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ON MULTIPLY TRANSITIVE GROUPS VII

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1. Introduction

Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$, and let P be a Sylow 2-subgroup of a stabilizer of four points in G . If $P=1$, then by a theorem of M. Hall [1. Theorem 5.8.1] G must be one of the following groups: S_4 , S_5 , A_6 , A_7 or M_{11} . From a recent result of H. Nagao [7] it follows that, if $P \neq 1$ is semi-regular and leaves exactly four or five points fixed, then G must be one of the following groups: S_6 , S_7 , A_8 , A_9 or M_{12} .

The purpose of this paper is to extend the result of H. Nagao. Namely we shall prove the following

Theorem. *Let G be a 4-fold transitive group. If a Sylow 2-subgroup of a stabilizer of four points in G is semi-regular and not identity, then G must be S_6 , S_7 , A_8 , A_9 , M_{12} or M_{23} .*

DEFINITION AND NOTATION. A permutation x is called semi-regular if there is no point fixed by x . A permutation group H is called semi-regular if every nonidentity element of H is semi-regular on the points actually moved by H . For a permutation group G on Ω , let $G_{i,j,\dots,r}$ denote the stabilizer of the points i, j, \dots, r in G . For a subset S of G we denote the normalizer (or centralizer) of S in G by $N_G(S)$ (or $C_G(S)$). Let $\alpha_i(x)$ denote the number of i -cycles of a permutation x . The totality of points left fixed by a set X of permutations is denoted by $I(X)$, and if a subset Δ of Ω is a fixed block of X , then the restriction of X on Δ is denoted by X^Δ .

2. Proof of the theorem

To prove the theorem we may assume that a stabilizer of four points in G fixes exactly four points (See [6]). In the proof of the theorem, we shall also make use of the fact [1. p. 80] that a 4-fold transitive group of degree less than 35 is one of the known groups.

Let P be a Sylow 2-subgroup of G_{1234} . Then $|I(P)|$ is four, five, six, seven or eleven, and $N_G(P)^{I(P)}$ is S_4 , S_5 , A_6 , A_7 or M_{11} (cf. [5]. Lemma 1). By the theorem of H. Nagao, we may treat only the last three cases.

Case I. $|I(P)| = 11$, $N_G(P)^{I(P)} = M_{11}$.

Let a be a central involution of P , and suppose that P has an involution b different from a . Then a and b fix the same eleven points and generate a four group. Therefore we may assume that

$$\begin{aligned} a &= (1)(2)\cdots(11)(ij)(kl)\cdots, \\ b &= (1)(2)\cdots(11)(ik)(jl)\cdots. \end{aligned}$$

Then $\langle a, b \rangle$ normalizes G_{ijkl} and hence it normalizes some Sylow 2-subgroup P' of G_{ijkl} . Since $a^{I(P')}$ is an involution of M_{11} , it fixes three points. Now $I(a) = I(b) = \{1, 2, \dots, 11\}$. Hence $b^{I(P')}$ fixes these three points and the four group $\langle a, b \rangle^{I(P')}$ is contained in a stabilizer of three points of M_{11} . But this is impossible, because a stabilizer of three points of M_{11} is a quaternion group.

Thus P has only one involution, and hence P must be a cyclic group or a generalized quaternion group [1. Theorem 12.5.2.]. By Theorem 1 in [8] P is not cyclic, and by the following unpublished result of H. Nagao we have a contradiction.

Lemma 1. (H. Nagao) *Let G be a 4-fold transitive group, and $P \neq 1$ a Sylow 2-subgroup of G_{1234} . If P fixes eleven points, then P is not a generalized quaternion group.*

Thus we have no group in this case.

Case II. $|I(P)| = 6$ or 7 , $N_G(P)^{I(P)} = A_6$ or A_7 .

In the proofs of the following series from i) to v) we assume that $N_G(P)^{I(P)} = A_6$, and we need the following result [8. Theorem 2] that any involution of G fixes exactly six points. The proofs in the case $N_G(P)^{I(P)} = A_7$ are similar.

