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# TWISTED ALEXANDER POLYNOMIAL OF A RIBBON 2-KNOT OF 1-FUSION

Taizo KANENOBU and Toshio SUMI

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**Abstract**

The twisted Alexander polynomial is defined as a rational function, not necessarily a polynomial. It is shown that for a ribbon 2-knot, the twisted Alexander polynomial associated to an irreducible representation of the knot group to $\mathrm{SL}(2,\mathbb{F})$ is always a polynomial. Furthermore, the twisted Alexander polynomial of a fibered ribbon 2-knot of 1-fusion has the lowest and highest degree coefficients 1 with breadth $2m-2$, where $m$ is the breadth of its Alexander polynomial.

## 1. Introduction

The twisted Alexander polynomial was introduced by Lin [19] and Wada [28], which not only are useful in classifying classical knots but have wide applications and generalizations; see [6, 17, 21]. In the previous papers [13, 14] we have seen the twisted Alexander polynomial are also useful in classifying the ribbon 2-knots. Kitano and Morifuji [18] proved that the twisted Alexander polynomial of a classical knot group associated to any nonabelian $\mathrm{SL}(2,\mathbb{F})$-representation is a polynomial, and as a corollary, they showed that it is always a monic polynomial of degree $4g-2$ for a fibered knot of genus $g$. In this paper we consider analogous properties of the twisted Alexander polynomial for a ribbon 2-knot.

A ribbon 2-knot is an embedded 2-sphere in $S^4$ obtained by adding $r$ 1-handles to a trivial 2-link with $r+1$ components for some $r$, which is called a ribbon 2-knot of $r$-fusion. Note that any classical knot group is a ribbon 2-knot group; cf. Proposition 2.1. We show the twisted Alexander polynomial of a ribbon 2-knot group associated to an irreducible $\mathrm{SL}(2,\mathbb{F})$-representation is a polynomial (Theorem 4.2). Then, we give a condition for the twisted Alexander polynomial of a ribbon 2-knot group associated to a reducible $\mathrm{SL}(2,\mathbb{C})$-representation to be a polynomial (Theorem 4.5). We give an example of ribbon 2-knot of 1-fusion having a nonabelian reducible representation to $\mathrm{SL}(2,\mathbb{C})$ whose twisted Alexander polynomial is not a Laurent polynomial (Example 6.2).

Next, we show that the twisted Alexander polynomial of a fibered ribbon 2-knot of 1-fusion associated with an irreducible $\mathrm{SL}(2,\mathbb{F})$-representation has the lowest and highest degree coefficients 1 with breadth $2m-2$, where $m$ is the breadth of its Alexander polynomial (Corollary 5.8). We give examples of non-fibered ribbon 2-knots whose Alexander polynomial has lowest and highest degree coefficients $\pm 1$ (Examples 6.5 and 6.6). They do not satisfy the conditions of the coefficients and the breadths of the twisted Alexander polynomials of a fibered ribbon 2-knot. It is known a classical knot is fibered if and only if

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the commutator subgroup of the knot group is finitely generated. Yoshikawa [31] showed a ribbon 2-knot of 1-fusion has a similar property (Proposition 5.3). Using this we have confirmed that for a ribbon 2-knot $K$ presented by virtual arcs with up to four crossings $K$ listed in [12, Table 2] is fibered if and only if its Alexander polynomial is nontrivial and the lowest and highest degree coefficients are $\pm 1$ (Theorem 6.4).

This paper is organized as follows: In Sect. 2 we review a ribbon 2-knot. In Sect. 3 we define the twisted Alexander polynomial for a ribbon 2-knot. In Sect. 4 we discuss the divisibility of the twisted Alexander polynomial for a ribbon 2-knot; we prove above-mentioned Theorems 4.2 and 4.5. In Sect. 5 we give some properties of the twisted Alexander polynomial for a ribbon 2-knot of 1-fusion. In Sect. 6 we give some examples of the twisted Alexander polynomials of ribbon 2-knots of 1-fusion related to Sects. 4 and 5.

## 2. Ribbon 2-knot

A ribbon 2-knot is an embedded 2-sphere in $S^4$ obtained by adding $r$ 1-handles to a trivial 2-link with $r+1$ components for some integer $r$; cf. [15, p. 178]. The fusion number of a ribbon 2-knot $K$ is the least number of $r$ possible for $K$. Yajima [29] characterized the knot group $G(K) = \pi_1(S^4 - K)$ of a ribbon 2-knot $K$.

**Proposition 2.1.** *A finitely presented group $G$ is the group of some ribbon 2-knot $K$, $G \cong G(K)$ if and only if* (i) $G/G' \cong \mathbb{Z}$*, where $G'$ is the commutator subgroup of $G$, and* (ii) *$G$ has a Wirtinger presentation of deficiency one.*

A group presentation $\langle\, x_1, \dots, x_n \,|\, r_1, \dots, r_m \,\rangle$ is called a *Wirtinger presentation*, if each relator is described in a form $w_{ij} x_i w_{ij}^{-1} x_j^{-1}$ where $w_{ij}$ is a word of the free group $\langle x_1, \dots, x_n \rangle$; cf. [4, p. 86]. Note that a classical knot group is a ribbon 2-knot group by Artin's spinning construction; cf. Sect. 3J in [26].

A ribbon 2-knot of 1-fusion is presented as a 2-knot $R(p_1, q_1, \dots, p_n, q_n)$ for some integers $p_1, q_1, \dots, p_n, q_n$, which is given by the moving pictures as follows: Let $L_0 = S^1_0 \cup S^1_1$ be a trivial 2-component link in $\mathbb{R}^3$. We add a band $B$ to $L_0$ as shown in Fig. 1, where $\tau_{p_1}, \dots, \tau_{p_n}$, $\sigma_{q_1}, \dots, \sigma_{q_n}$ are pairs $(D^3, a \cup \beta)$ of a 3-ball $D^3$ and a properly embedded arc $a$ and band $\beta$ as shown in Fig. 2. Regard the band $B$ as the image of an embedding $b: I \times I \to \mathbb{R}^3$, $B = b(I \times I)$, so that $S^1_i \cap b(I \times I) = b(I \times \{i\})$, $i = 0, 1$, where $I$ is the unit interval $[0, 1]$. We take disjoint 2-disks $D_0 \cup D_1$ in $\mathbb{R}^3$ so that $S^1_i = \partial D_i$, $i = 0, 1$. Let $K_0 = (L_0 - b(I \times \partial I)) \cup b(\partial I \times I)$. Then we define the ribbon 2-knot $R(p_1, q_1, \dots, p_n, q_n)$ in $S^4 = \mathbb{R}^4 \cup \{\infty\}$ by:

$$R(p_1, q_1, \dots, p_n, q_n) \cap (\mathbb{R}^3 \times \{t\}) = \begin{cases} K_0 & \text{for } |t| < 1; \\ K_0 \cup B = L_0 \cup B & \text{for } |t| = 1; \\ L_0 & \text{for } 1 < |t| < 2; \\ D_0 \cup D_1 & \text{for } |t| = 2; \\ \emptyset & \text{for } |t| > 2, \end{cases}$$

