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## ON POLYNOMIAL CURVES IN THE AFFINE PLANE

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### Abstract

A curve that can be parametrized by polynomials is called a polynomial curve. It is well-known that a polynomial curve has only one place at infinity. Let  $C$  be a curve with one place at infinity. Sathaye presented the following question raised by Abhyankar: Is there a polynomial curve associated with the semigroup generated by pole orders of  $C$  at infinity? In this paper, we give a negative answer to this question using Gröbner basis computation.

### 1. Introduction

Let  $C$  be an irreducible algebraic curve in the complex affine plane  $\mathbf{C}^2$ . We say that  $C$  has *one place at infinity*, if the closure of  $C$  intersects with the  $\infty$ -line in  $\mathbf{P}^2$  at only one point  $P$  and  $C$  is locally irreducible at that point  $P$ .

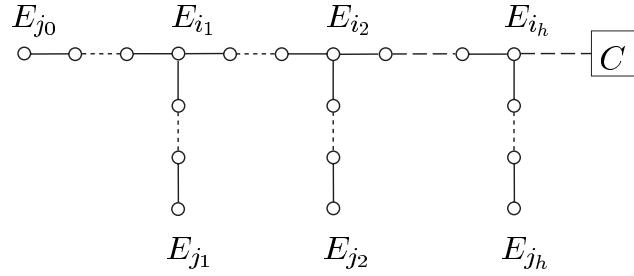
Abhyankar–Moh [1, 4, 5] investigated properties of  $\delta$ -sequences that are sequences of pole orders of *approximate roots* of curves with one place at infinity and obtained a criterion for a curve to have only one place at infinity. This result is called Abhyankar–Moh’s semigroup theorem. Sathaye–Stenerson [14] proved that, conversely, if a sequence  $S$  of natural numbers satisfies Abhyankar–Moh’s condition then there exists a curve with one place at infinity having its  $\delta$ -sequence  $S$ . Suzuki [16] made clear the relationship between the  $\delta$ -sequence and the dual graph of the minimal resolution of the singularity of the curve  $C$  at infinity, and gave an algebro-geometric proof of the semigroup theorem and its inverse theorem due to Sathaye–Stenerson. Fujimoto–Suzuki [6] gave an algorithm to compute the defining polynomial of the curve with one place at infinity from a given  $\delta$ -sequence.

A curve that can be parametrized by polynomials is called a *polynomial curve*. It is well-known that a polynomial curve has only one place at infinity. Let  $C$  be a curve with one place at infinity, and  $\Omega$  the semigroup generated by pole orders of  $C$  at infinity. Sathaye [13] presented the following question for curves with one place at infinity raised Abhyankar: Is there a polynomial curve associated with  $\Omega$ ? Sathaye–Stenerson [14] suggested a candidate for a negative answer to this question; however, they could not give an answer to the question since a root computation for a huge polynomial system was required.

We found a negative answer to the Abhyankar's question using a computer algebra system. In this paper, we give its details.

## 2. Preliminaries

Through this paper, we set  $\mathbf{N} = \{n \in \mathbf{Z} \mid n \geq 0\}$  and  $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ . Let  $C$  be a curve with one place at infinity defined by a polynomial equation  $f(x, y) = 0$  in the complex affine plane  $\mathbf{C}^2$ . Assume that  $\deg_x f = m$ ,  $\deg_y f = n$  and  $d = \gcd(m, n)$ . The dual graph corresponding to the minimal resolution of the singularity of  $C$  at infinity is of the following form [16]:

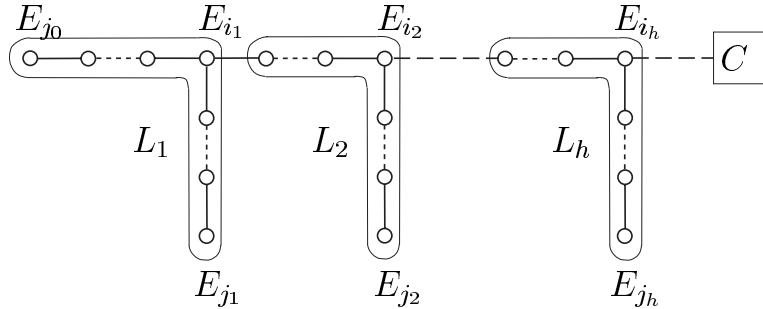


**DEFINITION 1** ( $\delta$ -sequence). Let  $f$  be a defining polynomial of a curve  $C$  with one place at infinity. Let  $\delta_k$  ( $0 \leq k \leq h$ ) be the order of the pole of  $f$  on the curves corresponding to the edge nodes  $E_{j_k}$  in the above dual graph. We call the sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$  the  $\delta$ -sequence of  $C$  (or of  $f$ ).