- i) *If an element a of G has a 4-cycle, then its order is an odd multiply of 4 or 8.*
- 1) *If a is of order 4, then $\alpha_2(a)=2$ and $\alpha_1(a)=2$. (When $N_G(P)^{I(P)} = A_7$, $\alpha_2(a)=2$ and $\alpha_1(a)=3$).*
- 2) *If a is of order $4t$ with t odd, then $\alpha_2(a)=2$ and $\alpha_1(a)=2$. (When $N_G(P)^{I(P)} = A_7$, $\alpha_2(a)=2$ and $\alpha_1(a)=0$ or 3).*
- 3) *If a is of order $8t$ with t odd, then $\alpha_4(a)=1$ and $\alpha_2(a)=1$.*

Proof. 1) Let a be an element of order 4. Then we may assume that

$$a = (1\ 2\ 3\ 4)\cdots.$$

Since a normalizes G_{1234} , a normalizes some Sylow 2-subgroup of G_{1234} . We may assume that a normalizes P . Since $N_G(P)^{I(P)} = A_6$, a must be of the following form

$$a = (1\ 2\ 3\ 4)(5\ 6)\cdots,$$

where $I(P) = \{1, 2, 3, 4, 5, 6\}$. Since a^2 is an involution and $\alpha_1(a^2) = 6, \alpha_1(a) = 0, 2$ or 4 . If $\alpha_1(a) = 0$, then $\alpha_2(a) = 3$ and a is of the form

$$a = (1\ 2\ 3\ 4)(5\ 6)(i_1 j_1)(i_2 j_2)\cdots.$$

From this form a normalizes some Sylow 2-subgroup P' of $G_{56i_1 j_1}$ and $a^{I(P')} = (5\ 6)(i_1 j_1)(i_2 j_2)$, which is contrary to $N_G(P')^{I(P')} = A_6$. Therefore $\alpha_1(a) \neq 0$. Since P is semi-regular, P is elementary abelian by Lemma 2 in [8]. Therefore $\alpha_1(a) \neq 4$. Hence we have that $\alpha_1(a) = 2$ and consequently $\alpha_2(a) = 2$.

2) Let a be of order $4t$ with t odd, then $\alpha_2(a^t) = 2$ and $\alpha_1(a^t) = 2$. Therefore $\alpha_2(a) = 2$ and $\alpha_1(a) = 2$.

3) Let a be of order 8. Then from 1) $\alpha_2(a^2) = 2$. Hence $\alpha_4(a) = 1$. Thus we may assume that a is of the following form

$$a = (1\ 2\ 3\ 4)\cdots.$$

Then a normalizes some Sylow 2-subgroup of G_{1234} . We may assume that a normalizes P . Since $N_G(P)^{I(P)} = A_6$, a must be of the following form

$$a = (1\ 2\ 3\ 4)(5\ 6)\cdots,$$

where $I(P) = \{1, 2, 3, 4, 5, 6\}$. Since $\alpha_1(a^2) = 2, \alpha_2(a) = 1$. This is also true for an element of order $8t$ with t odd.

Since an element of order 8 has only one 4-cycle, G has no element of order 16.

ii) P is an elementary abelian group of order 16.

Proof. By Lemma 2 in [8] P is elementary abelian. Therefore it suffices to prove that the order of P is 16. Let $a = (1\ 2)(3\ 4)\cdots$ be an involution of G . Then a normalizes a Sylow 2-subgroup of G_{1234} . We may assume that a normalizes P . Let $I(P) = \{1, 2, 3, 4, 5, 6\}$. Since $a^{I(P)}$ must be an even permutation, a is of the following form

$$a = (1\ 2)(3\ 4)(5\ 6)\cdots.$$

Let a fixes the point 7 and let Δ be the P -orbit containing 7. Then a fixes at most four points of Δ and P is regular on Δ . Therefore by Lemma in [4] we have that the order of P is at most 16.

