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# A STATIONARY APPROACH TO LONG-RANGE SCATTERING

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## Introduction

This paper deals with a stationary construction of modified wave operators for long-range scattering.

The time dependent construction of modified wave operators for long-range scattering, the study of which was begun by Dollard [4] for the Schrödinger operator with pure Coulmob potential, has been rather well established recently by Buslaev-Matveev [3], Alsholm-Kato [1], Alsholm [2] and others. But it seems that the time independent (or stationary) approach has not been tried yet. In the theory of short-range scattering, however, the stationary approach has played an important role (see e.g. [7], [8], [9] and [10]). So it is not too ridiculous to conceive that the stationary approach may be useful also in studying the long-range scattering. In fact, using our method developed below, we can prove that the invariance principle for modified wave operators holds under Assumption 1.1 (see §4). But the proof of this result will not be discussed in this work. It will be discussed elsewhere.

In this paper we shall construct the modified wave operators by a stationary method essentially following the line established by Kato and Kuroda [7] and [8]. But some modifications will be necessary (see the proof of Theorem 2.8).

Now we describe the outline of this paper and at the same time give a heuristic explanation of some notations which will be used in this paper. Consider the time dependent modified wave operator for long-range scattering:

$$W_D^{+}=\underset{t\to\infty}{\text{s-lim}}\, e^{itH_2}e^{-itH_1-iX(t)}$$

where $H_1=-\frac{1}{2}\Delta\left(-\frac{1}{2}\text{ Laplacian on } R^n\right)$, $H_2=H_1+V$ ($V$ denotes the long-range potential) and $X(t)=\mathcal{F}^{-1}\left[\int_0^t V(s\xi)ds\cdot\right]\mathcal{F}$ ($\mathcal{F}$ denotes the Fourier transform in $L^2(R^n)$). Here we have assumed that

$$|V(x)|\leq C_0(1+|x|)^{-\beta} \qquad \text{with } \frac{1}{2}<\beta<1,\ C_0>0\,,$$

$$|\partial V(x)| \leq C_0(1+|x|)^{-1-\beta},$$
$$|\partial^2 V(x)| \leq C_0(1+|x|)^{-2-\gamma} \quad \text{with } \frac{(1-\beta^2)}{\beta} < \gamma < 1,$$

where $\partial^k$ denotes any $k$-th order partial differentiation in $x$. Now, for $u, v \in L^2(R^n)$, we have

$$\begin{aligned}
&(W_D^+ u, v) \\
&= \lim_{t \to \infty} (e^{-itH_1 - iX(t)} u, e^{-itH_2} v) \\
&= \lim_{\nu \to +0} 2\nu \int_0^\infty e^{-2\nu t}(e^{-itH_1 - iX(t)} u, e^{-itH_2} v) dt .
\end{aligned}$$

By Parseval's relation this becomes equal to

$$\begin{aligned}
&\lim_{\nu \to +0} \frac{\nu}{\pi} \int_{-\infty}^{\infty} (S^+(\mu+i\nu)u, R_2(\mu+i\nu)v) d\mu \\
&= \lim_{\nu \to +0} \int_{-\infty}^{\infty} (G^+(\mu+i\nu)u, \delta_2(\mu+i\nu)v) d\mu .
\end{aligned}$$

Here
$$\begin{aligned}
S^+(\mu+i\nu)u &= i\int_0^\infty e^{it\mu-\nu t} e^{-itH_1 - iX(t)} u\, dt , \\
G^+(\mu+i\nu)u &= (H_2-(\mu+i\nu))S^+(\mu+i\nu)u , \\
\delta_2(\mu+i\nu) &= \frac{\nu}{\pi} R_2(\mu+i\nu)^* R_2(\mu+i\nu)
\end{aligned}$$
and
$$R_2(z) = (H_2 - z)^{-1} .$$

So, roughly speaking, we may only consider the boundary values of $G^+(\mu+i\nu)u$ and $\delta_2(\mu+i\nu)v$ when $\nu \to +0$ instead of the time limit $W_D^+$. The existence of the boundary value of $\delta_2(\mu+i\nu)v$ for $v \in \mathcal{D} = \mathcal{F}^{-1}(C_0^\infty(R^n - \{0\}))$ when $\nu \to +0$ is assured by the limiting absorption principle (see §2, Theorem 2.1). Here $C_0^\infty(S)$ for an open subset $S$ of $R^n$ is the set of all the infinitely differentiable functions on $S$ into $C$ with compact support in $S$. Thus it appears as if our problem remaining were only to prove the existence of the boundary value of $G^+(\mu+i\nu)u$. But, in order to apply the Kato-Kuroda method to our case, it is necessary, further, to prove that the approximate spectral form

$$f^+(\mu+i\nu; u, v) = \frac{\nu}{\pi}(S^+(\mu+i\nu)u, S^+(\mu+i\nu)v)$$

for $u, v \in \mathcal{D}$, converges in some sense to the spectral form

$$e_1(\mu; u, v) = \frac{1}{2\pi i}((R_1(\mu+i0) - R_1(\mu-i0))u, v) .$$

Here $R_1(\mu\pm i0)u$ denote the boundary values of $R_1(\mu\pm i\nu)u$ (where $R_1(z)=(H_1-z)^{-1}$), the existence of which is also assured by the limiting absorption principle. These two problems will be solved in §2 and §3.

Using those two results obtained in §2 and §3, we shall construct an isometric linear operator $\hat{G}^+$ from $\mathcal{M}_1$ into $\mathcal{M}_2$ in §2, where $\mathcal{M}_j$ is the so-called spectral representation space for $H_j(j=1, 2)$ (cf. Definition 2.6 and Proposition 2.7). This $\hat{G}^+$ plays a role of a link between $H_1$ and $H_2$. Then the stationary wave operator $W^+$ can be easily constructed from $\hat{G}^+$. This will be done also in §2.

In §4, some possible generalizations and some applications (*i.e.* invariance principle) will be mentioned without proof. The proof will be discussed elsewhere.

Here the author wishes to express his sincere appreciation to Professor Yoshimi Saitō for encouraging conversations with him.

Before entering into our main task, some notations are to be introduced.

1° $R^n$ denotes the $n$-dimensional Eucledean space. Moreover we shall use the conventional notation such as $R^n_x$, $R^n_\xi$, etc., to specify the variable $x$, $\xi$, etc., under consideration.
2° $\mathcal{F}$ denotes the Fourier transform from $L^2(R^n_x)$ onto $L^2(R^n_\xi)$, that is,

$$(\mathcal{F}u)(\xi)=(2\pi)^{-n/2}\operatorname*{l.i.m.}_{N\to\infty}\int_{|x|\leq N}e^{-ix\xi}u(x)\,dx \qquad \text{in } L^2(R^n_\xi)$$

for $u\in L^2(R^n_x)$. Furthermore, we also use the notation $\hat{u}=\mathcal{F}u$ for any $u\in L^2(R^n_x)$.
3° For every measurable function $F$ on $R^n$ into $\boldsymbol{C}$, we put $F(D)=\mathcal{F}^{-1}[F(\xi)\cdot]\mathcal{F}$, where $F(\xi)\cdot$ denotes the maximal multiplication operator in $L^2(R^n_\xi)$ defined by the function $F$.
4° $\boldsymbol{C}^\pm=\{z\,|\,z\in\boldsymbol{C},\ \operatorname{Im} z\gtrless 0\}$.
5° $L^2_\alpha(R^n)$, $\alpha\in R^1$, is a Hilbert space of all the measurable functions $g(x)$ on $R^n$ into $\boldsymbol{C}$ satisfying

$$||g||_{L^2_\alpha(R^n)}=\left[\int_{R^n}|g(x)|^2(1+|x|)^{2\alpha}dx\right]^{1/2}<\infty\,.$$

And the norm of this Hibert space is defined by this expression. The inner product of this Hibert space is given as follows:

$$(f, g)_{L^2_\alpha(R^n)}=\int_{R^n}f(x)\overline{g(x)}(1+|x|)^{2\alpha}dx\,.$$

When $\alpha$ is 0, $\alpha$ is omitted.
6° $C_0^\infty(S)$ ($S$ is an open subest of $R^n$) is a set of all the infinitely differentiable functions on $S$ into $\boldsymbol{C}$ with compact support in $S$.

7° For any Hilbert space $Y$, $(\ ,\ )_Y$ and $\|\ \|_Y$ denote the inner product and the norm of $Y$, respectively. But, when unnecessary, the subscript $Y$ is omitted.
8° $L^2(R^1; X)$ ($X$ is a Hilbert space) is a Hilbert space of all the strongly measurable functions $g(x)$ on $R^1$ into $X$ such that

$$\|g\|_{L^2(R^1;X)} = \left[\int_{R^1} \|g(x)\|_X^2 dx\right]^{1/2} < \infty .$$

The inner product of this Hilbert space is given as follows:

$$(f, g)_{L^2(R^1;X)} = \int_{R^1} (f(x), g(x))_X dx .$$

9° $H^2(C^\pm; X)$ ($X$ is a Hilbert space) denote the Hardy classes on $C^\pm$ into $X$, i.e. all of the functions $f$: $C^\pm \to X$ such that

(a) $f$ is holomorphic on $C^\pm$

and

(b) $\sup_{\nu \gtreqless 0} \left(\int_{-\infty}^{\infty} \|f(\mu+i\nu)\|_X^2 d\mu\right) < \infty .$

10° For any Borel subset $A$ of $R^m$ ($m \geq 1$), $|A|$ denotes the Lebesgue measure of $A$.
11° $\mathcal{D}(T)$ and $\mathcal{R}(T)$ denote the domain and the range of an operator $T$, respectively.
12° $\mathcal{H} = L^2(R^n)$.
13° $\chi_+(t) = \begin{cases} 1 & \text{for } t>0 \\ 0 & \text{for } t<0 , \end{cases}$

$\chi_-(t) = \begin{cases} 0 & \text{for } t>0 \\ 1 & \text{for } t<0 . \end{cases}$

## 1. Assumption and some definitions

We consider two self-adjoint operators $H_1 = -\frac{1}{2}\Delta$ (self-adjoint realization of $-\frac{1}{2}\Delta$ in $L^2(R^n)$[1]) and $H_2 = H_1 + U$ in a Hilbert space $\mathcal{H} = L^2(R^n)$, $n \geq 1$. Here $U$ denotes a bounded self-adjoint operator in $\mathcal{H}$ satisfying the following

**Assumption 1.1.** $U$ can be decomposed as $U = V_s + V$, where $V_s$ and $V$ denote the maximal multiplication operators defined by the functions $V_s(x)$ and $V(x)$, respectively.