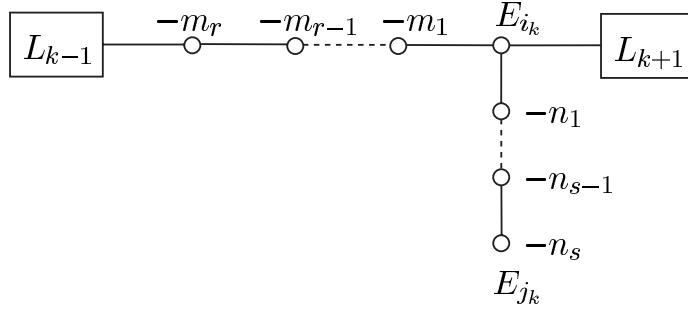
We have the following fact, since  $\deg_x f = m$  and  $\deg_y f = n$ .

FACT 1.  $\delta_0 = n$ ,  $\delta_1 = m$ .

We set  $L_k$  for each  $k$  ( $1 \leq k \leq h$ ), the linear branches as shown in the following figure:



DEFINITION 2 (( $p, q$ )-sequence). Now, we assume that the weights of  $L_k$  are of the following form:



We define the natural numbers  $p_k, q_k, a_k, b_k$  satisfying

$$(p_k, a_k) = 1, \quad (q_k, b_k) = 1, \quad 0 < a_k < p_k, \quad 0 < b_k < q_k,$$

$$\frac{p_k}{a_k} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}} \quad \text{and} \quad \frac{q_k}{b_k} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}}.$$

We call the sequence  $\{(p_1, q_1), (p_2, q_2), \dots, (p_h, q_h)\}$  the  $(p, q)$ -sequence of  $C$  (or of  $f$ ).

The following Abhyankar-Moh's semigroup theorem and its converse theorem by Sathaye-Stenerson are results for  $\delta$ -sequence.

**Theorem 1** (Abhyankar-Moh [1, 4, 5]). *Let  $C$  be an affine plane curve with one place at infinity. Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  be the  $\delta$ -sequence of  $C$  and  $\{(p_1, q_1), \dots, (p_h, q_h)\}$  the  $(p, q)$ -sequence of  $C$ . We set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h+1$ ). We have then,*

- (i)  $q_k = d_k/d_{k+1}$ ,  $d_{h+1} = 1$  ( $1 \leq k \leq h$ ),
- (ii)  $d_{k+1}p_k = \begin{cases} \delta_1 & (k=1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \leq k \leq h) \end{cases}$ ,
- (iii)  $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \dots + \mathbf{N}\delta_{k-1}$  ( $1 \leq k \leq h$ ).

**Theorem 2** (Sathaye-Stenerson [14]). *Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  ( $h \geq 1$ ) be a sequence of  $h+1$  natural numbers. We set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h+1$ ) and  $q_k = d_k/d_{k+1}$  ( $1 \leq k \leq h$ ). Furthermore, suppose that the following conditions are satisfied:*

- (1)  $\delta_0 < \delta_1$ ,
- (2)  $q_k \geq 2$  ( $1 \leq k \leq h$ ),
- (3)  $d_{h+1} = 1$ ,
- (4)  $\delta_k < q_{k-1}\delta_{k-1}$  ( $2 \leq k \leq h$ ),
- (5)  $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \cdots + \mathbf{N}\delta_{k-1}$  ( $1 \leq k \leq h$ ).

Then, there exists a curve with one place at infinity having the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$ .

Suzuki [16] gave an algebro-geometric proof of the above two theorems by a consideration of the resolution graph at infinity. Further, Suzuki gave an algorithm for mutual conversion of a dual graph and a  $\delta$ -sequence.