Now let K be the kernel of the natural homomorphism $N_G(P) \rightarrow N_G(P)^{I(P)}$. Then $K C_G(P) / C_G(P) \cong K / K \cap C_G(P)$, and $N_G(P) \supseteq K \cdot C_G(P) \supseteq K$. Since $N_G(P) / K = N_G(P)^{I(P)}$, $N_G(P) / K$ is a simple group. Therefore $N_G(P) = K \cdot C_G(P)$ or $K \cdot C_G(P) = K$. Since $G_{1234} \geq K \geq K \cap C_G(P) \geq P$ and P is a Sylow 2-subgroup of G_{1234} , $K / K \cap C_G(P)$ is of odd order. If $N_G(P) = K \cdot C_G(P)$, then

from $K C_G(P)/C_G(P) \cong K/K \cap C_G(P)$, $N_G(P)/C_G(P)$ is of odd order. Hence any 2-element of $N_G(P)$ belongs to $C_G(P)$. On the other hand there is an element x of order four, which is of the following form

$$x = (1\ 2\ 3\ 4) \cdots.$$

Then x normalizes G_{1234} , and hence we may assume that x normalizes P . From $N_G(P)^{I(P)} = A_6$, x must be of the form

$$x = (1\ 2\ 3\ 4)(5\ 6) \cdots,$$

where $I(P) = \{1, 2, 3, 4, 5, 6\}$. By 1) of i) x fixes two points of $\Omega - I(P)$. Since P is semi-regular on $\Omega - I(P)$, x commutes with exactly two elements of P . But by Theorem 1 in [8] $|P| > 2$. Therefore $x \notin C_G(P)$, which is a contradiction. Thus $K \cdot C_G(P) = K$.

Now $N_G(P)/C_G(P)$ is a subgroup of the automorphism group of P . From $N_G(P)/K \cong A_6$ and $K \geq C_G(P)$, the order of the automorphism group of P is not smaller than the order of A_6 . Since P is elementary abelian and its order is at most 16, the order of P must be 16.

Next we also need a theorem of G. Frobenius (See [3]. Proposition 14.5), which will be stated here as Lemma 2.

Lemma 2. (G. Frobenius) *Let $G \leq S_n$, then*

$$\sum_{x \in G} \binom{\alpha_1(x)}{\kappa} \binom{\alpha_2(x)}{\lambda} \cdots = \frac{m \cdot |G|}{1^{\kappa} \cdot \kappa! \cdot 2^{\lambda} \cdot \lambda! \cdots}.$$

Here m is an integer obtained in the following way. Let $\Omega^{(t)} = \{(i_1, \dots, i_{\kappa}, j_1, j_1', \dots, j_{\lambda}, j_{\lambda}', \dots)\}$ be a family of ordered sets consisting of $t (= \kappa + 2\lambda + \dots)$ points of Ω such that there is at least one element x of G of the form

$$x = (i_1) \cdots (i_{\kappa})(j_1 j_1') \cdots (j_{\lambda} j_{\lambda}') \cdots.$$

When G is regarded as a permutation group on $\Omega^{(t)}$ by setting

$$(a_1, \dots, a_t)^x = (a_1^x, \dots, a_t^x)$$

for $x \in G$ and $(a_1, \dots, a_t) \in \Omega^{(t)}$, m is the number of G -orbits in $\Omega^{(t)}$.

iii) *Let x be an involution of $N_G(P) - P$. Then any fixed point of an element $(\neq 1)$ of $\langle x, P \rangle^{\Omega - I(P)}$ is contained in exactly one orbit of P . The number of P -orbits in $\Omega - I(P)$ is odd.*

