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MIXED PROBLEM FOR THE WAVE EQUATION WITH AN OBLIQUE DERIVATIVE BOUNDARY CONDITION

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1. Introduction

Consider a mixed problem

$$(1.1)\qquad \begin{cases} \Box u = \left(\dfrac{\partial^2}{\partial t^2}-\dfrac{\partial^2}{\partial x^2}-\dfrac{\partial^2}{\partial y^2}\right)u(x,y,t) = f(x,y,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \Omega\times(0,T) \\ Bu = \left(b_1(x,y)\dfrac{\partial}{\partial x}+b_2(x,y)\dfrac{\partial}{\partial y}\right)u(x,y,t) = g(x,y,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad S\times(0,T) \\ u(x,y,0) = u_0(x,y) \\ \dfrac{\partial u}{\partial t}(x,y,0) = u_1(x,y) \end{cases}$$

where $S=\partial\Omega$ is a C^∞ simple and compact curve in R^2 and $b_i(x,y)$ $(i=1,2)$ are real-valued C^∞-functions defined on S. We assume that $(b_1(x,y), b_2(x,y))$ is not tangential to S, i.e. $b_1n_1+b_2n_2\neq 0$ on S, where $n(x,y)=(n_1(x,y), n_2(x,y))$ is the unit outer normal of S at $(x,y)\in S$. The boundary operator B is called an oblique derivative when

$$(1.2)\qquad b_1(x,y)n_2(x,y)-b_2(x,y)n_1(x,y)\neq 0 \qquad \text{on } S.$$

In this paper we consider the mixed problem (1.1) under the condition (1.2).

In recent years mixed problems for hyperbolic equations have been studied by many authors and the general theory developed (for example S. Agmon [1], T. Balaban [2], H.O. Kreiss [9], R. Sakamoto [10]). Concerning second order equations, the problems with the Dirichlet boundary condition and with the Neumann boundary condition are studied satisfactorily. The author showed the well-posedness in L^2-sense of the problems with a fairly general first order derivative boundary condition in [5]. But the problem (1.1) is not contained, under the condition (1.2), in the results of [1], [2], [5], [7], [9] or [10]. Concerning the problem (1.1), we showed its ill-posedness in L^2-sense when a domain is

R_+^2 in [6]. It seems that the ill-posedness is caused mainly by the following two facts: (i) $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ *with the boundary condition* $Bu=0$ *on* S *has no selfadjoint realization in* $L^2(\Omega)$. (ii) $\square$ *with the boundary operator* B *does not satisfy the complementary condition posed by S. Agmon* [1], *which is called the uniform Lopatinski condition.*[1)]

Therefore to consider the problem (1.1) it is necessary to treat it in a weaker topology than the ordinary L^2-sense. J. Chazarain [3] proved the well-posedness of (1.1) in the space of vector-valued ultra-distributions of t. On the other hand A. Inoue [8] showed a precise estimate for the solution of (1.1) in the case where the domain is a half-space and b_i is constant, but it seems to me that his method is not applicable to non half-space.

In this paper we will show the following:

Theorem 1. *For data* $\{u_0(x, y), u_1(x, y), f(x, y, t), g(x, y, t)\}$ *satisfying the compatibility condition of order* $m+N$, *the mixed problem* (1.1) *has a solution* $u(x, y, t)$ *in* $\mathcal{E}_t^0(H^{m+2}(\Omega))\cap\mathcal{E}_t^1(H^{m+1}(\Omega))\cap\cdots\cap\mathcal{E}_t^{m+2}(L^2(\Omega))$ $(m=0, 1, 2, \cdots)$ *where* N *is an integer determined by* B *and* S. *Furthermore* (1.1) *represents a propagation phenomenon with a finite velocity, which is majorated by*

$$\sup_{(x,y)\in S}\sqrt{1+(b_1n_2-b_2n_1)^2/(b_1^2+b_2^2)}\,.$$

Now we give the definition of the compatibility condition of order m.

DEFINITION 1.1. Data $\{u_0, u_1, f, g\}$ are said to satisfy the compatibility condition of order m when $u_0(x, y)\in H^{m+2}(\Omega)$, $u_1(x, y)\in H^{m+1}(\Omega)$, $f(x, y, t)\in H^{m+1}(\Omega\times(0, T))$, $g(x, y, t)\in H^{m+1}(S\times(0, T))$ and

$$Bu_p(x, y)=\frac{\partial^p g}{\partial t^p}(x, y, 0) \qquad \text{on } S$$

for $p=0, 1, 2, \cdots, m$, where $u_p(x, y)$ $(p=2, 3, \cdots, m)$ are defined successively by the formula

$$u_p(x, y)=\Delta u_{p-2}(x, y)+f^{(p-2)}(x, y, 0)\,.$$

We should like to remark that under the condition (1.2) the mixed problem (1.1) has a velocity larger than that of the Cauchy problem for $\square$; this fact is shown in the appendix. The mixed problems treated in [5] and [7] have the same velocity as the Cauchy problem, therefore we can say that the above fact is one of the characteristics of the L^2 ill-posed problems.

1) The Neumann condition does not satisfy the uniform Lopatinski condition, but the mixed problem with the Neumann boundary condition is well posed in L^2-sense since Δ has a self-adjoint realization in $L^2(\Omega)$.

To prove Theorem 1 we consider at first the mixed problem for an equation with variable coefficients in a domain of a half-space. We reduce this problem, by a Laplace transformation in t, to a boundary value problem with a parameter $s\in \boldsymbol{C}_+=\{\eta+i\xi: \eta>0, \xi\in R\}$. In the treatment we make use of pseudo-differential operators with a parameter $s\in \boldsymbol{C}_+$.

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2. The case where the domain is a half-space

Let R^2_+ be a half-space $\{(x, y); x>0, y\in R\}$. Consider a hyperbolic operator L_φ and a boundary operator B_φ such that

$$\begin{aligned} L_\varphi\Big(y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\Big) &= (1-a_{22}\varphi^2)\frac{\partial^2}{\partial t^2}-2a_{12}\varphi\frac{\partial^2}{\partial x\partial t}-2a_{22}\varphi\frac{\partial^2}{\partial y\partial t} \\ &\quad -\Big(a_{11}\frac{\partial^2}{\partial x^2}+2a_{12}\frac{\partial^2}{\partial x\partial y}+a_{22}\frac{\partial^2}{\partial y^2}\Big)+(\text{first order}) \\ &= L_{\varphi 0}\Big(y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\Big)+(\text{first order}) \end{aligned} \tag{2.1}$$

$$\begin{aligned} B_\varphi\Big(y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\Big) &= a_{11}\frac{\partial}{\partial x}+(a_{11}+b)\Big(\frac{\partial}{\partial y}+\varphi\frac{\partial}{\partial t}\Big)+c \\ &= B_{\varphi 0}+c \end{aligned} \tag{2.2}$$

where all the coefficients are real-valued C^∞-functions of y and

$$b(y)\neq 0 \qquad \text{for all } y\in R. \tag{2.3}$$

We consider in this section the following mixed problem

$$\begin{cases} L_\varphi u(x, y, t) = f(x, y, t) & \text{in } R^2_+\times(0, T) \\ B_\varphi u(x, y, t)|_{x=0} = g(y, t) & \\ u(x, y, 0) = u_0(x, y) & \\ \dfrac{\partial u}{\partial t}(x, y, 0) = u_1(x, y)\,. & \end{cases} \tag{2.4}$$

When we derive energy estimates of the solution or show the existence of solution, an essential role is played by an apriori estimate of the boundary value problem with a parameter $s=\eta+i\xi\in \boldsymbol{C}_+$

$$(2.5)\qquad \begin{cases} L_\varphi\left(y, \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, s\right)v(x, y) = p(x, y) & \text{in } R_+^2 \\ B_\varphi\left(y, \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, s\right)v(x, y)|_{x=0} = q(y)\,. \end{cases}$$

2.1. Pseudo-differential operators with a parameter $s\in C_+$

We denote $\frac{\partial}{\partial y}$ by ∂_y and $\frac{1}{i}\frac{\partial}{\partial y}$ by D_y.

Let $\mathcal{P}(y, \omega, s)$ be $k\times k$ matrix-valued $C^\infty(R\times R\times C_+)$ function. $\mathcal{P}\in S^m_{C+}$ means that

$$|\partial_y^\alpha\partial_\omega^\beta\mathcal{P}(y, \omega, s)| \leqslant C_{\alpha,\beta}(\omega^2+|s|^2)^{(m-|\beta|)/2}$$

holds for all $(y, \omega, s)\in R\times R\times C_+$. For $\mathcal{P}(y, \omega, s)\in S^m_{C+}$ we define a pseudo-differential operator $\mathcal{P}(y, D_y, s)$ by

$$\mathcal{P}(y, D_y, s)V(y) = \frac{1}{2\pi}\int e^{i\omega y}\mathcal{P}(y, \omega, s)\hat{V}(\omega)d\omega$$

for $V(y)\in\mathcal{S}(R)$[2], where

$$\hat{V}(\omega) = \int e^{-i\omega y}V(y)dy\,.$$

Lemma 2.1.

(i) *Let $\mathcal{P}(y, \omega, s)\in S^m_{C+}$ and $\mathcal{Q}(y, \omega, s)\in S^{m'}_{C+}$ then*

$$\begin{aligned}&\mathcal{P}(y, D_y, s)\mathcal{Q}(y, D_y, s)\\ &= \sum_{|\alpha|<N}\frac{1}{\alpha!}(\partial_\omega^\alpha\mathcal{P}\circ D_y^\alpha\mathcal{Q})(y, D_y, s)+\mathcal{R}_N(y, D_y, s)\end{aligned}$$

where $\mathcal{R}_N(y, \omega, s)\in S^{m+m'-N}_{C+}$.

(ii) *For $\mathcal{P}(y, \omega, s)\in S^m_{C+}$ there exists $\mathcal{P}^\sharp(y, \omega, s)\in S^m_{C+}$ such that*

$$(\mathcal{P}(y, D_y, s)V, W) = (V, \mathcal{P}^\sharp(y, D_y, s)W)$$

holds for all $V, W \in\mathcal{S}(R)$, and the following expansion

$$\mathcal{P}^\sharp(y, \omega, s) = \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\omega^\alpha D_y^\alpha\mathcal{P}^*(y, \omega, s)+\mathcal{R}_N(y, \omega, s)$$

holds where $\mathcal{P}^*(y, \omega, s)$ denotes the adjoint matrix of $\mathcal{P}(y, \omega, s)$ and $\mathcal{R}_N(y, \omega, s)\in S^{m-N}_{C+}$.

(iii) *When $\mathcal{P}(y, \omega, s)\in S^0_{C+}$, there exists a constant $C>0$ such that*

$$\|\mathcal{P}(y, D_y, s)V\|\leqslant C\|V\| \qquad \text{for all } V\in\mathcal{S}.$$

2) $\mathcal{S}(R)$ is the set of all rapidly decreasing functions defined in R.

(iv) *When* $\mathcal{P}(y, \omega, s) \in S_{C+}^{-1}$, *there exists a constant* $C>0$ *such that*

$$\|\mathcal{P}(y, D_y, s)V\| \leqslant \frac{C}{|s|}\|V\|$$

holds for all $V \in \mathcal{S}(R)$ *and* $s \in \boldsymbol{C}_+$.

(v) *Let* $\mathcal{P}(y, \omega, s) \in S_{C+}^{1}$ *and* $\inf_{(y,\omega)\in R^2} \operatorname{Re} \mathcal{P}(y, \omega, s) \geqslant c(s)$ *then there exists a constant* $C>0$ *and it holds that for all* $s \in \boldsymbol{C}_+$

$$\operatorname{Re}(\mathcal{P}(y, D_y, s)V, V) \geqslant c(s)\|V\|^2 - C\|V\|^2.$$

The proof of this lemma is not given here, because it can be proved without much difficulties only by using the method of H. Kumano-go, "An algera of pseudo-differential operators", J. Fac. Sci. Univ. Tokyo **17** (1970), 31–50.