### 3. Construction of defining polynomials of curves

We shall assume that  $f(x, y)$  is monic in  $y$ . We define approximate roots by Abhyankar's definition.

**DEFINITION 3** (approximate roots). Let  $f(x, y)$  be a defining polynomial, monic in  $y$ , of a curve with one place at infinity. Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  be the  $\delta$ -sequence of  $f$ . We set  $n = \deg_y f$ ,  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  and  $n_k = n/d_k$  ( $1 \leq k \leq h+1$ ). Then, for each  $k$  ( $1 \leq k \leq h+1$ ), a pair of polynomials  $(g_k(x, y), \psi_k(x, y))$  satisfying the following conditions is uniquely determined:

- (i)  $g_k$  is monic in  $y$  and  $\deg_y g_k = n_k$ ,
- (ii)  $\deg_y \psi_k < n - n_k$ ,
- (iii)  $f = g_k^{d_k} + \psi_k$ .

We call this  $g_k$  the  $k$ -th approximate root of  $f$ .

We can easily get the following fact from the definition of approximate roots.

**FACT 2.** We have

$$g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad g_{h+1} = f$$

where  $c_j \in \mathbf{C}$ ,  $p = \deg_x f/d$ ,  $q = \deg_y f/d$ ,  $d = \gcd\{\deg_x f, \deg_y f\}$  and  $\lfloor p/q \rfloor$  is the maximal integer  $l$  such that  $l \leq p/q$ .

**DEFINITION 4** (Abhyankar-Moh's condition). We call the conditions (1)–(5) concerning  $\{\delta_0, \delta_1, \dots, \delta_h\}$  in Theorem 2 Abhyankar-Moh's condition.

In [6], we presented the following theorem to give normal forms of defining polynomials of curves with one place at infinity, and detailed a method of construction of their defining polynomials by computer.

**Theorem 3** ([6]). *Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  ( $h \geq 1$ ) be a sequence of natural numbers satisfying Abhyankar-Moh's condition (see Definition 4). Set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h+1$ ) and  $q_k = d_k/d_{k+1}$  ( $1 \leq k \leq h$ ).*

(1) *We define  $g_k$  ( $0 \leq k \leq h+1$ ) as follows:*

$$\left\{ \begin{array}{l} g_0 = x, \\ g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbf{C}, \quad p = \frac{\delta_1}{d_2}, \quad q = \frac{\delta_0}{d_2}, \\ g_{i+1} = g_i^{q_i} + a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \dots g_{i-1}^{\bar{\alpha}_{i-1}} \\ \quad + \sum_{(\alpha_0, \alpha_1, \dots, \alpha_i) \in \Lambda_i} c_{\alpha_0 \alpha_1 \dots \alpha_i} g_0^{\alpha_0} g_1^{\alpha_1} \dots g_i^{\alpha_i}, \\ a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}} \in \mathbf{C}^*, \quad c_{\alpha_0 \alpha_1 \dots \alpha_i} \in \mathbf{C} \quad (1 \leq i \leq h), \end{array} \right.$$

where  $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{i-1})$  is the sequence of  $i$  non-negative integers satisfying

$$\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i, \quad \bar{\alpha}_j < q_j \quad (0 < j < i)$$

and

$$\Lambda_i = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_i) \in \mathbf{N}^{i+1} \left| \alpha_j < q_j \quad (0 < j < i), \quad \alpha_i < q_i - 1, \quad \sum_{j=0}^i \alpha_j \delta_j < q_i \delta_i \right. \right\}.$$

Then,  $g_0, g_1, \dots, g_h$  are approximate roots of  $f$  ( $= g_{h+1}$ ), and  $f$  is the defining polynomial, monic in  $y$ , of a curve with one place at infinity having the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$ .

(2) The defining polynomial  $f$ , monic in  $y$ , of a curve with one place at infinity having the  $\delta$ -sequence  $\{\delta_0, \delta_1, \dots, \delta_h\}$  is obtained by the procedure of (1), and the values of parameters  $\{a_{\bar{\alpha}_0 \bar{\alpha}_1 \dots \bar{\alpha}_{i-1}}\}_{1 \leq i \leq h}$  and  $\{c_{\alpha_0 \alpha_1 \dots \alpha_i}\}_{0 \leq i \leq h}$  are uniquely determined for  $f$ .