$V_s(x)$ is a real-valued measurable function on $R^n$ and satisfies

1) As usual, $H_1$ is a unique self-adjoint extension of $-\frac{1}{2}\Delta = -\frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ with its domain restricted to $C_0^\infty(R^n)$.

(S) $\quad |V_s(x)| \leq C_0(1+|x|)^{-1-\alpha} \quad$ with $0<\alpha<1$ and $C_0>0$.

$V(x)$ is a real-valued infinitely defferentiable function on $R^n$ and satisfies

(L0) $\quad |V(x)| \leq C_0(1+|x|)^{-\beta} \quad$ with $\frac{1}{2}<\beta<1$,

(L1) $\quad |\partial V(x)| \leq C_0(1+|x|)^{-1-\beta}$,

(L2) $\quad |\partial^2 V(x)| \leq C_0(1+|x|)^{-2-\gamma} \quad$ with $\frac{(1-\beta^2)}{\beta}<\gamma<1$,

(L3) $\quad |\partial^3 V(x)| \leq C_0(1+|x|)^{-2-\gamma}$,

where $\partial^k$ denotes any $k$-th order partial differentiation in $x$.

Under this assumption, we have $\mathcal{D}(H_1)=\mathcal{D}(H_2)=H^2(R^n)=$ the Sobolev space of order two.

REMARK 1.2. (L3) can be omitted without loss of generality. In this case we may only consider $V'(x)=(V*\omega)(x)\equiv\int_{R^n}V(x-y)\omega(y)dy$ and $V_s'(x)=V_s(x)+V(x)-V'(x)$, where $\omega\in C_0^\infty(R^n)$ and $||\omega||_{L^1(R^n)}=1$, instead of $V(x)$ and $V_s(x)$, respectively.

Now we make some definitions which will be useful later.

DEFINITION 1.3.

(1.1) $\quad X(t)=\int_0^t V(sD)ds \quad$ for $t\in R^1$.

(1.2) $\quad S^\pm(z)u=\pm i\int_{-\infty}^{\infty}\chi_\pm(t)e^{-iX(t)}e^{it(z-H_1)}u\,dt$,

(1.3) $\quad Q_1^\pm(z)u=V_sS^\pm(z)u$,

(1.4) $\quad Q_2^\pm(z)u=\pm i\int_{-\infty}^{\infty}\chi_\pm(t)(V-V(tD))e^{-iX(t)}e^{it(z-H_1)}u\,dt$,

(1.5) $\quad Q^\pm(z)u=Q_1^\pm(z)u+Q_2^\pm(z)u$

$$=\pm i\int_{-\infty}^{\infty}\chi_\pm(t)(U-V(tD))e^{-iX(t)}e^{it(z-H_1)}u\,dt,$$

(1.6) $\quad G^\pm(z)u=u+Q^\pm(z)u$,

where $u\in\mathcal{H}$ and $z\in \boldsymbol{C}^\pm$.

(1.7) $\quad R_j(z)=(H_j-z)^{-1}$,

(1.8) $\quad \delta_j(z)=\frac{|\mathrm{Im}\,z|}{\pi}R_j(z)^*R_j(z)$,

(1.9) $\quad e_j(z;u,v)=(\delta_j(z)u,v)_{\mathcal{H}}$,

where $z\in\rho(H_j)=$the resolvent set of $H_j$, $u, v\in\mathcal{H}$ and $j=1, 2$.

$$(1.10)\qquad \sigma^{\pm}(z)=\frac{|\operatorname{Im} z|}{\pi}S^{\pm}(z)^{*}S^{\pm}(z)\,,$$

$$(1.11)\qquad f^{\pm}(z;\,u,\,v)=(\sigma^{\pm}(z)u,\,v)_{\mathcal{H}}\,,$$

for $z\in \boldsymbol{C}^{\pm}$ and $u,\,v\in\mathcal{H}$.

**Proposition 1.4.**

(i) *$S^{\pm}(z)$, $Q_1^{\pm}(z)$, $Q_2^{\pm}(z)$, $Q^{\pm}(z)$, $G^{\pm}(z)$, $R_j(z)$, $\delta_j(z)$ and $\sigma^{\pm}(z)$ are all bounded operators in $\mathcal{H}$ for each $z\in\boldsymbol{C}^{\pm}$.*

(ii) *For every $z\in\boldsymbol{C}^{\pm}$ and $u,\,v\in\mathcal{H}$,*

$$(1.12)\qquad G^{\pm}(z)=(H_2-z)S^{\pm}(z)\,,$$

$$(1.13)\qquad f^{\pm}(z;\,u,\,v)=e_2(z;\,G^{\pm}(z)u,\,G^{\pm}(z)v)\,.$$

For the proof of this proposition, we prepare the following two lemmas.

**Lemma 1.5.** *Let $X$ be a Banach space and $B(X)$ be the Bancah space of all the bounded operators in $X$. Suppose that $I=[a,\,b]$, $-\infty<a<b<\infty$, and that $f(t)\colon I\to X$ and $A(t)\colon I\to B(X)$ are all strongly differentiable and their derivatives are Bochner integrable on $I$. Then, $A(t)f'(t)$ and $A'(t)f(t)$ are Bochner integrable in $X$ on $I$, and*

$$(1.14)\qquad \int_a^b A(t)f'(t)dt=A(b)f(b)-A(a)f(a)-\int_a^b A'(t)f(t)dt\,.$$

The proof is similar to the numerical function case, and hence we omit it.

**Lemma 1.6.** *Let $(S,\,\mathcal{B},\,m)$ be a $\sigma$-finite measure space, and $X$ be a Banach space. Let $T$ be a linear operator in $X$ and $f$ be a Bochner $m$-integrable function on $S$ into $X$. Suppose that $T$ and $f$ satisfy the following three conditions:*

(*a*) *$T$ is a closed operator in $X$.*

(*b*) *range of $f=f(S)\subset\mathcal{D}(T)$.*

(*c*) *$(Tf)(s)\colon S\to X$ is a Bochner $m$-integrable function.*

*Then $\int_S f(s)m(ds)\in\mathcal{D}(T)$ and*

$$(1.15)\qquad T\int_S f(s)m(ds)=\int_S (Tf)(s)\,m(ds)\,.$$

Proof. See Hille and Phillips [5], Theorem 3.7.12.

Proof of Proposition 1.4. (i) is obvious. (ii) (1.13) is obtained by (1.9)~(1.12). We proceed to the proof of (1.12). Let $u\in\mathcal{D}(H_1)=\mathcal{D}(H_2)$. In Lemma 1.6, let $S=R^1$ (with Lebesgue measure), $X=\mathcal{H}$, $f(t)=\chi_{\pm}(t)e^{-iX(t)}e^{it(z-H_1)}u$ and $T=H_1-z$, then the required conditions of Lemma 1.6 are satisfied. Thus, $S^{\pm}(z)u\in\mathcal{D}(H_1)$ and

$$(1.16)\qquad (H_1-z)S^{\pm}(z)u = \mp\int_{-\infty}^{\infty} \chi_{\pm}(t)e^{-iX(t)}\frac{d}{dt}(e^{it(z-H_1)}u)dt\,.$$

In Lemma 1.5, let $X=\mathcal{H}$, $I=[a, b]$, $f(t)=e^{it(z-H_1)}u$ and $A(t)=e^{-iX(t)}$. Then making $a\to-\infty$ or $b\to\infty$ in (1.14), we obtain

$$(1.17)\qquad (H_1-z)S^{\pm}(z)u = u\mp i\int_{-\infty}^{\infty} \chi_{\pm}(t)V(tD)e^{-iX(t)}e^{-it(z-H_1)}u\,dt$$

for each $u\in\mathcal{D}(H_1)=\mathcal{D}(H_2)$. The right-hand side of this equation defines bounded linear operators in $\mathcal{H}$. Denote them by $1-T^{\pm}(z)\in B(\mathcal{H})$. Then we obtain

$$(1.18)\qquad (H_1-z)S^{\pm}(z)u = (1-T^{\pm}(z))u$$

for each $u\in\mathcal{D}(H_1)=\mathcal{D}(H_2)$. Now, $\mathcal{D}(H_1)$ is dense in $\mathcal{H}$ and $H_1-z$ is a closed operator in $\mathcal{H}$. Therefore, for every $u\in\mathcal{H}$, we obtain

$$(1.19)\qquad (H_1-z)S^{\pm}(z)u = (1-T^{\pm}(z))u\,.$$

Thus, $S^{\pm}(z)\mathcal{H}\subset\mathcal{D}(H_1)=\mathcal{D}(H_2)$ and

$$(1.20)\qquad \begin{aligned}(H_2-z)S^{\pm}(z) &= US^{\pm}(z)+(H_1-z)S^{\pm}(z)\\ &= US^{\pm}(z)+(1-T^{\pm}(z)) = G^{\pm}(z).\end{aligned}$$

Q.E.D.

## 2. Construction of modified wave operators

Let $\Gamma=[a, b]$, $0<a<b<\infty$ be fixed in this section.

