = 3k - 1$, we have $s(P) = 12c_2(P) - 8(1 + b_2^+(X)) = 0$. On the other hand, for any of lower $SU(3)$-bundles $E^{(r)}$ with $c_2(E^{(r)}) = c_2(E) - r$ $(r > 0)$ the virtual dimension of the moduli space $\mathcal{M}_{E^{(r)}}(g)$ predicted by the Atiyah-Singer index formula is negative, and so the generic parts of these moduli spaces $\mathcal{M}_{E^{(r)}}(g)$ are generically empty. Note also that an argument as above shows that for a generic metric g the irreducible parts of these moduli spaces $\mathcal{M}_{E^{(r)}}(g)$ consist of only generic ASD connections.

As remarked at the beginning of the Subsection 4.4.1 in [2], a straightforward generalization of the compactness theorem 4.4.4 in [2] to more general gauge groups says that any infinite sequence $\{A_\alpha\}$ in $\mathcal{M}_P(g)$ whose terms are in our case generic ASD connections has a weekly convergent subsequence whose limit $(A, \{x_1, x_2, \ldots, x_r\})$ lies in $\mathcal{M}_{E^{(r)}}(g) \times \mathrm{Sym}^r(X)$ for some non-negative integer r, where $\mathrm{Sym}^r(X)$ is the r-th symmetric product. (See Subsection 4.4.1 in [2] for the definition of a weekly convergent sequence.) So, let $\rho_\alpha \colon E^{(r)}|_{X \setminus \{x_1, x_2, \ldots, x_r\}} \to$

$E|_{X\setminus\{x_1,x_2,\ldots,x_r\}}$ be bundle isomorphisms such that $\rho_\alpha^*(A_\alpha)$ converges in C^∞ on any compact subset of the punctured manifold $X\setminus\{x_1, x_2, \ldots, x_r\}$ to A in $E^{(r)}$, as in Subsection 4.4.1 in [2]. For the sake of simplicity, let A'_α to be $A_\alpha|_{X\setminus\{x_1,x_2,\ldots,x_r\}}$. Then an argument completely similar to Lemma 4.3.21 in [2] asserts that A'_α is irreducible on $E|_{X\setminus\{x_1,x_2,\ldots,x_r\}}$. Thus the restricted holonomy group of A'_α is isomorphic to $SO(3)$ or $SU(3)$. If the restricted holonomy group of A'_α is isomorphic to $SO(3)$ then the restricted holonomy group of $\rho_\alpha^*(A'_\alpha)$ is also isomorphic to $SO(3)$. Now extend $\rho_\alpha^*(A'_\alpha)$ to an ASD connection B_α on the $SU(3)$-bundle $E^{(s)}$ with $0 \le c_2(E^{(s)}) \le c_2(E)$ by the Removable Singularities Theorem of Uhlenbeck in [11]. Then clearly the holonomy group of B_α is isomorphic to $SO(3)$ or $SU(3)$. But the irreducible parts of those moduli spaces $\mathcal{M}_{E^{(s)}}(g)$ for $s > 0$ are generically empty, so the restricted holonomy group of A'_α should be isomorphic to $SU(3)$. It is also easy to see that by the same argument as above this case cannot occur, either. Thus such an infinite sequence does not exist for $r > 0$. It follows that $\mathcal{M}_P(g)$ is itself compact. Hence the moduli space $\mathcal{M}_P(g)$ is a finite set of points, each a transverse zero of F^+.

Now fix an orientation as above to give a sign to each point in $\mathcal{M}_P(g)$, and then define $q_k(X)$ to be the number of points in the moduli space, counted with signs. This will be independent of the metric by the same argument as above.

To finish the proof, we need to prove the mod 2 vanishing result of the theorem. Using the conjugate connections on P and the outer automorphism group $E(SU(3), SO(3))$, we have a compatible $\mathbb{Z}_2$ action on the moduli space of ASD $SU(3)$ connections on P. Since the fixed point set of the action of $E(SU(3), SO(3))$ on the moduli space of ASD connections on P consists of the homeomorphic images of the moduli space of irreducible ASD connections on $Q' \in \Gamma$ (Theorem 1.1), we have already shown that in our case the fixed point set is empty. Thus the invariants $q_k(X)$ must be zero modulo 2. This completes the proof.

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Department of Mathematics
Oklahoma State University
Stillwater, OK 74078-1058
USA

Current address:
Department of Mathematics
KAIST
Kusong-dong, Yusong-gu
Daejon 305-701, South Korea
e-mail: jinkim@math.kaist.ac.kr