#### 4. Abhyankar's question and Sathaye-Stenerson's conjecture

**DEFINITION 5** (planar semigroup). Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  ( $h \geq 1$ ) be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by  $\{\delta_0, \delta_1, \dots, \delta_h\}$  is said to be a planar semigroup.

**DEFINITION 6** (polynomial curve). Let  $C$  be an algebraic curve defined by  $f(x, y) = 0$ , where  $f(x, y)$  is an irreducible polynomial in  $\mathbf{C}[x, y]$ . We call  $C$  a polynomial curve, if  $C$  has a parametrization  $x = x(t)$ ,  $y = y(t)$ , where  $x(t)$  and  $y(t)$  are polynomials in  $\mathbf{C}[t]$ .

The following question was introduced by Sathaye [13].

**Abhyankar's Question.** Let  $\Omega$  be a planar semigroup. Is there a polynomial curve with a  $\delta$ -sequence generating  $\Omega$ ?

Moh [11] showed that there is no polynomial curve with the  $\delta$ -sequence  $\{6, 8, 3\}$ . But this is not a negative answer to the Abhyankar's question since there is a polynomial curve  $(x, y) = (t^3, t^8)$  with the  $\delta$ -sequence  $\{3, 8\}$  that generates the same semigroup as above. Sathaye-Stenerson [14] proved that the semigroup generated by  $\{6, 22, 17\}$  has no other  $\delta$ -sequence generating the same semigroup, and proposed the following conjecture on this question.

**Sathaye-Stenerson's Conjecture.** There is no polynomial curve having the  $\delta$ -sequence  $\{6, 22, 17\}$ .

By Theorem 3, the defining polynomial of the curve with one place at infinity having the  $\delta$ -sequence  $\{6, 22, 17\}$  is as follows:

$$\begin{aligned} f = & (g_2^2 + a_{2,1}x^2g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 \\ & + c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0} \end{aligned}$$

where

$$\begin{aligned} g_1 = & y + c_3x^3 + c_2x^2 + c_1x + c_0, \\ g_2 = & (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\ & + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\ & + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}. \end{aligned}$$

Since a curve has one place at infinity and genus zero if and only if it has polynomial parametrization (see [2] or [3]),  $\{6, 22, 17\}$  is a negative answer to the Abhyankar's question if it can be shown that the above type curve does not include a polynomial curve.

We summarize elementary facts about polynomial parametrizations (see [8], [9]).

**DEFINITION 7** (proper polynomial parametrization). A polynomial parametrization  $(x, y) = (u(t), v(t))$ , where  $u, v \in \mathbf{C}[t]$ , is called proper if and only if  $t$  may be expressed as a rational function in  $x, y$ .

**FACT 3.** Any polynomial curve has a proper polynomial parametrization.

**FACT 4.** Let  $C$  be a polynomial curve defined by an irreducible polynomial equation  $f(x, y) = 0$  in the complex affine plane  $\mathbf{C}^2$ . Let  $(x, y) = (u(t), v(t))$  be a proper polynomial parametrization of  $C$ . Then  $\deg_t u = \deg_y f$  and  $\deg_t v = \deg_x f$ .

Now we assume that there exists a polynomial curve having the  $\delta$ -sequence  $\{6, 22, 17\}$ . Thus, the defining polynomial  $f$  of  $C$  has the above form using the approximate roots  $g_1$  and  $g_2$ . By Fact 1 and Fact 4, this curve has the following polynomial parametrization:

$$\begin{cases} x = t^6 + a_1 t^5 + a_2 t^4 + a_3 t^3 + a_4 t^2 + a_5 t + a_6 \\ y = t^{22} + b_1 t^{21} + b_2 t^{20} + b_3 t^{19} + \cdots + b_{20} t^2 + b_{21} t + b_{22} \end{cases}$$

The following lemma presented in [14] plays a vital role to generate polynomial systems corresponding to  $\delta$ -sequences.

