UNIFORM APPROXIMATION BY ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Dedicated to Professor Yukinari Tôki on his 70th birthday

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Introduction. Let G be a holomorphically convex open subset of C^n and T a closed subset of G. We say that T is *totally real*, if it is the zero set of a non-negative C^2 function ρ which is strictly plurisubharmonic on T. It is known that a real C^1 submanifold M is totally real if and only if it has no complex tangents (cf. [3]). The problem of uniform approximation on totally real sub-manifols was studied to a great extent by many authors (cf. Wells [9], Hörmander and Wermer [4], Nirenberg and Wells [5], Harvey and Wells [2], [3] and Nunemacher [6]). The result of [6] states that if M is a totally real submanifold then there exists a holomorphically convex open neighborhood B such that every continuous function on M is uniformly approximated on M by functions holomorphic in B. In [8], the author extended this result to the case of totally real sets with C^∞ defining functions. (A totally real set is not necessarily a sub-manifold. The approximation theorem for totally real sets contains one for totally real analytic subvarieties which was conjectured by Wells [9].)

In this paper, we give a sufficient condition for T and G under which every continuous function on T is uniformly approximated on T by functions holomorphic in G. The theorem we prove contains the following result which is a straight generalization to higher dimensions of Carleman's theorem [1].

Every continuous function on R^n, canonically imbedded in C^n, is uniformly approximated on R^n by entire functions on n complex variables.

We shall make use of the L^2-method due to Hörmander and Wermer [4] and the swelling method similar to one used in [8].

1. Statements. Let S be a closed subset of an open set U of C^n. We denote by $H(S)$ (or $H(S, U)$) the algebra of uniform limits of restrictions of functions holomorphic in a neighborhood of S (or in U, resp.).

We use an abbreviation $L[u; \xi]$ for the Levi form of a C^∞ function u:

$$L[u; \xi] = \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k, \qquad \xi \in C^n.$$

By an exhaustion function σ of G we mean a positive C^∞ strictly plurisubharmonic

function which maps properly G into $\boldsymbol{R}$. We define a form

$$A[\sigma; \xi] = \frac{1}{2\sigma} L[\sigma^2; \xi]$$

$$= L[\sigma; \xi] + \frac{1}{\sigma} \left| \sum_j \frac{\partial \sigma}{\partial z_j} \xi_j \right|^2, \quad \xi \in \boldsymbol{C}^n.$$

Theorem. *Let G be a holomorphically convex open subset of $\boldsymbol{C}^n$ and σ be an exhaustion function of G. If T is the zero set of a nonnegative C^∞ function ρ on G satisfying*

$$(1) \qquad\qquad L[\rho; \xi] \geq cA[\sigma; \xi], \quad \xi \in \boldsymbol{C}^n,$$

for some constant $c > 0$, then $H(T, G) = C(T)$.

When G is $\boldsymbol{C}^n$, this is a uniform approximation theorem by entire functions. In this case, we can choose $\sigma(z) = |z|^2 + 1$ as an exhaustion function of $\boldsymbol{C}^n$ and we have $|\xi|^2 \leq A[\sigma; \xi] \leq 2|\xi|^2$, $\xi \in \boldsymbol{C}^n$. Therefore, we obtain

Corollary 1. *If T is the zero set of a nonnegative C^∞ function ρ on $\boldsymbol{C}^n$ satisfying*

$$(2) \qquad\qquad L[\rho; \xi] \geq c|\xi|^2, \quad \xi \in \boldsymbol{C}^n,$$

with some constant $c > 0$, then $H(T, \boldsymbol{C}^n) = C(T)$.

If we write $\boldsymbol{R}^n = \{z; y_j = 0, j = 1, \cdots, n\}$, then $\rho(z) = \sum_j |y_j|^2$ is a defining function of $\boldsymbol{R}^n$ satisfying (2). Thus we obtain the following corollary.

Corollary 2. $H(\boldsymbol{R}^n, \boldsymbol{C}^n) = C(\boldsymbol{R}^n)$.

The proof of Theorem is based on the following lemma essentially due to [4]. (For the proof, see Proposition 1 of [7].)

