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THE FIXED SUBRINGS OF A FINITE GROUP OF AUTOMORPHISMS OF $\aleph_0$-CONTINUOUS REGULAR RINGS

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Let R be an associative ring, G a finite group of automorphisms of R, and let R^G be the fixed subring of G on R. A. Page has proved that if R is a left self-injective regular ring and the order $|G|$ of G is invertible in R, then R^G is also a left self-injective regular ring [8]. This theorem is very useful when we investigate some structure of a nonsingular ring and the fixed subring of a finite group of automorphisms.

Recently D. Handelman has discovered an $\aleph_0$-continuous regular ring which coordinates the lattice of projections of a finite Rickart C^*-algebra as a subring of the maximal quotient ring of its C^*-algebra [4]. We shall prove in this note the following generalization of Page's theorem: if R is a left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring and $|G|$ is invertible in R, then R^G is again a left $\aleph_0$-continuous, $\aleph_0$-injective regular ring. We shall show as a corollary that if R is a left $\aleph_0$-continuous regular ring with $|G|^{-1}\in R$, R^G is a left $\aleph_0$-continuous regular ring and S^G is the maximal left $\aleph_0$-quotient ring of R^G, where S is the maximal left $\aleph_0$-quotient ring of R.

1. Skew group rings

DEFINITION [7]. Let R be a ring with identity element 1 and G a finite group of automorphisms of R. The skew group ring, $R*G$, is defined to be a free left R-module with basis $\{g: g\in G\}$ and multiplication given as follows: if $r, s\in R$ and $g, h\in G$, then $(rg)(sh)=rs^{g^{-1}}gh$.

DEFINITION [3]. A regular ring R is left $\aleph_0$-continuous if the lattice of principal left ideals of R is upper $\aleph_0$-continuous. A ring T is left $\aleph_0$-injective if every homomorphism from a countably generated left ideal of T into T is extendable to a T-module endomorphism of T. For modules A and B, $A\subset_e B$ implies that A is an essential submodule of B.

A regular ring R has a maximal left $\aleph_0$-quotient ring S which is a quotient ring defined by the filter-like set $\mathfrak{X}$ consisting of all countably generated, essential left ideals of R [3, § 14]. An element x in the maximal left quotient ring of R is contained in S if and only if there exists some $J \in \mathfrak{X}$ such that $Jx \subset R$. Let $g \in G$. Then J^g is also contained in $\mathfrak{X}$ and we define $x^g: J^g \to R$ by setting $rx^g = (r^{g^{-1}}x)^g$ for any $r \in J^g$. Then x^g determines a left R-homomorphism from J^g to R and thus x^g is a uniquely determined element of S.

K.R. Goodearl has proved the following fundamental result.

Lemma 1 [3, Th. 14.12]. *Let R be a left $\aleph_0$-continuous regular ring, and let S be the maximal left $\aleph_0$-quotient ring of R. Then S is a left $\aleph_0$-continuous, left $\aleph_0$-injective, regular ring and R contains all the idempotents of S.*

It is well-known that if R is a left self-injective regular ring and $|G|$ is invertible in R, then $R*G$ is a left self-injective regular ring. We shall show an analogous result for left $\aleph_0$-continuous, left $\aleph_0$-injective regular rings. Of course, left $\aleph_0$-continuous, left $\aleph_0$-injective regular rings are not necessarily self-injective (See for example, [3, p. 174]).

Theorem 1. *Let S be a left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring and G a finite group of automorphisms of S with $|G|^{-1} \in S$. Then the skew group ring $S*G$ is a left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring.*

Proof. By [5, Th. 3.2], $S*G$ is already a regular ring. First we shall show the $\aleph_0$-injectivity. Let I be any countably generated left ideal of $S*G$ and ϕ any homomorphism from I to $S*G$. I is countably generated as an S-module. Then there exists an S-endomorphism ψ of $S*G$ such that ψ is equal to ϕ on I by [3, Prop. 14.19]. Define $\bar{\psi}(x) = |G|^{-1}\sum_g g\psi(g^{-1}x)$ for any x in $S*G$. One easily checks that $\bar{\psi}$ is an $S*G$-homomorphism and it is an extension of ϕ. Since S is left $\aleph_0$-continuous and left $\aleph_0$-injective, any matrix ring over S is also a left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring by [3, Prop. 14.19]. Therefore the lattice consisting of all finitely generated S-submodules of $S*G$ is upper $\aleph_0$-continuous. Now let J be any countably generated left ideal of $S*G$. Then we have finitely generated S-submodule A of $S*G$ such that $J \subset_e A$ as an S-module. Put $B = \bigcap_g gA$, then it is finitely generated as an S-module and a left ideal of $S*G$. As B is a direct summand as S-module, B is a direct summand of $S*G$ as $S*G$-module by Maschke's Theorem (See for example, [7, Th. 0.1]). Since $J \subset_e B$ as an S-module, we have $J \subset_e B$ as an $S*G$-module. Now our assertion follows by [3, Cor. 14.4].