2.2. Apriori estimate of a solution of (2.5)

In this paragraph, the subscript φ of L_φ and B_φ is dropped for the simplicity, and we denote by $\|\cdot\|_i$ the norm of the space $H^i(R_+^2)$ and by $\langle\cdot\rangle_i$ that of $H^i(R)$. Hereafter we assume that all the coefficients depend on y in $|y| \leqslant d$ and that

$$a_{ij}(0) = \delta_{ij} \qquad (i, j=1, 2). \tag{2.6}$$

We should like to treat the boundary value problem (2.5) in an equivalent system by putting

$$V(x, y, s) = \begin{bmatrix} v_1(x, y, s) \\ v_2(x, y, s) \end{bmatrix} = \begin{bmatrix} i(D_y^2 + \xi^2)^{1/2} v(x, y) \\ \dfrac{\partial v}{\partial x}(x, y) \end{bmatrix} \qquad (s=\eta+i\xi)$$

$$\begin{cases} \text{(i)} & \dfrac{\partial}{\partial x} V(x, y, s) = \mathcal{M}(y, D_y, s)V(x, y, s) + P(x, y) \\ \text{(ii)} & \mathcal{B}(y, D_y, s)V(x, y, s)|_{x=0} = q(y) \end{cases} \tag{2.7}$$

where

$$P(x, y) = \begin{bmatrix} 0 \\ p(x, y) \end{bmatrix}$$

$$\mathcal{M}(y, \omega, s) = \mathcal{M}_0(y, \omega, s) + \mathcal{M}_1(y, \omega, s)$$

$$\mathcal{M}_0(y, \omega, s) = \begin{bmatrix} 0 & i(\omega^2+\xi^2)^{1/2} \\ \dfrac{(1-a_{22}\varphi^2)s^2 + a_{22}\omega^2 - 2ia_{22}\varphi\omega s}{i(\omega^2+\xi^2)^{1/2}a_{11}} & \dfrac{-2ia_{12}\varphi\omega - 2a_{12}s}{a_{11}} \end{bmatrix}$$

$$\mathcal{M}_1(y, \omega, s) \in S_{C+}^0$$

and

$$\mathcal{B}(y, \omega, s) = \begin{bmatrix} \dfrac{(a_{12}+b)(\varphi s + i\omega) + c}{i(\omega^2+\xi^2)^{1/2}} & a_{11} \end{bmatrix}$$

$$= \mathscr{B}_0(y, \omega, s) + \begin{bmatrix} \dfrac{c}{i(\omega^2+\xi)^{21/2}} & 0 \end{bmatrix}.$$

Remark that the eigenvalues of $\mathscr{M}(y, \omega, s)$ ($\mathscr{M}_0(y, \omega, s)$) coinside with the roots of the equation $L(y, \kappa, i\omega, s)=0$ ($L_0(y, \kappa, i\omega, s)=0$) in κ.

Lemma 2.2. *There exists a constant* η_1 *such that if* $\mathrm{Re}\, s \geqslant \eta_1$, *the equation in* κ

$$L(y, \kappa, i\omega, s) = 0$$

has a root κ_+ *with a positive real part and a root* κ_- *with a negative real part.*

Proof. Remark that for purely imaginary κ, $L(y, \kappa, i\omega, s)$ is never equal to zero if $\mathrm{Re}\, s \geqslant \eta'$ for some constant $\eta' > 0$. Indeed, there exists a constant c such that $|\mathrm{Re}\, s(y, \kappa, i\omega)| \leqslant c$ for any root $s(y, \kappa, i\omega)$ of $L(y, \kappa, i\omega, s)=0$ for all $\left(y, \frac{1}{i}\kappa, \omega\right) \in R^3$, therefore the root κ is never purely imaginary. On the other hand

$$\frac{1}{\lambda^2} L(0, 0, \lambda\kappa, \lambda s) = 0$$

has a root with positive real part and a root with negative real part when λ is sufficiently large. These two facts prove Lemma. Q.E.D.

Let us denote by $\mathring{\kappa}_+(y, \omega, s)$ ($\mathring{\kappa}_-(y, \omega, s)$) the root with positive real part (negative real part) of $L_0(y, \kappa, i\omega, s)=0$ for $\mathrm{Re}\, s>0$,[3] and we have

$$\mathring{\kappa}_\pm(y, \omega, s) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} \kappa_\pm(y, \lambda\omega, \lambda s)$$

$$\mathring{\kappa}_\pm(y, \lambda\omega, \lambda s) = \lambda \mathring{\kappa}_\pm(y, \omega, s) \qquad \text{for } \lambda>0.$$

We have the following lemma from the hyperbolicity of L.

Lemma 2.3. *There exists a constant* $c>0$ *such that*

$$\mathrm{Re}\, \mathring{\kappa}_+(y, \omega, s) \geqslant c\, \mathrm{Re}\, s$$

$$\mathrm{Re}\, \mathring{\kappa}_-(y, \omega, s) \leqslant -c\, \mathrm{Re}\, s$$

for all $\mathrm{Re}\, s \geqslant \eta_1$.

Define $\mathring{\kappa}_\pm(y, \omega, i\xi)$ by $\lim\limits_{\eta \to +0} \mathring{\kappa}_\pm(y, \omega, \eta+i\xi)$ and set

$$\Gamma(y, \omega, s) = B_0(y, \mathring{\kappa}_-(y, \omega, s), i\omega, s)^{4)}$$

3) $L_0=0$ has a root with positive real part and a root with negative real part if $\mathrm{Re}\, s \neq 0$.

4) $\Gamma(y, \omega, s)$ is called Lopatinski determinant and the uniform Lopatinski condition means that $\Gamma(y, \omega, s) \neq 0$ for all $\mathrm{Re}\, s \geqslant 0$.

for Re $s \geqslant 0$. Evidently

$$\Gamma(y, \lambda\omega, \lambda s) = \lambda\Gamma(y, \omega, s) \qquad \text{for } \lambda > 0.$$

Let us assume

ASSUMPTION I. $\quad a_{11}a_{22} - a_{12}^2 > 0 \qquad$ for all y

$\qquad (1 - a_{22}\varphi^2) > 0 \qquad$ for all y

and

$$\sup_{y\in R} |\varphi(y)| < \inf_{y\in R} \sqrt{a_{11}(y)/(a_{11}(y)a_{22}(y) - a_{12}(y)^2 + b(y)^2)}\ .$$

Lemma 2.4. *Under Assumption I, $\Gamma(y, \omega, s)$ vanishes only for purely imaginary s.*

Since $\Gamma(y, \omega, s)$ can be written explicitly, Lemma is proved by an elementary calculas.

Hereafter we assume that

$$b(y) > 0 \qquad \text{for all } y \in R.$$

Lemma 2.5. *There exist two points (ω_0, ξ_0) on the sphere $\{(\omega, \xi);\ \omega^2 + \xi^2 = 1\}$ such that $\Gamma(0, \omega_0, i\xi_0) = 0$. And we have*

$$\mathring{\kappa}_+(0, \omega_0, i\xi_0) \neq \mathring{\kappa}_-(0, \omega_0, i\xi_0)$$

$$\operatorname{Im} \frac{\partial \mathring{\kappa}_-}{\partial \omega}(0, \omega_0, i\xi_0) \neq 0\ .$$

Proof. Set $k(y) = a_{11}(y)a_{22}(y) - a_{12}(y)^2 + b(y)^2$. $\Gamma(0, \omega_0, i\xi_0) = 0$ means that

$$\sqrt{\tilde{\omega}_0^2 - \xi_0^2} = ib(0)\tilde{\omega}_0$$

where $\tilde{\omega}_0 = \omega_0 + \varphi(0)\xi_0$. From this it follows that

$$\tilde{\omega}_0 = \pm\sqrt{\frac{1}{k(0)}}\,\xi_0\,,$$

and by taking account of the definition of $\kappa_-(y, \omega_0, i\xi_0)$ and $b(0) > 0$ we get

$$\tilde{\omega}_0 = \sqrt{\frac{1}{k(0)}}\,\xi_0\,.$$

From the explicit form we have

$$\mathring{\kappa}_+(0, \omega_0, i\xi_0) - \mathring{\kappa}_-(0, \omega_0, i\xi_0) = i2b\tilde{\omega}_0 \neq 0$$

and

$$\operatorname{Im} \frac{\partial \mathring{\kappa}_-}{\partial \omega}(0, \omega_0, i\xi_0) = -\frac{\tilde{\omega}_0}{\sqrt{\tilde{\omega}_0^2 - \xi_0^2}} = \frac{i}{b(0)} \neq 0\,.$$

Q.E.D.

Since $\Gamma(0, \omega, s)$ equals zero at only two points in $\{(\omega, s);\ \omega^2+|s|^2=1,\ \mathrm{Re}\, s\geqslant 0\}$ we have that for some positive constant $0<d_2<d_1$ and γ_i $(i=1, 2)$

$$|\Gamma(0, \omega, i\xi)|\geqslant 2\gamma_1$$

on $\{(\omega, \xi);\ \omega^2+\xi^2=1 \text{ and } d_2\leqslant|\omega-\omega_0|+|\xi-\xi_0|\leqslant d_1\}$ and

$$\left|\mathrm{Im}\, \frac{\partial\kappa_-}{\partial\omega}(0, \omega, i\xi)\right|\geqslant 2\gamma_2$$

on $\{(\omega, \xi);\ \omega^2+\xi^2=1 \text{ and } |\omega-\omega_0|+|\xi-\xi_0|\leqslant d_1\}$.

Let us suppose

ASSUMPTION II. It holds that

$$|\mathring{\kappa}_+(y, \omega, s)-\mathring{\kappa}_-(y, \omega, s)|\geqslant\gamma_0 \tag{2.8}$$

$$\left|\mathrm{Im}\, \frac{\partial\mathring{\kappa}_-}{\partial\omega}(y, \omega, s)\right|\geqslant\gamma_2 \tag{2.9}$$

for all $y\in R$, $s=\eta+i\xi$ and ω such that $0\leqslant\eta\leqslant\eta_2$ and $(\omega, \xi)\in\{(\omega, \xi);\ |\omega-\omega_0|+|\xi-\xi_0|\leqslant d_1 \text{ and } \omega^2+\xi^2=1\}$ and that

$$|\Gamma(y, \omega, s)|\geqslant\gamma_1 \tag{2.10}$$

for all $y\in R$, $s=\eta+i\xi$ and ω such that $0\leqslant\eta\leqslant\eta_1$ and $(\omega, \xi)\in\{(\omega, \xi);\ d_2\leqslant|\omega-\omega_0|+|\xi-\xi_0|\leqslant d_1 \text{ and } \omega^2+\xi^2=1\}$, where η_2 and γ_0 are positive constants.

Take d_3 as $d_2<d_3<d_1$ and set

$$\Delta_\xi^{(i)}=\{\omega';\ |\omega'-\omega_0|+|\xi'-\xi_0|\leqslant d_i\} \qquad (i=1, 2, 3)$$

where $(\omega', \xi')=(\omega, \xi)/(\omega^2+\xi^2)^{1/2}$, and take a real-valued C^∞-function $\chi_0(\omega, \xi)$ such that

$$\chi_0(\omega, \xi)=\begin{cases} 1 & \omega\in\Delta_\xi^{(3)} \\ 0 & \omega\notin\Delta_\xi^{(1)}\,. \end{cases}$$

Operate $\chi_0(D_y, \xi)$ to (i) of (2.7) and we have

$$\begin{aligned} \frac{\partial}{\partial x}\chi_0 V &= \mathcal{M}\chi_0 V+[\chi_0, \mathcal{M}]V+\chi_0 P \\ &= \mathcal{M}\chi_0 V+P_0\,. \end{aligned} \tag{2.11}$$

Put

$$\tilde{\kappa}_\pm(y, \omega, s)=(\xi^2+\omega^2)^{-1/2}\kappa_\pm(y, \omega, s)\,,$$

then by changing the value of $\mathcal{M}(y, \omega, s)$ in the outside of $\Delta_\xi^{(1)}$ we may assume that

(2.12) $$|\tilde{\kappa}_+(y, \omega, s) - \tilde{\kappa}_-(y, \omega, s)| \geqslant \gamma_1' > 0$$

for all $(y, \omega) \in R^2$ when ξ is sufficiently large since for $\omega \in \Delta_\xi^{(1)}$ (2.12) follows from (2.8).

