Oka manifolds and ellipticity

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Abstract. A complex manifold is called an Oka manifold if the Oka principle for maps from Stein spaces holds. On the other hand, ellipticity is opposite to Kobayashi–Eisenman–Brody hyperbolicity, and it means the existence of many dominating holomorphic maps from complex Euclidean spaces. In this thesis, we investigate the relationship between these notions. We first establish the characterization of Oka manifolds by convex ellipticity which implies Gromov's conjecture. As an application, the localization principle for Oka manifolds is proved. By using this principle, we show that there exists a nonelliptic Oka manifold which negatively answers a long-standing question of Gromov.

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1. Introduction

The purpose of this thesis is to investigate the relationship between the Oka property and ellipticity. Roughly speaking, a complex manifold is said to enjoy the Oka property if continuous maps from Stein spaces can be deformed into holomorphic maps with approximation and interpolation. Such a complex manifold is called an *Oka manifold* (see Definition 2.5). This notion has its roots in the work of Oka [39] on the *Oka principle*. The heuristic Oka principle states that analytic problems on Stein spaces admit analytic solutions if there are no topological obstructions. Thus we can say that a complex manifold is Oka if and only if the Oka principle for maps from Stein spaces holds. The following Oka principle is one of the most important theorems in Oka theory.

Theorem 1.1 (Forstnerič's Oka principle [11]). *A complex manifold Y is Oka if and only if it enjoys the following* Convex Approximation Property (CAP)*: Every holomorphic map from an open neighborhood of a compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) to Y can be approximated uniformly on K by holomorphic maps $\mathbb{C}^n \to Y$.*

It follows from the result of Grauert [21] that every complex Lie group and, more generally, every complex homogeneous manifold is Oka (Proposition 2.7). In general, however, it is difficult to verify CAP directly except these examples. Gromov's Oka principle gives a sufficient condition, called *ellipticity*, for a manifold to be Oka. Ellipticity is opposite to Kobayashi–Eisenman–Brody hyperbolicity, and hence it should mean the existence of *many* dominating holomorphic maps from $\mathbb{C}^N$ ($N \in \mathbb{N}$). The following is the definition of Gromov's ellipticity.

Definition 1.2. A complex manifold Y is said to be *elliptic* if there exists a holomorphic map $s : E \to Y$ from the total space of a holomorphic vector bundle over Y such that $s(0_y) = y$ and $s|_{E_y} : E_y \cong \mathbb{C}^N \to Y$ is a submersion at 0_y for each $y \in Y$.

The following Oka principle is the main result of Gromov's paper [23] in 1989.

Theorem 1.3 (Gromov's Oka principle [23]). *Every elliptic manifold is Oka.*

In the same paper, Gromov also proved the converse for Stein manifolds [23, Remark 3.2.A][1]. Then he asked a question whether the converse holds for *all* complex manifolds [23, Question 3.2.A″].

Question 1.4 (Gromov [23, Question 3.2.A″]). Is every Oka manifold elliptic?

There is another ellipticity condition introduced by Gromov [22] which is called *Condition* Ell_1.

Definition 1.5. A complex manifold Y satisfies *Condition* Ell_1 if for any Stein space X and any holomorphic map $f : X \to Y$ there exist $N \in \mathbb{N}$ and a holomorphic map $s : X \times \mathbb{C}^N \to Y$ such that $s(x, 0) = f(x)$ and $s(x, \cdot) : \mathbb{C}^N \to Y$ is a submersion at 0 for each $x \in X$.

Gromov conjectured that Condition Ell_1 implies a Runge type approximation theorem for holomorphic maps from Stein manifolds and hence the Oka property [23, §1.4.E″] (see also [22, p. 72]). Since the Oka property implies Condition Ell_1 (Proposition 3.4), Gromov's conjecture can be rephrased as follows.

Conjecture 1.6 (cf. Gromov [23, §1.4.E″]). A complex manifold is an Oka manifold if and only if it satisfies Condition Ell_1.

[1] The condition Ell_∞ in [23, Remark 3.2.A] is equivalent to the Oka property (cf. [14, §5.15]).

The goal of this thesis is to solve Question 1.4 and Conjecture 1.6. After the brief reviews of Stein spaces in Section 2 and dominating sprays in Section 3, we prove that Conjecture 1.6 is true.

Theorem 1.7. *A complex manifold is an Oka manifold if and only if it satisfies Condition* Ell_1*.*

Theorem 1.7 is proved in Section 4. In fact, we introduce the new ellipticity condition called *convex ellipticity* and show that it characterizes Oka manifolds.