**Lemma 1.** *Let  $C$  be a polynomial curve defined by  $f(x, y) = 0$  having the proper polynomial parametrization  $(u(t), v(t))$  and the  $\delta$ -sequence  $\{\delta_0, \delta_1, \delta_2\}$ . Let  $g_2$  be the second approximate root of  $f$ . Then  $\deg_t g_2(u(t), v(t)) = \delta_2$ .*

Proof. This follows immediately from the form of  $f$  ( $= g_3$ ) obtained by Theorem 3.  $\square$

By this lemma, all formal terms with  $t$ -degree more than 17 in  $g_2(x(t), y(t))$  must be eliminated. We get the polynomial system  $I$  from the coefficients of these terms. Furthermore, we can successively eliminate some variables by using polynomials with the form:  $cz - h(w_1, w_2, \dots, w_s)$  in  $I$ , where  $c \in \mathbf{C}^*$ ,  $z, w_1, w_2, \dots, w_s$  are variables and  $h \in \mathbf{C}[w_1, w_2, \dots, w_s]$ . As a result, we obtain the polynomial system with 11 variables and 17 polynomials.

$\{6, 22, 17\}$  is a negative answer to the Abhyankar's question if the polynomial system  $I$  does not have a root. For such a huge polynomial system it is suitable to compute the Gröbner basis of the ideal. However, it has been impossible to compute the Gröbner basis of  $I$  under *well-known* term orderings, even using a computer with 8 GB of memory.

## 5. A negative answer to Abhyankar's question

We find a *lighter* candidate for a negative answer to the Abhyankar's question. Let  $C$  be a curve with one place at infinity defined by a polynomial equation  $f(x, y) = 0$  in the complex affine plane  $\mathbf{C}^2$ . Let  $M$  be the surface obtained by the minimal resolution of the singularity of  $C$  at infinity, and  $E$  the exceptional curve on  $M$ . We assume that  $E_0, E_1, \dots, E_{i_h}$  are irreducible components of  $E$ , where the numbering of indices is by the ordering generated in the process to get  $M$ . The holomorphic 2-form  $\omega = dx \wedge dy$  in  $\mathbf{C}^2$  extends to a meromorphic 2-form on  $M$ . The canonical divisor  $K = (\omega)$  has the support on  $E$ . We get  $K = \sum_{l=0}^{i_h} k_l E_l$ , where  $k_l$  is the zero order of  $\omega$  on  $E_l$ . We call the zero order  $k_{i_h}$  of  $\omega$  on  $E_{i_h}$  *k-number*. We obtain the following fact, since the proper transform of  $C$  intersects only  $E_{i_h}$  on  $M$ .

FACT 5.  $K \cdot C = k_{i_h}$ .

The  $k$ -number corresponding to the  $\delta$ -sequence  $\{6, 22, 17\}$  is 20. We classified  $\delta$ -sequences with genus  $\leq 50$  into groups that generate the same semigroups. Furthermore, we listed  $\delta$ -sequences with the following three properties: (i) There is no other  $\delta$ -sequence that generates the same semigroup. (ii) The number of generators is 3. (iii)  $k$ -number  $\geq -1$ . Then, we obtained  $\{6, 15, 4\}, \{4, 14, 9\}, \{6, 15, 7\}, \{6, 21, 4\}, \{6, 10, 11\}, \{4, 18, 13\}, \dots$ . We got  $\{6, 21, 4\}$  as a negative answer to the Abhyankar's question using Gröbner basis computations for polynomial systems corresponding to these  $\delta$ -sequences. We show its details below.

First, we need to prove the uniqueness of  $\{6, 21, 4\}$  since the above-mentioned classification is for genus  $\leq 50$ . Let  $\{\delta_0, \delta_1, \dots, \delta_h\}$  be a sequence of natural numbers satisfying Abhyankar-Moh's condition, where  $h \geq 1$ . Set  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$  ( $1 \leq k \leq h+1$ ) and  $q_k = d_k/d_{k+1}$  ( $1 \leq k \leq h$ ).