Remark that $\mathcal{M}(y, \omega, s)$ can be written as

$$\begin{bmatrix} 0 & i(\omega^2+\xi^2)^{1/2} \\ -\dfrac{\kappa_+(y, \omega, s)\kappa_-(y, \omega, s)}{i(\omega^2+\xi^2)^{1/2}} & \kappa_+(y, \omega, s)+\kappa_-(y, \omega, s) \end{bmatrix}.$$

Then for a matrix

$$\mathcal{N}(y, \omega, s) = \begin{bmatrix} a(y, \omega, s)\tilde{\kappa}_-(y, \omega, s) & -ia(y, \omega, s) \\ \tilde{\kappa}_+(y, \omega, s) & -i \end{bmatrix}$$

we get

$$\mathcal{N}(y, \omega, s)\mathcal{M}(y, \omega, s) = \begin{bmatrix} \kappa_+(y, \omega, s) & 0 \\ 0 & \kappa_-(y, \omega, s) \end{bmatrix} \mathcal{N}(y, \omega, s)$$

where $a(y, \omega, s)$ is an arbtrary function and when $a(y, \omega, s)$ is not zero $\mathcal{N}(y, \omega, s)$ is a non-singular matrix for large ξ from (2.12). Denote the above diagonal matrix by $\mathcal{K}(y, \omega, s)$. (i) of Lemma 2.1 shows

(2.13)
$$\begin{aligned} &\mathcal{N}(y, D_y, s)\mathcal{M}(y, D_y, s) \\ &= \{\mathcal{K}(y, D_y, s) + (\partial_\omega \mathcal{N} \circ D_y \mathcal{M} - \partial_\omega \mathcal{K} \circ D_y \mathcal{N}) \circ \mathcal{N}^{-1}(y, D_y, s)\} \mathcal{N}(y, D_y, s) \\ &\quad + \mathcal{R}_{-1}(y, D_y, s) \\ &= (\mathcal{K}(y, D_y, s) + \mathcal{T}(y, D_y, s)) \cdot \mathcal{N}(y, D_y, s) + \mathcal{R}_{-1}(y, D_y, s) \end{aligned}$$

where $\mathcal{T}(y, \omega, s) \in S^0_{C+}$ and $\mathcal{R}_{-1}(y, \omega, s) \in S^{-1}_{C+}$. Set $\mathcal{T}(y, \omega, s) = [t_{ij}(y, \omega, s)]_{i,j=1,2}$ and choose $a(y, \omega, s) \in S^0_{C+}$ as $t_{12}(y, \omega, s)$ is zero in $\Delta_\xi^{(1)}$. The (1, 2) entry of $((\partial_\omega \mathcal{N} \circ D_y \mathcal{M} - \partial_\omega \mathcal{K} \circ D_y \mathcal{N}) \circ \mathcal{N}^{-1})(y, \omega, s)$ equals

$$\frac{\partial_y \tilde{\kappa}_-}{i(\tilde{\kappa}_+ - \tilde{\kappa}_-)} \{(\kappa_+ - \kappa_-)\partial_\omega a - a \cdot \partial_\omega \kappa_+\},$$

then we define $a(y, \omega, s)$ by

$$a(y, \omega, s) = \exp\left(\int_{\omega_\xi}^{\omega} \left(\frac{\partial_\omega \kappa_+}{\kappa_+ - \kappa_-}\right)(y, \zeta, s)d\zeta\right)$$

for $\omega \in \Delta_\xi^{(1)}$ where $\omega_\xi = \omega_0 \xi/\xi_0$ and we define suitably in the outside of $\Delta_\xi^{(1)}$ as $a(y, \omega, s) \in S^0_{C+}$ and $|a(y, \omega, s)| \geqslant c_0$. Then we have

$$\mathcal{T}(y, \omega, s) \in S^0_{C+}$$

and

(2.14) $$t_{12}(y, \omega, s) = 0 \qquad \text{for } \omega \in \Delta_\xi^{(1)}.$$

Put $\kappa_-(y, \omega, s) = \kappa_1(y, \omega, s) + i\kappa_2(y, \omega, s)$.

Let us assume

ASSUMPTION III. There exists a real-valued C^∞-function $\psi(y, \omega, s) \in S^1_{C+}$ satisfying

$$\frac{\partial \kappa_2}{\partial y}\frac{\partial \psi}{\partial \omega} - \frac{\partial \kappa_2}{\partial \omega}\frac{\partial \psi}{\partial y} = 0 \tag{2.15}$$

$$\psi(y, \omega, s) = |\xi| \qquad \text{for } \omega \notin \Delta_\xi^{(3)}, \tag{2.16}$$

$$\psi(y, \omega, s) = 0 \qquad \text{for } \omega \in \Delta_\xi^{(2)}. \tag{2.17}$$

Set

$$\mathcal{D}(y, D_y, s) = \begin{bmatrix} \beta|\xi|^2 & 0 \\ 0 & -(\psi(y, D_y, s)+\alpha)^*(\psi(y, D_y, s)+\alpha) \end{bmatrix}$$

where α is a positive constant such that $\|(\psi+\alpha)w\| \geqslant \|w\|$ for all $w \in L^2(R^2_+)$ and β is a positive constant which will be determined later.

Operate $\mathcal{N}(y, D_y, s)$ to the both sides of (2.11) and we have from (2.13)

$$\frac{\partial}{\partial x}\mathcal{N}\chi_0 V = \mathcal{K}\mathcal{N}\chi_0 V + \mathcal{I}\mathcal{N}\chi_0 V + \mathcal{R}_{-1}\chi_0 V + \mathcal{N}P_0 .$$

Set $W(x, y, s) = \mathcal{N}(y, D_y, s)\chi_0(D_y, \xi)V(x, y, s)$.

$$\begin{aligned}
&2\,\mathrm{Re}\,(\mathcal{D}(y, D_y, s)W(x, y, s), -(\mathcal{R}_{-1}\chi_0 V + \mathcal{N}P_0))\\
&= 2\,\mathrm{Re}\Big(\mathcal{D}W, -\frac{\partial}{\partial x}W\Big) + 2\,\mathrm{Re}\,(\mathcal{D}W, \mathcal{K}W) + 2\,\mathrm{Re}\,(\mathcal{D}W, \mathcal{I}W)\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{I} &= 2\,\mathrm{Re}\,\beta|\xi|^2\Big(w_1(x, y, s), -\frac{\partial}{\partial x}w_1(x, y, s)\Big)\\
&\quad -2\,\mathrm{Re}\Big((\psi+\alpha)^*(\psi+\alpha)w_2(x, y, s), -\frac{\partial}{\partial x}w_2(x, y, s)\Big)\\
&= \beta|\xi|^2\langle w_1(0, y, s)\rangle^2 - \langle(\psi(y, D_y, s)+\alpha)w_2(0, y, s)\rangle^2\\
\mathrm{II} &= \beta|\xi|^2 2\,\mathrm{Re}\,(w_1(x, y, s), \kappa_+(y, D_y, s)w_1(x, y, s))\\
&\quad +2\,\mathrm{Re}((\psi+\alpha)^*(\psi+\alpha)w_2(x, y, s), (-\kappa_-)(y, D_y, s)w_2(x, y, s))\\
&= \beta|\xi|^2 2\,\mathrm{Re}\,(w_1(x, y, s), \kappa_+(y, D_y, s)w_1(x, y, s))\\
&\quad +2\,\mathrm{Re}\,((\psi+\alpha)w_2(x, y, s), (-\kappa_-)(y, D_y, s)(\psi+\alpha)w_2(x, y, s))\\
&\quad +2\,\mathrm{Re}\,((\psi+\alpha)w_2(x, y, s), [\kappa_-, \psi]w_2(x, y, s))
\end{aligned}$$

by using (v) of Lemma 2.1 and Lemma 2.3

$$\begin{aligned}
&\geqslant (c\eta - C)\{\beta|\xi|^2\|w_1(x, y, s)\|^2 + \|(\psi+\alpha)w_2(x, y, s)\|^2\}\\
&\quad +2\,\mathrm{Re}\,((\psi+\alpha)w_2(x, y, s), [\kappa_-, \psi]w_2(x, y, s)) .
\end{aligned}$$

$$\begin{aligned}|\mathrm{III}| &= \beta|\xi|^2|(w_1, t_{11}w_1+t_{12}w_2)| + |((\psi+\alpha)^*(\psi+\alpha)w_2, t_{21}w_1+t_{22}w_2)| \\ &\leqslant C(|\xi|^2\|w_1\|^2+\|(\psi+\alpha)w_2\|^2)+\beta|\xi|^2\cdot|(w_1, t_{12}(y, D_y\, s)w_2)|.\end{aligned}$$

From (2.14), we have for any integer N

$$\|t_{12}(y, D_y, s)w_2\| \leqslant \frac{C_N}{|s|^N}\|\chi_0 V\|.$$

Therefore we get

$$|\mathrm{III}| \leqslant C\{|\xi|^2\|w_1\|^2+\|(\psi(y, D_y, s)+\alpha)w_2\|^2+\|\chi_0 V\|^2\}.$$

Now let us estimate $2\,\mathrm{Re}\,((\psi+\alpha)w_2, [\kappa_-, \psi]w_2)$. Put

$$\kappa_-(y, \omega, s) = \kappa_1(y, \omega, s)+i\kappa_2(y, \omega, s)+(\kappa_- - \mathring{\kappa}_-)(y, \omega, s),$$

evidently $\kappa_- - \mathring{\kappa}_- \in S^0_{C+}$ and by an elementary calculas $\kappa_1(y, \omega, s)$ can be represented as

$$\kappa_1(y, \omega, s) = \eta\kappa_3(y, \omega, s), \qquad \kappa_3(y, \omega, s) \in S^0_{C+}$$

in $\Delta^{(1)}_\xi$.
Thus we have

$$\|[\kappa_1(y, D_y, s)+(\kappa_- - \mathring{\kappa}_-)(y, D_y, s), \psi(y, D_x, s)]w_2\| \leqslant C\eta\|w_2\|.$$

And

$$\begin{aligned}&i[\kappa_2(y, D_y, s), \psi(y, D_y, s)] \\ &= \left(\frac{\partial\kappa_2}{\partial\omega}\frac{\partial\psi}{\partial y}-\frac{\partial\kappa_2}{\partial y}\frac{\partial\psi}{\partial\omega}\right)(y, D_y, s)+\mathcal{R}_0(y, D_y, s)\end{aligned}$$

by taking account of (2.15)

$$= \mathcal{R}_0(y, D_y, s)$$

where $\mathcal{R}_0(y, \omega, s) \in S^0_{C+}$. Thus we get

$$\begin{aligned}&|2\,\mathrm{Re}\,((\psi+\alpha)w_2, [\psi, i\kappa_2]w_2)| \\ &\leqslant C\eta\|(\psi+\alpha)w_2\|\,\|w_2\|.\end{aligned}$$

On the other hand

$$\begin{aligned}&|2\,\mathrm{Re}\,(\mathcal{D}W, -\mathcal{R}_{-1}\chi_0 V+\mathcal{N}P_0)| \\ &\leqslant C(|\xi|\,\|w_1\|+\|(\psi+\alpha)w_2\|)(\|\chi_0 V\|+|\xi|\,\|P_0\|).\end{aligned}$$

We get

$$\begin{aligned}&(c\eta-C)\{\beta|\xi|^2\|w_1\|^2+\|(\psi+\alpha)w_2\|^2\}-C\eta\|(\psi+\alpha)w_2\|\,\|w_2\| \\ &\quad -C\|\chi_0 V\|^2+\beta|\xi|^2\langle w_1(0, y, s)\rangle^2-\langle(\psi+\alpha)w_2(0, y, s)\rangle^2 \\ &\leqslant C|\xi|^2\|P_0\|^2.\end{aligned}$$

Remark that

$$\frac{1}{C}\|W\| \leqslant \|\chi_0 V\| \leqslant C\|W\| \qquad \text{if } |s| \geqslant |s_0|$$

since

$$\begin{aligned}\|\chi_0 V\| &= \|(\mathcal{N}^{-1}\circ\mathcal{N})\chi_0 V\| \\ &= \|\mathcal{N}^{-1}(y, D_y, s)\mathcal{N}(y, D_y, s)\chi_0 V + (\text{order } -1)\chi_0 V\| \\ &\leqslant C\|W\| + C\frac{1}{|s|}\|\chi_0 V\|,\end{aligned}$$

and the left-hand side is evident.

Thus it holds that, if we take α sufficiently large,

$$\begin{aligned}&c\eta\{\beta|\xi|^2\|w_1\|^2 + \|(\psi+\alpha)w_2\|^2\} \\ &\quad + \beta|\xi|^2\langle w_1(0, y, s)\rangle^2 - \langle(\psi+\alpha)w_2(0, y, s)\rangle^2 \\ &\leqslant C|\xi|^2\|P_0\|^2\end{aligned} \tag{2.18}$$

for $\eta \geqslant \eta_2'$.