As an application of Theorem 1.7, we prove the following localization principle for Oka manifolds in Section 5. It can be considered as the analytic version of Gromov's localization principle [23, §3.5.B] (see also [9, Proposition 1.3]). Here, a subset of Y is said to be *Zariski open* if its complement is a closed complex subvariety.

Theorem 1.8. *Let* Y *be a complex manifold. Assume that each point of* Y *has a Zariski open Oka neighborhood. Then* Y *is an Oka manifold.*

Theorem 1.8 gives new examples of Oka manifolds. Some of them are given in Section 5. The existence of a nonelliptic Oka manifold is also proved by using this localization principle. Andrist, Shcherbina and Wold [2] showed that in a Stein manifold of dimension at least three every compact holomorphically convex[2] set with an infinite derived set has a nonelliptic complement. However, there has been no example of an Oka complement of this type. In Section 6, as an application of the localization principle, we obtain such an example which negatively answers Question 1.4.

Theorem 1.9. *For any* $n \geq 3$*, the complement* $\mathbb{C}^n \setminus ((\overline{\mathbb{N}^{-1}})^2 \times \{0\}^{n-2})$ *is a non-elliptic Oka manifold where* $\mathbb{N}^{-1} = \{j^{-1} : j \in \mathbb{N}\} \subset \mathbb{C}$*.*

There is a weaker variant of ellipticity called *weak subellipticity* (Definition 3.5) which is also a sufficient condition to be Oka (see Section 3). In fact, we prove that the complement in Theorem 1.9 is not even weakly subelliptic (Corollary 6.4).

In Section 7, we mention the results obtained in [34, 36] which generalize and improve the above theorems. In particular, the relative version (Theorem 7.4) of our characterization of Oka manifolds and further counterexamples (Corollary 7.10) to Question 1.4 are given.

2. Stein spaces and the Oka property

In this section, we briefly review the basic properties of Stein spaces and the Oka property of a complex manifold. Throughout this thesis, all complex spaces are taken to be reduced and finite dimensional.

Definition 2.1 (Stein [42], Grauert [20])**.** A second countable complex space X is a *Stein space* if it satisfies the following conditions:

[2]The holomorphic convexity was not assumed in [2] but used in the proof implicitly.

(1) (holomorphic separability) For any pair of distinct points $x, x' \in X$ there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $f(x) \neq f(x')$.
(2) (holomorphic convexity) For any compact set $K \subset X$ its $\mathcal{O}(X)$-*hull*

$$\widehat{K}_{\mathcal{O}(X)} = \left\{ x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(X) \right\}$$

is also compact.

The notion of a Stein space generalizes the classes of open Riemann surfaces, domains of holomorphy and affine algebraic varieties as follows.

Example 2.2. (1) (Behnke–Stein [4]) A connected Riemann surface is Stein if and only if it is an open Riemann surface.
(2) (Cartan–Thullen [6]) A domain in $\mathbb{C}^n$ ($n \in \mathbb{N}$) is Stein if and only if it is a domain of holomorphy.
(3) By definition, every closed complex subvariety of a Stein space is Stein. In particular, every affine algebraic variety is Stein.

The following is one of the most important theorems about Stein spaces. In fact, Cartan's Theorem B characterizes Stein spaces.

Theorem 2.3 (Cartan [5])**.** *Let X be a Stein space and $\mathcal{F}$ be a coherent analytic sheaf on X.*
(1) (Cartan's Theorem A) Every stalk $\mathcal{F}_x$ ($x \in X$) is generated as an $\mathcal{O}_{X,x}$-module by global sections of $\mathcal{F}$.
(2) (Cartan's Theorem B) $H^q(X, \mathcal{F}) = 0$ for all $q \geq 1$.

In Oka theory, Siu's theorem on the existence of Stein neighborhoods is also very important.

Theorem 2.4 (Siu [41])**.** *Every locally closed Stein subvariety of a complex space admits a basis of open Stein neighborhoods.*

Let us recall the definition of the Oka property of a complex manifold. Here, a compact set $K \subset X$ is said to be $\mathcal{O}(X)$-*convex* if $K = \widehat{K}_{\mathcal{O}(X)}$ (see Definition 2.1). For a subset A of a topological space X, let $\operatorname{Op} A = \operatorname{Op}_X A$ denote a non-specified open neighborhood of A in X.