Corollary. *Let G be a finite group of automorphisms of a left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring S. Assume that $|G|$ is invertible in S. Then S^G is again a left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring.*

Proof. As in [7], consider S as an $S*G$-S^G-bimodule. As a left $S*G$-

module, S is projective and isomorphic to the principal left ideal $(S*G)e$, where $e=|G|^{-1}\sum_g g$. Since $S*G$ is left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring, $End_{S*G}(S)$ is a left $\aleph_0$-continuous, left $\aleph_0$-injective regular ring by [3, Prop. 14.19]. On the other hand we have $S^G\cong \mathrm{End}_{S*G}(S)$ by [7, Prop. 0.3] and the proof is complete.

2. The fixed subring in an $\aleph_0$-continuous regular ring

Let R be a left $\aleph_0$-continuous regular ring, Q the maximal left quotient ring of R and S the maximal left $\aleph_0$-quotient ring of R. A finite group G acting on R may be extended to automorphisms on Q and on S as well. We assume that $|G|^{-1}\in R$. Then Q^G is the maximal left quotient ring of R^G by [8, Th. 3.6]. Hence it is natural to ask whether S^G is the maximal left $\aleph_0$-quotient ring of R^G. This is true. We need next two lemmas for its proof.

Lemma 2. *Let I be an essential, countably generated left ideal of R^G. Then RI is an essential, countably generated left ideal of R.*

Proof. Since RI is a countably generated left ideal of R, there exists a principal left ideal J such that $RI\subset_e J$ by [3, Cor. 14.4]. Put $Ra=\bigcap_g J^g$, where $a=a^2$, then $RI\subset_e Ra$. Since Ra is G-invariant, $(1-a)R$ is also G-invariant. If $a\neq 1$, then $(1-a)R\cap R^G\neq 0$ by Bergman-Isaak Theorem [1, Prop. 2.3]. Choose some $y\neq 0\in(1-\mathrm{a})R\cap R^G$. We have $ay=0$ and so $Iy=0$. Then y must be zero since $I\subset_e R^G$. This is a contradiction and the proof is complete.

Lemma 3. *For any countably generated, essential, G-invariant left ideal I, there exists a countably generated, essential, left ideal A of R^G such that $A\subset I\cap R^G$.*

Proof. Put $M=I*G$. Then M is a countably generated, essential left ideal of $R*G$. Let $M_1\subset\cdots\subset M_n\subset\cdots$ be an increasing sequence of finitely generated left ideals such that $M=\bigcup_n M_n$. Put $T=e(R*G)e$, where $e=|G|^{-1}\sum_g g$. Each M_ne is a direct summand of $(R*G)e$. Let ϕ_n be a projection from $(R*G)e$ onto M_ne. We have $\phi_n(e)\in T$ for all n. We claim that $\sum_n T\phi_n(e)$ is an essential left ideal of T. Let Ta be any non-zero principal left ideal of T, where $a^2=a$. Since $Me\subset_e(R*G)e$, we have a non-zero principal left ideal $(R*G)y\subset Me\cap(R*G)a$. Let ψ be a projection from $(R*G)e$ onto $(R*G)y$. Then we have $\psi(e)a=\psi(e)$. Since $(R*G)y\subset M_ne$ for some n, we have $\psi(e)=\phi_n(\psi(e)e)=\psi(e)\phi_n(e)$. Thus $Ta\cap\sum_n T\phi_n(e)$ contains a non-zero element $\psi(e)$. Next consider the well-known isomorphism $\theta\colon e(R*G)e\to R^G$ given by $\theta[e(\sum_g r_g g)e]=\sum_g t(r_g)$, where $t(a)=|G|^{-1}\sum a^g$ ([6, Lemma 0.1]). Put $A=\theta(\sum_n T\phi_n(e))$, then A is a countably generated, essential left ideal of R^G. We claim that $A\subset I\cap R^G$. In fact each $\phi_n(e)$ is contained in $eMe=e(I*G)e$. Since I is G-invariant, we have $t(r)\in I$ for all $r\in I$ and thus $\theta(e(I*G)e)\subset I\cap R^G$. Consequently we have $A\subset I\cap R^G$.

Now we shall prove our main theorem.

Theorem 2. *Let R be a left $\aleph_0$-continuous regular ring, G a finite group of automorphisms of R and S the maximal left $\aleph_0$-quotient ring of R. Assume $|G|$ is invertible in R. Then R^G is a left $\aleph_0$-continuous regular ring and S^G is the maximal left $\aleph_0$-quotient ring of R^G.*

Proof. All idempotents of S^G are contained in R^G by Lemma 1. Then the lattice of principal left ideals of R^G is isomorphic to that of S^G. Since S^G is left $\aleph_0$-continuous regular ring by the Corollary of § 1, R^G is a $\aleph_0$-continuous regular ring. Let s be any element in S^G. There exists a countably generated left ideal $J\subset_e R$ such that $Js\subset R$ by [3, Prop. 14.11]. Put $I=\bigcap_g J^g$, then I is again countably generated, essential left ideal of R by [3, Lemma 14.10] and is G-invariant. By Lemma 3, we find a countably generated, left ideal $A\subset_e R^G$ such that $A\subset I\cap R^G$. Therefore we have $As\subset(I\cap R^G)s\subset R^G$. On the other hand, let x be any element in Q^G such that $Ix\subset R^G$ for some countably generated, essential left ideal I of R^G. By Lemma 2, RI is countably generated and $RI\subset_e R$. Since $RIx\subset R$, x is contained in S^G. We complete the proof.

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