Next we estimate the boundary term. Operate $\chi_0(D_y, \xi)$ to (ii) of (2.7) and we have

$$\mathcal{B}(y, D_y, s)\chi_0(D_y, \xi)V(0, y, s) = [\mathcal{B}, \chi_0]V(0, y, s) + \chi_0 q\,.$$

$$\mathcal{B}(y, D_y, s) = (\mathcal{B}\circ\mathcal{N}^{-1})(y, D_y, s)\cdot\mathcal{N}(y, D_y, s) + \mathcal{B}_{-1}(y, D_y, s)$$

where $\mathcal{B}_{-1}(y, \omega, s) \in S_{C+}^{-1}$. Then

$$\begin{aligned}&b_1(y, D_y, s)w_1(0, y, s) + b_2(y, D_y, s)w_2(0, y, s) \\ &\quad = \mathcal{B}_{-1}\chi_0 V(0, y, s) + [\mathcal{B}, \chi_0]V(0, y, s) + \chi_0(D_y, s)q(y)\end{aligned} \tag{2.19}$$

where $b_i(y, \omega, s) \in S_{C+}^0$ $(i=1, 2)$ and

$$b_2(y, \omega, s) = -i(\omega^2+\xi^2)^{-1/2}\Gamma(y, \omega, s)\,.$$

We have $|\Gamma(y, \omega, s)| \geqslant c\eta$ from Lemma 2.3, and from (2.10) $|b_2(y, \omega, s)| \geqslant \gamma_1$ when $\omega \notin \Delta_\xi^{(2)}$.

$$\begin{aligned}&\langle \xi b_2(y, D_y, s)w_2(0, y, s)\rangle \\ &\geqslant c\eta\langle w_2(0, y, s)\rangle - C\langle w_2(0, y, s)\rangle \\ &\langle \psi(y, D_y, s)b_2(y, D_y, s)w_2(0, y, s)\rangle \\ &\geqslant \langle b_2(y, D_y, s)\psi w_2(0, y, s)\rangle - C\langle w_2(0, y, s)\rangle \\ &\geqslant \gamma_1\langle \psi(y, D_y, s)w_2(0, y, s)\rangle - C\{|s|^{-1}\langle \psi(y, D, s)w_2(0, y, s)\rangle \\ &\qquad\qquad + \langle w_2(0, y, s)\rangle\}\,.\end{aligned}$$

Then we get

$$
\begin{aligned}
& c\eta\langle w_2(0, y, s)\rangle+\gamma_1\langle\psi w_2(0, y, s)\rangle-C\langle w_2(0, y, s)\rangle \\
& \leqslant C|\xi|\langle b_2(y, D_y, s)w_2(0, y, s)\rangle \\
& \leqslant C|\xi|\langle b_1(y, D_y, s)w_1(0, y, s)\rangle+C|\xi|\{\langle\chi_0 q\rangle+\langle[\mathscr{B}, \chi_0]V\rangle+\langle\mathscr{B}_{-1}\chi_0 V\rangle\} \\
& \leqslant C|\xi|\langle w_1(0, y, s)\rangle+C|\xi|\{\langle\chi_0 q\rangle+\langle[\mathscr{B}, \chi_0]V\rangle+\langle\mathscr{B}_{-1}\chi_0 V\rangle\}.
\end{aligned}
$$

By choosing β sufficiently large, it holds that

$$
\begin{aligned}
(2.20)\qquad & \beta|\xi|^2\langle w_1(0, y, s)\rangle^2-\langle(\psi+\alpha)w_2(0, y, s)\rangle^2 \\
& \geqslant|\xi|^2\langle w_1(0, y, s)\rangle^2+\langle(\psi+\alpha)w_2(0, y, s)\rangle^2 \\
& \quad -C\,|\xi|^2\{\langle\chi_0 q\rangle^2+\langle[\mathscr{B}, \chi_0]V\rangle^2\},
\end{aligned}
$$

here we used the estimate $\langle\mathscr{B}_{-1}\chi_0 V\rangle\leqslant\frac{C}{|\xi|}\langle\chi_0 V\rangle$ and $\langle\chi_0 V\rangle\leqslant C\langle W\rangle$ holds when $|s|\geqslant|s_0|$.

Therefore by combining the estimates (2.18) and (2.20), we have

Proposition 2.6. *Under Assumptions* I, II *and* III, *the estimate*

$$
\begin{aligned}
(2.21)\qquad & \eta\|(\psi(y, D_y, s)+\alpha)\chi_0(D_y, \xi)V(x, y, s)\|^2 \\
& +\langle(\psi(y, D_y, s)+\alpha)\chi_0(D_y, \xi)V(0, y, s)\rangle^2 \\
& \leqslant C\{\|sp(x, y)\|^2+\|s[\chi_0, \mathscr{M}]V(x, y, s)\|^2 \\
& \quad +\langle sq(y)\rangle^2+\langle s[\chi_0, \mathscr{B}]V(0, y, s)\rangle^2\}
\end{aligned}
$$

holds for all $s=\eta+i\xi$ *such that* $\eta_3\leqslant\eta\leqslant\eta_2|\xi|$.

Next let us consider Assumptions.

Proposition 2.7. *Assumptions* II *and* III *are satisfied when the variation of the coefficients and d are sufficiently small.*

Proof. It is evident that Assumption II is satisfied when the variation of the coefficients is so small. Then let us consider Assumption III.

The equation (2.15) (as ψ is unknown) in y and ω is hyperbolic, and since $\frac{\partial\kappa_2}{\partial\omega}\neq 0$ there exists a unique global solution when $\psi(0, \omega, s)$ is given. Take d_4, d_5 as $d_2<d_4<d_5<d_3$ and define $\Delta_\xi^{(4)}$ and $\Delta_\xi^{(5)}$ as the other $\Delta_\xi^{(i)}$. Let $\psi_0(\omega, \xi)$ be a real-valued C^∞-function such that

$$
\psi_0(\omega, \xi)=\begin{cases} |\xi| & \omega\notin\Delta_\xi^{(5)} \\ 0 & \omega\in\Delta_\xi^{(4)} \end{cases}
$$

$$
\psi_0(\lambda\omega, \lambda\xi) = \lambda\psi_0(\omega, \xi) \qquad \text{for any } \lambda>0\,.
$$

We take as $\psi(y, \omega, s)$ the solution of (2.15) for the initial condition $\psi(0, \omega, s)=\psi_0(\omega, \xi)$. Remarke that $\psi(y, \omega, s)$ is determined for all $(y, \omega, s)\in R\times R\times C_+$, and $C^\infty(R\times R\times C_+)$. To show (2.16) and (2.17), we make use of the bicharacteristic curve of the equation (2.15). Consider a curve in (y, ω)-space with a parameter s defined by

$$\frac{dy(l)}{dl}=\frac{\partial\kappa_2}{\partial\omega}(y(l), \omega(l), s) \qquad y(0)=0$$

$$\frac{d\omega(l)}{dl}=-\frac{\partial\kappa_2}{\partial y}(y(l), \omega(l), s) \qquad \omega(0)=\omega$$

$$(-\infty<l<\infty).$$

Let $\omega\in\Delta_\xi^{(3)}$ and suppose that $\omega(l)\in\Delta_\xi^{(1)}$ for $l\in[-l_0, l_0]$, then (2.9) shows that $\left|\frac{dy(l)}{dl}\right|\geqslant\gamma_2$. Let $\frac{dy(l)}{dl}>\gamma_2$ then we have $y(l)-y(l')\geqslant\gamma_2(l-l')$ for $-l_0\leqslant l'<l\leqslant l_0$. On the other hand, $(\partial_y\kappa_2)(y(l), \omega(l), s)\neq 0$ only when $|y(l)|\leqslant d$. Therefore we may assume that $\frac{d\omega}{dl}(l)=0$ if $l\notin[l_1, l_2]$, where $[l_1, l_2]$ is an interval such that $l_2-l_1\leqslant\frac{2d}{\gamma_2}$. From this it follows that an estimate

$$\begin{aligned}|\omega(l)-\omega(l')| &\leqslant\frac{2d}{\gamma_2}\sup|\partial_y\kappa_2|\\ &\leqslant C\frac{2d}{\gamma_2}|\xi|\end{aligned}$$

holds for any $l, l'\in[-l_0, l_0]$. Therefore when d is so amall as

$$2d<\frac{\gamma_2}{C}\inf(d_4-d_2, d_3-d_5),$$

$\omega(0)\notin\Delta_\xi^{(4)}$ leads $\omega(l)\notin\Delta_\xi^{(2)}$ for all $l\in R$ and if $\omega(0)\in\Delta_\xi^{(5)}$ we get $\omega(l)\in\Delta_\xi^{(3)}$ for all $l\in R$. Thus (2.16) and (2.17) follow immediately from the above fact with the aid of

$$\psi(y(l), \omega(l), s)=\psi(\omega(0), \xi) \qquad \text{for all } l. \tag{2.22}$$

The relation (2.22) shows that $\psi(y, \omega, s)=\psi(d, \omega, s)$ for all $y>d$ or $\psi(y, \omega, s)=\psi(-d, \omega, s)$ for all $y<-d$. Evidently $\psi(y, \lambda\omega, \lambda s)=\lambda\psi(y, \omega, s)$ for $\lambda>0$. Then by taking account of (2.16) and (2.17), $\psi(y, \omega, s)\in S_C^1$ is derived. Q.E.D.

Next consider a neighborhood of (ω_1, s_1) such that $\omega_1^2+|s_1|^2=1$ and $L_0(0, \kappa, \omega_1, s_1)=0$ has a purely imaginary double root, which occures when $(i\omega+\varphi(0)s)^2=s^2$. Assumption I leads that $s_1=i\xi_1$. Remark that

$$\Gamma(0, \omega_1, s_1)=ib\omega_1\neq 0. \tag{2.23}$$

We construct 2×2 matrix-valued C^∞-function $\mathscr{D}(y, \omega', s')$, defined for

$(y, \omega', s')\in R\times\{(\omega, \eta+i\xi);\ \omega^2+\xi^2=1,\ 0\leqslant\eta\leqslant\eta_2$ and $|\omega-\omega_1|+|\xi-\xi_1|\leqslant d_6\}$ $=R\times U$ where d_6 is some positive constant, with the following properties:

(i) $\mathcal{D}(y, \omega', s')$ *is symmetric*

(ii) $2\,\mathrm{Re}\,\mathcal{D}(y, \omega', s')\mathcal{M}_0(y, \omega', s')\geqslant\eta'=\mathrm{Re}\,s'$

(iii) $\mathcal{B}(y, \omega', s')V=q$ *implies that* $\mathcal{D}(y, \omega', s')V\cdot V\geqslant|V|^2-C|q|^2$.

This method is completely due to Kreiss [9]. When $\mathcal{D}(y, \omega', s')$ with the above properties is constructed we can prove the following

Proposition 2.8. *There exist positive constants* η_4 *and* C *such that*

$$\begin{aligned}(2.24)\qquad &\eta\|\chi_1(D_y, \xi)V(x, y, s)\|^2+\langle\chi_1(D_y, \xi)V(0, y, s)\rangle^2\\ &\leqslant C\{\|p\|^2+\|[\chi_1, \mathcal{M}]V\|^2+\langle q\rangle^2+\langle[\chi_1, \mathcal{B}]V\rangle^2\}\end{aligned}$$

holds for all $s=\eta+i\xi$ *such that* $\eta_4\leqslant\eta\leqslant\eta_2|\xi|$, *where* $\chi_1(\omega, \xi)$ *is a* C^∞*-function such that* $\chi_1(\lambda\omega, \lambda\xi)=\chi_1(\omega, \xi)$ *for* $\lambda>0$ *and*

$$\chi_1(\omega, \xi)=\begin{cases}1 & \textit{when } |\omega'-\omega_1|+|\xi'-\xi_1|\leqslant d_6-\varepsilon\\ 0 & \textit{when } |\omega'-\omega_1|+|\xi'-\xi_1|\geqslant d_6\end{cases}$$

where $\varepsilon>0$.