We first mention a theorem due to Ikebe-Saitō [6].

**Theorem 2.1.** *Let Assumption 1.1 be satisfied. Let* $K_{\Gamma}^{\pm}=\{z=\mu\pm i\nu \mid \mu\in\Gamma, \nu>0\}$ *and let* $\delta>\frac{1}{2}$ *be fixed.*

(i) *Then the mapping*

$$\begin{array}{ccc} K_{\Gamma}^{\pm}\times L^2_{\delta}(R^n) & \longrightarrow & L^2_{-\delta}(R^n) \\ \Cup & & \Cup \\ (z, u) & \longmapsto & R_j(z)u \end{array}$$

*can be extended uniquely to a continuous mapping*

$$\overline{K_{\Gamma}^{\pm}}\times L^2_{\delta}(R^n)\longrightarrow L^2_{-\delta}(R^n)\,.$$

*Moreover there exists a constant* $C>0$ *such that for every* $(z, u)\in\overline{K_{\Gamma}^{\pm}}\times L^2_{\delta}(R^n)$,

$$(2.1)\qquad \|R_j(z)u\|_{L^2_{-\delta}(R^n)}\leq C\|u\|_{L^2_{\delta}(R^n)}\qquad (j=1, 2)\,.$$

*Here* $R_j(z)=(H_j-z)^{-1}$.

(ii) *The mapping*

$$\begin{array}{ccc} (K_\Gamma^+ \cup K_\Gamma^-) \times L_\delta^2(R^n) \times L_\delta^2(R^n) & \longrightarrow & \boldsymbol{C} \\ \Downarrow & & \Downarrow \\ (z, u, v) & \longmapsto & e_j(z; u, v) \end{array}$$

*can be extended uniquely to a continuous mapping*

$$(K_\Gamma^+ \cup \Gamma \cup K_\Gamma^-) \times L_\delta^2(R^n) \times L_\delta^2(R^n) \longrightarrow \boldsymbol{C}$$

*and satisfies the following three conditions:*

(a) $e_j(z; \cdot, \cdot)$ *is a non-negative Hermitian form on* $L_\delta^2(R^n) \times L_\delta^2(R^n)$ *for each* $z \in K_\Gamma^+ \cup \Gamma \cup K_\Gamma^-$ $(j=1, 2)$.

(b) *For every* $u, v \in L_\delta^2(R^n)$, $e_j(\cdot\,; u, v) \in L^1(\Gamma)$ *and*

$$\int_\Delta e_j(\mu; u, v) d\mu = (E_{j,ac}(\Delta)u, v)_{\mathcal{H}} \tag{2.2}$$

*for any Borel subset* $\Delta$ *of* $\Gamma$. *Here* $E_{j,ac}(\Delta)$ *for a Borel subset* $\Delta$ *of* $R^1$ *denotes the absolutely continuous part of the spectral measure* $E_j(\Delta)$ *associated with* $H_j$ $(j=1, 2)$.

(c) *There exists a constant* $C>0$ *such that for each* $z \in K_\Gamma^+ \cup \Gamma \cup K_\Gamma^-$ *and every* $u, v \in L_\delta^2(R^n)$

$$|e_j(z; u, v)| \leq C \|u\|_{L_\delta^2(R^n)} \cdot \|v\|_{L_\delta^2(R^n)}, \qquad (j=1, 2). \tag{2.3}$$

REMARK 2.2. In the following, the notations $R_j(\mu \pm i0)u$, $\delta_j(\mu)u$ and $e_j(\mu; u, v)$ $(j=1, 2)$ for $\mu \in \Gamma$ and $u, v \in L_\delta^2(R^n)$ will be used as denoting the boundary values of $R_j(\mu \pm i\nu)u$, $\delta_j(\mu \pm i\nu)u$ and $e_j(\mu \pm i\nu; u, v)$ when $\nu \to +0$, respectively, the existence of which is assured by the above theorem.

**Proposition 2.3.** *Put* $\mathcal{D} = \mathcal{F}^{-1}(C_0^\infty(R^n - \{0\}))$ *and let* $u \in \mathcal{D}$ *be fixed. Then the following three propositions hold good.*

(i) *For any* $\delta$ *such that* $0 \leq \delta < \frac{1}{2} + \alpha$, $Q_1^\pm(z)u \in L_\delta^2(R^n)$ *for each* $z \in \boldsymbol{C}^\pm$, *and*

$$Q_1^\pm(\cdot)u \in H^2(\boldsymbol{C}^\pm; L_\delta^2(R^n)). \tag{2.4}$$

(ii) *Put* $\rho = \min(\beta\gamma - (1-\beta)^2, 2\beta - 1, \gamma^2) > 0$. *Then for any* $\delta$ *such that* $0 \leq \delta < \frac{1}{2} + \min\left(\rho, \frac{1}{2}\right)$, $Q_2^\pm(z)u \in L_\delta^2(R^n)$ *for each* $z \in \boldsymbol{C}^\pm$, *and*

$$Q_2^\pm(\cdot)u \in H^2(\boldsymbol{C}^\pm; L_\delta^2(R^n)). \tag{2.5}$$

(iii) *For any* $\delta$ *such that* $0 \leq \delta < \frac{1}{2} + \min\left(\alpha, \rho, \frac{1}{2}\right)$, $Q^\pm(z)u \in L_\delta^2(R^n)$ *for each* $z \in \boldsymbol{C}^\pm$, *and*

(2.6) $$Q^{\pm}(\cdot)u \in H^2(C^{\pm}; L^2_\delta(R^n)) .$$

*Especially, there exist* $\underset{\nu\to+0}{\text{l.i.m.}}\, Q^{\pm}(\mu\pm i\nu)u$ *in* $L^2(R^1_\mu; L^2_\delta(R^n))$.

For the definitions of $Q_1^{\pm}(z)$, $Q_2^{\pm}(z)$ and $Q^{\pm}(z)$, see (1.3)~(1.5).

The proof of this proposition will be given in §3.

**Proposition 2.4.** *For every* $u, v \in \mathscr{D}$, *there exist the limits* $\underset{\nu\to+0}{\text{l.i.m.}}\, f^{\pm}(\mu\pm i\nu; u, v)$ *in* $L^2(R^1_\mu)$ *and*

(2.7) $$\underset{\nu\to+0}{\text{l.i.m.}}\, f^{\pm}(\mu\pm i\nu; u, v) = e_1(\mu; u, v) \qquad \text{for a.e. } \mu\in\Gamma .$$

*Here the exceptional null set may depend on u and v.*

For the definitions of $e_1$ and $f^{\pm}$, see (1.9) and (1.11).

The proof of this proposition also will be given in §3.

**Theorem 2.5.** *Let* $\frac{1}{2} < \delta < \frac{1}{2} + \min\left(\alpha, \rho, \frac{1}{2}\right)$. *For any* $u \in \mathscr{D}$, *put* $\tilde{u}^{\pm}(\mu) = u + \underset{\nu\to+0}{\text{l.i.m.}}\, Q^{\pm}(\mu\pm i\nu)u \in L^2(\Gamma; L^2_\delta(R^n))$. *Then, for every* $u, v \in \mathscr{D}$,

(2.8) $$e_2(\mu; \tilde{u}^{\pm}(\mu), \tilde{v}^{\pm}(\mu)) = e_1(\mu; u, v) \qquad \text{for a.e. } \mu\in\Gamma .$$

*Here the exceptional null set may depend on u and v.*

Proof. From the definitions of $\tilde{u}^{\pm}$ and $\tilde{v}^{\pm}$, and Proposition 2.4, we see that there exist a sequence $\{\nu_n\}$ and a Borel subset $A$ of $\Gamma$ such that

(2.9) $$\nu_n > 0 , \quad \lim_{n\to\infty} \nu_n = 0 , \quad |\Gamma - A| = 0$$

and

(2.10) $$\begin{cases} \left.\begin{aligned} &\lim_{n\to\infty} G^{\pm}(\mu\pm i\nu_n)u = \tilde{u}^{\pm}(\mu) \\ &\lim_{n\to\infty} G^{\pm}(\mu\pm i\nu_n)v = \tilde{v}^{\pm}(\mu) \end{aligned}\right\} \quad \text{in } L^2_\delta(R^n) \\ \lim_{n\to\infty} f^{\pm}(\mu\pm i\nu_n; u, v) = e_1(\mu; u, v) \end{cases}$$

for each $\mu\in A$. Thus, from (ii) of Theorem 2.1 and (ii) of Proposition 1.4, we obtain

$$\begin{aligned} & e_2(\mu; \tilde{u}^{\pm}(\mu), \tilde{v}^{\pm}(\mu)) \\ &= \lim_{n\to\infty} e_2(\mu\pm i\nu_n; G^{\pm}(\mu\pm i\nu_n)u, G^{\pm}(\mu\pm i\nu_n)v) \\ &= \lim_{n\to\infty} f^{\pm}(\mu\pm i\nu_n; u, v) \\ &= e_1(\mu; u, v) \end{aligned}$$

for every $\mu\in A$. Q.E.D.

In the following, we fix $\delta$ as $\frac{1}{2}<\delta<\frac{1}{2}+\min\left(\alpha, \rho, \frac{1}{2}\right)$, and put $\mathcal{X}_1=\mathcal{D}$ endowed with the topology of $L^2_\delta(R^n)$ and $\mathcal{X}_2=L^2_\delta(R^n)$.

By $\mathcal{X}_1\subset L^2_\delta(R^n)$ and Theorem 2.1, $e_j(\cdot\,;\cdot,\cdot)$ $(j=1, 2)$ are spectral forms on $\Gamma\times\mathcal{X}_j\times\mathcal{X}_j$ in the terminology of Kato and Kuroda (see [7], p. 103), that is, $e_j(\cdot\,;\cdot,\cdot)$ satisfies (a) and (b) of Theorem 2.1.