Definition 2.5. A complex manifold Y is an *Oka manifold* if it enjoys the following *Oka property*[3]: For any Stein space X, any closed complex subvariety $X' \subset X$, any compact $\mathcal{O}(X)$-convex set $K \subset X$, any compact Hausdorff spaces $P_0 \subset P$[4] and any (continuous) family of continuous maps $f_0 : P \times X \to Y$ such that $f_0|_{P_0 \times X}$,

[3]This property is usually called the *Parametric Oka Property with Approximation and Jet Interpolation (POPAJI)* (cf. [14, §5.15]).

[4]In fact, the parameter spaces $P_0 \subset P$ in Forstnerič's Oka principle (Theorem 1.1) were restricted to be Euclidean in [11]. This restriction was removed in [34].

$f_0|_{P\times X'}$ and $f_0|_{P\times \operatorname{Op} K}$ are families of holomorphic maps, there exists a homotopy $f_t : P \times X \to Y$ ($t \in [0,1]$) such that the following hold for each $t \in [0,1]$;

(1) $f_t = f_0$ on $(P_0 \times X) \cup (P \times X')$,
(2) $f_t|_{P\times \operatorname{Op} K}$ is a family of holomorphic maps which approximates f_0 uniformly on $P \times K$, and
(3) $f_1 : P \times X \to Y$ is a family of holomorphic maps.

If, in addition, $f_0|_{P\times \operatorname{Op} X'}$ is a family of holomorphic maps, then the homotopy f_t can be chosen tangent to f_0 to a given finite order along X'.

The most basic examples of Oka manifolds are complex Euclidean spaces $\mathbb{C}^n$ ($n \in \mathbb{N}$). This is a consequence of the classical theory of Stein spaces. For the proof, we refer the reader to [14, Theorem 2.8.4].

Theorem 2.6 (Cartan–Oka–Weil Theorem with Parameters)**.** *For any $n \in \mathbb{N}$, the complex Euclidean space $\mathbb{C}^n$ is Oka.*

By using Forstnerič's Oka principle (Theorem 1.1), we can verify the Oka property of a complex homogeneous manifold. Recall that a complex manifold Y is called a *complex homogeneous manifold* if there exists a complex Lie group acting holomorphically and transitively on Y.

Proposition 2.7 (Grauert [21])**.** *Every complex homogeneous manifold is Oka.*

Proof. Let Y be a complex homogeneous manifold. By definition, there exists a complex Lie group G acting holomorphically and transitively on Y. By Forstnerič's Oka principle (Theorem 1.1), it suffices to prove that Y enjoys CAP. Take a compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) and a holomorphic map $f : \operatorname{Op} K \to Y$. We may assume that K contains the origin $0 \in \mathbb{C}^n$. Consider the homotopy $f_t : \operatorname{Op} K \to Y$ ($t \in [0,1]$) defined by $f_t(z) = f(tz)$. Then $f_1 = f$ and f_0 is a constant map. Thus f_0 can be extended to $\mathbb{C}^n \to Y$ holomorphically. Let $\mathfrak{g} = \mathrm{T}_1 G$ denote the complex Lie algebra of G. Note that the holomorphic map $s : Y \times \mathfrak{g} \to Y$, $s(y,v) = \exp v \cdot y$ satisfies that for each $y \in Y$ the restriction $s(y,\cdot) : \mathfrak{g} \to Y$ is a submersion at 0. Thus if $N \in \mathbb{N}$ is large enough, there exist holomorphic maps $\varphi_j : \operatorname{Op} K \to \mathfrak{g}$ ($j = 1,\dots,N$) such that $f_{j/N} = \exp \varphi_j \cdot f_{(j-1)/N}$ for all $j = 1,\dots,N$ (see Lemma 3.8). By Theorem 2.6, there exist holomorphic maps $\widetilde{\varphi}_j : \mathbb{C}^n \to \mathfrak{g}$ ($j = 1,\dots,N$) which approximate φ_j uniformly on K. Then

$$\tilde{f} = \exp \varphi_N \cdots \exp \varphi_1 \cdot f_0 : \mathbb{C}^n \to Y$$

approximates f uniformly on K. □

Recall that ellipticity also implies the Oka property by Gromov's Oka principle (Theorem 1.3). We give a few examples of elliptic manifolds.

Example 2.8. (1) Every complex homogeneous manifold Y is elliptic. Indeed, the holomorphic map $s : Y \times \mathfrak{g} \to Y$ in the proof of Proposition 2.7 satisfies the condition in the definition of ellipticity (Definition 1.2).