**Lemma 2.** *For any  $k$  ( $1 \leq k \leq h$ ),  $d_{k+1} \neq \delta_k$ .*

Proof. Assume that there exists a natural number  $k$  ( $1 \leq k \leq h$ ) such that  $d_{k+1} = \delta_k$ . We get  $q_k \delta_k = (d_k/d_{k+1}) \delta_k = d_k$ . From this and  $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ ,  $q_k \mid \delta_i$  for each  $i$  ( $0 \leq i \leq k-1$ ). By Abhyankar-Moh's condition (5), it follows that there exists an integer  $k_0$  ( $0 \leq k_0 \leq k-1$ ) such that  $q_k \delta_k = \delta_{k_0}$ . However, it must be  $k_0 = k-1$  from  $q_k \delta_k = d_k$  and Abhyankar-Moh's condition (2). Thus, we obtain  $d_k = \delta_{k-1}$  and  $\delta_{k-1} > \delta_k$ . We get  $\delta_0 > \delta_1 > \dots > \delta_{k-1} > \delta_k$ , using the above result inductively, which is contradictory to Abhyankar-Moh's condition (1).  $\square$

**DEFINITION 8** (primitive). An element of a semigroup is called primitive if it is not a sum of two nonzero elements of the semigroup.

**Lemma 3** ([14]). *Let  $\Omega$  be a semigroup and  $\{\delta_0, \delta_1, \dots, \delta_h\}$  a generators of  $\Omega$ . If  $x$  is a primitive element of  $\Omega$ , there exists a integer  $k$  ( $0 \leq k \leq h$ ) such that  $x = \delta_k$ .*

Proof. By the definition of primitive elements, this assertion is clear.  $\square$

**Proposition 1.** *The planar semigroup generated by  $\{6, 21, 4\}$  has no other sequence satisfying Abhyankar-Moh's condition.*

Proof. Let  $\Omega$  be the planar semigroup generated by  $\{6, 21, 4\}$ . 6, 21 and 4 are primitive elements of  $\Omega$ . Thus, by Lemma 3, 6, 21 and 4 belong to any generating set of  $\Omega$ . There are six possible cases for the order of 6, 21 and 4.

(i)  $\{\dots, 6, \dots, 21, \dots, 4, \dots\}$ : By  $\gcd\{6, 21, 4\} = 1$  and Abhyankar-Moh's condition (2), 4 is the last element of the sequence. By  $\gcd\{6, 21\} = 3$ ,  $\gcd\{6, 21, 4\} = 1$  and Abhyankar-Moh's condition (2), there is no element between of 6 and 21, and

also between of 21 and 4. Furthermore, by Lemma 2, 6 is the first element of the sequence. Thus, we get  $\{6, 21, 4\}$ .

(ii)  $\{\dots, 21, \dots, 6, \dots, 4, \dots\}$ : We get  $\{21, 6, 4\}$  in the same way as (i). But this is contradictory to Abhyankar-Moh's condition (1).

(iii)  $\{\dots, 4, \dots, 21, \dots, 6, \dots\}$ : By  $\gcd\{4, 21\} = 1$ , this case is impossible.

(iv)  $\{\dots, 21, \dots, 4, \dots, 6, \dots\}$ : By  $\gcd\{21, 4\} = 1$ , this case is impossible.

(v)  $\{\dots, 6, \dots, 4, \dots, 21, \dots\}$ : We get  $\{6, 4, 21\}$  in the same way as (i). But this is contradictory to Abhyankar-Moh's condition (1).

(vi)  $\{\dots, 4, \dots, 6, \dots, 21, \dots\}$ : We get  $\{4, 6, 21\}$  in the same way as (i). From  $d_1 = 4$ ,  $d_2 = \gcd\{4, 6\} = 2$ ,  $q_1 = d_1/d_2 = 2$ . Thus,  $q_1\delta_1 = 12 < \delta_2$ . But this is contradictory to Abhyankar-Moh's condition (4).