Now, we make some definitions following Kato and Kuroda [7].

DEFINITION 2.6. Let $j=1$ or 2 be fixed.

$e_j(\mu; x)=e_j(\mu; x, x)$ for $x\in\mathcal{X}_j$, $\mu\in\Gamma$.

$\mathcal{N}_j(\mu)=\{x\in\mathcal{X}_j\,|\,e_j(\mu; x)=0\}$ for $\mu\in\Gamma$.

$\mathcal{X}_j(\mu)=\mathcal{X}_j/\mathcal{N}_j(\mu)$ for $\mu\in\Gamma$. (This becomes a pre-Hilbert spece with respect to the inner product induced by $e_j(\mu;\cdot,\cdot)$.)

$\tilde{\mathcal{X}}_j(\mu)=$the completion of $\mathcal{X}_j(\mu)$ for $\mu\in\Gamma$.

$(\ ,\ )_{j,\mu}$ and $\|\ \|_{j,\mu}$ are the inner product and the norm of $\tilde{\mathcal{X}}_j(\mu)$ for $\mu\in\Gamma$, respectively.

$J_j(\mu)$: $\mathcal{X}_j\to\tilde{\mathcal{X}}_j(\mu)$ is a canonical homomorphism. (This is a continuous linear operator by (c) in (ii) of Theroem 2.1.)

$\tilde{\mathcal{X}}_j=\prod_{\mu\in\Gamma}\tilde{\mathcal{X}}_j(\mu)$ is an algebraic direct product of $\{\tilde{\mathcal{X}}_j(\mu)\}_{\mu\in\Gamma}$.

$\mathcal{S}_j=\{h\colon \Gamma\to\mathcal{X}_j\,|\,h(\mu)=\sum_{k=1}^{n}\chi_{\Delta_k}(\mu)x_k$, where $\Delta_k\subset\Gamma$ (Borel subset) and $x_k\in\mathcal{X}_j\}$. (Here $\chi_\Delta$ denotes the characteristic function for $\Delta\subset R^1$.)

$g\in\tilde{\mathcal{X}}_j$ is called $e_j$-measurable, if for some sequence $\{h_n\}\subset\mathcal{S}_j$, $g$ satisfies $\lim_{n\to\infty}\|g(\mu)-J_j(\mu)h_n(\mu)\|_{j,\mu}=0$ for *a.e.* $\mu\in\Gamma$.

For $g_1, g_2\in\tilde{\mathcal{X}}_j$, $g_1\sim_j g_2$ means that $g_1(\mu)=g_2(\mu)$ for *a.e.* $\mu\in\Gamma$.

For $g\in\tilde{\mathcal{X}}_j$, $[g]$ denotes the equivalence class of $g$ by $\sim_j$.

$\mathcal{M}_j=\left\{[g]\in\tilde{\mathcal{X}}_j/\sim_j \,\middle|\, [g] \text{ is } e_j\text{-measurable and } \|[g]\|^2_{\mathcal{M}_j}\equiv\int_\Gamma\|g(\mu)\|^2_{j,\mu}d\mu<\infty\right\}$.

($\mathcal{M}_j$ becomes a Hilbert space with respect to the norm $\|\ \|_{\mathcal{M}_j}$ and $J_j\mathcal{S}_j/\sim_j$ is a dense linear subspace of $\mathcal{M}_j$[2].)

$\mathring{\mathcal{L}}_{j,ac}(\Gamma)=\{\sum_{k=1}^{n}E_{j,ac}(\Delta_k)x_k\,|\,\Delta_k\subset\Gamma$ (Borel subset), $x_k\in\mathcal{X}_j\}$.

$\mathcal{L}_{j,ac}(\Gamma)=(\mathring{\mathcal{L}}_{j,ac}(\Gamma))^a$, where $(M)^a$ means the closure of $M\subset\mathcal{H}$ in $\mathcal{H}$. (In our case, $\mathcal{L}_{j,ac}(\Gamma)=\mathcal{H}_{j,ac}(\Gamma)\equiv E_{j,ac}(\Gamma)\mathcal{H}$.)

The next proposition gives a spectral representation for $H_j$ $(j=1, 2)$.

**Proposition 2.7.** *Let $j=1$ or 2. Then there exists a unitary operator $\pi_j$ from $\mathcal{L}_{j,ac}(\Gamma)$ onto $\mathcal{M}_j$ which satisfies the following two conditions:*

---

2) For the proof of these two facts, see Propositions 1.9 and 1.10 of Kato and Kuroda [7].

(a) $\pi_j E_{j,ac}(\Delta)u = \{\chi_\Delta(\mu)(\pi_j u)(\mu)\}_{\mu\in\Gamma}$ *for* $\Delta\subset R^1$ *(Borel subset) and* $u\in \mathcal{L}_{j,ac}(\Gamma)$.

(b) $\pi_j E_{j,ac}(\Gamma)x = \{J_j(\mu)x\}_{\mu\in\Gamma}$ *for* $x\in\mathcal{X}_j$.

*Especially, for every* $u=\sum_{k=1}^{n} E_{j,ac}(\Delta_k)x_k \in \mathring{\mathcal{L}}_{j,ac}(\Gamma)$, *we have*

$$\pi_j u = \{J_j(\mu)\sum_{k=1}^{n}\chi_{\Delta_k}(\mu)x_k\}_{\mu\in\Gamma} \in J_j\mathcal{S}_j/\sim_j .$$

Proof. See Kato and Kuroda [7], p. 106.

In the above, we have constructed two spectral representations exactly in the same way as Kato and Kuroda [7]. Now, we are in a position to construct $\hat{G}^\pm$ which play a role of links between $H_1$ and $H_2$.

**Theorem 2.8.** *Let Assumption* 1.1 *be satisfied. Then, there exist unique isometric linear operators* $\hat{G}^\pm$ *from* $\mathcal{M}_1$ *into* $\mathcal{M}_2$ *which satisfy the following two conditions:*

(a) $\hat{G}^\pm\{\chi_\Delta(\mu)u(\mu)\}_{\mu\in\Gamma} = \{\chi_\Delta(\mu)(\hat{G}^\pm u)(\mu)\}$ *for any Borel subset* $\Delta$ *of* $\Gamma$ *and any* $u\in\mathcal{M}_1$.

(b) $\hat{G}^\pm\{J_1(\mu)x\}_{\mu\in\Gamma} = \{J_2(\mu)\tilde{x}^\pm(\mu)\}_{\mu\in\Gamma}$ *for every* $x\in\mathcal{X}_1$, *where* $\tilde{x}^\pm(\mu) = x + \operatorname{l.i.m.}_{\nu\to+0} Q^\pm(\mu\pm i\nu)x$ (*cf.* Theorem 2.5).

Proof. We have to prove the existence and the uniqueness of isometric linear operators from $J_1\mathcal{S}_1/\sim_1$ into $\mathcal{M}_2$ which satisfy conditions (a) and (b).

Take any element $\varphi$ of $J_1\mathcal{S}_1/\sim_1$. Then it can be represented as $\varphi=[J_1 h]$ for some $h\in\mathcal{S}_1$. Put $F=$range of $h$. Then $F$ is a finite set $\{h_1,\cdots,h_m\}$ ($m\geq 0$, $m$ is an integer), where $h_j\in\mathcal{X}_1$ ($j=1,2,\cdots,m$) and $h_j\neq h_k$ for $j\neq k$. If we put $B_j=\{\mu\in\Gamma \mid h(\mu)=h_j\}$ ($j=1,2,\cdots,m$), then $B_j$ becomes a Borel subset of $\Gamma$ since $h\in\mathcal{S}_1$. Furthermore, $\Gamma=\bigcup_{j=1}^{m} B_j$ and $B_j\cap B_k=$empty for $j\neq k$.

Set $\tilde{h}_j^\pm(\mu)=h_j+\operatorname{l.i.m.}_{\nu\to+0} Q^\pm(\mu\pm i\nu)h_j\in L^2(\Gamma;\mathcal{X}_2)$ ($j=1,2,\cdots,m$) and $\tilde{h}^\pm(\mu)=\tilde{h}_j^\pm(\mu)$ for $\mu\in B_j$ ($j=1,2,\cdots,m$). Then $\tilde{h}^\pm(\mu)$ are uniquely defined for *a.e.* $\mu\in\Gamma$ because $\Gamma=\bigcup_{j=1}^{m}B_j$ and $B_j\cap B_k=$empty for $j\neq k$. So the mappings $\tilde{h}^\pm\colon\Gamma\to\mathcal{X}_2$ are well defined. Moreover, since $B_j$ is a Borel subset of $\Gamma$ and $\tilde{h}_j^\pm\in L^2(\Gamma;\mathcal{X}_2)$ for $j=1,2,\cdots,m$, these mappings $\tilde{h}^\pm$ are strongly measurable. Thus, $J_2\tilde{h}^\pm\in\tilde{\mathcal{X}}_2$ are $e_2$-measurable because $J_2(\mu)\colon\mathcal{X}_2\to\tilde{\mathcal{X}}_2(\mu)$ is continuous by Theorem 2.1.

On the other hand, using Theorem 2.5, we have for *a.e.* $\mu\in B_j$ ($j=1$, $2,\cdots,m$),

$$\begin{aligned}&\|J_2(\mu)\tilde{h}^\pm(\mu)\|_{2,\mu}^2\\ &= e_2(\mu;\tilde{h}^\pm(\mu))\end{aligned}$$

$$
\begin{aligned}
&= e_2(\mu;\, \tilde{h}_j^{\pm}(\mu)) \\
&= e_1(\mu;\, h_j) \\
&= e_1(\mu;\, h(\mu)) \\
&= \|J_1(\mu)h(\mu)\|_{1,\mu}^2 .
\end{aligned}
$$

Therefore, we have

$$
\|[J_2\tilde{h}^{\pm}]\|_{\mathcal{M}_2} = \|[J_1h]\|_{\mathcal{M}_1} = \|\varphi\|_{\mathcal{M}_1} < \infty \tag{2.11}
$$

and $[J_2\tilde{h}^{\pm}] \in \mathcal{M}_2$.