Proof. Since the order of P is 16, by Lemma in [4] x commutes with four elements of P . Since P is semi-regular and x fixes four points of $\Omega - I(P)$, these points must be contained in the same P -orbit, say Δ . Put $Q = \langle x, P \rangle$. The

order of Q is 32 and Q fixes Δ . For an element a of P if x commutes with a , then xa is of order 2, and if not, then xa is of order 4, since $(xax)a$ belongs to P and it is not the identity. Let xa be of order 4, then by 1) of i) $\alpha_1(xa)=2$ and xa has no fixed point on $\Omega-I(P)$. Therefore

$$\begin{aligned}\sum_{y \in Q} \alpha_1(y^\Delta) &= \alpha_1(1^\Delta) + \sum_{y'}' \alpha_1(y'^\Delta) \\ &= 16 + \sum_{y'}' \alpha_1(y'^\Delta),\end{aligned}$$

where y' ranges over all involutions of $Q-P$. On the other hand from Lemma 2

$$\sum_{y \in Q} \alpha_1(y^\Delta) = |Q^\Delta| = 32.$$

Hence $\sum_{y'}' \alpha_1(y'^\Delta) = 16$. Since $Q-P$ has four involutions and these involutions have four fixed points in $\Omega-I(P)$ respectively, these 16 points are all contained in Δ . Hence Q is semi-regular on $\Omega-\{I(P) \cup \Delta\}$, in which any Q -orbit contains exactly two P -orbits. Thus the number of P -orbits in $\Omega-I(P)$ is odd.

iv) G has an element of order 8.

Proof. Let a be an element of order four and of the following form

$$a = (1\ 2\ 3\ 4) \cdots.$$

Then a normalizes a Sylow 2-subgroup of G_{1234} . We may assume that a normalizes P . From $N_G(P)^{I(P)} = A_6$, a must be of the following form

$$a = (1\ 2\ 3\ 4)(5\ 6) \cdots,$$

where $I(P) = \{1, 2, 3, 4, 5, 6\}$. By 1) of i) a fixes two points of $\Omega-I(P)$, and these points are contained in a P -orbit, say Δ . Put $Q = \langle P, a \rangle$. Then the order of Q is $4 \cdot 16$ and Δ is a Q -orbit. Suppose that Q has no element of order 8. From iii) any fixed point of an element ($\neq 1$) of $\langle P, a^2 \rangle^{Q-I(P)}$ is contained in Δ . Let a' be any element of Pa or Pa^{-1} . Then a' is of the following form

$$(1\ 2\ 3\ 4)(5\ 6) \cdots \text{ or } (1\ 4\ 3\ 2)(5\ 6) \cdots.$$

We assumed that Q has no element of order 8. Hence a' is of order 4, and a' has exactly two fixed points. Since Q/P is a cyclic group of order 4, a'^2 belongs to $\langle P, a^2 \rangle$. Therefore a' fixes two points of Δ .

From $Q = P + Pa + Pa^2 + Pa^{-1}$

$$\begin{aligned}\sum_{x \in Q} \alpha_1(x^\Delta) &\geq \sum_{x \in P} \alpha_1(x^\Delta) + \sum_{x \in Pa} \alpha_1(a^\Delta) + \sum_{x \in Pa^{-1}} \alpha_1(x^\Delta) \\ &= 16 + 2 \cdot 16 + 2 \cdot 16 = 5 \cdot 16.\end{aligned}$$

On the other hand by Lemma 2

$$\sum_{x \in Q} \alpha_1(x^A) = |Q| = 4 \cdot 16,$$

which is a contradiction. Thus Q has an element of order 8.

Since G is 4-fold transitive, by Lemma 2

$$\sum_{x \in G} \alpha_1(x) = \frac{1}{4}g,$$

and

$$\sum_{x \in G} \alpha_2(x) \cdot \alpha_4(x) = \frac{m \cdot g}{2 \cdot 4},$$

where $g = |G|$. From i) if $\alpha_4(x) \neq 0$, then $\alpha_4(x) \cdot \alpha_2(x) = 2 \cdot \alpha_4(x)$ or $\alpha_4(x)$. Since there is an element of order 8, from i) we have

$$\sum_{x \in G} \alpha_4(x) < \sum_{x \in G} \alpha_2(x) \cdot \alpha_4(x) < 2 \cdot \sum_{x \in G} \alpha_4(x)$$