The knot group of $K = R(p_1, q_1, \dots, p_n, q_n)$ has a presentation

$$\langle\, x, y \mid w x w^{-1} y^{-1} \,\rangle, \quad w = x^{p_1} y^{q_1} \cdots x^{p_n} y^{q_n}, \tag{1}$$

where $x$ and $y$ are meridians of $S^1_0$ and $S^1_1$, respectively. We denote this group presentation

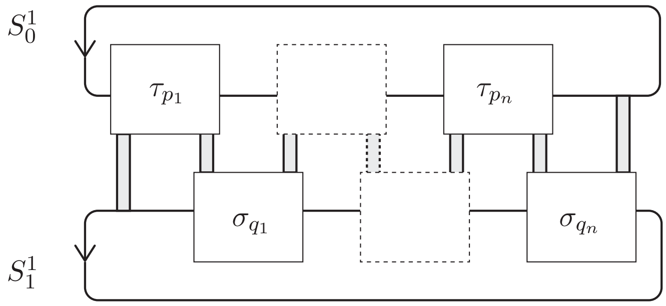


Fig. 1. Adding a band $B$ to a trivial link $L_0 = S_0^1 \cup S_1^1$.

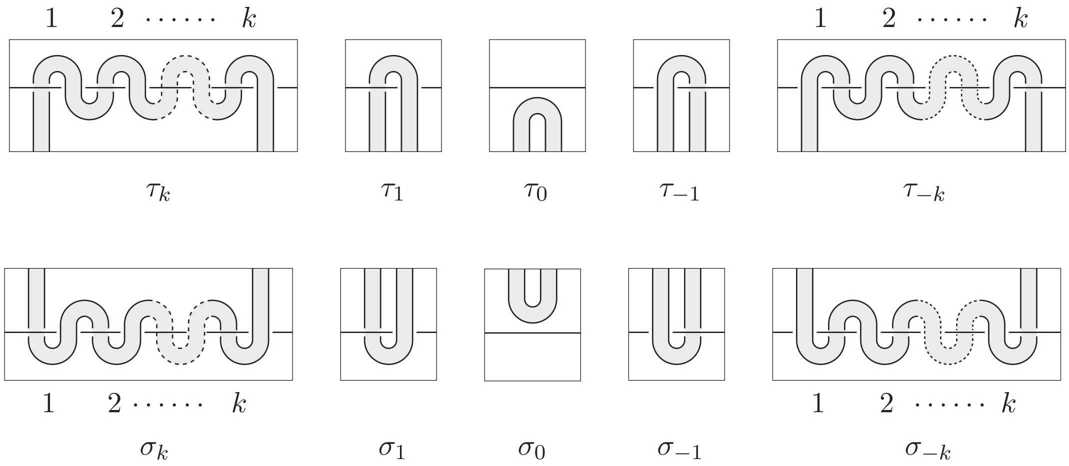


Fig. 2. Embedded arc and band, $\tau_p$ and $\sigma_q$.

by $G(p_1, q_1, \ldots, p_n, q_n)$.

The Alexander polynomial of a ribbon 2-knot $K$, $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$, is defined up to $\pm t^n$. The Alexander polynomial of the ribbon 2-knot $R(p_1, q_1, \ldots, p_n, q_n)$ is given as follows:

$$
\begin{aligned}
\Delta_K(t) &= t^{-q_1-q_2-\cdots-q_n}\bigl(1 - t^{p_1} + t^{p_1+q_1} - t^{p_1+q_1+p_2} + \cdots \\
&\qquad\qquad - t^{p_1+q_1+p_2+\cdots+p_n} + t^{p_1+q_1+p_2+\cdots+p_n+q_n}\bigr) \\
&= t^{p_n+p_{n-1}+\cdots+p_1}\bigl(1 - t^{-q_n} + t^{-q_n-p_n} - t^{-q_n-p_n-q_{n-1}} + \cdots \\
&\qquad\qquad - t^{-q_n-p_n-q_{n-1}-\cdots-q_1} + t^{-q_n-p_n-q_{n-1}-\cdots-q_1-p_1}\bigr),
\end{aligned} \tag{2}
$$

where we normalize so that $\Delta_K(1) = 1$ and $(d/dt)\Delta_K(1) = 0$; cf. [11, 16, 20].

## 3. Twisted Alexander polynomial

Let $K$ be a ribbon 2-knot with knot group $G = G(K)$ and $\rho : G \to \mathrm{SL}(n, \mathbb{F})$ a representation, where $\mathbb{F}$ is a field. Suppose $G$ is presented by a Wirtinger presentation $\langle x_0, x_1, \ldots, x_m \mid r_1, \ldots, r_m \rangle$, where $x_i$ are meridians. Let $\phi \colon F \to G$ be the canonical homomorphism, which induces the ring homomorphism $\tilde{\phi} : \mathbb{Z}F \to \mathbb{Z}G$, where $F = \langle x_0, x_1, \ldots, x_m \rangle$ is the free group with free basis $\{x_0, x_1, \ldots, x_m\}$. Let $\alpha : G \to G/G' \cong \langle t \rangle$ be the abelianization given by $\alpha(x_i) = t$, $0 \leq i \leq m$. Then $\rho$ and $\alpha$ induce the ring homomorphisms $\tilde{\alpha} : \mathbb{Z}G \to \mathbb{Z}[t, t^{-1}]$ and

$\tilde{\rho} : \mathbb{Z}G \to M(n, \mathbb{F})$, where $M(n, \mathbb{F})$ is the set of $n \times n$ matrices over $\mathbb{F}$. Now, we define a ring homomorphism $\Phi_\rho = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi}$ as follows:

(3)
$$\begin{aligned} \Phi_\rho : \mathbb{Z}F &\xrightarrow{\quad\tilde{\phi}\quad} \mathbb{Z}G \xrightarrow{\quad\tilde{\rho}\otimes\tilde{\alpha}\quad} M(n, \mathbb{F}[t, t^{-1}]) \\ \frac{\partial r_i}{\partial x_j} &\longmapsto \sum_{g \in G} \mu_{i,g} g \longmapsto \sum_{g \in G} \mu_{i,g} \alpha(g) \rho(g), \end{aligned}$$

where $\partial/\partial x_j$ denotes the Fox derivation and $\mu_{i,g} \in \mathbb{Z}$. Let $A_{\rho,x_0} = \left(\Phi_\rho(\partial r_i/\partial x_j)\right)_{1 \le i,j \le m} \in M(mn, \mathbb{F}[t, t^{-1}])$. Denote by $I_n$ the identity matrix of order $n$. Then the twisted Alexander polynomial of $K$ associated to the representation $\rho$ is defined to be a rational function

(4)
$$\Delta_{G,\rho}(t) = \frac{\det A_{\rho,x_0}}{\det(I_n - t\rho(x_0))},$$

which is well-defined up to a factor $(\pm 1)^n t^{nk}$, $k \in \mathbb{Z}$. We also denote this by $\Delta_{K,\rho}(t)$. In general, $\det A_{\rho,x_0}/\det(I_n - t\rho(x_0))$ is not a Laurent polynomial. If it is a Laurent polynomial, the twisted Alexander polynomial is presented as a polynomial. See [17, 28] for more details.