Lemma 1. *Let δ be a nonnegative function defined in an open set V in $\boldsymbol{C}^n$. Suppose K is a compact subset of V satisfying the following condition: There exists a constant $\eta > 0$ such that for every sufficiently small $\varepsilon > 0$, we can find a holomorphically convex open set V_ε satisfying*

$$\{z: dist(z, K) < \varepsilon\} \subset V_\varepsilon \subset \{z: \delta(z) < \varepsilon\eta\}.$$

If F is a a C^∞ function on V satisfying

$$|\bar{\partial} F(z)| \leq c\delta(z)^{n+1}, \quad z \in V,$$

then $F|_K$ belongs to $H(K)$.

2. Construction of an exhaustion $\{K_m\}$ of G. Let σ and ρ be func-

tions satsifying the assumption of the theorem. For every positive number r, the open set $G_r = \{z \in G_r : \sigma(z) < r\}$ is relatively compact in G.

Let λ be a C^∞ function: $\mathbf{R} \to [0, 1]$ such that $\lambda(t) = 1$ ($t < 0$) and $\lambda(t) = 0$ ($t > 2$). For every positive number m, we set

$$\lambda_m(z) = \lambda(\sigma(z)/m).$$

Then we have

$$L[\lambda_m : \xi] = \frac{1}{m} \left\{ \lambda' L[\sigma; \xi] + \frac{\lambda''}{m} \left| \sum_j \frac{\partial \sigma}{\partial z_j} \xi_j \right|^2 \right\}$$

$$\leq \frac{a}{m} A[\sigma; \xi], \quad \xi \in \mathbf{C}^n,$$

with $a = \sup \{ |\lambda'| + 2|\lambda''| + 1 \}$, since $\lambda_m(z) = 0$ for $z \in G \setminus G_{2m}$.

We set $\rho_0 = \rho$ and $\rho_m = \rho - m\lambda_m$ for $m > 1$. Since we may assume that $L[\rho; \xi] \geq 2aA[\sigma; \xi]$, $\xi \in \mathbf{C}^n$, multiplying ρ by a constant if necessary, we have

$$L[\rho_m; \xi] \geq aA[\sigma; \xi], \quad \xi \in \mathbf{C}^n.$$

For each nonnegative integer m, we define the compact set $K_m = \{z \in \bar{G}_{2m+3} : \rho_m(z) \leq 0\}$. It is easy to show that $K_m \subset K_{m+1}$ and $\bigcup_m K_m = G$.

3. Approximation on K_m.

In this section, we fix a nonnegative integer m. We shall prove the following lemma.

Lemma 2. *If f is a C^∞ function, then $f|_{K_0} \in H(K_0, G)$. If f is a C^∞ function which is holomorphic in an open neighborhood of $\bar{G}_{2m}$, $m > 0$, then $f|_{K_m} \in H(K_m, G)$.*

Proof. Since ρ_m is strictly plurisubharmonic in G and since $K_m = \{\rho_m \leq 0\} \cap \{\sigma \leq 2m+3\}$, K_m is $\mathcal{O}_G$-convex and therefore we have $H(K_m) = H(K_m, G)$. It suffices to prove that $f|_{K_m} \in H(K_m)$.

Let ψ be a C^∞ function satsifying $\psi = 1$ in an open neighborhood of $\bar{G}_{2m}$ and $\psi = 0$ in $G \setminus \bar{G}_{2m+1}$. We consider the function

$$\delta_m = \psi \rho_m + (1 - \psi) \sum_\nu \left| \frac{\partial \rho}{\partial z_\nu} \right|^2.$$

If $z \in G_{2m}$, we have $L[\delta_m; \xi] = L[\rho_m : \xi]$, $\xi \neq 0$. If $z \in T \setminus G_{2m}$ then $\rho_m = \rho = 0$ and $d\rho = 0$. Hence we have

$$L[\delta_m; \xi] \geq \psi L[\rho; \xi] + (1 - \psi) L[\rho; \xi] |\xi|^{-2} > 0, \quad \xi \neq 0.$$