(2) Gromov proved that the complement $\mathbb{C}^n \setminus A$ of a closed algebraic subvariety A of codimension at least two in $\mathbb{C}^n$ is elliptic [23, §0.5.B]. Forstnerič and Prezelj generalized Gromov's result to a tame closed complex subvariety $A \subset \mathbb{C}^n$ of codimension at least two [17, Theorem 1.6]. Here, a closed complex subvariety $A \subset \mathbb{C}^n$ is said to be *tame* if there exists a holomorphic automorphism φ of $\mathbb{C}^n$ such that the closure of $\varphi(A)$ in the projective space $\mathbb{P}^n \supset \mathbb{C}^n$ does not contain the hyperplane at infinity $\mathbb{P}^n \setminus \mathbb{C}^n$. On the other hand, Rosay and Rudin constructed a discrete set in $\mathbb{C}^n$ $(n > 1)$ whose complement is not Oka [40, Theorem 4.5].

The above examples and the following fact yield many examples of Oka manifolds. In the case when π is a holomorphic covering map, this is an easy consequence of Theorem 1.1.

Proposition 2.9 (cf. [14, Theorem 5.6.5])**.** *Assume that B is a complex manifold and $\pi : Y \to B$ is a holomorphic fiber bundle whose fiber is Oka (e.g. a holomorphic covering map). Then Y is an Oka manifold if and only if B is an Oka manifold.*

3. Dominating sprays and ellipticity conditions

Next, we recall the notion of a dominating spray which plays a fundamental role in Oka theory.

Definition 3.1. Let X be a complex space and Y be a complex manifold.

(1) A *(local) spray* over a holomorphic map $f : X \to Y$ is a holomorphic map $s : U \to Y$ from an open neighborhood of the zero section of a holomorphic vector bundle $E \to X$ such that $s(0_x) = f(x)$ for all $x \in X$. Particularly in the case of $U = E$, s is also called a *global spray*.

(2) A family of sprays $s_j : U_j \to Y$ $(j = 1, \ldots, k)$ over a holomorphic map $f : X \to Y$ is *dominating on a subset* $A \subset X$ if $\sum_{j=1}^k (\mathrm{d}s_j)_{0_x}(\mathrm{T}_{0_x} U_{j,x}) = \mathrm{T}_{f(x)} Y$ for each $x \in A$. In the case of $A = X$, it is simply said to be *dominating*.

Note that the holomorphic map $s : E \to Y$ in Definition 1.2 (resp. $s : X \times \mathbb{C}^N \to Y$ in Definition 1.5) is nothing but a dominating global spray over the identity map $\mathrm{id}_Y : Y \to Y$ (resp. $f : X \to Y$).

Remark 3.2. Assume that X is a Stein space, Y is a complex manifold and $s : U \to Y$ is a spray over a holomorphic map $f : X \to Y$. Since every holomorphic vector bundle over a Stein space is the image of a holomorphic vector bundle morphism from a trivial vector bundle by Cartan's Theorem A (Theorem 2.3), there exist $N \in \mathbb{N}$, an open neighborhood $\widetilde{U} \subset X \times \mathbb{C}^N$ of the zero section and a spray $\tilde{s} : \widetilde{U} \to Y$ such that $(\mathrm{d}\tilde{s})_{0_x}(\mathrm{T}_{0_x} \widetilde{U}_x) = (\mathrm{d}s)_{0_x}(\mathrm{T}_{0_x} U_x)$ for all $x \in X$.

The following lemma ensures the existence of dominating local sprays.

Lemma 3.3 (cf. [14, Lemma 5.10.4])**.** *Assume that X is a Stein space and Y is a complex manifold. Then for any holomorphic map $f : X \to Y$ there exists a dominating local spray over f.*

Proof. Take an arbitrary holomorphic map $f : X \to Y$. Consider the graph $\Gamma_f \subset X \times Y$ of f. Then by Siu's theorem (Theorem 2.4) there exists an open Stein neighborhood $U \subset X \times Y$ of Γ_f. It follows from Cartan's Theorem A (Theorem 2.3) that there exist finitely many holomorphic vector fields V_j $(j = 1, \ldots, k)$ on U which span the relative tangent bundle of the projection $X \times Y \to X$ at every point of Γ_f. Let φ_t^j denote the local flow of V_j for each $j = 1, \ldots, k$. Then

$$s(x, t_1, \ldots, t_k) = \varphi_{t_1}^1 \circ \cdots \circ \varphi_{t_k}^k \circ f(x)$$

defines a dominating local spray s over f from an open neighborhood of the zero section $X \times \{0\}^N \subset X \times \mathbb{C}^N$. $\square$

By using Lemma 3.3, we can prove the following implication in Theorem 1.7.