Proof. Take $\mathcal{D}_1(y, \omega, s)\in S^0_{C+}$ such that $\mathcal{D}_1(y, \omega, s)=\mathcal{D}(y, \omega', s')$ when $(\omega', s')\in U$, and symmetric in $R^2\times \boldsymbol{C}_+$. Operate $\chi_1(D_y, s)$ to the both sides of (i) of (2.7) and we have

$$\begin{aligned}\frac{\partial}{\partial x}\chi_1V&=\mathcal{M}\chi_1V+[\chi_1, \mathcal{M}]V+\chi_1P\\ &=\mathcal{M}\chi_1V+P_1\,.\end{aligned}$$

Put $V_1(x, y, s)=\chi_1(D_y, s)V(x, y, s)$, then

$$\begin{aligned}&2\,\mathrm{Re}\,(\mathcal{D}_1(y, D_y, s)V_1, -P_1)\\ &=2\,\mathrm{Re}\Big(\mathcal{D}_1(y, D_y, s)V_1, -\frac{\partial}{\partial x}V_1\Big)+2\,\mathrm{Re}\,(\mathcal{D}_1V_1, \mathcal{M}V)\\ &=\mathrm{I}+\mathrm{II}.\\ \mathrm{I}&=2\,\mathrm{Re}\,\langle\mathcal{D}_1(y, D_y, s)V_1(0, y, s), V_1(0, y, s)\rangle\\ &\quad+2\,\mathrm{Re}\Big((\mathcal{D}_1(y, D_y, s)-\mathcal{D}_1(y, D_y, s)^*)\frac{\partial V_1}{\partial x}, V\Big)\\ &\geqslant 2\,\mathrm{Re}\,\langle\mathcal{D}_1(y, D_y, s)V_1(0, y, s), V_1(0, y, s)\rangle\\ &\quad-C\|V_1\|\Big(\|V_1\|+\frac{1}{|s|}\|P_1\|\Big),\end{aligned}$$

here we used $\mathcal{D}_1(y, \omega, s)-\mathcal{D}_1^*(y, \omega, s)\in S^{-1}_{C+}$ since $\mathcal{D}_1(y, \omega, s)$ is symmetric.

$$\mathrm{II} = ((\mathcal{D}_1^* \mathcal{M} + \mathcal{M}^* \mathcal{D}_1) V_1, V_1)$$

from Lemma 2.2

$$= ((2 \operatorname{Re} \mathcal{D}_1 \mathcal{M}_0)(y, D_y, s) V_1, V_1) + (\mathcal{R}_0 V_1, V_1)$$

where $\mathcal{R}_0(y, \omega, s) \in S^0_{C+}$, then

$$\geqslant \eta \|V_1\|^2 - C\|V_1\|^2.$$

Thus we get

$$(2.25) \qquad \begin{aligned} &\eta \|V_1\|^2 + 2 \operatorname{Re} \langle \mathcal{D}_1(y, D_y, s) V_1(0, y, s), V_1(0, y, s) \rangle \\ &\quad - C\|V_1\|^2 - C\|P_1\|^2 \\ &\leqslant C(\|V_1\|^2 + \|P_1\|^2) . \end{aligned}$$

On the other hand $\mathcal{B} V_1 = \chi_1 q + [\mathcal{B}, \chi_1] V$, and from the property (iii) we have

$$\begin{aligned} &2 \operatorname{Re} \langle \mathcal{D}_1(y, D_y, s) V_1(0, y, s), V_1(0, y, s) \rangle \\ &\geqslant \langle V_1(0, y, s) \rangle^2 - \frac{C}{|s|} \langle V_1(0, y, s) \rangle^2 \\ &\quad - C\{\langle \chi_1 q \rangle^2 + \langle [\mathcal{B}, \chi_1] V \rangle^2\} . \end{aligned}$$

Inserting this estimate into (2.25) we have (2.24). Q.E.D.

Now let us construct $\mathcal{D}(y, \omega', s')$ with the properties (i)$\sim$(iii). Put

$$\kappa_0 = \kappa_\pm(0, \omega_1, i\xi_1)$$

$$\mathcal{U}_0 = \begin{bmatrix} 2i\kappa_0 + 1 & 2 \\ i\kappa_0 & 1 \end{bmatrix}$$

and

$$\tilde{\mathcal{M}}(y, \omega, s) = \mathcal{U}_0 \mathcal{M}(y, \omega, s) \mathcal{U}_0^{-1} .$$

Then

$$\tilde{\mathcal{M}}_0(0, \omega_1, i\xi_1) = \mathcal{U}_0 \mathcal{M}_0(0, \omega_1, i\xi_1) \mathcal{U}_0^{-1} = \begin{bmatrix} \kappa_0 & i \\ 0 & \kappa_0 \end{bmatrix} .$$

Let

$$\lim_{\eta \to +0} \frac{\partial \tilde{\mathcal{M}}_0}{\partial \eta}(y, \omega, \eta + i\xi) = \mathcal{H}(y, \omega, \xi) = [h_{ij}(y, \omega, \xi)]_{i,j=1,2} .$$

It is easily seen that

$$a_{11}(0) h_{21}(0, \omega_1, \xi_1) = \left(\frac{\partial L_0}{\partial s}\right)(0, \kappa_0, i\omega_1, i\xi_1) .$$

Remark that the regularly hyperbolicity of L with respect to t assures

$$|L_0(y, \kappa, i\omega, s)| \geqslant c_0 \operatorname{Re} s$$

for all κ purely imaginary. Then from

$$\begin{aligned} &L_0(0, \kappa_0, i\omega_1, \eta+i\xi_1) \\ &= L_0(0, \kappa_0, i\omega_1, i\xi_1)+\eta\frac{\partial L_0}{\partial s}(0, \kappa_0, i\omega_1, i\xi_1)+0(\eta^2) \end{aligned}$$

it follows

$$\left|\frac{\partial L_0}{\partial s}(0, \kappa_0, i\omega_1, i\xi_1)\right| \geqslant c_0 ,$$

which shows

$$|h_{21}(0, \omega_1, \xi_1)| \geqslant 2c .$$

Now we pose

ASSUMPTION IV. $|h_{21}(y, \omega, \xi)| \geqslant c$ for all $y \in R$, $(\omega, \xi) \in \{(\omega, \xi);\ |\omega-\omega_1| + |\xi-\xi_1| \leqslant d_7,\ \omega^2+\xi^2=1\}$ where c and d_7 are positive constants.

$$\begin{aligned} \tilde{\mathcal{M}}_0(y, \omega', \eta'+i\xi') &= \tilde{\mathcal{M}}_0(0, \omega_1, i\xi_1)+\tilde{\mathcal{M}}_0(y, \omega', \eta'+i\xi') \\ &\quad -\tilde{\mathcal{M}}_0(y, \omega', i\xi')+\tilde{\mathcal{M}}_0(y, \omega', i\xi')-\tilde{\mathcal{M}}_0(0, \omega_1, i\xi_1) \\ &= \begin{bmatrix} \kappa_0 & i \\ 0 & \kappa_0 \end{bmatrix}+\eta'\mathcal{H}(y, \omega', \xi')+0(\eta'^2)+i\mathcal{E}(y, \omega', \xi') \end{aligned} \tag{2.26}$$

where

$$\mathcal{E}(y, \omega', \xi') = \frac{1}{i}\{\tilde{\mathcal{M}}_0(y, \omega', \xi')-\tilde{\mathcal{M}}_0(0, \omega_1, i\xi_1)\} .$$

Notice that all the entries of $\mathcal{E}(y, \omega', \xi')$ are real-valued and

$$|\mathcal{E}(y, \omega', \xi')| \leqslant C\{|\omega'-\omega_1|+|\xi'-\xi_1|+\text{variation of coefficients of } L_0\} . \tag{2.27}$$

Lemma 2.9. *The matirx*

$$\left(\begin{bmatrix} 0 & k_1 \\ k_1 & k_2 \end{bmatrix}+\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}\right) = (\mathcal{D}_0+\mathcal{A})(C_0+\mathcal{E})$$

is symmetric if all entries are real, $1+e_2 \neq 0$ *and*

$$a = \frac{k_1e_1+k_2e_3-k_1e_4}{1+e_2} .$$

Of course $a=0(\sum_{i=1}^{4}|e_i|)$ *when* $\sum_{i=1}^{4}|e_i| \to 0$.

Let us put

$$\tilde{\mathscr{D}}(y, \omega', s') = \begin{bmatrix} 0 & k_1 \\ k_1 & k_2 \end{bmatrix} + \begin{bmatrix} a(y, \omega', s') & 0 \\ 0 & 0 \end{bmatrix} - i\eta' \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix}$$
$$= \mathscr{D}_0 + \mathscr{A}(y, \omega', s') - i\eta' \mathscr{F}$$

where $a(y, \omega', s')$ is determined by Lemma 2.9 from $\mathscr{E}(y, \omega', s')$ and $k_i (i=1, 2)$ and f are constants which will be determined later.

$$2 \operatorname{Re} \tilde{\mathscr{D}}(y, \omega', s') \tilde{\mathscr{M}}_0(y, \omega', s')$$
$$= \eta' \{2 \operatorname{Re} (\mathscr{F} C_0 + \mathscr{D}_0 \mathscr{H}) + 0(\eta' + |\mathscr{E}|)\}.$$

Since

$$\mathscr{F} C_0 + \mathscr{D}_0 \mathscr{H} = \begin{bmatrix} k_1 h_{21} & k_1 h_{12} \\ k_1 h_{11} + k_2 h_{21} & k_1 h_{12} + k_2 h_{22} + f \end{bmatrix}$$

we can make $\operatorname{Re} (\mathscr{F} C_0 + \mathscr{D}_0 \mathscr{H}) \geqslant 1$ by choosing as $k_1 h_{21} \geqslant 2$ (from Assumption IV) and f sufficiently large. Then when $|\mathscr{E}|$ is not so large we get

$$2 \operatorname{Re} \tilde{\mathscr{D}}(y, \omega', s') \tilde{\mathscr{M}}_0(y, \omega', s') \geqslant \eta'$$

for $0 \leqslant \eta' \leqslant \eta_5$. $|\mathscr{E}|$ becomes small according to d_6 and the variation of coefficients. Put

$$\mathscr{D}(y, \omega,' s') = \mathscr{U}_0^* \tilde{\mathscr{D}}(y, \omega', s') \mathscr{U}_0$$

then it satisfies (i) and (ii).

$$(2.28) \qquad (\mathscr{D}(y, \omega', s')V, V) = (\tilde{\mathscr{D}}(y, \omega', s') \tilde{V}, \tilde{V})$$
$$\geqslant 2 \operatorname{Re} (k_1 \tilde{v}_2, \tilde{v}_1) + k_2 (\tilde{v}_2, \tilde{v}_2) - 0(|\mathscr{E}| + \eta') |\tilde{V}|^2,$$

where $\tilde{V} = (\tilde{v}_1, \tilde{v}_2) = \mathscr{U}_0 V$,

$$\geqslant -2k_1 |\tilde{v}_1|^2 + \frac{k_2}{2} |\tilde{v}_2|^2.$$
$$\mathscr{B}(y, \omega', s') = b_1(y, \omega', s') \tilde{v}_1 + b_2(y, \omega', s') \tilde{v}_2$$

then $b_1(0, \omega_1', s_1') = \Gamma(0, \omega_1', s_1')$. From (2.23) when d_6 and the variation of the coefficients are not so large $|\Gamma(y, \omega', s')| \geqslant c$, this implies that

$$|\tilde{v}_1|^2 \leqslant c |b_2 \tilde{v}_2 - q|^2$$
$$\leqslant C |\tilde{v}_2|^2 + |q|_2.$$

Inserting this into (2.28) we have

$$(\mathscr{D}(y, \omega', s)V, V) \geqslant |\tilde{V}|^2 - C |q|^2$$
$$\geqslant C |V|^2 - C |q|^2$$

by choosing k_2 large. Thus (iii) is proved.

Now, remark that Assumption IV is satisfied when the variation of the coefficients and d_7 are sufficiently small.

The number of the points (ω_0, ξ_0) considered in Proposition 2.6 are two and the number of the points (ω_1, ξ_1) considered in Proposition 2.8 are four. Let us denote by $\chi(\omega, \xi)$ the sum of $\chi_0(\omega, \xi)$ and $\chi_1(\omega, \xi)$ of all the points (ω_0, ξ_0) and (ω_1, ξ_1). Then it is easy to get the estimate of the type of (2.24) for $\chi_2(D_y, \xi)V(x, y, s)=(1-\chi(D_y, \xi))V(x, y, s)$. Indeed, we can use the method of Proposition 2.6 by taking

$$\mathcal{N}(y, \omega, s) = \begin{bmatrix} \kappa_-(y, \omega', s') & -i \\ \kappa_+(y, \omega', s') & -i \end{bmatrix}$$

and

$$\mathcal{D}(y, \omega, s) = \begin{bmatrix} \beta & 0 \\ 0 & -1 \end{bmatrix}.$$

Remark that it holds that

$$|s|^2 \sum_{i=0}^{2} \|[\mathcal{M}, \chi_i]V\|^2$$
$$\leqslant C\{\|(\psi(D_y, \xi)+\alpha)\chi_0 V\|^2 + |s|^2 \sum_{i=1}^{2} \|\chi_i V\|^2 + \|V\|\}^2$$

and

$$|s|^2 \sum_{i=0}^{2} \langle [\mathcal{B}, \chi_i]V\rangle^2$$
$$\leqslant C\{\langle(\psi(D_y, \xi)+\alpha)\chi_0\rangle^2 + |s|^2 \sum_{i=1}^{2} \langle \chi_i V\rangle^2 + \langle V\rangle^2\}.$$

Then we have

Theorem 2.10. *When the variation of the coefficients and d are sufficiently small, there exist positive constants C_0 and η_0 such that for any solution $v(x, y)$ of (2.5) in $H^2(R^2_+)$, the estimate*

$$\eta\{\|v(x, y)\|_1^2 + \|sv(x, y)\|^2\} + \langle v(0, y)\rangle_1^2 + \langle sv(0, y)\rangle^2 + \left\langle \frac{\partial v}{\partial x}(0, y)\right\rangle^2 \tag{2.29}$$
$$\leqslant C_0\{\|sp(x, y)\|^2 + \langle sq(y)\rangle^2\}$$

holds when Re $s = \eta \geqslant \eta_0$.