Note that we can use relations instead of relators in this definition. In fact, let $r_i = r_{i1} r_{i2}^{-1}$ in $F$ for each $i$. Then,

(5)
$$\frac{\partial r_i}{\partial x_j} = \frac{\partial r_{i1}}{\partial x_j} + r_{i1} \frac{\partial r_{i2}^{-1}}{\partial x_j} = \frac{\partial r_{i1}}{\partial x_j} - r_i \frac{\partial r_{i2}}{\partial x_j},$$

and so

(6)
$$\Phi_\rho\left(\frac{\partial r_i}{\partial x_j}\right) = \Phi_\rho\left(\frac{\partial r_{i1}}{\partial x_j}\right) - \Phi_\rho(r_i)\Phi_\rho\left(\frac{\partial r_{i2}}{\partial x_j}\right) = \Phi_\rho\left(\frac{\partial (r_{i1} - r_{i2})}{\partial x_j}\right).$$

## 4. Divisibility of the twisted Alexander polynomials of a ribbon 2-knot

For the twisted Alexander polynomial of a classical knot Kitano and Morifuji showed the following [18, Theorem 1.1].

**Proposition 4.1.** *Let $K$ be a classical knot and $G = \pi_1(S^3 - K)$. If $\rho : G \to \mathrm{SL}(2, \mathbb{F})$ is a nonabelian representation, then the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is a Laurent polynomial with coefficients in $\mathbb{F}$.*

We have an analogous result for a ribbon 2-knot with $\rho$ an irreducible representation.

**Theorem 4.2.** *Let $K$ be a ribbon 2-knot and $\rho : G(K) \to \mathrm{SL}(2, \mathbb{F})$ an irreducible representation. Then the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is a Laurent polynomial with coefficients in $\mathbb{F}$.*

In Example 6.2 we will give a ribbon 2-knot of 1-fusion having a nonabelian reducible representation to $\mathrm{SL}(2, \mathbb{C})$ whose twisted Alexander polynomial is not a Laurent polynomial. Theorem 4.2 follows from Proposition 2.1 and Propositions 4.3 and 4.4 below.

**Proposition 4.3.** *Let $G$ be a finitely presentable group with $G/G' \cong \mathbb{Z}$. If $\rho : G \to \mathrm{SL}(2, \mathbb{F})$ is an irreducible representation, then there exists an element $c$ of $G'$ whose image $\rho(c)$ does not have the eigenvalue 1.*

**Proposition 4.4.** *Suppose that a group $G$ admits a finite presentation of deficiency one with $G/G' \cong \mathbb{Z}$. Let $\rho : G \to \mathrm{SL}(n, \mathbb{F})$ be a representation such that there exists an element $c$ of the commutator subgroup $G'$ of $G$, whose image $\rho(c)$ does not have the eigenvalue 1. Then the twisted Alexander polynomial $\Delta_{G,\rho}(t)$ is a Laurent polynomial with coefficients in $\mathbb{F}$.*

For the proof of Proposition 4.3, see the first and second paragraphs of the proof of Theorem 3.1 in [18]. The proof of Proposition 4.4 is parallel to that of Proposition 8 in [28].

For a ribbon 2-knot $K$ with $\rho : G(K) \to \mathrm{SL}(2, \mathbb{C})$ a reducible representation we have the following.

**Theorem 4.5.** *Let $K$ be a ribbon 2-knot and $\rho\colon G(K) \to \mathrm{SL}(2, \mathbb{C})$ a reducible representation. Then the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is a Laurent polynomial if and only if either* (i) *there exists $c \in G(K)'$ such that $\rho(c)$ does not have the eigenvalue* 1*, or* (ii) $\Delta_K(e^2) = 0$ *for any eigenvalue $e$ of $\rho(x)$, where $x$ is a meridian.*

*In particular, if the Alexander polynomial $\Delta_K(t)$ is reciprocal and $\rho$ is nonabelian, then $\Delta_{K,\rho}(t)$ is a Laurent polynomial.*

We divide the proof of Theorems 4.2 and 4.5 into a sequence of lemmas. Let $K$ be a ribbon 2-knot and $G = G(K)$ presented by a deficiency one Wirtinger presentation

$$\langle\, x_0, x_1, \dots, x_m \mid r_1, \dots, r_m \,\rangle, \tag{7}$$

where each relator $r_k$ is of the form $x_i^{-1} x_l^{\epsilon} x_j x_l^{-\epsilon}$, $\epsilon = \pm 1$. In fact, any group with a Wirtinger presentation given in Sect. 2 can be deformed into this form by a sequence of Tietze transformations. We use the notation introduced in Sect. 2. Suppose that $\rho : G \to \mathrm{SL}(2, \mathbb{F})$ is a representation such that the twisted Alexander polynomial $\Delta_{G,\rho}(t)$ is not a Laurent polynomial. By Propositions 4.3 and 4.4, $\rho$ is reducible and 1 is an eigenvalue of $\rho(c)$ for any $c \in G'$. Then we may assume that

$$\rho(x_i) = \begin{pmatrix} s_i & u_i \\ 0 & s_i^{-1} \end{pmatrix} \tag{8}$$

for $i = 0, 1, \dots, m$. Furthermore, since $x_i x_j^{-1} \in G(K)'$, we have $s_0 = s_1 = \cdots = s_n$, which we write $s$. The Alexander polynomial $\Delta_K(t)$ is given by

$$\det\left(\tilde{\alpha}\tilde{\phi}\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1\le i,j\le m}. \tag{9}$$

Then we have the following.

**Lemma 4.6.**

$$\Delta_{G,\rho}(t) = \frac{\Delta(st)\Delta(s^{-1}t)}{(s-t)(s^{-1}-t)}. \tag{10}$$

Proof. Interchanging rows and columns appropriately, the matrix $A_{\rho,x_0}$ becomes

$$\begin{pmatrix} \tilde{\alpha}_s\tilde{\phi}(\partial r_i/\partial x_j) & * \\ 0 & \tilde{\alpha}_{s^{-1}}\tilde{\phi}(\partial r_i/\partial x_j) \end{pmatrix}_{1\le i,j\le m}, \tag{11}$$

where $\tilde{\alpha}_c$ is a linear expansion of $x_i \mapsto c\alpha(x_i) = ct$. The time of interchanging columns and that of interchanging rows are the same, and so we have $\det A_{\rho,x_0} = \Delta(st)\Delta(s^{-1}t)$. Since $\det\left(I - t\begin{pmatrix} s & u_0 \\ 0 & s^{-1}\end{pmatrix}\right) = (1-st)(1-s^{-1}t) = (s-t)(s^{-1}-t)$, we obtain Eq. (10). □

Since $\Delta_{G,\rho}(t)$ is not a Laurent polynomial, Lemma 4.6 immediately implies the following.