Proposition 3.4 (cf. [14, Corollary 8.8.7])**.** *Every Oka manifold enjoys Condition* Ell_1.

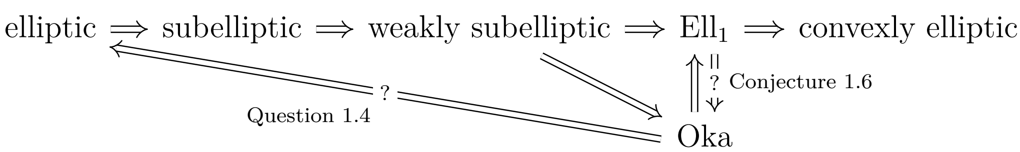
Proof. Let Y be an Oka manifold, and take a Stein space X and a holomorphic map $f : X \to Y$. By Lemma 3.3, there exist $N \in \mathbb{N}$, an open neighborhood $U \subset X \times \mathbb{C}^N$ of the zero section and a dominating local spray $s : U \to Y$ over f. After shrinking U if necessary, we can extend s to a continuous map $s : X \times \mathbb{C}^N \to Y$. Since Y is Oka, there exists a holomorphic map $\tilde{s} : X \times \mathbb{C}^N \to Y$ which agrees with s to the second order along $X \times \mathbb{C}^N$. Then $\tilde{s}$ is a dominating global spray over f. $\square$

Let us recall the following variants of ellipticity and Condition Ell_1.

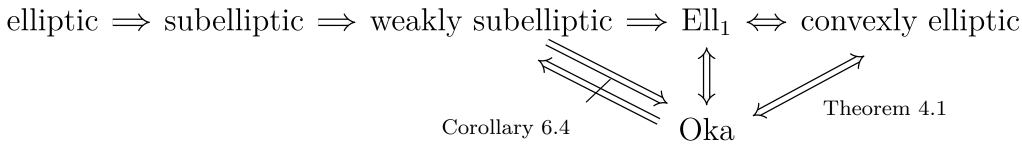
Definition 3.5 (cf. [14, Definition 5.6.13], [34, Definition 1.2])**.** Let Y be a complex manifold.

(1) Y is *subelliptic* if there exists a finite dominating family of global sprays over the identity map id_Y.
(2) Y is *weakly subelliptic* if for any point $y \in Y$ there exists a family of global sprays over the identity map id_Y dominating on y.
(3) Y is *convexly elliptic* if for any compact convex set $K \subset \mathbb{C}^n$ $(n \in \mathbb{N})$ and any holomorphic map $f : \mathrm{Op}\, K \to Y$ there exists a dominating global spray over f.

In fact, Gromov's Oka principle (Theorem 1.3) was generalized to weakly subelliptic manifolds by Forstnerič (cf. [14, Corollary 5.6.14]). Note that Condition Ell_1 implies convex ellipticity since every compact convex set admits a basis of open Stein neighborhoods. Thus we have the following:



Our main results can be summarized as follows:



Remark 3.6. (1) For a Stein manifold Y, it can be easily seen that ellipticity, the Oka property and Condition Ell_1 are equivalent. Indeed, if Y satisfies Condition Ell_1, then there exists a dominating global spray over the identity map id_Y, hence Y is elliptic.
(2) The implication from ellipticity to Condition Ell_1 can be verified without Gromov's Oka principle as follows. Let $s : E \to Y$ be a dominating global spray over the identity map id_Y, X be a Stein space and $f : X \to Y$ be a holomorphic map. Consider the pullback of the holomorphic vector bundle $E \to Y$ by f:

$$\begin{array}{ccccc} f^*E & \xrightarrow{\tilde{f}} & E & \xrightarrow{s} & Y \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{f} & Y & & \end{array}$$

Then the composition $s \circ \tilde{f} : f^*E \to Y$ is a dominating global spray over f. By the similar argument, we can also prove that subellipticity (resp. weak subellipticity) implies Condition Ell_1 (resp. convex ellipticity) directly (cf. [14, Proof of Proposition 8.8.11 (b)]).

In the rest of this section, we recall a few facts which are needed in the proof of Theorem 1.7. Recall that a *Cartan pair* in a complex space X is a pair (A, B) of compact subsets of X such that each of $A, B, A \cap B$ and $A \cup B$ has a basis of open Stein neighborhoods in X and the separation condition $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$ is satisfied. The following is one of the main tools used in the proof of Forstnerič's Oka principle (Theorem 1.1).