and hence $1 < \frac{m}{2} < 2$. Thus $m = 3$, and

$$\sum_{x \in G} \alpha_2(x) \cdot \alpha_4(x) = \frac{3}{8}g.$$

From two equations above, we obtain

$$\begin{aligned} \sum_y' \alpha_4(y) + \sum_{y'}' \alpha_4(y') &= \frac{1}{4}g, \\ \sum_y' 2 \cdot \alpha_4(y) + \sum_{y'}' \alpha_4(y') &= \frac{3}{8}g, \end{aligned}$$

where y and y' range over all elements of order $4t$ and $8t$ (t : odd) respectively. Hence

$$\sum_y' \alpha_4(y) = \frac{1}{8}g.$$

On the other hand

$$\sum_{x \in G} \binom{\alpha_2(x)}{2} \cdot \alpha_4(x) = \frac{m' \cdot g}{2^2 \cdot 2 \cdot 4}.$$

Since an element of order $8t$ with t odd has only one 2-cycle, and an element of order $4t$ with t odd has two 2-cycles,

$$\sum_{x \in G} \binom{\alpha_2(x)}{2} \cdot \alpha_4(x) = \sum_y' \alpha_4(y) = \frac{1}{8}g,$$

Therefore

$$\frac{m' \cdot g}{2^2 \cdot 2 \cdot 4} = \frac{1}{8} g,$$

hence

$$m' = 4.$$

From the remark of Lemma 2, G has four orbits on $\Omega^{(8)} = \{(i_1, i_2, j_1, j_2, k_1, k_2, k_3, k_4) \mid x = (i_1 i_2)(j_1 j_2)(k_1 k_2 k_3 k_4) \cdots \in G, x \text{ is of order } 4\}$. Since G is 4-fold transitive on Ω , $G_{1234} = H$ has four orbits on $\Omega^{(4)} = \{(k_1, k_2, k_3, k_4) \mid a = (1 2)(3 4)(k_1 k_2 k_3 k_4) \cdots \in G, a \text{ is of order } 4\}$.

When H is regarded as a permutation group on $\Omega^{(4)}$, we denote it by H^* .

If $(k_1, k_2, k_3, k_4) \in \Omega^{(4)}$, then there is an element a of G of the form

$$a = (1 2)(3 4)(k_1 k_2 k_3 k_4) \cdots.$$

Since $a^{-1} = (1 2)(3 4)(k_1 k_4 k_3 k_2) \cdots \in G$, we have eight points (k_1, k_2, k_3, k_4) , (k_2, k_3, k_4, k_1) , (k_3, k_4, k_1, k_2) , (k_4, k_1, k_2, k_3) , (k_1, k_4, k_3, k_2) , (k_4, k_3, k_2, k_1) , (k_3, k_2, k_1, k_4) and (k_2, k_1, k_4, k_3) of $\Omega^{(4)}$.

v) Let $(k_1, k_2, k_3, k_4) \in \Omega^{(4)}$. Then $(k_1, k_2, k_3, k_4)^{H^*}$ and $(k_2, k_3, k_4, k_1)^{H^*}$ are the different H^* -orbits.

Proof. Since any 2-element of H is of order 2, H has no element as follows:

$$\begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \cdots \\ k_2 & k_3 & k_4 & k_1 & \cdots \end{pmatrix} = (k_1 k_2 k_3 k_4) \cdots.$$

Therefore $(k_1, k_2, k_3, k_4)^{H^*} \neq (k_2, k_3, k_4, k_1)^{H^*}$.

From now on we treat two cases $N_G(P)^{I(P)} = A_6$ and A_7 separately. For the proofs in these cases the result that the number of H^* -orbits is four is important.

$$\text{A) } N_G(P)^{I(P)} = A_6.$$

Let $I(P) = \{1, 2, 3, 4, 5, 6\}$. and $G_{1234} = H$. Then the points 5 and 6 are contained in H -orbits of odd length. Put $5^H = \Delta_1$ and $6^H = \Delta_2$.