**Lemma 4.7.** *It holds that* $\Delta_K(s^2) \neq 0$ *or* $\Delta_K(s^{-2}) \neq 0$.

**Lemma 4.8.** *If* $\rho$ *is nonabelian, then* $\Delta_K(s^2) = 0$.

Proof. Let $M(t) = \left(\tilde{\alpha}\tilde{\phi}(\partial r_i/\partial x_j)\right)$ be the Alexander $m \times (m+1)$ matrix. By the fundamental property

$$w - 1 = \sum_{j=0}^{m} \frac{\partial w}{\partial x_j}(x_j - 1), \tag{12}$$

we see $0 = (t-1)\sum_{j=0}^{m} \tilde{\alpha}\tilde{\phi}(\partial r_i/\partial x_j)$ and then $0 = \sum_{j=0}^{m} \tilde{\alpha}\tilde{\phi}(\partial r_i/\partial x_j)$. Thus, $M(t)\mathbf{1} = 0$, where $\mathbf{1} = {}^t(1,\dots,1)$. If $r_i = x_k^{-1}x_j^{\epsilon}x_l x_j^{-\epsilon}$, then $\rho(x_j^{\epsilon}x_l) = \rho(x_k x_j^{\epsilon})$, which implies

$$(1 - s^{2\epsilon})u_j - u_k + s^{2\epsilon}u_l = 0, \tag{13}$$

and on the other hand, we see

$$\left(\tilde{\alpha}\tilde{\phi}\left(\frac{\partial r_i}{\partial x_0}\right), \dots, \tilde{\alpha}\tilde{\phi}\left(\frac{\partial r_i}{\partial x_m}\right)\right)\boldsymbol{u} = t^{-1}((1-t^{\epsilon})u_j - u_k + t^{\epsilon}u_l), \tag{14}$$

where $\boldsymbol{u} = {}^t(u_0, u_1, \dots, u_m)$. Therefore, $M(s^2)\boldsymbol{u} = 0$. If $\rho$ is nonabelian, $\mathbf{1}$ and $\boldsymbol{u}$ are linearly independent, rank $M(s^2) < m$, and we obtain $\Delta_K(s^2) = 0$. □

## 5. Twisted Alexander polynomials of a fibered ribbon 2-knot of 1-fusion

For a classical fibered knot, Kitano and Morifuji proved the following [18, Theorem 3.2]; cf. [2, 5, 7, 8, 9, 10]. In this section, we prove an analogous theorem for a fibered ribbon 2-knot of 1-fusion. For a *fibered knot* we refer the reader to Sect. 10H in [26].

**Proposition 5.1.** *Let* $\rho : G(K) \to \mathrm{SL}(2,\mathbb{F})$ *be a nonabelian representation of a genus* $g$ *fibered classical knot* $K$*. Then the twisted Alexander polynomial* $\Delta_{K,\rho}(t)$ *is presented as a monic polynomial of degree* $4g-2$.

REMARK 5.2. A polynomial $c_0 + c_1 t + \cdots + c_m t^m \in \mathbb{F}[t]$ is called *monic* if the highest degree coefficient $c_m$ is one. In Proposition 5.1, the lowest degree coefficient $c_0$ is also one. In fact, if a knot $K$ is fibered, then so is its mirror image $K!$. There is a nonabelian representation $\rho' : G(K!) \to \mathrm{SL}(2,\mathbb{F})$ such that $\Delta_{K!,\rho'}(t)$ is equal to $\Delta_{K,\rho}(t^{-1})$ up to a factor $t^{2k}$ ($k \in \mathbb{Z}$), which is also monic.

For a classical knot it was shown by Neuwirth [22] and Stallings [27] that a knot $K$ is fibered if and only if the commutator subgroup of the knot group is finitely generated or, equivalently, free. For a fibered ribbon 2-knot of 1-fusion we have an analogous theorem due to Yoshikawa [31]; cf. [1].

**Proposition 5.3.** *Let* $K$ *be a ribbon* 2*-knot of* 1*-fusion. Then* $K$ *is fibered if and only if the commutator subgroup of the knot group is finitely generated or, equivalently, free.*

Suppose that a ribbon 2-knot of 1-fusion $K$ is fibered. Then the commutator subgroup of the knot group $G(K)$ is finitely generated, and by [31, Lemma] (cf. [23]) the knot group $G(K)$ is presented by

$$\langle x, a_0, \ldots, a_{m-1} \mid xa_ix^{-1} = a_{i+1}\ (i = 0, \ldots, m-2),\ xa_{m-1}x^{-1} = w\ \rangle, \tag{15}$$

where $w$ is a word in $a_0, \ldots, a_{m-1}$ and the abelianized group $G(K)/G(K)'$ is the infinite cyclic group $\langle x\rangle$, and the commutator subgroup $G(K)'$ is the free group $\langle a_0, \ldots, a_{m-1}\rangle$. Also, the map $a_i \mapsto a_{i+1}$ $(i = 0, \ldots, m-2)$, $a_{m-1} \mapsto w$ induces an automorphism $\varphi$ of $G(K)'$; see [31, Remark in Sect. 2]. Let $M$ be the $m \times m$ integral matrix corresponding to the induced automorphism $G(K)'/G(K)'' \to G(K)'/G(K)''$ of the free abelian group of rank $m$. Then, the Alexander polynomial of $K$, $\Delta_K(t)$, is given by $\det(M - tI)$, where $I$ is the identity matrix. Thus, we obtain:

**Proposition 5.4.** *Let $K$ be a fibered ribbon 2-knot of 1-fusion. Then the lowest and highest degree coefficients of $\Delta_K(t)$ are $\pm 1$, and the breadth of $\Delta_K(t)$ coincides with the rank of the free commutator subgroup of the knot group.*

This immediately implies:

**Corollary 5.5.** *If a non-trivial ribbon 2-knot of 1-fusion has trivial Alexander polynomial, then it is not a fibered knot.*

Proof. If a fibered ribbon 2-knot of 1-fusion $K$ has trivial Alexander polynomial, then by Proposition 5.4 the commutator subgroup of its knot group is trivial, and so the knot group is infinite cyclic group. Since a ribbon 2-knot of 1-fusion is trivial if and only if its knot group is infinite cyclic ([20]), $K$ is trivial. □

Thus, for example, all the knots in the family of ribbon 2-knots of 1-fusion with trivial Alexander polynomial which we have classified in [13] are non-fibered.