Lemma 3.7 (cf. [14, Remark 5.9.4 (C)])**.** *Let X be a complex space, Y be a complex manifold, (A, B) be a Cartan pair in X, $W \subset \mathbb{C}^N$ be an open neighborhood of the origin, and $s_A : \mathrm{Op}\, A \times W \to Y$, $s_B : \mathrm{Op}\, B \times W \to Y$ be sprays over holomorphic maps $f_A : \mathrm{Op}\, A \to Y$, $f_B : \mathrm{Op}\, B \to Y$, respectively. Assume that $s_A|_{\mathrm{Op}(A\cap B)\times W}$ is dominating, and s_B is sufficiently close to s_A on $\mathrm{Op}(A \cap B) \times W$. Then there exists a holomorphic map $f : \mathrm{Op}(A \cup B) \to Y$ which is close to f_A on A. If, in addition, $s_B|_{\mathrm{Op}\, K_0 \times W}$ is sufficiently close to a given spray $s_0 : \mathrm{Op}\, K_0 \times W \to Y$ over $f_0 : \mathrm{Op}\, K_0 \to Y$, where $K_0 \subset B$ is a compact set, then f can be chosen close to f_0 on K_0.*

The following lemma is also well-known, but we give its proof for the reader's convenience.

Lemma 3.8. *Let X be a Stein space, Y be a complex manifold and $s : U \to Y$ be a dominating spray over a holomorphic map $f : X \to Y$ where U is an open* *neighborhood of the zero section of a holomorphic vector bundle* $p : E \to X$*. Then for the fiber-preserving holomorphic map* $(p, s) : U \to X \times Y$ *there exists a fiber-preserving holomorphic map* $\iota : \operatorname{Op}\Gamma_f \to U$ *from an open neighborhood of the graph* Γ_f *of* f *such that* $(p, s) \circ \iota = \mathrm{id}_{\operatorname{Op}\Gamma_f}$*.*

Proof. Let E' denote the holomorphic vector subbundle of E with fibers

$$E'_x = \ker(\mathrm{d}(s|_{U_x})_{0_x} : E_x = \mathrm{T}_{0_x}U_x \to \mathrm{T}_{f(x)}Y) \quad (x \in X).$$

Since X is Stein, there exists another holomorphic vector subbundle E'' of E such that $E = E' \oplus E''$ by an easy application of Cartan's Theorem B (cf. [14, Corollary 2.6.6]). Then the map $(p, s)|_{U\cap E''} : U \cap E'' \to X \times Y$ restricts to a biholomorphic map from an open neighborhood $U'' \subset U \cap E''$ of the zero section onto an open neighborhood of the graph Γ_f. Its inverse holomorphic map ι finishes the proof. □

4. Elliptic characterization of Oka manifolds

The goal of this section is to prove the following theorem which implies Theorem 1.7.

Theorem 4.1. *A complex manifold is Oka if and only if it is convexly elliptic.*

In fact, we prove the equivalence between CAP (see Theorem 1.1) and convex ellipticity. In the proof, we use the technique established in our previous paper [30] where we proved other characterizations of Oka manifolds. By approximating compact convex sets by convex polyhedra, we can reduce the approximation problem in CAP to the following lemma (cf. [30, Proof of Theorem 3.2]; see also [14, Lemma 5.15.4]).

Lemma 4.2. *A complex manifold* Y *is Oka if and only if for any compact convex set* $K \subset \mathbb{C}^n$ *and any affine linear function* $\lambda : \mathbb{C}^n \cong \mathbb{R}^{2n} \to \mathbb{R}$*, every holomorphic map from an open neighborhood of* $K_0 = \{z \in K : \lambda(z) \leq 0\}$ *to* Y *can be approximated uniformly on* K_0 *by holomorphic maps* $\operatorname{Op} K \to Y$*.*

Let $\mathrm{pr}_{X_{\lambda_0}} : \prod_{\lambda\in\Lambda} X_\lambda \to X_{\lambda_0}$ denote the projection map to X_{λ_0} $(\lambda_0 \in \Lambda)$.