Proof. (2.29) is derived for s such that $\eta_0 \leqslant \eta \leqslant c_0|\xi|$ by combining Propositions 2.6 and 2.8 and the above remarks. On the other hand when $\eta \geqslant c_0|\xi|$, (2.29) is already known in the general theory of boundary value problems for elliptic equations. Q.E.D.

Corollary of Theorem 2.10.

$$(2.30) \qquad \eta\sum_{i=0}^{2}||s^i v(x, y)||_{2-i}^2 \leqslant c_0\{||s^2 p(x, y)||^2 + \langle s^2 q(y)\rangle^2 + \langle q(y)\rangle_{1/2}^2\}$$

holds when Re $s \geqslant \eta_0$.

Proof. Recall an apriori estimate concerning an elliptic boundary value problem

$$\begin{cases} a_2(y, D_x, D_y)u(x, y) = (a_{11}D_x^2 + 2a_{12}D_xD_y + a_{22}D_y^2)u(x, y) \\ \qquad\qquad = f(x, y) \qquad \text{in } R_+^2 \\ \left(a_{11}\dfrac{\partial}{\partial x} + (a_{12}+b(y))\dfrac{\partial}{\partial y}\right)u(x, y)\,|_{x=0} = g(y), \end{cases}$$

namely for some positive constant C

$$(2.31) \qquad ||u(x, y)||_2^2 \leqslant C(||f||^2 + \langle g\rangle_{1/2}^2 + ||u||^2).$$

Apply it for

$$\begin{cases} a_2 v = p - s a_1 v + s^2 v \\ \left(a_{11}\dfrac{\partial}{\partial x} + (a_{12}+b)\dfrac{\partial}{\partial y}\right)v = g + s\varphi(a_{12}+b)v \end{cases}$$

and we have

$$||v||_2^2 \leqslant C\{||p||^2 + ||sv||_1^2 + ||s^2 v||^2 + \langle sv\rangle_{1/2}^2 + \langle g\rangle_{1/2}^2\}.$$

From (2.29) we have

$$\eta\{||sv||_1^2 + \langle sv\rangle_1^2 + ||s^2 v||^2\} \leqslant C\{||s^2 p||^2 + \langle s^2 g\rangle\}^2,$$

insert this into the above, (2.30) follows. Q.E.D.

Let us denote by $\mathcal{L}_\varphi(s)$ the operator from $H^2(R_+^2)$ into $L^2(R_+^2)\times H^{1/2}(R)$ defined by

$$\mathcal{L}_\varphi(s)u = \{L_\varphi(s)u, B_\varphi(s)u\,|_{x=0}\}.$$

Theorem 2.11. *$\mathcal{L}_\varphi(s)$ is a bijective mapping when* Re $s \geqslant \eta_0$ *and $\mathcal{L}_\varphi(s)^{-1}\{f(x, y, s), g(y, s)\}$ is $H^2(R_+^2)$ valued homolorphic function when f and g are vector valued holomorphic function in $L^2(R_+^2)$ and $H^{1/2}(R)$ respectively.*

Proof. $L_\varphi^*(s)$ the formal adjoint of $L_\varphi(s)$ has the principal part

$$L_{\varphi 0}(y, -\partial_x, -\partial_y, \bar{s}).$$

Then for a boundary operator $\tilde{B}$ with the principal part

$$B_0(y, -\partial_x, -\partial_y, \bar{s})$$

we have for all $u, v \in H^2(R_+^2)$

$$(L_\varphi(s)u,\ v)-(u,\ L_\varphi^*(s)v)=\int(B(s)u\bar{v}+u\overline{\tilde{B}(s)v})dy\,.$$

Since the apriori estimate for $L_\varphi^*(s)$, $\tilde{B}(s)$ of the type (2.30) holds for $\mathrm{Re}\, s\geqslant\eta_0^*$, we see that $\mathcal{L}_\varphi(s)$ is a bijective mapping. The last part of Theorem is easily proved with the aid of (2.30).

2.3. Energy estimate

Hereafter, in this section, we assume that the operators (2.1) and (2.2) satisfy Assumptions I$\sim$IV posed in the previous paragraph.

Proposition 2.12. *For a solution* $u(x,y,t)\in\mathcal{E}_t^0(H^3(R_+^2))\cap\mathcal{E}_t^1(H^2((R_+^2))\cap\mathcal{E}_t^2(H^1(R_+^2))\cap\mathcal{E}_t^3(L^2(R_+^2))$ *of* (2.4), *the energy estimate holds*:

$$\begin{aligned}(2.32)\qquad &\int_0^t\Big\{\|u(x,y,t)\|_1^2+\|u'(x,t,y)\|^2+\langle u(0,y,t)\rangle_1^2\\&\quad+\Big\langle\frac{\partial u}{\partial x}(0,y,t)\Big\rangle^2+\langle u'(0,y,t)\rangle^2\Big\}dt\\&\leqslant C_T\Big\{\|u(x,y,0)\|_3^2+\|u'(x,y,0)\|_2^2+\|f(x,y,0)\|_1^2\\&\quad+\int_0^t(\|f'(x,y,t)\|^2+\langle g'(y,t)\rangle^2)dt\Big\}\qquad \text{for } t\in[0,T]\end{aligned}$$

where C_T depends on L_φ, B_φ and T and is independent of u.

Proof. At first assume that $u(x,y,0)=u'(x,y,0)=0$, $f(x,y,0)=0$ and $g(y,0)=0$. Take a function $\chi(t)\in C^\infty(R)$ such that

$$\chi(t)=\begin{cases}1 & t<T\\ 0 & t>T+\delta\ \ (\delta>0).\end{cases}$$

Then

$$(2.33)\qquad L_\varphi\chi(t)u(x,y,t)=\chi(t)f(x,y,t)-[L_\varphi,\ \chi]u$$

$$(2.34)\qquad B_\varphi\chi(t)u(x,y,t)=\chi(t)g(y,t)-[B_\varphi,\ \chi]u\,.$$

Put $\chi(t)u(x,y,t)=v(x,y,t)$ and the right-hand side of (2.33) and (2.34) as f_0 and g_0 respectively. Evidently v and f_0 are in $L^2(R_+,\ L^2(R_+^2))$ and g_0 is in $L^2(R_+,\ L^2(R))$. Define $f_i(x,y,t)$ ($i=1,2$) by

$$f_1(x,y,t)=\begin{cases}\dfrac{\partial f_0}{\partial t}(x,y,t) & \text{for } t\leqslant t_0\\[2mm] 0 & \text{for } t>t_0\end{cases}$$

$$f_2(x,y,t)=\frac{\partial f_0}{\partial t}(x,y,t)-f_1(x,t,y),$$

and g_i $(i=1, 2)$ by the same way. Laplace transformation with respect to t gives

$$L_\varphi(y, \partial_x, \partial_y, s)\tilde{v}(x, y, s) = \tilde{f}_0(x, y, s)$$
$$B_\varphi(y, \partial_x, \partial_y, s)\tilde{v}(x, y, s)|_{x=0} = \tilde{g}_0(y, s)$$

where $\tilde{v}$, $\tilde{f}_0$ and $\tilde{g}_0$ are the Laplace image of v, f_0 and g_0 respectively. Then for all $\operatorname{Re} s \geqslant \eta_0$

$$\tilde{v}(x, y, s) = \mathcal{L}_\varphi(s)^{-1}(\tilde{f}_0(x, y, s), \tilde{g}_0(y, s)),$$

and

$$\tilde{f}_0(x, y, s) = \frac{1}{s}(\tilde{f}_1(x, y, s)+\tilde{f}_2(x, y, s))$$

$$\tilde{g}_0(y, s) = \frac{1}{s}(\tilde{g}_1(y, s)+\tilde{g}_2(y, s)).$$

Put

$$\tilde{v}_i(x, y, s) = \frac{1}{s}\mathcal{L}_\varphi(s)^{-1}(\tilde{f}_i(x, y, s), \tilde{g}_i(y, s))$$

and they are holomorphic in $\operatorname{Re} s \geqslant \eta_0$, moreover

$$||\tilde{v}_2(x, y, s)|| \leqslant Ce^{-t_0 \operatorname{Re} s}$$

holds since $||\tilde{f}_2(x, y, s)|| \leqslant Ce^{-t_0 \operatorname{Re} s}$ and $\langle \tilde{g}_2(y, s)\rangle \leqslant Ce^{-t_0 \operatorname{Re} s}$. Therefore $v_2(x, y, t)$ the inverse image of $\tilde{v}_2(x, y, s)$ has the support in $[t_0, \infty)$, then $u(x, y, t)=v_1(x, y, t)$ for $t \in [0, t_0)$. The Parseval's equality shows

$$\begin{aligned}
&\int_0^\infty e^{-2\eta t}(||v_1(x, y, t)||_1^2+||v_1'(x, y, t)||^2)dt \\
&=\int_{-\infty}^\infty (||\tilde{v}_1(x, y, \eta+i\xi)||_1^2+||s\tilde{v}_1(x, y, \eta+i\xi)||^2)d\xi \\
&\leqslant \frac{C}{\eta}\int_{-\infty}^\infty (||\tilde{f}_1(x, y, \eta+\xi i)||^2+\langle \tilde{g}_1(y, \eta+i\xi)\rangle^2)d\xi \\
&=\frac{C}{\eta}\int_0^{t_0} e^{-2\eta t}(||(f'(x, y, t)||^2+\langle g'(y, t)\rangle^2)dt,
\end{aligned}$$

thus (2.32) holds by taking $C_T=Ce^{2\eta T}$.

Next let us prove the general case. Take $w(x, y, t) \in \mathcal{E}_t^0(H^3(R_+^2)) \cap \mathcal{E}_t^1(H^2(R_+^2)) \cap \mathcal{E}_t^2(H^1(R_+^2)) \cap \mathcal{E}_t^3(L^2(R_+^2))$ as

$$\begin{cases}
L_\varphi w = f(x, y, 0) & \text{in } R_+^2 \times (0, T) \\
w(x, y, 0) = u(x, y, 0) & \text{in } R_+^2 \\
\dfrac{\partial w}{\partial t}(x, y, 0) = u'(x, y, 0) & \text{in } R_+^2
\end{cases}$$

then $v(x, y, t)=u(x, y, t)-w(x, y, t)$ satisfies the condition assumed in the first. Applying the just obtained result we have

$$\int_0^t (||u(x, y, t)-w(x, y, t)||_1^2+||u'(x, y, t)-w'(x, y, t)||^2)dt$$
$$\leqslant C_T\int_0^t (||f'(x, y, t)||^2+\langle g'(y, t)+(Bw)'(0, y, t)\rangle^2)dt\,.$$

Remark that

$$||w(x, y, t)||_1^2+||w'(x, y, t)||^2$$
$$\leqslant C_T(||u(x, y, 0)||_1^2+||u'(x, y, 0)||^2+||f(x, y, 0)||^2)\,,$$

and

$$\langle Bw'(y, t)\rangle\leqslant C_T\{||u(x, y, 0)||_3^2+||u'(x, y, 0)||_2^2+||f(x, y, 0)||_1^2\}\,,$$

which inserting into the above, (2.32) follows. Q.E.D.

Define $|||u(x, y, t)|||_{k,R_+^2}$ and $\langle\!\langle g(y, t)\rangle\!\rangle_{k,R}$ $(k=1, 1, 2, \cdots)$ by

$$|||u(x, y, t)|||_{k,R_+^2}^2=\sum_{i=0}^k\left\|\left(\frac{\partial}{\partial t}\right)^i u(x, y, t)\right\|_{k-i,L^2(R_+^2)}^2$$
$$\langle\!\langle g(y, t)\rangle\!\rangle_{k,R}^2=\sum_{i=0}^k\left\langle\left(\frac{\partial}{\partial t}\right)^i (y, t)\right\rangle_{k-i,L^2(R)}^2$$

respectively.