REMARK 5.6. Yanagawa [30] has shown that every ribbon 2-knot bounds $W-$(open 3-disk) in $S^4$, where $W$ is a connected sum of copies of $S^1 \times S^2$; cf. [3].

Let $\varphi$ be an automorphism of the free group $\langle a_1, \ldots, a_m\rangle$. Put $G = \langle x, a_1, \ldots, a_m \mid xa_ix^{-1} = \varphi(a_i)\ (i = 1, \ldots, m)\ \rangle$. Suppose that $G/G' \cong \mathbb{Z} = \langle t\rangle$ and the abelianization $\alpha$ sends $x$ to $t$. Then $\alpha(a_i) = 1$ for any $i$.

**Theorem 5.7.** *With the above notation the twisted Alexander polynomial $\Delta_{G,\rho}(t)$ associated to a representation $\rho\colon G \to \mathrm{SL}(n, \mathbb{F})$ is presented as a rational function of the form:*

$$\Delta_{G,\rho}(t) = \frac{c_0 + c_1t + \cdots + c_{nm-1}t^{nm-1} + t^{nm}}{(-1)^n + d_1t + \cdots + d_{n-1}t^{n-1} + t^n}, \tag{16}$$

*where $c_0 = (-1)^{nm}$. In particular, if $\Delta_{G,\rho}(t)$ is a Laurent polynomial presented by*

$$\Delta_{G,\rho}(t) = e_0 + e_1t + \cdots + e_\mu t^\mu, \tag{17}$$

*with $e_0e_\mu \neq 0$, then $\mu = n(m-1)$, $e_0 = (-1)^{n(m-1)}$, and $e_\mu = 1$.*

Proof. If we put $x_0 = x$, $x_1 = a_1$, …, $x_m = a_m$, then

$$\det A_{\rho,x} = \det\left(\Phi_\rho\left(\frac{\partial(xa_i - \varphi(a_i)x)}{\partial a_j}\right)\right)_{1\le i,j\le m} \tag{18}$$

$$= \det\left(\delta_{ij}\Phi_\rho(x) - \Phi_\rho\left(\frac{\partial\varphi(a_i)}{\partial a_j}\right)\right)_{1\le i,j\le m}$$
$$= \det(t(\rho(x)\otimes I_m) - M)$$
$$= (-1)^{nm}\det(M) + c_1 t + \cdots + t^{nm},$$

where

$$M = \left(\Phi_\rho\left(\frac{\partial\varphi(a_i)}{\partial a_j}\right)\right)_{1\le i,j\le m} = \left(\tilde{\rho}\tilde{\phi}\left(\frac{\partial\varphi(a_i)}{\partial a_j}\right)\right)_{1\le i,j\le m}. \tag{19}$$

If we put $b_i = \varphi(a_i)$, $i = 1,\dots,m$, then we obtain another presentation of $G$ as follows.

$$G = \left\langle x, b_1,\dots,b_m \mid x\varphi^{-1}(b_i)x^{-1} = b_i \ (i = 1,\dots,m)\right\rangle. \tag{20}$$

By using this presentation, we obtain the twisted Alexander polynomial

$$\Delta' = \frac{\det A'}{\det(I_n - t\rho(x))}, \tag{21}$$

where

$$A' = \left(\Phi_\rho\left(\frac{\partial(xa_i - b_i x)}{\partial b_j}\right)\right)_{1\le i,j\le m}. \tag{22}$$

Since $(\partial a_i/\partial b_j)_{1\le i,j\le g}(\partial b_i/\partial a_j)_{1\le i,j\le m} = I_m$, we see that

$$\begin{aligned}\det A' &= \Delta' \det(I_n - t\rho(x)) \\ &= \det\left(\Phi_\rho\left(x\frac{\partial a_i}{\partial b_j} - \delta_{ij}\right)\right)_{1\le i,j\le m} \\ &= \det\left(t\rho(x)\tilde{\rho}\tilde{\phi}\left(\frac{\partial a_i}{\partial b_j}\right) - \delta_{ij}I_n\right)_{1\le i,j\le m} \\ &= \det\left(t(\rho(x)\otimes I_m)\left(\tilde{\rho}\tilde{\phi}\left(\frac{\partial a_i}{\partial b_j}\right)\right)_{1\le i,j\le m} - I_{nm}\right) \\ &= \det\left(t(\rho(x)\otimes I_m) - M\right)\det(M^{-1}) \\ &= \det(M)^{-1}\det A_{\rho,x}\end{aligned} \tag{23}$$

Since both $\det A'$ and $\det A_{\rho,x}$ are monic polynomials with the same degree, we obtain $\det(M) = 1$. □

**Corollary 5.8.** *Let $K$ be a fibered ribbon 2-knot of 1-fusion whose Alexander polynomial has breadth $m$ and $\rho\colon G(K) \to \mathrm{SL}(2,\mathbb{F})$ be an irreducible representation. Then the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is presented as a polynomial of breadth $2m-2$ whose lowest and highest degree coefficients are 1.*

Proof. The knot group $G(K)$ is presented by (15). By Theorem 4.2, $\Delta_{K,\rho}(t)$ is a Laurent polynomial. By Theorem 5.7, multiplying $t^{2k}$ for some $k \in \mathbb{Z}$, $\Delta_{K,\rho}(t)$ is a monic polynomial of degree $2m-2$ such that the constant term is 1. □

## 6. Examples

In this section, we give several examples of ribbon 2-knots of 1-fusion and their twisted Alexander polynomials associated to the representations to $\mathrm{SL}(2,\mathbb{C})$. Since the twisted Alexander polynomial is well-defined up to a factor of $t^{2k}$, $k \in \mathbb{Z}$, we ignore a factor of $t^{2k}$ in the equality of the twisted Alexander polynomial. Let us consider the presentation Eq. (1) of the group $G$ of the ribbon 2-knot $K = R(p_1, q_1, \ldots, p_n, q_n)$ of 1-fusion. Then since $x$ and $y$ are conjugate, any nonabelian representation $G \to \mathrm{SL}(2,\mathbb{C})$ is conjugate to a representation $\rho : G \to \mathrm{SL}(2,\mathbb{C})$ given by $\rho(x) = X$ and $\rho(y) = Y$, where

$$X = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \quad Y = \begin{pmatrix} s & 0 \\ u & s^{-1} \end{pmatrix} \tag{24}$$

for some $s, u \in \mathbb{C}$ with $s \neq 0$ and $(s, u) \neq (\pm 1, 0)$. Such a representation $\rho$ is parametrized by the trace $s + s^{-1}$ and $u$. See [24, 25]; cf. [13, Proposition 3.1]. Furthermore, the nonabelian representation $\rho$ is reducible if and only if $u + (s + s^{-1})^2 = 4$ or $u = 0$; cf. [13, Lemma 3.2]. In this section $I$ stands for the identity matrix of order 2. We will use the following to find a representation $\rho$.