Proof of Theorem 4.1. We have already seen in Proposition 3.4 that the Oka property implies Condition Ell_1 and hence convex ellipticity, but we prove here that CAP implies convex ellipticity without Forstnerič's Oka principle (Theorem 1.1). Assume that Y is an Oka manifold, and take a compact convex set $K \subset \mathbb{C}^n$ and a holomorphic map $f : \operatorname{Op} K \to Y$. By Lemma 3.3, there exists a dominating local spray $s : \operatorname{Op} K \times \mathbb{B}^N \to Y$ over f. Since Y enjoys CAP, there exists a dominating global spray $\tilde{s} : \operatorname{Op} K \times \mathbb{C}^N \to Y$ which is close to s on $\operatorname{Op} K \times \mathbb{B}^N$. Then Rouché's theorem (cf. [7, p. 110]) implies the existence of a holomorphic map $\varphi : \operatorname{Op} K \to \mathbb{C}^N$ which is close to 0 such that $\tilde{s} \circ (\mathrm{id}_{\operatorname{Op} K}, \varphi) = f$ (see [33, Theorem 1.2]). Then

$$s \circ (\mathrm{pr}_{\operatorname{Op} K}, \mathrm{pr}_{\mathbb{C}^N} + \varphi \circ \mathrm{pr}_{\operatorname{Op} K}) : \operatorname{Op} K \times \mathbb{C}^N \to Y$$

is a dominating global spray over f.

Next, we prove the implication from convex ellipticity to the Oka property. Assume that Y is a convexly elliptic manifold. We shall verify the condition in Lemma 4.2. Let $K \subset \mathbb{C}^n$ be a compact convex set and $\lambda : \mathbb{C}^n \to \mathbb{R}$ be an affine linear function. Set $K_0 = \{z \in K : \lambda(z) \leq 0\}$. We may assume that K_0 contains the origin $0 \in \mathbb{C}^n$. Take a holomorphic map $f : \operatorname{Op} K_0 \to Y$ and consider the homotopy $f_t : \operatorname{Op} K_0 \to Y$ ($t \in [0,1]$) defined by $f_t(z) = f(tz)$. Then $f_1 = f$ and f_0 is a constant map. Let us show that

$$I = \{t \in [0,1] : f_t \text{ can be approximated on } K_0 \text{ by holomorphic maps } \operatorname{Op} K \to Y\}$$

coincides with $[0,1]$. Since $0 \in I$ and I is closed by definition, it suffices to prove that I is open in $[0,1]$. Take $t_0 \in I$ arbitrarily. By assumption, there exist an open neighborhood $\Omega \subset \mathbb{C}^n$ of K_0 and a dominating global spray $s : \Omega \times \mathbb{C}^N \to Y$ over f_{t_0} (see Remark 3.2). After shrinking $\Omega \supset K_0$ if necessary, we may assume that Ω is a bounded convex domain. By Lemma 3.8, there exists a fiber-preserving holomorphic map $\iota : U \to \Omega \times \mathbb{C}^N$ from an open neighborhood U of the graph $\Gamma_{f_{t_0}} \subset \Omega \times Y$ such that $(\mathrm{pr}_\Omega, s) \circ \iota = \mathrm{id}_U$. Then there exists a small number $\delta > 0$ such that $(\mathrm{id}, f_t)(K_0) \subset U$ for all $t \in [t_0 - \delta, t_0 + \delta]$. Let us consider the homotopy

$$\varphi_t = \mathrm{pr}_{\mathbb{C}^N} \circ \iota \circ (\mathrm{id}, f_t) : \operatorname{Op} K_0 \to \mathbb{C}^N \quad (t \in [t_0 - \delta, t_0 + \delta]).$$

Then it satisfies $s \circ (\mathrm{id}, \varphi_t) = f_t$ for all $t \in [t_0 - \delta, t_0 + \delta]$. Thus it suffices to show that for any holomorphic map $\varphi : \operatorname{Op} K_0 \to \mathbb{C}^N$ the holomorphic map $s \circ (\mathrm{id}, \varphi) : \operatorname{Op} K_0 \to Y$ can be approximated uniformly on K_0 by holomorphic maps $\operatorname{Op} K \to Y$.

Since $t_0 \in I$, there exists a holomorphic map $g : \operatorname{Op} K \to Y$ which satisfies $(\mathrm{id}, g)(K_0) \subset U$. Take a small $\varepsilon > 0$ such that

$$\{z \in K : \lambda(z) \leq 2\varepsilon\} \subset \Omega \cap (\mathrm{id}, g)^{-1}(U).$$

Set

$$A = \{z \in K : \lambda(z) \geq \varepsilon\}, \quad B = \{z \in K : \lambda(z) \leq 2\varepsilon\}.$$