Theorem 2.13. *For a solution* $u(x, y, t)\in\mathcal{E}_t^0(H^{m+2}(R_+^2))\cap\mathcal{E}_t^1(H^{m+1}(R_+^2))\cap\cdots\cap\mathcal{E}_t^{m+2}((L^2(R_+^2))$ *the energy estimate*

$$\text{(2.35)}\qquad \int_0^t |||u(x, y, t)|||_m^2 dt$$
$$\leqslant C_{T,m}\Big\{||u(x, y, 0)||_{m+2}^2+||u'(x, y, 0)||_{m+1}^2+|||f(0)|||_{m,R_+^2}^2$$
$$+\int_0^t (|||f'(x, y, t)|||_{m,R_+^2}^2+\langle\!\langle g'(y, t)\rangle\!\rangle_{m,R}^2)dt\Big\}$$
$$\textit{for all } t\in[0, T]$$

holds for $m=1, 2, 3, \cdots$.

Proof. The $(m-1)$-times differentiation in t the both sides of (2.4) gives

$$L_\varphi(u^{(m-1)}(x, y, t))=f^{(m-1)}(x, y, t)$$
$$B_\varphi(u^{(m-1)}(x, y, t))|_{x=0}=g^{(m-1)}(y, t)\,.$$

It follows from Proposition 2.12 that

$$\int_0^t |||u^{(m-1)}(x, y, t)|||_1^2 dt$$
$$\leqslant C_T\Big\{||u^{(m-1)}(x, y, 0)||_3^2+||u^{(m)}(x, y, 0)||_2^2+||f^{(m-1)}(x, y, 0)||_1^2$$
$$+\int_0^t (||f^{(m)}(x, y, t)||^2+\langle g^{(m)}(y, t)\rangle^2)dt\Big\}.$$

With the aid of (2.31) we get an estimate

$$|||u(x, y, t)|||_m^2\leqslant C_m\{|||u^{(m-1)}(x, y, t)|||_1^2+|||f(x, y, t)|||_{m-2}^2 + \langle\!\langle g(y, t)\rangle\!\rangle_{m-1}^2\}, \tag{2.36}$$

from $L_\varphi u=f$, $B_\varphi u|_{x=0}=g$. Then (2.35) is derived by inserting the above estimate and by using

$$||u^{(m-1)}(x, y, 0)||_3^2+||u^{(m)}(x, y, 0)||_2^2$$
$$\leqslant C(||u(x, y, 0)||_{m+2}^2+||u'(x, y, 0)||_{m+1}^2+|||f(x, y, 0)|||_m^2).$$

Q.E.D.

2.4. Existence and regularity of the solution

Theorem 2.14. *For given data u_0, u_1, f and g, if they satisfy the compatibility condition of order $m+2$, there exists a solution $u(x, y, t)$ of (2.3) uniquely in $H^{m+2}(R_+^2\times(0, T))$.*

Proof. Consider at first the case of $m=0$. Assume that $u_0(x, y)=u_1(x, y)=0$, $f(x, y, 0)=f'(x, y, 0)=0$ and $g(y, 0)=g'(y, 0)=0$. Put

$$\tilde{u}(x, y, s)=\frac{1}{s^2}\mathcal{L}_\varphi^{-1}(s)(\tilde{f}''(x, y, s), \tilde{g}''(y, s)).$$

Then Corollary of Theorem 2.10 shows

$$||\tilde{u}(x, y, s)||_2^2+||s\tilde{u}(x, y, s)||_1^2+||s^2\tilde{u}(x, y, s)||^2$$
$$\leqslant\frac{C}{\eta}(||\tilde{f}''(x, y, s)||^2+\langle\tilde{g}''(y, s)\rangle^2).$$

Of couse $\tilde{u}(x, y, s)$ is holomorphic in $\operatorname{Re} s\geqslant\eta_0$. The inverse Laplace image $u(x, y, t)$ of $\tilde{u}(x, y, s)$ exists as $L^2(R_+^2)$-valued distribution and from the above estimate $e^{-\eta t}u(x, y, t)\in H^2(R_+^2\times R)$ and $u(x, y, t)=0$ for $t<0$. Evidently

$$L_\varphi u=f \quad \text{in } R_+^2\times(0, \infty)$$
$$B_\varphi u|_{x=0}=g(y, t) \quad \text{in } R\times(0, \infty).$$

This means that $u(x, y, t)\in H^2(R_+^2\times(0, T))$ is the desired solution of (2.4). Next let us consider the case of non zero initial data. Take a function

$$v(x, y, t) = u_0(x, y) + tu_1(x, y) + \frac{t^2}{2} u_2(x, y),$$

then

$$(f - L_\varphi v)(x, y, 0) = \frac{\partial}{\partial t}(f - L_\varphi v)(x, y, 0) = 0$$

$$(g - B_\varphi v)(y, 0) = \frac{\partial}{\partial t}(g - L_\varphi v)(y, 0) = 0$$

follow from the compatibility condition of order 2. The just obtained result shows the existence of $w(x, y, t) \in H^2(R^2_+ \times (0, T))$ satisfying

$$\begin{aligned} L_\varphi[w] &= f - L_\varphi[v] \\ B_\varphi[w] &= g - B_\varphi[v] \\ w(x, y, 0) &= w'(x, y, 0) = 0 . \end{aligned}$$

Then $u(x, y, t) = w(x, y, t) + v(x, y, t)$ is the required solution.

Now we prove Theorem for $m \geqslant 1$. Set

$$u(x, y, t) = u_0(x, y) + tu_1(x, y) + \cdots + \frac{t^{m-1}}{(m-1)!} u_{m-1}(x, y)$$
$$+ \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} w_m(x, y, \tau) d\tau$$

where $w_m(x, y, t) \in H^2(R^2_+ \times (0, T))$ is the solution of

$$\begin{aligned} L_\varphi[w_m] &= f^{(m)}(x, y, t) \\ B_\varphi[w_m] &= g^{(m)}(y, t) \\ w_m(x, y, 0) &= u_m(x, y) \\ w'_m(x, y, 0) &= u_{m+1}(x, y), \end{aligned}$$

whose existence is assured by the result for $m=0$ since $u_m(x, y)$, $u_{m+1}(x, y)$, $f^{(m)}(x, y, t)$ and $g^{(m)}(x, t)$ satisfy the compatibility condition of order 2. It is easy to see that $u(x, y, t)$ is a solution of (2.3) for u_0, u_1, f and g. Now we get $u(x, y, t) \in H^{m+2}(R^2_+ \times (0, T))$ from $u^{(m)}(x, y, t) = w_m(x, y, t) \in H^2(R^2_+ \times (0, T))$ with the aid of (2.36). Q.E.D.

2.5. Finiteness of the propagation speed

Lemma 2.15. *Let* $u(x, y, t) \in \mathcal{E}^0_t(H^2(R^2_+)) \cap \mathcal{E}^2_t(H^1(R^2_+)) \cap \mathcal{E}^2_t(L^2(R^2_+))$ *be a solution of* (2.4). *If*

$$\begin{aligned} &f(x, y, t) = 0 \qquad \text{for } x + v_\varphi t \leqslant \delta,\ x, t \geqslant 0 \\ &g(y, t) = 0 \qquad \text{for } v_\varphi t \leqslant \delta,\ t \geqslant 0 \\ &u_0(x, y) = u_1(x, y) = 0 \qquad \text{for } 0 \leqslant x \leqslant \delta, \end{aligned}$$

then $u(x, y, t)=0$ *for* $x+v_{\varphi}t\leqslant\delta$, *where* v_{φ} *denotes the maximam propagation speed of the Cauchy problem for the hyperbolic operator* L_{φ}.

Proof. Let $v(x, y, t)$ be the solution of the Cauchy problem $L_{\varphi}u=f$ for the initial data $\{u_0(x, y), u_1(x, y)\}$, here $u_i(x, y)$ is extend to $x<0$ by 0. The finiteness of propagation speed shows that $v(x, y, t)=0$ when $x+v_{\varphi}t\leqslant\delta$, $t\geqslant 0$, therefore $v(x, y, t)$ satisfies (2.4) when $v_{\varphi}t\leqslant\delta$. By applying (2.32) for $u(x, u, t)-v(x, y, t)$ we have $u(x, y, t)=v(x, y, t)$ for $0\leqslant v_{\varphi}t\leqslant\delta$, this proves Lemma. Q.E.D.

Lemma 2.16. *Let* $u(x, y, t)\in\mathcal{E}_t^0(H^2(R_+^2))\cap\mathcal{E}_t^1(H^1(R_+^2))\cap\mathcal{E}_t^2(L^2(R_+^2))$ *be a solution of* (2.4) *and* U *be a neighborhood of* $t=y=0$ *in* $\{(y, t)\colon y\in R, t>0\}$. *If*

$$f(x, y, t)=0 \qquad \text{for } (x, y, t)\in R_+\times U$$
$$g(y, t)=0 \qquad \text{for } (y, t)\in U$$
$$u_0(x, y)=u_1(x, y)=0 \qquad \text{for } (x, y)\in R_+\times\{U\cap(t=0)\},$$

then there exists a neighborhood U_0 *of* $y=t=0$ *in* $\{(y, t);\ y\in R, t>0\}$ *such that* $u(x, y, t)=0$ *in* $R_+\times U_0$.

Proof. Consider the Folmgren transformation in (y, t) space

$$t'=t+y^2$$
$$y'=y\,.$$

Define $\tilde{u}(x, y', t')$ by $\tilde{u}(x, y, t+y^2)=u(x, y, t)$ for $t'-y'^2>0$ and equals zero for $t'-y'^2\leqslant 0$. $\tilde{f}$ and $\tilde{g}$ are defined by the same way. From the condition posed on u_0, u_1 we have $\tilde{u}(x, y', t')\in\mathcal{E}_t^0(H^2(R_+^2))\cap\mathcal{E}_t^1(H^1(R_+^2))\cap\mathcal{E}_t^2(L^2(R_+^2))$ when $t'\leqslant\delta_0$ for some constant $\delta_0>0$ and $\tilde{u}(x, y', 0)=\tilde{u}'(x, y', 0)=0$. Evidently $\tilde{f}(x, y', t')=0$ and $\tilde{g}(y', t')=0$ when $t'\leqslant\delta_0$. And it holds that

$$\begin{cases} \tilde{L}_{\varphi}\tilde{u}=\tilde{f} & \text{in } R_+^2\times(0, \delta_0) \\ \tilde{B}_{\varphi}\tilde{u}\,|_{x=0}=\tilde{g} & \text{in } R\times(0, \delta_0), \end{cases} \tag{2.37}$$

where

$$\tilde{L}_{\varphi 0}=(1-a_{22}\varphi^2)\frac{\partial^2}{\partial t'^2}-2a_{12}\varphi\frac{\partial^2}{\partial x\partial t'}-2a_{22}\varphi\Big(\frac{\partial}{\partial t'}+2y'\frac{\partial}{\partial y'}\Big)\frac{\partial}{\partial t'}$$
$$-\Big\{a_{11}\frac{\partial^2}{\partial x^2}+2a_{12}\Big(\frac{\partial}{\partial t'}+2y'\frac{\partial}{\partial y'}\Big)\frac{\partial}{\partial x}+a_{22}\Big(\frac{\partial}{\partial t'}+2y'\frac{\partial}{\partial y'}\Big)^2\Big\}$$
$$\tilde{B}_{\varphi 0}=a_{11}(y')\frac{\partial}{\partial x}+(a_{12}(y')+b(y'))\Big\{\frac{\partial}{\partial y'}+\varphi\Big(\frac{\partial}{\partial t'}+2y'\frac{\partial}{\partial y'}\Big)\Big\}^{5)}.$$

5) $\tilde{L}_{\varphi 0}$ and $\tilde{B}_{\varphi 0}$ are not of the form (2.1) and (2.1), but Theorem 2.10 (therefore Theorem 2.13) holds for any operators which are of the form (2.1) and (2.1) at $y=0$ and whose variation of the coefficients and d are so small.