**Lemma 6.1.** *For $W \in \mathrm{SL}(2,\mathbb{C})$, let $R = WX - YW$. Then, $R = O$ if and only if $R_{12} = R_{22} = 0$, where $R_{ij}$ is the $(i, j)$ entry of $R$.*

Proof. Let $W_{ij}$ be the $(i, j)$ entry of $W$. Then we have

$$R_{11} = 0; \tag{25}$$

$$R_{12} = W_{11} - (s - s^{-1})W_{12}; \tag{26}$$

$$R_{21} = -uW_{11} + (s - s^{-1})W_{21}; \tag{27}$$

$$R_{22} = W_{21} - uW_{12}, \tag{28}$$

and thus $R_{21} = (s - s^{-1})R_{22} - uR_{12}$, which completes the proof. □

EXAMPLE 6.2. Let $K = R(1, -1)$. The Alexander polynomial is $\Delta_K(t) = 2t - t^2$ from Eq. (2). We calculate the twisted Alexander polynomial. For the matrices $X$ and $Y$ in Eq. (24) let $R = \tilde{\rho}((xy^{-1})x - y(xy^{-1})) = (XY^{-1})X - Y(XY^{-1})$. Then from

$$R = \begin{pmatrix} 0 & -s^2 - u + 2 \\ -u\left(-s^{-2} - u + 2\right) & -\left(s + s^{-1}\right)u \end{pmatrix} = O, \tag{29}$$

we have 3 cases:

$$(s + s^{-1}, u) = (0, 3), \quad \left(\sqrt{2} + \frac{1}{\sqrt{2}}, 0\right), \quad \left(-\sqrt{2} - \frac{1}{\sqrt{2}}, 0\right). \tag{30}$$

We denote the corresponding representations by $\rho_1$, $\rho_2$, $\rho_3$, respectively. Then, $\rho_1$ is irreducible, and $\rho_2$ and $\rho_3$ are reducible. For $r = wx - yw = xy^{-1}x - yxy^{-1}$, we have

$$\begin{aligned} \det A_{\rho,y} &= \Phi_\rho(\partial r/\partial x) \\ &= \Phi_\rho(1 + xy^{-1} - y) \\ &= \det(I + XY^{-1} - tY) \end{aligned} \tag{31}$$

$$= 4 - u + (s + s^{-1})(u - 2)t + t^2;$$

$$\det(I - tY) = 1 - (s + s^{-1})t + t^2, \tag{32}$$

and so we obtain the twisted Alexander polynomial $\Delta_{K,\rho}(t) = \det A_{\rho,y}/\det(I-tY)$ as follows:

$$\Delta_{K,\rho_1}(t) = \frac{1+t^2}{1+t^2} = 1; \tag{33}$$

$$\Delta_{K,\rho_2}(t) = \frac{(2\sqrt{2}-t)(\sqrt{2}-t)}{(1/\sqrt{2}-t)(\sqrt{2}-t)} = \frac{2\sqrt{2}-t}{1/\sqrt{2}-t} = \frac{4-\sqrt{2}t}{1-\sqrt{2}t}; \tag{34}$$

$$\Delta_{K,\rho_3}(t) = \frac{(2\sqrt{2}+t)(\sqrt{2}+t)}{(1/\sqrt{2}+t)(\sqrt{2}+t)} = \frac{2\sqrt{2}+t}{1/\sqrt{2}+t} = \frac{4+\sqrt{2}t}{1+\sqrt{2}t}. \tag{35}$$

If $\rho$ is reducible, then by Lemma 4.6 we have:

$$\Delta_{K,\rho}(t) = \frac{\Delta_K(st)}{s-t}\cdot\frac{\Delta_K(s^{-1}t)}{s^{-1}-t} = \frac{2st-s^2t^2}{s-t}\cdot\frac{2s^{-1}t-s^{-2}t^2}{s^{-1}-t} = \frac{(2s^{-1}-t)(2s-t)}{(s-t)(s^{-1}-t)}, \tag{36}$$

up to a factor $t^2$, which yields Eqs. (34) ($s = \sqrt{2}$) and (35) ($s = -\sqrt{2}$), respectively.

EXAMPLE 6.3. Let $K = R(1,-2)$. Then $K$ is a fibered knot since the commutator subgroup $G(K)'$ of the knot group $G(K)$ of $K$ is a free group of rank 2.

Proof. Putting $y = ax$ in

$$G(K) = \langle\, x, y \mid (xy^{-2})x(xy^{-2})^{-1}y^{-1}\,\rangle,$$

we obtain $(xy^{-2})x(xy^{-2})^{-1}y^{-1} = a^{-1}(x^{-1}a^{-1}x)a(xax^{-1})a^{-1}$. Letting $a_i = x^iax^{-i}$, we have $a_i^{-1}a_{i-1}^{-1}a_ia_{i+1}a_i^{-1} = 1$, $i \in \mathbb{Z}$, and so

$$G(K) = \langle\, x, a_0, a_1 \mid xa_0x^{-1} = a_1,\ xa_1x^{-1} = a_1^{-1}a_0a_1^2\,\rangle.$$

Then the commutator subgroup $G(K)'$ is a free group with free basis $a_0, a_1$. In fact, the map $a_0 \mapsto a_1$, $a_1 \mapsto a_1^{-1}a_0a_1^2$ defines an automorphism $\varphi$ of a free group with basis $a_0, a_1$.

The matrix corresponding to the induced isomorphism $\varphi_*$ is $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and so we obtain the Alexander polynomial as follows: $\Delta_K(t) = \det\begin{pmatrix} -t & 1 \\ 1 & 1-t \end{pmatrix} = -1 - t + t^2$, which is also obtained from Eq. (2). □

Next, we calculate the twisted Alexander polynomial associated to a nonabelian representation to $\mathrm{SL}(2,\mathbb{C})$. For the matrices $X$ and $Y$ in Eq. (24) let $R = (XY^{-2})X - Y(XY^{-2})$. By Lemma 6.1 we consider $R_{12}$ and $R_{22}$, the $(1,2)$ and $(2,2)$ entries of $R$:

$$R_{12} = -s^3 - (s + s^{-1})(u - 1); \tag{37}$$

$$R_{22} = -(s^{-1} - 1 + s)(s^{-1} + 1 + s)u. \tag{38}$$

From $R_{22} = 0$ we obtain $s + s^{-1} = \epsilon = \pm 1$ or $u = 0$. If $s + s^{-1} = \epsilon$, then $s^3 = -\epsilon$, and so $R_{12} = 0$ implies $u = 2$. If $u = 0$, then from $R_{12} = 0$ we obtain $s + s^{-1} = \pm c_1, \pm c_2$, where $c_1 = \sqrt{2+\sqrt{5}}$ and $c_2 = \sqrt{2-\sqrt{5}}$. So, we have 6 representations $\rho_i$, $1 \le i \le 6$, parametrized by:

$$(s+s^{-1},u)=(1,2),\ (-1,2),\ (c_1,0),\ (-c_1,0),\ (c_2,0),\ (-c_2,0), \tag{39}$$

respectively; $\rho_1$ and $\rho_2$ are irreducible and the others are reducible.