Although $\varphi \in J_1\mathcal{S}_1/\sim_1$ does not determine $h \in \mathcal{S}_1$ uniquely, using (2.11), we can easily show that $[J_2\tilde{h}^{\pm}]$ does not depend on the choice of $h \in \mathcal{S}_1$. Thus we can define isometrical mappings

$$
\begin{array}{ccc}
\hat{G}^{\pm}\colon J_1\mathcal{S}_1/\sim_1 & \longrightarrow & \mathcal{M}_2 \\
\Cup & & \Cup \\
[J_1h] & \longmapsto & [J_2\tilde{h}^{\pm}] .
\end{array}
$$

The linearity of these mappings is obvious. Moreover, as can be easily shown, these mappings satisfy (a) and (b).

Now there remains only to prove the uniqueness of these operators. Let $\hat{H}^{\pm}$ be the isometric linear operators from $\mathcal{M}_1$ into $\mathcal{M}_2$ which satisfy (a) and (b). Then for any $h(\mu) = \sum_{k=1}^{n} \chi_{\Delta_k}(\mu)x_k \in \mathcal{S}_1$ ($\Delta_k \subset \Gamma$ (Borel subset), $x_k \in \mathfrak{X}_1$),

$$
\begin{aligned}
\hat{H}^{\pm}J_1h &= \hat{H}^{\pm}\{J_1(\mu)h(\mu)\}_{\mu\in\Gamma} \\
&= \sum_{k=1}^{n} \{\chi_{\Delta_k}(\mu)(\hat{H}^{\pm}\{J_1(\mu)x_k\}_{\mu\in\Gamma})(\mu)\}_{\mu\in\Gamma} \\
&= \sum_{k=1}^{n} \{\chi_{\Delta_k}(\mu)J_2(\mu)\tilde{x}_k^{\pm}(\mu)\}_{\mu\in\Gamma}
\end{aligned}
$$

by (a) and (b). Similarly,

$$
\hat{G}^{\pm}J_1h = \sum_{k=1}^{n} \{\chi_{\Delta_k}(\mu)J_2(\mu)\tilde{x}_k^{\pm}(\mu)\}_{\mu\in\Gamma} .
$$

Thus we obtain $\hat{H}^{\pm} = \hat{G}^{\pm}$ on $J_1\mathcal{S}_1/\sim_1$. Q.E.D.

Now we can define the modified wave operators $W_{\Gamma}^{\pm}$.

DEFINITION 2.9.

$$
W_{\Gamma}^{\pm}u = \begin{cases} \pi_2^{-1}\hat{G}^{\pm}\pi_1u & \text{for } u \in \mathcal{L}_{1,ac}(\Gamma) , \\ 0 & \text{for } u \in \mathcal{H} \ominus \mathcal{L}_{1,ac}(\Gamma) . \end{cases}
$$

Here, note that $\mathcal{L}_{1,ac}(\Gamma) = \mathcal{H}_{1,ac}(\Gamma)$ in our case.

**Theorem 2.10.** *Let Assumption* 1.1 *be satisfied. Then* $W_{\Gamma}^{\pm}$ *are partially* *isometric operators in $\mathcal{H}$ with the initial sets $\mathcal{H}_{1,ac}(\Gamma)$ and the final sets contained in $\mathcal{H}_{2,ac}(\Gamma)$. And for any Borel subset $\Delta$ of $R^1$, we have*

$$W_\Gamma^\pm E_1(\Delta)=E_2(\Delta)W_\Gamma^\pm . \tag{2.12}$$

*Moreover, for any $x\in\mathcal{X}_1$, $y\in\mathcal{X}_2$ and Borel subsets $\Delta_1$, $\Delta_2$ of $\Gamma$,*

$$(W_\Gamma^\pm E_{1,ac}(\Delta_1)x,\ E_{2,ac}(\Delta_2)y)_{\mathcal{H}}=\int_{\Delta_1\cap\Delta_2}(\tilde{x}^\pm(\mu),\ \delta_2(\mu)y)_{\mathcal{H}}d\mu . \tag{2.13}$$

Proof. It is obvious that $W_\Gamma^\pm$ are partially isometric operators with the initial sets $\mathcal{H}_{1,ac}(\Gamma)$ and the final sets contained in $\mathcal{H}_{2,ac}(\Gamma)$.

Let a Borel set $\Delta$ of $R^1$ be fixed. If $u\perp\mathcal{H}_{1,ac}(\Gamma)$, then $W_\Gamma^\pm E_1(\Delta)u=0=E_2(\Delta)W_\Gamma^\pm u$ by Definition 2.9. If $u\in\mathcal{H}_{1,ac}(\Gamma)$, then $W_\Gamma^\pm E_1(\Delta)u$ and $E_2(\Delta)W_\Gamma^\pm u$ both belong to $\mathcal{H}_{2,ac}(\Gamma)$. Thus we have only to prove that $\pi_2W_\Gamma^\pm E_1(\Delta)u=\pi_2E_2(\Delta)W_\Gamma^\pm u$. This is proved as follows:

$$\begin{aligned}
&\pi_2W_\Gamma^\pm E_1(\Delta)u\\
&=\hat{G}^\pm\pi_1E_1(\Delta)u && \text{(by Definition 2.9)}\\
&=\hat{G}^\pm\pi_1E_{1,ac}(\Delta)u\\
&=\hat{G}^\pm\chi_\Delta\pi_1u && \text{(by (a) of Proposition 2.7)}\\
&=\chi_\Delta\hat{G}^\pm\pi_1u && \text{(by (a) of Theorem 2.8)}\\
&=\pi_2E_{2,ac}(\Delta)\pi_2^{-1}\hat{G}^\pm\pi_1u && \text{(by (a) of Proposition 2.7)}\\
&=\pi_2E_2(\Delta)W_\Gamma^\pm u && \text{(by Definition 2.9).}
\end{aligned}$$

The last part of this theorem is proved as follows:

$$\begin{aligned}
&(W_\Gamma^\pm E_{1,ac}(\Delta_1)x,\ E_{2,ac}(\Delta_2)y)_{\mathcal{H}}\\
&=(\hat{G}^\pm\pi_1E_{1,ac}(\Delta_1)x,\ \pi_2E_{2,ac}(\Delta_2))_{\mathcal{M}_2}\\
&=(\hat{G}^\pm\chi_{\Delta_1}J_1x,\ \chi_{\Delta_2}J_2y)_{\mathcal{M}_2} && \text{(by (a) of Proposition 2.7)}\\
&=(\chi_{\Delta_1}\hat{G}^\pm J_1x,\ \chi_{\Delta_2}J_2y)_{\mathcal{M}_2} && \text{(by (a) of Theorem 2.8)}\\
&=(\chi_{\Delta_1}J_2\tilde{x}^\pm,\ \chi_{\Delta_2}J_2y)_{\mathcal{M}_2} && \text{(by (b) of Theorem 2.8)}\\
&=\int_{\Delta_1\cap\Delta_2}(J_2(\mu)\tilde{x}^\pm(\mu),\ J_2(\mu)y)_{2,\mu}d\mu\\
&=\int_{\Delta_1\cap\Delta_2}e_2(\mu;\ \tilde{x}^\pm(\mu),\ y)d\mu\\
&=\int_{\Delta_1\cap\Delta_2}(\tilde{x}^\pm(\mu),\ \delta_2(\mu)y)_{\mathcal{H}}d\mu .
\end{aligned}$$

Q.E.D.

## 3. Proof of Propositions 2.3 and 2.4

To begin with, we prepare some lemmas which will be used in the proof of Propositions 2.3 and 2.4. As the first two lemmas are found in Alsholm-Kato's paper [1], the proof will be omitted.

**Lemma 3.1.** *For any $t\in R^1$ and $u\in \mathcal{H}=L^2(R^n)$, we have*

$$(3.1)\qquad \begin{aligned} &e^{itH_1}(V-V(tD))e^{-itH_1}\\ &=\int_0^1\Big[\frac{it}{2}e^{it\lambda^{-1}H_1}(\Delta V)(\lambda x)e^{-it\lambda^{-1}H_1}u\\ &\quad+e^{it\lambda^{-1}H_1}\langle(\nabla V)(\lambda x)e^{-it\lambda^{-1}H_1}, x\rangle u\Big]d\lambda\,, \end{aligned}$$

*where* $\langle x, y\rangle=\sum_{k=1}^n x_k y_k$ *for* $x, y\in C^n$ *and* $(\nabla V)(x)=\left(\frac{\partial V}{\partial x_1}(x), \cdots, \frac{\partial V}{\partial x_n}(x)\right)$.

**Lemma 3.2.** *Put* $\mathcal{D}_a=\{u\in\mathcal{D}\,|\,\mathrm{supp}\,\hat{u}\subset E_a\equiv\{\xi|\ |\xi|\geq a\}\}$ *for* $a>0$ *and*

$$(3.2)\qquad Y=Y(t, \lambda)=t\lambda^{-1}H_1+X(t)\,.$$

*Then, for any $a>0$, there exists a postitve constant $C_a$ such that for any $u\in\mathcal{D}_a$, $t\in R^1$ and $1\geq\lambda>0$, the next four estimates hold good:*

(i) $\|(1+|\lambda x|)^{-k}e^{-iY}u\|\leq C_a(1+|t|)^{-k}\|(1+|\lambda x|)^k u\|$, $(k=0, 1, 2)$.