As a consequence, the generating sequence of  $\Omega$  satisfying Abhyankar-Moh's condition is only  $\{6, 21, 4\}$ .  $\square$

We assume that there exists a polynomial curve having the  $\delta$ -sequence  $\{6, 21, 4\}$ . The defining polynomial of this curve is as follows:

$$f = g_2^3 + a_{2,0}x^2 + c_{1,0,1}xg_2 + c_{1,0,0}x + c_{0,0,1}g_2 + c_{0,0,0}$$

where

$$\begin{aligned} g_2 &= g_1^2 + a_7x^7 + c_{6,0}x^6 + c_{5,0}x^5 + c_{4,0}x^4 + c_{3,0}x^3 \\ &\quad + c_{2,0}x^2 + c_{1,0}x + c_{0,0}, \\ g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0. \end{aligned}$$

By the substitution of  $g_1$  for  $g_2$  and changing parameters, we get

$$\begin{aligned} g_2 &= y^2 + a_7x^7 + y(c_{3,1}x^3 + c_{2,1}x^2 + c_{1,1}x + c_{0,1}) \\ &\quad + c_{6,0}x^6 + c_{5,0}x^5 + c_{4,0}x^4 + c_{3,0}x^3 + c_{2,0}x^2 + c_{1,0}x + c_{0,0}. \end{aligned}$$

We can set  $a_7 = -1$  by the automorphism of  $\mathbf{C}[x, y]$ ,  $x \mapsto -a^{-1/7}x$ ,  $y \mapsto y$ . By  $x \mapsto x + c_{6,0}/7$ , we can remove the term  $c_{6,0}x^6$ . Further, by  $y \mapsto y - (c_{3,1}x^3 + c_{2,1}x^2 + c_{1,1}x + c_{0,1})/2$ , we can remove the terms  $y(c_{3,1}x^3 + c_{2,1}x^2 + c_{1,1}x + c_{0,1})$ . The proper polynomial parametrization of this curve is of the following form:

$$\begin{cases} x = t^6 + a_1t^5 + a_2t^4 + a_3t^3 + a_4t^2 + a_5t + a_6 \\ y = t^{21} + b_1t^{20} + b_2t^{19} + b_3t^{18} + \dots + b_{19}t^2 + b_{20}t + b_{21} \end{cases}$$

By the automorphism of  $\mathbf{C}[t]$ ,  $t \mapsto t - a_1/6$ , we may remove the term  $a_1t^5$  in  $x(t)$ . By Lemma 1, we get  $\deg_t g_2(x(t), y(t)) = 4$ . All formal terms with  $t$ -degree more than 4 in  $g_2(x(t), y(t))$  must be eliminated. We obtain the polynomial system  $J$  from the coefficients of these terms. Furthermore, we can successively eliminate the variables

$b_1, c_{5,0}, c_{4,0}, c_{3,0}, c_{2,0}, c_{1,0}, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{13}, b_{14}, b_{15}, b_{16}, b_{17}, b_{19}, b_{20}$  and  $b_{21}$  in this order by using polynomials with the form:  $cz - h(w_1, w_2, \dots, w_s)$  in  $J$ , where  $c \in \mathbf{C}^*$ ,  $z, w_1, w_2, \dots, w_s$  are variables and  $h \in \mathbf{C}[w_1, w_2, \dots, w_s]$ . As a result, we can get the polynomial system with 7 variables  $\{a_2, a_3, a_4, a_5, a_6, b_{12}, b_{18}\}$  and 13 polynomials. We denote the obtained polynomial system by the same character  $J$ .

We used total degree reverse lexicographic ordering (DRL) with  $a_2 > a_3 > a_4 > a_5 > a_6 > b_{12} > b_{18}$  to the Gröbner basis computation. The CPU time for the computation was 3 hours 40 minutes and the required memory 850 MB. The computation was conducted on a dual AMD AthlonMP 2200+ (1.8 GHz) machine with 4 GB memory running FreeBSD 4.7. The computer algebra system used was Risa/Asir [12].

The obtained Gröbner basis  $G_{\{6,21,4\}}$  of the ideal  $Id(J)$  was not  $\{1\}$ . However, the normal form of the coefficient  $p$  of the term with  $t$ -degree = 4 in  $g_2(x(t), y(t))$  with respect to  $G_{\{6,21,4\}}$  is 0. By the property of Gröbner bases for ideal membership, this shows that  $p \in Id(J)$ . Thus, we get  $\deg_t g_2(x(t), y(t)) < 4$ . Since this is contradictory to  $\deg_t g_2(x(t), y(t)) = 4$ , there is no polynomial curve having the  $\delta$ -sequence  $\{6, 21, 4\}$ . Consequently,  $\{6, 21, 4\}$  is a negative answer to the Abhyankar's question.