For $r = wx - yw = xy^{-2}x - yxy^{-2}$, we have

$$\begin{aligned}\det A_{\rho,y} &= \Phi_\rho(\partial r/\partial x) \\ &= \Phi_\rho(1 + xy^{-2} - y) \\ &= \det(I + t^{-1}XY^{-2} - tY);\end{aligned} \tag{40}$$

$$\det(I - tY) = (s - t)(s^{-1} - t). \tag{41}$$

For $\rho = \rho_1$ or $\rho_2$, we have

$$\det A_{\rho,y} = (1 + t^2)(1 - \epsilon t + t^2)t^{-2}; \tag{42}$$

$$\det(I - tY) = 1 - \epsilon t + t^2, \tag{43}$$

where $\epsilon = 1$ if $\rho = \rho_1$ and $\epsilon = -1$ if $\rho = \rho_2$. This yields the twisted Alexander polynomial as follows:

$$\Delta_{K,\rho}(t) = t^{-2} + 1. \tag{44}$$

If $\rho = \rho_i$, $i = 3, 4, 5, 6$, then $u = 0$, and so

$$\det A_{\rho,y} = (s^2 + st - t^2)(s^{-2} + s^{-1}t - t^2)t^{-2}, \tag{45}$$

which yields the twisted Alexander polynomial as follows:

$$\Delta_{K,\rho}(t) = \frac{(s^2 + st - t^2)(s^{-2} + s^{-1}t - t^2)}{(s - t)(s^{-1} - t)} = \frac{1 + pt + (3 - p^2)t^2 - pt^3 + t^4}{1 - pt + t^2}, \tag{46}$$

where $p = s + s^{-1}$. This is also given by Lemma 4.6.

The ribbon 2-knots presented by virtual arcs with up to four crossings listed in [12, Table 2] are all of 1-fusion, and among them there are 30 nontrivial knots whose Alexander polynomial is nontrivial and the lowest and highest degree coefficient are $\pm 1$, where we do not count both a knot and its mirror image. By calculating the commutator subgroup of the group of these knots as in Example 6.3 we see they are all fibered. Thus, we have the following.

**Theorem 6.4.** *Let $K$ be a ribbon 2-knot presented by virtual arcs with up to four crossings, and let $\Delta_K(t)$ be its Alexander polynomial. Then $K$ is fibered if and only if $\Delta_K(t)$ is nontrivial and the lowest and highest degree coefficients of $\Delta_K(t)$ are $\pm 1$.*

Goda, Kitano, and Morifuji [9, Example 4.3] gave an example of a non-fibered classical knot with monic Alexander polynomial, whose non-fiberedness can be detected by the twisted Alexander polynomial; cf. [18, Example 4.3]. The following example is an analogous example for a ribbon 2-knot of 1-fusion.

EXAMPLE 6.5. Let $K_0 = R(1, -3, 1, 2)$. The Alexander polynomial is $\Delta_{K_0}(t) = t^{-1} - 1 + t$. Suppose $K_0$ were a fibered knot. Then by Theorem 5.7 if the twisted Alexander polynomial $\Delta_{K_0,\rho}(t)$ associated to a representation $\rho : G(K_0) \to \mathrm{SL}(2, \mathbb{C})$ is a Laurent polynomial, then the lowest and highest degree coefficients are one and the breadth is 2. However, one of the twisted Alexander polynomials Eqs. (54) and (57) below detects $K_0$ is not fibered.

We calculate the twisted Alexander polynomials $\Delta_{K_0,\rho}(t)$. For the matrices $X$ and $Y$ in Eq. (24) let $R = (XY^{-3}XY^2)X - Y(XY^{-3}XY^2)$. By Lemma 6.1 we consider $R_{12}$ and $R_{22}$, the $(1,2)$ and $(2,2)$ entries of $R$:

$$R_{12} = -s^{-3}(-p^2+3-u) + pu(p^2-1)(-p^2+4-u); \tag{47}$$

$$R_{22} = u(p^2-1)(-p^2+3-u), \tag{48}$$

where $p = s + s^{-1}$. From $R_{22} = 0$ we obtain $p = \epsilon = \pm 1$, $u = -p^2 + 3$, or $u = 0$. Then $R_{12} = 0$ implies

$$(s + s^{-1}, u) = (0,3),\ (1,2),\ (-1,2),\ (\sqrt{3},0),\ (-\sqrt{3},0). \tag{49}$$

We denote the corresponding representations by $\rho_i$, $1 \le i \le 5$, respectively; $\rho_4$ and $\rho_5$ are reducible and the others are irreducible. For $r = wx - yw = xy^{-3}xy^2x - yxy^{-3}xy^2$, we have

$$\begin{aligned} \det A_{\rho,y} &= \Phi_\rho(\partial r/\partial x) \\ &= \Phi_\rho(1 + t^{-2}xy^{-3} + txy^{-3}xy^2 - ty - t^{-1}yxy^{-3}) \\ &= \det(I + t^{-2}XY^{-3} + tXY^{-3}XY^2 - tY - t^{-1}YXY^{-3}) \end{aligned} \tag{50}$$

$$\det(I - tY) = (s-t)(s^{-1}-t). \tag{51}$$

For $\rho = \rho_1$, we have

$$\det A_{\rho,y} = (1+t^2)(1+t^2+3t^4)t^{-4}; \tag{52}$$

$$\det(I - tY) = 1 + t^2, \tag{53}$$

which yields the twisted Alexander polynomial as follows:

$$\Delta_{K,\rho}(t) = 1 + t^2 + 3t^4. \tag{54}$$

For $\rho = \rho_2$ or $\rho_3$, we have

$$\det A_{\rho,y} = (1 - \epsilon t + t^2)(1 - t^2 + 2t^4)t^{-4}; \tag{55}$$

$$\det(I - tY) = 1 - \epsilon t + t^2, \tag{56}$$

where $\epsilon = 1$ if $\rho = \rho_2$ and $\epsilon = -1$ if $\rho = \rho_3$. This yields the twisted Alexander polynomial as follows:

$$\Delta_{K,\rho}(t) = 1 - t^2 + 2t^4. \tag{57}$$

For $\rho = \rho_4$ or $\rho_5$, we have

$$\det A_{\rho,y} = (1+t^2)(1 - \sqrt{3}\epsilon t + t^2)t^{-4}; \tag{58}$$

$$\det(I - tY) = 1 - \sqrt{3}\epsilon t + t^2, \tag{59}$$

where $\epsilon = 1$ if $\rho = \rho_4$ and $\epsilon = -1$ if $\rho = \rho_5$. This yields the twisted Alexander polynomial as follows:

$$\Delta_{K,\rho}(t) = 1 + t^2. \tag{60}$$

This is also given by Lemma 4.6.