Therefore we can find an open neighborhood Ω_m of K_m so that δ_m is strictly plurisubharmonic in Ω_m. There exists a constant $\eta > 0$ such that $\delta_m(z) \leq \eta \, \text{dist}(z, K_m)$ and $\sigma(z) \leq 2m+3+\eta \, \text{dist}(z, G_{2m+3})$. If we set $\delta(z) = \max \{0, \delta_m(z)\}$

and $V_\varepsilon = \{z \in \Omega_m : \delta_m(z) < \varepsilon\eta\} \cap G_{2m+3+\varepsilon}$, then, for sufficiently small $\varepsilon > 0$, V_ε is holomorphically convex and satisfies

$$\{z : \mathrm{dist}\,(z, K_m) < \varepsilon\} \subset V_\varepsilon \subset \{z : \delta(z) < \varepsilon\eta\}\ .$$

We can now find a C^∞ extension F of $f|_T$ on G which satisfies

$$|\bar\partial F(z)| \leq c\delta(z)^{n+1}\ , \quad z \in V\backslash K_m$$

for an open neighborhood V of K_m and for some positive constant c. The way of construction of F is the same as one in Lemma 6 of [7]. We note that, if f is holomorphic in an open neighborhood U of $\bar{G}_{2m}$, then F is holomorphic in U. By Lemma 1, we have $f|_{K_m} = F|_{K_m} \in H(K_m)$, which proves the lemma.

4. Global approximation. Let f be an arbitrary function in $C^\infty(G)$ and let ε be any positive number. We shall construct a sequence $\{f_m\}$ of functions holomorphic in G and satisfying

$$|f_m - f_{m-1}| < 2^{-m-1}\varepsilon \quad\text{on}\quad K_m$$

and

$$|f_m - f| < \sum_{\nu=1}^{m+1} 2^{-\nu}\varepsilon \quad\text{on}\quad T \cap \bar{G}_{2m+3}\ .$$

We define the function $f_\varepsilon = \lim f_m$. A standard argument shows that f_ε is holomorphic in G and that $|f_\varepsilon - f| < \varepsilon$ on T.

The construction of $\{f_m\}$ is as follows. By Lemma 2, we can find a function f_0 holomorphic in G such that

$$|f_0 - f| < 2^{-1}\varepsilon \quad\text{on}\quad K_0 = T \cap \bar{G}_3$$

Suppose f_j, $j = 1, \cdots, m-1$ are already defined. Let ψ be a C^∞ function: $G \to [0, 1]$ satisfying $\psi = 1$ in an open neighborhood U of $\bar{G}_{2m}$ and $\psi = 0$ in $G \backslash G_{2m+1}$. Set $g = \psi f_{m-1} + (1-\psi)f$. Then g is holomorphic in U. By Lemma 2, we can find a function f_m holomorphic in G so that

$$|f_m - g| < 2^{-m-1}\varepsilon \quad\text{on}\quad K_m\ .$$

Since $g = f_{m-1}$ in U and $K_m \subset U$, we have

$$|f_m - f_{m-1}| < 2^{-m-1}\varepsilon \quad\text{on}\quad K_m\ .$$

Since $|g - f| = \psi|f_{m-1} - f| < \sum_{\nu=1}^{m} 2^{-\nu}\varepsilon$ on $T \cap \bar{G}_{2m+1}$ and since $g = f$ on $T \backslash G_{2m+1}$, we have

$$|f_m - f| < (2^{-m-1} + \sum_{\nu=1}^{m} 2^{-\nu})\varepsilon \quad\text{on}\quad T \cap \bar{G}_{2m+3}\ .$$

This completes the proof of the theorem.

REMARK 1. The question arises whether the same conclusion as Theorem can be obtained under the condition that ρ is C^∞ strictly plurisubharmonic in G. (There is a simple example of T such that every defining function of T is not strictly plurisubharmonic in G and such that $H(T, G) \neq C(T)$.) When T is compact this condition is sufficient. This follows at once from Theorem 2 of [7] and the fact that T is then $\mathcal{O}_G$-convex. We do not know whether it is true even when T is not assumed to be compact.

REMARK 2. It is reasonable to conjecture that the theorem will be valid even when a defining function ρ of T is of class C^2. In fact, when T is a submanifold, C^2-diffrentiability of ρ is sufficient to derive the approximation by functions holomorphic in a neighborhood of T (c.f. Harvey-Wells [2] and Nunemacher [6]). The C^∞ differentiability assumption in this paper was necessary because of the L^2-method we employed.

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