**REMARK.** We computed the Gröbner bases corresponding to the  $\delta$ -sequences  $\{6, 15, 4\}$ ,  $\{4, 14, 9\}$  and  $\{6, 15, 7\}$ , and obtained the normal forms of the coefficients of terms with  $t$ -degree  $\delta_2$  in  $g_2(x(t), y(t))$  with respect to them. However, they were not 0 unlike the case of  $\{6, 21, 4\}$ .

## 6. Gröbner basis computation using weighted ordering

It is well-known that Gröbner basis computation is accelerated by setting weights if the input polynomial system is quasi homogeneous (see [10]). The polynomial system  $J$  corresponding to the  $\delta$ -sequence  $\{6, 21, 4\}$  is quasi homogeneous by the constructing method, and  $J$  become homogeneous by setting the indices of each variable as weights. We get the following weighted ordering:  $b_{18} > b_{12} > a_6 > a_5 > a_4 > a_3 > a_2$  with weights  $\{18, 12, 6, 5, 4, 3, 2\}$ .

After various trials and errors, we obtained the Gröbner basis of the ideal  $Id(J)$  by lexicographic ordering (LEX) with the above setting in a very short time and only 11 MB of memory. For verification of the results obtained by Asir and a comparison of computation time, we used another computer algebra system Singular 2.0.4 [7]. The results obtained by Singular coincided with Asir. The computation times are as follows:

$\delta$ -seq.	System	DRL	Sawada	Sawada weight DRL	Sawada weight LEX	Weight DRL	Weight LEX
$\{6, 21, 4\}$	Asir	5884	2.17	0.28	0.26	0.24	0.17
	Singular	53 h	—	0.35	0.34	0.31	0.17

‘h’ means hour. The time unit of values without ‘h’ are seconds. The line ‘—’ means out of memory. ‘Sawada’ is an automatic block ordering by Dr. Sawada in AIST (see [15]). Sawada ordering is obtained by a heuristic algorithm.

We tried to compute the Gröbner basis of  $\{6, 22, 17\}$ -type by using weighted ordering. Let  $I$  be the polynomial system corresponding to the  $\delta$ -sequence  $\{6, 22, 17\}$  (see Section 4).  $I$  has 11 variables  $\{a_2, a_3, a_4, a_5, a_6, b_2, b_8, b_{12}, b_{14}, b_{18}, b_{20}\}$  and 17 polynomials. Further,  $I$  is also quasi homogeneous, and becomes homogeneous by setting the indices of each variable as weights. As the above, we get the following weighted ordering:  $b_{20} > b_{18} > b_{14} > b_{12} > b_8 > a_6 > a_5 > a_4 > a_3 > b_2 > a_2$  with weights  $\{20, 18, 14, 12, 8, 6, 5, 4, 3, 2, 2\}$ . We obtained the Gröbner basis of the ideal  $Id(I)$  by LEX with the above setting. The memory used was 116 MB. The computation times were as follows:

$\delta$ -seq.	System	DRL	Sawada	Sawada weight DRL	Sawada weight LEX	Weight DRL	Weight LEX
$\{6, 22, 17\}$	Asir	—	—	303.8	382.8	2368	285.7
	Singular	—	—	92 h	92 h	326 h	78 h

Let  $G_{\{6, 22, 17\}}$  be the obtained Gröbner basis of  $Id(I)$ . Let  $q$  be the coefficient of the term with  $t$ -degree = 17 in  $g_2(x(t), y(t))$ . Further, let  $\bar{q}$  be the normal form of  $q$  with respect to  $G_{\{6, 22, 17\}}$ . We got that the normal form of  $\bar{q}^3$  with respect to  $G_{\{6, 22, 17\}}$  is 0 by Asir and Singular. This shows that  $q \in \sqrt{Id(I)}$ . This is contradictory to  $\deg_t g_2(x(t), y(t)) = 17$ . Consequently, the Sathaye-Stenerson’s conjecture is also true.

The data files for polynomial systems that appeared in this paper are available from <http://www.fukuoka-edu.ac.jp/~fujimoto/abh2/>.

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