We extend Example 6.5 as follows.

EXAMPLE 6.6. Let $K_k = R(1, -3, 1-3k, 2+3k)$ be a ribbon 2-knot of 1-fusion. Then, $K_k$ is fibered if and only if $k < 0$.

Proof. The Alexander polynomial is $\Delta_{K_k}(t) = -t^{-6k} + t^{-1-3k} + t^{1-3k}$. For the matrices $X$ and $Y$ in Eq. (24) with $(s+s^{-1}, u) = (\epsilon, 2)$, $\epsilon = \pm 1$, $X^3 = Y^3 = -\epsilon I$ and $XYX = YXY$, we have $W = XY^{-3}X^{1-3k}Y^{2+3k} = -\epsilon X^{-1}Y^{-1}$ and $WX - YW = O$. Thus there is an irreducible representation $\rho : G(K_k) \to \mathrm{SL}(2, \mathbb{C})$ sending $x, y$ to $X, Y$, respectively.

We calculate the twisted Alexander polynomial $\Delta_{K_k,\rho}(t)$ of $K_k$ for $k > 0$ associated to the representation $\rho$ which is parametrized by $(s+s^{-1}, u) = (1, 2)$. Since $\rho$ is irreducible, $\Delta_{K_k,\rho}(t)$ is a Laurent polynomial by Theorem 4.2. For $r = wx - yw$, $w = xy^{-3}x^{1-3k}y^{2+3k}$, we have

$$
\begin{aligned}
\det A_{\rho,y} &= \Phi_\rho(\partial r/\partial x) \\
&= \Phi_\rho\Big(1 - xy^{-3}(x^{-1} + x^{-2} + \cdots + x^{1-3k}) + xy^{-3}x^{1-3k}y^{2+3k} \\
&\qquad -y + yxy^{-3}(x^{-1} + x^{-2} + \cdots + x^{1-3k})\Big) \\
&= \det M,
\end{aligned} \tag{61}
$$

where

$$
\begin{aligned}
M = &I - t^{-2}XY^{-3}(t^{-1}X^{-1} + t^{-2}X^{-2} + \cdots + t^{1-3k}X^{1-3k}) + tXY^{-3}X^{1-3k}Y^{2+3k} \\
&- tY + t^{-1}YXY^{-3}(t^{-1}X^{-1} + t^{-2}X^{-2} + \cdots + t^{1-3k}X^{1-3k}).
\end{aligned}
$$

Then in each entry of $M$ the lowest and highest terms appear in $t^{-1-3k}XY^{-3}X^{1-3k} = (-\epsilon)^k t^{-1-3k}X^{-1}$ and $t(-Y + XY^{-3}X^{1-3k}Y^{2+3k}) = -t(Y + \epsilon X^2Y^2)$, respectively. Since $\det(-\epsilon)^k X^{-1} = 1$ and $\det(-Y - \epsilon X^2Y^2) = \det(\epsilon X^2(X - Y)Y) = \det(X - Y) = 2$, the lowest and highest terms of $\det M$ are $t^{-2-6k}$ and $2t^2$, respectively. Then, since $\det(I - tY) = 1 - t + t^2$, the lowest and highest terms of the twisted Alexander polynomial $\Delta_{K_k,\rho}(t) = A_{\rho,y}/\det(I - tY)$ are $t^{-2-6k}$ and 2, respectively, $\Delta_{K_k,\rho}(t) = t^{-2-6k} + \cdots + 2$. So, if $k > 0$, then the breadth of $\Delta_{k,\rho}(t)$ is $2 + 6k\,(> 6k)$, and hence $K_k$ is not fibered.

Let $\tilde{K}_m = R(1, -3, m, 3-m)$; so, $K_k = \tilde{K}_{1-3k}$. We show if $m \geq 4$, then the commutator subgroup of the knot group $G(\tilde{K}_m)$ is a free group of finite rank, and thus $\tilde{K}_m$ is fibered by Proposition 5.3.

Putting $y = ax$ in

$$
G(\tilde{K}_m) = \langle\, x, y \mid (xy^{-3}x^m y^{3-m})x(y^{m-3}x^{-m}y^3x^{-1})y^{-1} \,\rangle, \tag{62}
$$

we obtain

$$
\begin{aligned}
&x(ax)^{-3}x^m(ax)^{3-m}x(ax)^{m-3}x^{-m}(ax)^3x^{-1}(ax)^{-1} \\
=&a^{-1}x^{-1}a^{-1}x^{-1}a^{-1}x^{m-1}(a^{-1}x^{-1})^{m-3}x(xa)^{m-3}x^{1-m}axaxax^{-1}a^{-1} \\
=&a^{-1}(x^{-1}a^{-1}x)(x^{-2}a^{-1}x^2)(x^{m-3}a^{-1}x^{-m+3})(x^{m-4}a^{-1}x^{-m+4})\cdots(x^2a^{-1}x^{-2})(xa^{-1}x^{-1}) \\
&\times (x^2ax^{-2})(x^3ax^{-3})\cdots(x^{m-2}ax^{-m+2})(x^{-1}ax)a(xax^{-1})a^{-1}.
\end{aligned} \tag{63}
$$

By putting $a_i = x^i a x^{-i}$, this becomes

$$
a_0^{-1}a_{-1}^{-1}a_{-2}^{-1}(a_{m-3}^{-1}a_{m-4}^{-1}\cdots a_2^{-1}a_1^{-1})(a_2a_3\cdots a_{m-2})a_{-1}a_0a_1a_0^{-1}. \tag{64}
$$

Then the commutator subgroup $G(\tilde{K}_m)'$ is presented as follows:

$$G(\tilde{K}_m)' = \langle\, a_i \ (i \in \mathbb{Z}) \mid r_j \ (j \in \mathbb{Z}) \,\rangle, \tag{65}$$

where

$$r_j = a_{j+2}^{-1} a_{j+1}^{-1} a_j^{-1} (a_{j+m-1}^{-1} a_{j+m-2}^{-1} \cdots a_{j+4}^{-1} a_{j+3}^{-1})(a_{j+4} a_{j+5} \cdots a_{j+m}) a_{j+1} a_{j+2} a_{j+3} a_{j+2}^{-1}. \tag{66}$$

Then since $a_j$ is expressed in a word of $a_{j+1}, a_{j+2}, \ldots, a_{j+m}$, and $a_{j+m}$ is expressed in a word of $a_j, a_{j+1}, \ldots, a_{j+m-1}$, $G(\tilde{K}_m)'$ is a free group of rank $m$, completing the proof. □

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Taizo Kanenobu
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku, Osaka, 558–8585
Japan
e-mail: kanenobu@sci.osaka-cu.ac.jp

Toshio Sumi
Faculty of Arts and Science, Kyushu University
Motooka 744, Nishi-ku, Fukuoka, 819–0395
Japan
e-mail: sumi@artsci.kyushu-u.ac.jp