By changing the values of coefficients of $\tilde{L}_\varphi$ and $\tilde{B}_\varphi$ in $\{y'; |y'|\geqslant\delta_1\}$ and by taking δ_1 sufficiently small, we can assume $\tilde{L}_\varphi$ and $\tilde{B}_\varphi$ satisfy Assumptions I$\sim$IV. And we see that $\tilde{L}_\varphi\tilde{u}=0$ and $\tilde{B}_\varphi\tilde{u}=0$ for $0\leqslant t'\leqslant\delta_1$ by taking account of the fact $u(x, y', t')=0$ when $y'^2-t'>0$. Apply the energy estimate (2.32) for (2.37) and we get $\tilde{u}(x, y', t')=0$ for $t'\leqslant\delta_1$. This shows that $u(x, y, t)=0$ when $y^2+t\leqslant\delta_1$. Q.E.D.

The above lemma derives that the propagation speed of the tangential direction of (2.3) for $\varphi=0$ is majorated by

$$\sup_{y\in R}\sqrt{\{a_{11}(y)a_{22}(y)-a_{12}(y)^2+b(y)\}^2/a_{11}(y)}$$

with the aid of the sweeping out method of F. John. Thus we get

Proposition 2.17. *The propagation speed of* (2.4) *for* $\varphi=0$ *is majorated by*

$$\sup_{y\in R}\sqrt{\{a_{11}(y)a_{22}(y)-a_{12}(y)^2+b(y)^2\}/a_{11}(y)}$$

in the tangential direction and

$$\sup_{y\in R}\sqrt{\{a_{11}(y)a_{22}(y)-a_{12}(y)^2\}/a_{11}(y)}$$

in the normal direction.

3. Proof of main theorem

Let $s_0=(x_0, y_0)\in S$ and the outer unit normal of S at s_0 be $(-1, 0)$. Consider *a* transformation M

$$\text{(3.1)}\qquad \begin{cases} x' = x-\mu(y) \\ y' = y-y_0\,. \end{cases}$$

where $x=\mu(y)$ represents an equation of S near s_0. M maps a neigborhood of s_0 in Ω into R_+^2. Then (1.1) is transformed by M into the equations

$$\text{(3.2)}\qquad \begin{cases} L^{(\mu)}[\tilde{u}] = \tilde{f} \\ B^{(\mu)}[\tilde{u}] = \tilde{g} \\ \tilde{u}(x', y', 0) = \tilde{u}_0(x', y') \\ \tilde{u}'(x', y', 0) = \tilde{u}_1(x', y')\,, \end{cases}$$

where

$$L^{(\mu)} = \frac{\partial^2}{\partial t^2}-\Big((1+\mu'(y')^2)\frac{\partial^2}{\partial x'^2}-2\mu'(y')\frac{\partial^2}{\partial x'\partial y'}+\frac{\partial^2}{\partial y'^2}\Big)+(\text{first order})$$

$$B^{(\mu)} = (1-\mu'(y))\frac{\partial}{\partial x'}+b(y')\frac{\partial}{\partial y'}\,.$$

Since $\mu'(0)=0$ by defining suitably the coefficients in $\{y'; |y'|>\delta\}$ and by choosing δ sufficiently small we can assume that $L^{(\mu)}$ and $B^{(\mu)}$ satisfy Assumptions I$\sim$IV of the previous section. The finiteness of the propagation speed of (2.4) derives the finiteness of the propagation speed of (1.1) in a neighborhood of s_0 in $\Omega\times R_+$. This fact holds for arbitrary $s_0\in S$. On the other hand the finiteness of the propagation speed in the interior of $\Omega\times R_+$ is already known. Thus the finiteness of the propagation speed of (1.1) is proved. And by taking account of Proposition 2.17 we see that the propagation speed of (1.1) is majorated by $v_{\max}=\sup_{y\in R}\sqrt{1+\frac{(b_1n_2-b_2n_1)^2}{b_1^2+b_2^2}}$.

Put

$$C(x_0, y_0, t_0) = \{(x, y, t);\ |x-x_0|+|y-y_0|\leqslant v_{\max}(t-t_0)\}\,.$$

then for $u(x, y, t)\in\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$ a sloution of (1.1), if

$$\begin{aligned} f(x, y, t) &= 0 && \text{in } C(x_0, y_0, t_0)\cap(\Omega\times(0, \infty))\\ g(x, y, t) &= 0 && \text{in } C(x_0, y_0, t_0)\cap(S\times(0, \infty))\\ u_0(x, y) = u_1(x, y) &= 0 && \text{in } C(x_0, y_0, t_0)\cap(\Omega\times\{t=0\}), \end{aligned}$$

then $u(x, y, t)=0$ in $C(x_0, y_0, t_0)\cap(\Omega\times(0, \infty))$.

Remark that when the given data $\{u_0, u_1, f, g\}$ satisfy the compatibility condition for (1.1) of order m, $\{\tilde{u}_0, \tilde{u}_1, \tilde{f}, \tilde{g}\}$ satisfy the compatibility condition or order m for (2.4) by changing the values in $\{y'; |y'|>\delta\}$.

Let d be a positive constant such that any $s_0\in S$, if a solution $\tilde{u}(x', y', t)$ satisfy (3.2) in $R_+^2\times(0, T)$, $u(x, y, t)$ satisfy (1.1) in $(\Omega\cap\{(x, y); |(x, y)-s_0|<d\})\times(0, T)$. Set $\Omega_d=\bigcup_{s_0\in S}\{(x, y); (x, y)\in\Omega \text{ and } |(x, y)-s_0|\leqslant d\}$. Define $u(x, y, t)$ for (x, y, t) such that $C(x, y, t)\cap\{t=0\}\subset\Omega_d$ by $u(x, y, t)=\tilde{u}(x', y', t)$, and for (x, y, t) such that $C(x, y, t)\cap(S\times(0, T))=\phi$ $u(x, y, t)$ equals the solution of the Cauchy problem $Lu=f$, $u(x, y, 0)=u_0(x, y)$ and $u'(x, y, 0)=u_1(x, y)$. We see that, by taking account of the above remarks, by this definition $u(x, y, t)$ is well defined and satisfies (1.1) for $0\leqslant t\leqslant\frac{d}{v_{\max}}=t_0$.

If the given data satisfy the compatibility condition of order $m+2$, it follows that $u(x, y, t)\in H^{m+2}(\Omega\times(0, t_0))$ from Theorem 2.14. Then we have

Proposition 3.1. *When the data $\{u_0, u_1, f, g\}$ satisfy the compatibility condition of order $m+3$, there exists a solution $u(x, y, t)$ in $\mathcal{E}_t^0(H^{m+1}(\Omega))\cap\mathcal{E}_t^1(H^m(\Omega))\cap\cdots\cap\mathcal{E}_t^{m+1}(L^2(\Omega))$ of* (1.1) *for $t\in[0, t_0]$.*

By applying this proposition step by step we see for any T, when the given data satisfy the compatibility condition of order $3\left(\left[\frac{T}{t_0}\right]+1\right)+m$, there exists a solution of (1.1) in $\mathcal{E}_t^0(H^{m+2}(\Omega)) \cap \mathcal{E}_t^1(H^{m+1}(\Omega)) \cap \cdots \cap \mathcal{E}_t^{m+2}(L^2(\Omega))$ for $t \in [0, T]$. This proves our Theorem 1.

Appendix

Consider a mixed problem

$$(\mathrm{A.1}) \qquad \begin{cases} \Box u = f(x, y, t) & \text{in } R_+^2 \times (0, T) \\ Bu = \left(-\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y}\right) u(x, y, t)\,|_{x=0} = g(y, t) \\ u(x, y, 0) = u_0(x, y) \\ \dfrac{\partial u}{\partial t}(x, y, 0) = u_1(x, y)\,, \end{cases}$$

where b is a positive constant.

Theorem A. (A.1) *has a propagation speed* $\sqrt{1+b^2}$.

This theorem shows that when $b \neq 0$, (A.1) has a propagation speed larger than Cauchy problem since that of Cauchy problem is 1.

Let us denote by $E^m(\Box, B)$ the set of data $\Psi = (u_0, u_1, f, g) \in H^{m+2}(R_+^2) \times H^{m+1}(R_+^2) \times H^{m+1}(R_+^2 \times (0, T)) \times H^{m+1}(R \times (0, T))$ satisfying the compatibility condition of order m for $\Box$ and B, for which we equip the following norm

$$\|\Psi\|_m^2 = ||u_0(x, y)||^2_{m+2, L^2(R_+^2)} + ||u_1(x, y)||^2_{m+1, L^2(R_+^2)} + ||f(x, y, t)||^2_{m+1, L^2(R_+^2 \times (0,T))} + ||g(y, t)||^2_{m+1, L^2(R \times (0,T))}\,.$$

Theorem 2.14 shows that the mapping from $E^m(\Box, B)$ to $H^m(R_+^2 \times (0, T))$ defined by

$$\Psi \to u(x, y, t) \text{ the solution of (A.1)}$$

is continuous. Therefore for any fixed point (x_0, y_0, t_0), the mapping from $E^2(\Box, B)$ to C defined by

$$\Psi \to u(x_0, y_0, t_0)$$

also continuous, namely for any $\Psi = (u_0, u_1, f, g) \in E^2(\Box, B)$ it holds

$$(\mathrm{A.2}) \qquad |u(x_0, y_0, t_0)| \leqslant C\|\Psi\|_2$$

where C does not depend on (x_0, y_0, t_0).

Assume that the maximam propagation speed of (A.1) is v_0, and set

$$C_0(x_0, y_0, t_0) = \{(x, y, t);\ |x - x_0| + |y - y_0| \leqslant v_0(t_0 - t)\}\,.$$

Remark that $u(x_0, y_0, t_0)$ is invariant with any change of values of data in the outside of $C_0(x_0, y_0, t_0)$.

Put

$$
\begin{aligned}
u_n(x, y, t) &= \exp\{n(-bx+\sqrt{1+b^2}\,t-y)\}\\
u_{n0}(x, y) &= u_n(x, y, 0)h(ny)\\
u_{n1}(x, y) &= u_n'(x, y, 0)h(ny)\\
f_n(x, y, t) &= k(nt)\square\{u_n(x, y, t)h(ny)\}\\
g_n(y, t) &= Bu_n(x, y, t)h(ny)k(nt)\,|_{x=0}
\end{aligned}
$$

where $h(y)$ is a C^∞-function such that $h(y)=0$ for $y\leqslant 0$, $h(y)=1$ for $y\geqslant 1$ and $k(t)$ is a C^∞-function such that $k(t)=1$ for $t\leqslant 1$, $k(t)=0$ for $t\geqslant 2$. Evidently $\Psi_n=(u_{n0}, u_{n1}, f_n, g_n)\in\bigcap_{m=1}^{\infty}E^m(\square, B)$ and

$$\text{(A.3)}\qquad \|\Psi_n\|_2\leqslant \text{const}\, n^4\,.$$

Let us denote by $\tilde{u}_n(x, y, t)$ the solution of (A.1) for the data Ψ_n, then by taking account of the definition of v_0 we have

$$\text{(A.4)}\qquad u_n(x, y, t)=\tilde{u}_n(x, y, t) \qquad \text{in } \left\{(x, y, t);\, x\geqslant 0,\, y-v_0t\geqslant\frac{1}{n} \text{ and } t\geqslant 0\right\}.$$

since $\square u_n(x, y, t)=0$ in $R_+^2\times[0, \infty)$, $Bu_n(0, y, t)=0$ in $R\times[0, \infty)$ and $u_{n0}(x, y)=u_n(x, y, 0)$, $u_{n1}(x, y)=u_n'(x, y, 0)$, $f_n(x, y, t)=0$, $g_n(y, t)=0$ hold if $y\geqslant\dfrac{1}{n}$.

Now we prove Theorem A. Assume that $v_0<\sqrt{1+b^2}$. Take a point such (y_0, t_0) that $t_0>0$, $y_0-v_0t_0>0$ and $y_0-\sqrt{1+b^2}\,t_0<0$. From (A.4) we have

$$\text{(A.5)}\qquad \tilde{u}_n(0, y_0, t_0)=\exp\{n(\sqrt{1+b^2}\,t_0-y_0)\}$$

for sufficiently large n. On the other hand from (A.2) and (A.3) it holds

$$\text{(A.6)}\qquad |\tilde{u}_n(0, y_0, t_0)|\leqslant \text{const}\, n^4\,.$$

Then (A.5) and (A.6) shows that

$$\exp\{n(\sqrt{1+b^2}\,t_0-y_0)\}\leqslant \text{const}\, n^4$$

holds for any sufficiently large n, this is a contradiction since $\sqrt{1+b^2}\,t_0-y_0>0$. Thus we have

$$v_0\geqslant\sqrt{1+b^2}$$

By combining the just obtained result and Proposition 2.17, Theorem A is proved.

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