Suppose that $\Delta_1 = \Delta_2$. Since P -orbits in $\Omega - I(P)$ are of even length, the length of Δ_1 is even, which is a contradiction. Therefore $\Delta_1 \neq \Delta_2$. Furthermore the other H -orbits in $\Omega - I(H)$ are all of even lengths.

From $N_G(P)^{I(P)} = A_6$ there is an element x of the following form

$$x = (1 2)(3)(4)(5 6) \cdots.$$

Since $x \in N_G(H)$, $\Delta_1^x = \Delta_2$. Hence $|\Delta_1| = |\Delta_2|$. Suppose that H -orbits in $\Omega - I(H)$ are Δ_1 and Δ_2 . From iii) the number of P -orbits in $\Omega - I(P)$ is odd,

and all P -orbits in $\Omega - I(P)$ are of the same length. Hence $|\Delta_1| \neq |\Delta_2|$, which is a contradiction. Therefore H has at least three orbits in $\Omega - I(H)$.

If $|\Delta_1| = |\Delta_2| = 1$, then $|I(H)| = 6$, contradicting the assumption that $|I(H)| = 4$. Therefore $|\Delta_1| = |\Delta_2| > 1$.

Let (k_1, k_2, k_3, k_4) be a point of $\Omega^{(4)}$. Then there is an element

$$a = (1\ 2)(3\ 4)(k_1\ k_2\ k_3\ k_4)\cdots$$

in G . We may assume that $a \in N_G(P)$ and a is of the form

$$a = (1\ 2)(3\ 4)(5\ 6)(k_1\ k_2\ k_3\ k_4)\cdots.$$

By the assumption a is of order four, and from 1) of i) any point in $\Omega - \{1, 2, \dots, 6\}$ appears in some 4-cycle of a . Since $\Delta_1^a = \Delta_1$ and $\Delta_2^a = \Delta_2$, we may assume that $\{i_1, i_2, i_3, i_4\} \subset \Delta_1$ and $\{j_1, j_2, j_3, j_4\} \subset \Delta_2$, where $a = (i_1\ i_2\ i_3\ i_4)(j_1\ j_2\ j_3\ j_4)\cdots$. Then from v) $(i_1, i_2, i_3, i_4)^{H^*}$, $(i_2, i_3, i_4, i_1)^{H^*}$, $(j_1, j_2, j_3, j_4)^{H^*}$ and $(j_2, j_3, j_4, j_1)^{H^*}$ are all different H^* -orbits. Thus we have four H^* -orbits. But H has at least three orbits in $\Omega - I(H)$. Hence there is a 4-cycle $(l_1\ l_2\ l_3\ l_4)$ of a such that $\{l_1, l_2, l_3, l_4\} \not\subset \Delta_1 \cup \Delta_2$. Therefore $(l_1, l_2, l_3, l_4)^{H^*}$ is the different H^* -orbit from these four H^* -orbits. Hence we have five H^* -orbits, which is a contradiction.

Thus we have no group in this case.

$$\text{B)} \quad N_G(P)^{I(P)} = A_7.$$

Let $I(P) = \{1, 2, 3, 4, 5, 6, 7\}$ and $H = G_{1234}$. Then from $N_G(P)^{I(P)} = A_7$, there is an element

$$x = (1)(2)(3)(4)(5\ 6\ 7)\cdots.$$

Since $x \in H$, three points 5, 6 and 7 belong to the same H -orbit, say Δ_1 . Then Δ_1 is the only H -orbit in $\Omega - I(H)$ of odd length. If H has only one orbit in $\Omega - I(H)$, namely, H is transitive on $\Omega - I(H)$, then a stabilizer of one point in G satisfies the assumption of Case II. A), which is a contradiction. Therefore H has at least two orbits, say Δ_1 and Δ_2 , in $\Omega - I(H)$.