(ii) $\|(1+|\lambda x|)^{-3}e^{-iY}u\|\leq C_a(1+|t|)^{-2-\gamma}\|(1+|\lambda x|)^3 u\|$,

(iii) $\|(1+|\lambda x|)^{-k}e^{-iY}(\nabla X)(t)u\|$
$\leq C_a(1+|t|)^{1-k-\beta}\|(1+|\lambda x|)^k u\|$, $(k=0, 1)$.

(iv) $\|(1+|\lambda x|)^{-2}e^{-iY}(\nabla X)(t)u\|$
$\leq C_a(1+|t|)^{-1-\min(\beta,\gamma)}\|(1+|\lambda x|)^2 u\|$.

Here $(\nabla X)(t)=\mathcal{F}^{-1}\left[\nabla_\xi\int_0^t V(s\xi)ds\cdot\right]\mathcal{F}$.

**Lamma 3.3.** *For $t\in R^1$ and $1\geq\lambda>0$, put*

$$(3.3)\qquad A_1(t, \lambda)=\frac{it}{2}e^{-it(1-\lambda^{-1})H_1}(\Delta V)(\lambda x)e^{-it\lambda^{-1}H_1-iX(t)}\,,$$

$$(3.4)\qquad A_2(t, \lambda)=e^{-it(1-\lambda^{-1})H_1}\langle(\nabla V)(\lambda x)e^{-it\lambda^{-1}H_1-iX(t)}, x\rangle\,,$$

$$(3.5)\qquad A_3(t, \lambda)=e^{-it(1-\lambda^{-1})H_1}\langle(\nabla V)(\lambda x)e^{-it\lambda^{-1}H_1-iX(t)}, (\nabla X)(t)\rangle\,.$$

*Then, for any $a>0$ and $u\in\mathcal{D}_a$, there exist positive constants $C_a$ and $C(u)$ such that for any $t\in R^1$ and $\lambda$ with $1\geq\lambda>0$, the following four estimates hold good:*

(i) $\|(1+|\lambda x|)^k A_1(t, \lambda)u\|\leq C_a(1+|t|)^{k-1-\gamma^2}C(u)$ $(k=0, 1)$.

(ii) $\|(1+|\lambda x|)^k A_2(t, \lambda)u\|\leq C_a(1+|t|)^{k-1-\beta}C(u)$ $(k=0, 1)$.

(iii) $\|(1+|\lambda x|)^k A_3(t, \lambda)u\|\leq C_a(1+|t|)^{k-1-\min(\beta\gamma-(1-\beta)^2, 2\beta-1)}C(u)$ $(k=0, 1)$.

(iv) $\|(1+|\lambda x|)^k A_m(t, \lambda)u\|\leq C_a(1+|t|)^{k-1-\rho}C(u)$ *for* $k=0, 1$ *and* $m=1$, 2, 3, *where* $\rho=\min(\beta\gamma-(1-\beta)^2, 2\beta-1, \gamma^2)>0$.

Proof. Let $a>0$ and $u\in\mathcal{D}_a$ be arbitrarily fixed. (iv) is the direct consequence of (i)~(iii). So we only prove (i)~(iii).

Case 1) $k=0$ in (i)~(iii). This case can be easily shown using (3.3)~(3.5) and Lemma 3.2. For example, let us prove (iii) for $k=0$. From (3.5) we have

$$||A_3(t, \lambda)u||\leq C_0\sum_{j=1}^{n}||(1+|\lambda x|)^{-1-\beta}e^{-iY(t,\lambda)}(\partial_j X)(t)u||,$$

where $C_0$ is the constant given in Assumption 1.1. Thus, by interpolating the estimates (iii) and (iv) of Lemma 3.2, we obtain (iii) for $k=0$. $\left(\text{Here } (\partial_j X)(t)=\mathcal{F}^{-1}\left[\frac{\partial}{\partial\xi_j}\int_0^t V(s\xi)ds\cdot\right]\mathcal{F}.\right)$

Case 2) $k=1$ in (i)~(iii). Because

$$||(1+|\lambda x|)A_m(t, \lambda)u||\leq||A_m(t, \lambda)u||+\sum_{j=1}^{n}||\lambda x_j A_m(t, \lambda)u||,$$

we have only to estimate $||\lambda x_j A_m(t, \lambda)u||$ ($j=1, 2, \cdots, n$, $m=1, 2, 3$).

From the identity

$$\lambda x_j e^{-it(1-\lambda^{-1})H_1}=e^{-it(1-\lambda^{-1})H_1}(\lambda x_j+t(\lambda-1)D_j),\quad D_j=-i\frac{\partial}{\partial x_j},$$

we obtain the following three equations:

$$\begin{aligned}(3.6)\qquad &\lambda x_j A_1(t, \lambda)u\\ &=\frac{it}{2}e^{-it(1-\lambda^{-1})H_1}[\lambda x_j(\Delta V)(\lambda x)+t\lambda(\lambda-1)(D_j\Delta V)(\lambda x)\\ &\qquad+t(\lambda-1)(\Delta V)(\lambda x)D_j]e^{-iY(t,\lambda)}u.\end{aligned}$$

$$\begin{aligned}(3.7)\qquad &\lambda x_j A_2(t, \lambda)u\\ &=e^{-it(1-\lambda^{-1})H_1}\langle[\lambda x_j(\nabla V)(\lambda x)+t\lambda(\lambda-1)(D_j\nabla V)(\lambda x)\\ &\qquad+t(\lambda-1)(\nabla V)(\lambda x)D_j]e^{-iY(t,\lambda)}, x\rangle u.\end{aligned}$$

$$\begin{aligned}(3.8)\qquad &\lambda x_j A_3(t, \lambda)u\\ &=e^{-it(1-\lambda^{-1})H_1}\langle[\lambda x_j(\nabla V)(\lambda x)+t\lambda(\lambda-1)(D_j\nabla V)(\lambda x)\\ &\qquad+t(\lambda-1)(\nabla V)(\lambda x)D_j]e^{-iY(t,\lambda)}, (\nabla X)(t)\rangle u.\end{aligned}$$

Thus, by direct computation and by interpolating the estimates of Lemma 3.2, we obtain the required results. Q.E.D.

Proof of Proposition 2.3. (i) Let $0\leq\delta<\frac{1}{2}+\alpha$ be fixed. Then $-2<-1-\alpha\leq-1-\alpha+\delta<-\frac{1}{2}$ and

(3.12) $$|(1+|x|)^{\delta}V_s(x)|\leq C_0(1+|x|)^{-1-\alpha+\delta}.$$

And hence $(1+|x|)^{\delta}V_s(x)\in B(\mathcal{H})$. Thus,

$$\begin{aligned}&(1+|x|)^{\delta}Q_1^{\pm}(z)u\\&=(1+|x|)^{\delta}V_sS^{\pm}(z)u\\&=\pm i\int_{-\infty}^{\infty}\chi_{\pm}(t)\{(1+|x|)^{\delta}V_se^{-iX(t)-itH_1}u\}e^{itz}dt.\end{aligned}$$

Moreover, by (3.12) and (i) of Lemma 3.2, we obtain

$$\|(1+|x|)^{\delta}V_se^{-iX(t)-itH_1}u\|\leq C(u)(1+|t|)^{-1-\alpha+\delta}$$

for some constant $C(u)>0$. Therefore $(1+|x|)^{\delta}V_se^{-iX(t)-itH_1}u\in L^2(R_t^1;\mathcal{H})$. Thus $(1+|x|)^{\delta}Q_1^{\pm}(z)u$ belong to the Hardy-class $H^2(C^{\pm};\mathcal{H})$ as the one-sided Laplace transforms of $L^2(R_t^1;\mathcal{H})$-functions.

Next we prove (ii). Let $0\leq\delta<\frac{1}{2}+\min\left(\rho,\frac{1}{2}\right)$, where $\rho=\min(\beta\gamma-(1-\beta)^2, 2\beta-1, \gamma^2)>0$. Then,

(3.13) $$-1-\rho\leq-1-\rho+\delta<-\frac{1}{2}-\rho+\min\left(\rho,\frac{1}{2}\right)<-\frac{1}{2}.$$

Moreover, by Lemma 3.1 and the equation

$$xe^{-iX(t)}=e^{-iX(t)}(x+(\nabla X)(t)),$$

we obtain

(3.14) $$Q_2^{\pm}(z)u=\pm i\int_{-\infty}^{\infty}\chi_{\pm}(t)\int_0^1\sum_{m=1}^{3}A_m(t,\lambda)u\,d\lambda\,e^{itz}dt.$$

Here we note that $(1+|x|)^{\delta}A_m(t,\lambda)ue^{itz}$ is a Bochner integrable function from $R^1\times(0,1]$ into $\mathcal{H}$. In fact this function is continuous by (3.6)~(3.8). Moreover $\|(1+|x|)^{\delta}A_m(t,\lambda)ue^{itz}\|$ is integrable over $R^1\times(0,1]$ because $\mathrm{Im}\,z\neq0$ and

(3.15) $$\begin{aligned}&\|(1+|x|)^{\delta}A_m(t,\lambda)u\|\\&\leq\lambda^{-\delta}\|(1+|\lambda x|)^{\delta}A_m(t,\lambda)u\|\\&\leq\lambda^{-\delta}(1+|t|)^{-1-\rho+\delta}C(u)\end{aligned}$$

for some constant $C(u)>0$ by (iv) of Lemma 3.3. Thus in Lemma 1.6, putting $S=R^1\times(0,1]$, $X=\mathcal{H}$, $f(t,\lambda)=\chi_{\pm}(t)A_m(t,\lambda)ue^{itz}$ and $T=(1+|x|)^{\delta}\cdot$, we obtain $Q_2^{\pm}(z)u\in L_{\delta}^2(R^n)=\mathcal{D}((1+|x|)^{\delta}\cdot)$ and

$$\begin{aligned}&(1+|x|)^{\delta}Q_2^{\pm}(z)\\&=\pm i\int_{-\infty}^{\infty}\chi_{\pm}(t)\left[\int_0^1\sum_{m=1}^{3}(1+|x|)^{\delta}A_m(t,\lambda)u\,d\lambda\right]e^{itz}dt.\end{aligned}$$

On the other hand, by (3.15), we obtain

$$\begin{aligned}&\left\|\int_0^1 \sum_{m=1}^3 (1+|x|)^\delta A_m(t, \lambda) u \, d\lambda\right\| \\ &\leq \sum_{m=1}^3 \int_0^1 \|(1+|x|)^\delta A_m(t, \lambda) u\| d\lambda \\ &\leq \sum_{m=1}^3 \int_0^1 \lambda^{-\delta} d\lambda (1+|t|)^{-1-\rho+\delta} C(u) .\end{aligned}$$

Therefore, by (3.13), $(1+|x|)^\delta Q_2^\pm(z) u$ belong to the Hardy-class $H^2(C^\pm; \mathcal{H})$ as the one-sided Laplace transforms of $L^2(R_t^1; \mathcal{H})$-functions.

(iii) is obvious by $Q^\pm(z)=Q_1^\pm(z)+Q_2^\pm(z)$. Q.E.D.

**Lemma 3.4.** *Put*

(3.16) $$Z^\pm = Z^\pm(t, s) = tH_1 \mp \operatorname{sgn}(t)\{X(\pm|s|)-X(\pm(|s|+|t|))\}$$

*for* $t, s \in R^1$, *where*

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{for } t>0 , \\ -1 & \text{for } t<0 . \end{cases}$$

*Then, for any* $u, v \in \mathcal{H}$, $\nu>0$ *and* $\mu \in R^1$, *we obtain*

(3.17) $$\begin{aligned}&f^\pm(\mu \pm i\nu; u, v) \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{it\mu-\nu|t|} 2\nu \int_{-\infty}^\infty \chi_\pm(s) e^{-2\nu|s|} (e^{-iZ^\pm(t,s)} u, v)_{\mathcal{H}} ds \, dt .\end{aligned}$$

Proof. First of all, we explain some notations which will be used in this proof. We shall use the notation $\mathcal{F}$ as the Fourier transform from $L^2(R_t^1; \mathcal{H})$ onto $L^2(R_\mu^1; \mathcal{H})$, that is,

$$(\mathcal{F} f)(\mu) = (2\pi)^{-1/2} \underset{N\to\infty}{\text{l.i.m.}} \int_{|t|\leq N} e^{-i\mu t} f(t) dt \qquad \text{in } L^2(R_\mu^1; \mathcal{H})$$

for any $f \in L^2(R_t^1; \mathcal{H})$. Moreover we shall use $\mathcal{F}_1^{-1}$ as the inverse Fourier transform from $L^2(R_t^1)$ onto $L^2(R_\mu^1)$.

Now let us prove Lemma 3.4. Let $u, v \in \mathcal{H}$ and $\nu>0$ be fixed. By definition,

$$f^\pm(\mu \pm i\nu; u, v) = \frac{\nu}{\pi} (S^\pm(\mu \pm i\nu) u, S^\pm(\mu \pm i\nu) v) .$$

Here

$$\begin{aligned} S^\pm(\mu \pm i\nu) u &= \pm\sqrt{2\pi}\, i (\mathcal{F}^{-1} g_\nu^\pm)(\mu) , \\ S^\pm(\mu \pm i\nu) v &= \pm\sqrt{2\pi}\, i (\mathcal{F} h_\nu^\pm)(\mu) , \end{aligned}$$

where

$$\left\{\begin{array}{l} g_\nu^\pm(t) = \chi_\pm(t)e^{-iX(t)}e^{-\nu|t|-itH_1}u \\ h_\nu^\pm(t) = \chi_\mp(t)e^{-iX(-t)}e^{-\nu|t|+itH_1}v \end{array}\right\} \in L^1(R_t^1; \mathcal{H}) \cap L^2(R_t^1; \mathcal{H}).$$

Then, for *a.e.* $\mu \in R^1$,

$$\begin{aligned} (3.18) \qquad & f^\pm(\mu \pm i\nu; u, v) \\ &= 2\nu((\mathcal{F}^{-1}g_\nu^\pm)(\mu), (\mathcal{F}h_\nu^\pm)(\mu))_{\mathcal{H}} \\ &= 2\nu(2\pi)^{-1/2}\Big(\mathcal{F}_1^{-1}\Big(\int_{-\infty}^\infty (g_\nu^\pm(t-s), h_\nu^\pm(s))_{\mathcal{H}}ds\Big)\Big)(\mu). \end{aligned}$$

Here we have

$$\begin{aligned} (3.19) \qquad & \int_{-\infty}^\infty (g_\nu^\pm(t-s), h_\nu^\pm(s))_{\mathcal{H}}ds \\ &= \int_{-\infty}^\infty \chi_\pm(t-s)\chi_\mp(s)e^{-\nu|t-s|-\nu|s|}(e^{-itH_1-iX(t-s)+iX(-s)}u, v)_{\mathcal{H}}ds \\ &= e^{-\nu|t|}\int_{-\infty}^\infty \chi_\pm(s)e^{-2\nu|s|}(e^{-iZ^\pm(t,s)}u, v)_{\mathcal{H}}ds. \end{aligned}$$

Then combining (3.18) and (3.19) we obtain the desired formula. Q.E.D.

**Lemma 3.5.** *For any $a>0$, there exists some constant $C_a>0$ such that for all $u \in \mathcal{D}_a$ and $t, s \in R^1$,*

$$(3.20) \qquad \|(1+|x|)^{-1}e^{-iZ^\pm(t,s)}u\| \le C_a(1+|t|)^{-1}\|(1+|x|)u\|.$$

Proof. We first list up some notations which will be used in this proof:

$$\begin{aligned} & X(t;\xi) = \int_0^t V(s\xi)ds, \\ & Q^\pm(t,s;\xi) = |\xi|^2 \mp \frac{1}{t}\operatorname{sgn}(t)\Big[|\xi|\Big(\frac{\partial}{\partial|\xi|}X\Big)(\pm|s|;\xi) \\ & \qquad\qquad - |\xi|\Big(\frac{\partial}{\partial|\xi|}X\Big)(\pm(|s|+|t|);\xi)\Big], \\ & Q^\pm(t,s) = \mathcal{F}^{-1}[Q^\pm(t,s;\xi)\cdot]\mathcal{F}, \\ & Z^\pm(t,s;\xi) = t\frac{|\xi|^2}{2} \mp \operatorname{sgn}(t)\{X(\pm|s|;\xi) - X(\pm(|s|+|t|);\xi)\}. \end{aligned}$$

Notice that $X(t) = \mathcal{F}^{-1}[X(t;\xi)\cdot]\mathcal{F}$ and $Z^\pm(t,s) = \mathcal{F}^{-1}[Z^\pm(t,s;\xi)\cdot]\mathcal{F}$.

Let $a>0$ be fixed. We divide the proof into four steps.

1st step) We prove the identity

$$(3.21) \qquad \langle x, D\rangle e^{-iZ^\pm(t,s)} - e^{-iZ^\pm(t,s)}\langle x, D\rangle = te^{-iZ^\pm(t,s)}Q^\pm(t,s),$$

where $D=(D_1, \cdots, D_n)$. This is proved as follows:

$$\mathcal{F}\langle x, D\rangle e^{-iZ^{\pm}(t,s)} = \langle i\nabla_{\xi}, \xi\rangle e^{-iZ^{\pm}(t,s;\xi)}\mathcal{F}$$
$$= (e^{-iZ^{\pm}(t,s;\xi)}\langle i\nabla_{\xi}, \xi\rangle - e^{-iZ^{\pm}(t,s;\xi)}\langle i\nabla_{\xi}(-iZ^{\pm}(t, s; \xi)), \xi\rangle)\mathcal{F}.$$

Here we get by straightforward computation

$$\langle i\nabla_{\xi}(-iZ^{\pm}(t, s; \xi)), \xi\rangle = tQ^{\pm}(t, s; \xi).$$

Thus we obtain

$$\mathcal{F}\langle x, D\rangle e^{-iZ^{\pm}(t,s)} = \mathcal{F}e^{-iZ^{\pm}(t,s)}\langle x, D\rangle - \mathcal{F}te^{-iZ^{\pm}(t,s)}Q^{\pm}(t, s).$$

This completes the proof of (3.21).

2nd step) Next we prove the following proposition:

There exists a positive constant $B_a$ such that for any $\xi \in E_a = \{\xi \in R^n \mid |\xi| \geq a\}$, $|t| \geq B_a$ and $s \in R^1$,

$$(3.22) \qquad |Q^{\pm}(t, s; \xi)| \geq \frac{|\xi|^2}{2}.$$

Put $f(r) = |\xi|\left(\frac{\partial}{\partial|\xi|}X\right)(r; \xi) = \sum_{j=1}^{n}\xi_j\left(\frac{\partial}{\partial\xi_j}X\right)(r; \xi)$. Then

$$(3.23) \qquad Q^{\pm}(t, s; \xi) - |\xi|^2 = \mp\frac{1}{t}\operatorname{sgn}(t)\{f(\pm|s|) - f(\pm(|s|+|t|))\}.$$

By the mean-value theorem, we get

$$(3.24) \qquad |f(\pm|s|) - f(\pm(|s|+|t|))| \leq nC_0(1+|s\xi|)^{-\beta}\cdot|t|.$$

Thus, combining (3.23) and (3.24), we obtain the following consequence: There exists some constant $A_a > 0$ such that for any $\xi \in E_a$, $t \in R^1$ and $|s| \geq A_a$,

$$|Q^{\pm}(t, s; \xi) - |\xi|^2| \leq \frac{a^2}{2}$$

and hence

$$(3.25) \qquad |Q^{\pm}(t, s; \xi)| \geq \frac{|\xi|^2}{2}.$$

On the other hand, we have by direct computation,

$$|f(r)| \leq C_1|\xi|^{-\beta}|r|^{1-\beta}$$

with $C_1 = nC_0/(1-\beta)$. Thus, we get

$$(3.26) \qquad |f(\pm|s|) - f(\pm(|s|+|t|))| \leq C_1|\xi|^{-\beta}(|s|^{1-\beta} + (|s|+|t|)^{1-\beta}).$$

So, combining (3.23) and (3.26), we obtain the following result: There exists some constant $B_a>0$ such that for any $\xi\in E_a$, $|t|\geq B_a$ and $|s|\leq A_a$,

$$|Q^{\pm}(t, s; \xi)|\geq\frac{|\xi|^2}{2}. \tag{3.27}$$

Thus by (3.25) and (3.27), we obtain the desired result.