Suppose that $|\Delta_1| > 3$. Let $(k_1, k_2, k_3, k_4) \in \Omega^{(4)}$. Then there is an element

$$a = (1\ 2)(3\ 4)(k_1\ k_2\ k_3\ k_4)\cdots$$

of G . By the assumption a is of order four, and from 1) of i) the cycles of a are all 4-cycles except two 2-cycles and three 1-cycles. Since $a \in N_G(H)$, Δ_2^a is an H -orbit. Assume that $\Delta_2^a \neq \Delta_2$. Since the length of Δ_2 is even, $\Delta_2^a \neq \Delta_1$. We may assume that $k_1 \in \Delta_2$, and hence $k_2 \notin \Delta_1 \cup \Delta_2$. Then we shall show that $(k_1, k_2, k_3, k_4)^{H^*}$, $(k_2, k_3, k_4, k_1)^{H^*}$, $(k_1, k_4, k_3, k_2)^{H^*}$ and $(k_4, k_3, k_2, k_1)^{H^*}$ are all different H^* -orbits. From v) $(k_1, k_2, k_3, k_4)^{H^*} \neq (k_2, k_3, k_4, k_1)^{H^*}$ and $(k_1, k_4, k_3, k_2)^{H^*} \neq (k_4, k_3, k_2, k_1)^{H^*}$.

If $(k_1, k_2, k_3, k_4)^{H^*} = (k_1, k_4, k_3, k_2)^{H^*}$, then H has a following element

$$x = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 \cdots \\ k_1 & k_4 & k_3 & k_2 \cdots \end{pmatrix} = (k_1) (k_3) (k_2 k_4) \cdots .$$

Since the order of x is even, there is a Sylow 2-subgroup of H fixing the point k_1 . By the conjugacy of Sylow 2-subgroups of H , k_1 must be contained in Δ_1 , which is a contradiction. Therefore $(k_1, k_2, k_3, k_4)^{H^*} \neq (k_1, k_4, k_3, k_2)^{H^*}$. In the same way $(k_2, k_3, k_4, k_1)^{H^*} \neq (k_4, k_3, k_2, k_1)^{H^*}$.

If $(k_2, k_3, k_4, k_1)^{H^*} = (k_1, k_4, k_3, k_2)^{H^*}$, then H has a following element

$$\begin{pmatrix} k_2 & k_3 & k_4 & k_1 \cdots \\ k_1 & k_4 & k_3 & k_2 \cdots \end{pmatrix} = (k_1 k_2) (k_3 k_4) \cdots$$

But this is impossible, for $k_1 \in \Delta_2$ and $k_2 \notin \Delta_2$. Therefore $(k_2, k_3, k_4, k_1)^{H^*} \neq (k_1, k_4, k_3, k_2)^{H^*}$.

Since $a^2 = (1)(2)(3)(4)(k_1 k_3)(k_2 k_4) \cdots$ belongs to H and $(k_4, k_3, k_2, k_1)^{a^2} = (k_2, k_1, k_4, k_3)$, $(k_4, k_3, k_2, k_1)^{H^*} = (k_2, k_1, k_4, k_3)^{H^*}$. Therefore in the same way $(k_1, k_2, k_3, k_4)^{H^*} \neq (k_2, k_1, k_4, k_3)^{H^*} = (k_4, k_3, k_2, k_1)^{H^*}$.

Thus we have four H^* -orbits. On the other hand since $|\Delta_1| > 3$ and $\Delta_1^a = \Delta_1$, there is a 4-cycle of a , say $(j_1 j_2 j_3 j_4)$, such that $\{j_1, j_2, j_3, j_4\} \subset \Delta_1$. Then $(j_1, j_2, j_3, j_4)^{H^*}$ is different from these four H^* -orbits. Thus we have five H^* -orbits which is a contradiction. Therefore $\Delta_2^a = \Delta_2$.

If H has an orbit different from Δ_1 and Δ_2 in $\Omega - I(H)$, then as proved above a fixes these three orbits respectively. By v) H^* has at least six orbits, which is a contradiction. Therefore H -orbits in $\Omega - I(H)$ are Δ_1 and Δ_2 .

Now we shall show that $|\Delta_1| > 3$ leads to a contradiction. Since $\Delta_1^a = \Delta_1$ and $\Delta_2^a = \Delta_2$, we may assume that $\{i_1, i_2, i_3, i_4\} \subset \Delta_1$ and $\{j_1, j_2, j_3, j_4\} \subset \Delta_2$, where $a = (1 2)(3 4)(i_1 i_2 i_3 i_4)(j_1 j_2 j_3 j_4) \cdots$. Since $\Delta_1 \neq \Delta_2$, by v) $(i_1, i_2, i_3, i_4)^{H^*}$, $(i_2, i_3, i_4, i_1)^{H^*}$, $(j_1, j_2, j_3, j_4)^{H^*}$ and $(j_2, j_3, j_4, j_1)^{H^*}$ are all different. We shall show that $(i_1, i_4, i_3, i_2)^{H^*}$ is different from these four H^* -orbits. From $\Delta_1 \neq \Delta_2$, $(i_1, i_4, i_3, i_2)^{H^*} \neq (j_1, j_2, j_3, j_4)^{H^*}$ and $(j_2, j_3, j_4, j_1)^{H^*}$. If $(i_1, i_2, i_3, i_4)^{H^*} = (i_1, i_4, i_3, i_2)^{H^*}$, then H has a following element

$$x = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \cdots \\ i_1 & i_4 & i_3 & i_2 \cdots \end{pmatrix} = (i_1) (i_3) (i_2 i_4) \cdots .$$

Since x is of order even, there is a Sylow 2-subgroup of H fixing 1, 2, 3, 4 i_1 and i_3 . Thus any Sylow 2-subgroup of $H_{i_1 i_3}$ is a Sylow 2-subgroup of H . On the other hand $a^2 = (1)(2)(3)(4)(i_1 i_3)(i_2 i_4) \cdots$ normalizes $H_{i_1 i_3}$. Hence a^2 normalizes a Sylow 2-subgroup P' of $H_{i_1 i_2}$, and $a^{I(P')} = (1)(2)(3)(4)(i_1 i_3) \cdots$, contrary to $N_G(P')^{I(P')} = A_7$. Thus $(i_1, i_2, i_3, i_4)^{H^*} \neq (i_1, i_4, i_3, i_2)^{H^*}$. If $(i_2, i_3, i_4, i_1)^{H^*} = (i_1, i_4, i_3, i_2)^{H^*}$, then H has a following element

$$h = \begin{pmatrix} i_2 & i_3 & i_4 & i_1 & \cdots \\ i_1 & i_4 & i_3 & i_2 & \cdots \end{pmatrix} = (i_1 i_2) (i_3 i_4) \cdots$$

Then

$$ah = (1\ 2) (3\ 4) (i_1) (i_3) (i_2 i_4) \cdots$$

Since $\alpha_2(ah) \geq 3$, from 1) ah is of order $2t$ with t odd. Put $b = (ah)^t$. Then b normalizes a Sylow 2-subgroup P'' of H . Since Δ_2 contains an odd number of P'' -orbits, from iii) the four points of $I(b)$ are contained in Δ_2 and the three points i_1, i_3 and some one of $I(b)$ are contained in Δ_1 . Since $b \in N_G(P'')$, $I(P'') \supset \{1, 2, 3, 4, i_1, i_3\}$. Therefore $a^2 = (1\ 2) (3\ 4) (i_1 i_3) \cdots$ normalizes a Sylow 2-subgroup of $H_{i_1 i_3}$, which is also a Sylow 2-subgroup P''' of H . Thus $(a^2)^{I(P''')} = (1\ 2) (3\ 4) (i_1 i_3) \cdots$, which contradicts the assumption $N_G(P''')^{I(P''')} = A_7$. Therefore $(i_2, i_3, i_4, i_1)^{H^*} \neq (i_1, i_4, i_3, i_2)^{H^*}$.

Thus H^* has at least five orbits, which is a contradiction. Therefore $|\Delta_1| = 3$.

In the proof of Case II of [5. Theorem 2] we needed only the following condition: The number of the fixed points of an involution is seven, and every Sylow 2-subgroup of H fixes the same points. Therefore in the same way we have that G is M_{23} .

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