3rd step) In this step we prove the following estimates:

There exists a positive constant $C_a'$ such that for any $u\in\mathcal{D}_a$, $|t|\geq B_a$ and $s\in R^1$,

$$\|(1+|x|)^{-1}e^{-iZ^{\pm}(t,s)}u\|\leq C_a'|t|^{-1}\|(1+|x|)u\|. \tag{3.28}$$

From (3.21) and (3.22), we have

$$\begin{aligned}&e^{-iZ^{\pm}(t,s)}u\\&=t^{-1}[\langle x, D\rangle e^{-iZ^{\pm}(t,s)}Q^{\pm}(t, s)^{-1}u-e^{-iZ^{\pm}(t,s)}\langle x, D\rangle Q^{\pm}(t, s)^{-1}u]\end{aligned}$$

for any $u\in\mathcal{D}_a$, $|t|\geq B_a$ and $s\in R^1$. Therefore, using the triangle inequality and the Plancherel's theorem, we get

$$\begin{aligned}&\|(1+|x|)^{-1}e^{-iZ^{\pm}(t,s)}u\|\\&\leq|t|^{-1}\Big[n\|Q^{\pm}(t, s; \xi)^{-1}|\xi|\hat{u}\|+n\|Q^{\pm}(t, s; \xi)^{-1}\hat{u}\|\\&\quad+\sum_{j=1}^{n}\Big\||\xi|Q^{\pm}(t, s; \xi)^{-1}\frac{\partial\hat{u}}{\partial\xi_j}\Big\|+\sum_{j=1}^{n}\Big\|\xi_j\frac{\partial}{\partial\xi_j}(Q^{\pm}(t, s; \xi)^{-1})u\Big\|\Big].\end{aligned}$$

All terms in the parentheses [ ] except the fourth one can be easily estimated by (3.22) and $u\in\mathcal{D}_a$. And hence we have

$$\begin{aligned}&\|(1+|x|)^{-1}e^{-iZ^{\pm}(t,s)}u\|\\&\leq|t|^{-1}\Big[2n(a^{-1}+a^{-2})\|u\|+2na^{-1}\|\,|x|u\|\\&\quad+\sum_{j=1}^{n}\Big\||\xi|Q^{\pm}(t, s; \xi)^{-2}\frac{\partial}{\partial\xi_j}(Q^{\pm}(t, s; \xi))\hat{u}\Big\|\Big].\end{aligned}$$

Thus we have only to estimate $\left|\frac{\partial}{\partial\xi_j}(Q^{\pm}(t, s; \xi))\right|$. For this purpose let us compute $\frac{\partial}{\partial\xi_j}(Q^{\pm}(t, s; \xi))$. Then we have

$$\begin{aligned}&\frac{\partial}{\partial\xi_j}(Q^{\pm}(t, s; \xi))\\&=2\xi_j\mp\frac{1}{t}\operatorname{sgn}(t)\Big[\Big\{\Big(\frac{\partial}{\partial\xi_j}X\Big)(\pm|s|; \xi)-\Big(\frac{\partial}{\partial\xi_j}X\Big)(\pm(|s|+|t|); \xi)\Big\}\\&\quad+\sum_{k=1}^{n}\Big\{\xi_k\Big(\frac{\partial^2}{\partial\xi_j\partial\xi_k}X\Big)(\pm|s|; \xi)-\xi_k\Big(\frac{\partial^2}{\partial\xi_j\partial\xi_k}X\Big)(\pm(|s|+|t|); \xi)\Big\}\Big].\end{aligned}$$

Applying the mean-value theorem as in the 2nd step to $n+1$ terms in the parentheses [ ], we get

$$\left|\frac{\partial}{\partial\xi_j}(Q^{\pm}(t, s; \xi))\right|\leq 2|\xi|+C_0|\xi|^{-1}+nC_0|\xi|^{-2}.$$

Thus we have from $u\in\mathcal{D}_a$ and (3.22),

$$\left\||\xi|Q^{\pm}(t, s; \xi)^{-2}\frac{\partial}{\partial\xi_j}(Q^{\pm}(t, s; \xi))\hat{u}\right\|\leq 2(2a^{-2}+C_0a^{-4}+nC_0a^{-5})\|u\|.$$

So, setting $C'_a=4n(a^{-1}+2a^{-2}+C_0a^{-4}(1+na^{-1}))$, we have the desired estimates.

4th step) Let us prove Lemma 3.5. If $u\in\mathcal{D}_a$, $|t|\leq B_a$ and $s\in R^1$, then we have

$$\|(1+|x|)^{-1}e^{-iZ^{\pm}(t,s)}u\|\leq(1+B_a)(1+|t|)^{-1}\|(1+|x|)u\|. \tag{3.29}$$

Thus, putting $C_a=\max\{(1+B_a), B_a^{-1}(1+B_a)C'_a\}$, we obtain (3.20) from (3.28) and (3.29). Q.E.D.

Proof of Proposition 2.4. By Lemma 3.5, there exists some constant $C>0$, such that for any $\nu>0$ and $t\in R^1$,

$$\left|2\nu\int_{-\infty}^{\infty}\chi_{\pm}(t)e^{-2\nu|s|}(e^{-iZ^{\pm}(t,s)}u, v)_{\mathcal{H}}ds\right|$$
$$\leq C(1+|t|)^{-1}\|(1+|x|)u\|\cdot\|(1+|x|)v\|.$$

The right-hand side of this inequality belongs to $L^2(R^1_t)$.

On the other hand, for any $t\in R^1$, we have

$$\begin{aligned}\lim_{\nu\to+0}2\nu\int_{-\infty}^{\infty}\chi_{\pm}(s)e^{-2\nu|s|}(e^{-iZ^{\pm}(t,s)}u, v)_{\mathcal{H}}ds\\ =\lim_{s\to\pm\infty}(e^{-iZ^{\pm}(t,s)}u, v)_{\mathcal{H}}\\ =(e^{-itH_1}u, v)_{\mathcal{H}}.\end{aligned} \tag{3.30}$$

Here $(e^{-itH_1}u, v)_{\mathcal{H}}\in L^1(R^1_t)\cap L^2(R^1_t)$ by (i) of Lemma 3.2.

Therefore, by the domianted convergence theorem of Lebesgue, there exist the limits

$$\underset{\nu\to+0}{\text{l.i.m.}}\, e^{-\nu|t|}2\nu\int_{-\infty}^{\infty}\chi_{\pm}(s)e^{-2\nu|s|}(e^{-iZ^{\pm}(t,s)}u, v)_{\mathcal{H}}ds \qquad \text{in } L^2(R^1_t)$$

and these are equal to $(e^{-itH_1}u, v)_{\mathcal{H}}$ in $L^2(R^1_t)$. Thus by Lemma 3.4 there exist the limits $\underset{\nu\to+0}{\text{l.i.m.}}\, f^{\pm}(\mu\pm i\nu; u,v)$ in $L^2(R^1_\mu)$ and we have

$$\underset{\nu\to+0}{\text{l.i.m.}}\, f^{\pm}(\mu\pm i\nu; u, v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{it\mu}(e^{-itH_1}u, v)_{\mathcal{H}}dt \qquad \text{in } L^2(R^1_\mu).$$

The right-hand side of this equality is equal to $e_1(\mu; u, v)$ for $\mu \in \Gamma$. Q.E.D.

## 4. Concluding remarks

In the above, we have constructed the modified wave operators by a stationary method under Assumption 1.1. But, if we replace $X(t)$ by some appropriate self-adjoint operator $X'(t)$, this assumption can be weakened in the direction that it admits longer range potentials. This will be discussed elsewhere.

Next we remark on the time limits and the invariance principle. Using our method, we can show that there exist the time limits

$$W_D^{\pm}(\Gamma) = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_2} e^{-itH_1 - iX(t)} E_{1,ac}(\Gamma)$$

and that $W_D^{\pm}(\Gamma) = W_\Gamma^{\pm}$. Furthermore we can prove that the invariance principle holds under Assumption 1.1. This result is formulated as the following

**Theorem 4.1.** *Let Assumption* 1.1 *be satisfied. Let* $\varphi \in C^\infty(R^1)$ *be real-valued and put*

$$W_\varphi^{as}(t) = e^{it\varphi(H_2)} e^{-it\varphi(H_1) - iX(t\varphi'(H_1))}$$

*for every* $t \in R^1$. *Suppose, further, that* $\varphi' > 0$ *and* $\varphi'' \neq 0$ *on some open neighbourhood of* $\Gamma$. *Then the limits*

$$W_\varphi^{\pm,as}(\Gamma) = \underset{t\to\pm\infty}{\text{s-lim}}\, W_\varphi^{as}(t) E_{1,ac}(\Gamma)$$

*exist and the following relations hold:*

$$W_\varphi^{\pm,as}(\Gamma) = W_\Gamma^{\pm}.$$

This theorem also will be discussed elsewhere together with the existence of the time limits mentioned above.

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