Clearly, (A, B) is a Cartan pair in $\mathbb{C}^n$ such that $A \cup B = K$. By Lemma 3.3, there exist open neighborhoods $A \subset \Omega_A \subset \mathbb{C}^n$ and $0 \in W \subset \mathbb{C}^L$, and a dominating local spray $s_A : \Omega_A \times W \to Y$ over $g|_{\Omega_A}$. Since $(\mathrm{id}, g)(A \cap B) \subset U$, after shrinking $W \ni 0$ if necessary, we may assume that there exists an open neighborhood $\Omega_{A\cap B} \subset \Omega_A \cap \Omega$ of $A \cap B$ such that $(\mathrm{id}, s_A(\cdot, w))(\Omega_{A\cap B}) \subset U$ for each $w \in W$. Then we can consider the local spray

$$\tilde{s}_A = \mathrm{pr}_{\mathbb{C}^N} \circ \iota \circ (\mathrm{pr}_{\Omega_{A\cap B}}, s_A) : \Omega_{A\cap B} \times W \to \mathbb{C}^N$$

which satisfies $s \circ (\mathrm{pr}_{\Omega_{A\cap B}}, \tilde{s}_A) = s_A|_{\Omega_{A\cap B} \times W}$.

Take an arbitrary holomorphic map $\varphi : \operatorname{Op} K_0 \to \mathbb{C}^N$ and assume that φ_0 is defined on an open neighborhood $\Omega_0 \subset \Omega$ of K_0. Consider the local spray $s_0 = \varphi \circ \mathrm{pr}_{\Omega_0} : \Omega_0 \times W \to \mathbb{C}^N$ over φ. After shrinking $\Omega_0 \supset K_0$, $\Omega_{A\cap B} \supset A \cap B$ and

$W \ni 0$ if necessary, we may assume that they are convex and $\Omega_0 \cap \Omega_{A\cap B} = \emptyset$. Recall that the union of two disjoint compact convex sets in $\mathbb{C}^{n+L}$ is polynomially convex (i.e. $\mathcal{O}(\mathbb{C}^{n+L})$-convex) since these sets can be separated by a linear functional. Thus by Theorem 2.6, there exists a spray $\tilde{s}_B : \Omega \times W \to \mathbb{C}^N$ which is sufficiently close to $\tilde{s}_A$ on $\Omega_{A\cap B} \times W$ and sufficiently close to s_0 on $\Omega_0 \times W$. Then the local spray $s_B = s \circ (\mathrm{pr}_\Omega, \tilde{s}_B) : \Omega \times W \to Y$ is sufficiently close to s_A on $\Omega_{A\cap B} \times W$ and sufficiently close to the local spray $s \circ (\mathrm{pr}_\Omega, s_0)$ over $s \circ (\mathrm{id}, \varphi) : \Omega_0 \to Y$ on $\Omega_0 \times W$. Then by Lemma 3.7, there exists a holomorphic map from an open neighborhood of $K = A \cup B$ to Y which approximates $s \circ (\mathrm{id}, \varphi)$ uniformly on K_0. Thus it follows that $I \subset [0,1]$ is open and hence $I = [0,1] \ni 1$. By the definition of I, the holomorphic map $f = f_1 : \mathrm{Op}\, K_0 \to Y$ can be approximated uniformly on K_0 by holomorphic maps $\mathrm{Op}\, K \to Y$. □

The above proof shows the following approximation theorem which may be of independent interest. In our paper [34], this theorem was generalized to the relative setting [34, Theorem 3.12] (see Theorem 7.5).

Theorem 4.3. *Let $K \subset \mathbb{C}^n$ be a compact convex set, $\lambda : \mathbb{C}^n \to \mathbb{R}$ be an affine linear function and Y be a complex manifold. Set $K_0 = \{z \in K : \lambda(z) \leq 0\}$ and assume that $f_t : \mathrm{Op}\, K_0 \to Y$ $(t \in [0,1])$ is a homotopy of holomorphic maps such that*

(1) f_0 can be approximated on K_0 by holomorphic maps $\mathrm{Op}\, K \to Y$, and
(2) there exists a dominating global spray over f_t for each $t \in [0,1]$.

Then f_1 can also be approximated on K_0 by holomorphic maps $\mathrm{Op}\, K \to Y$.

In the rest, we give a few more characterizations of Oka manifolds by using Theorem 4.1. The following characterizes Oka manifolds by a variant of weak subellipticity.

Corollary 4.4. *A complex manifold is Oka if and only if for any Stein space X, any holomorphic map $f : X \to Y$ and any point $x \in X$ there exists a family of global sprays over f dominating on x.*

Proof. It suffices to prove the "if" part. Let $K \subset \mathbb{C}^n$ be a compact convex set and $f : \mathrm{Op}\, K \to Y$ be a holomorphic map. Choose an arbitrary point $z_0 \in K$. By assumption, there exists a global spray $s_1 : \mathrm{Op}\, K \times \mathbb{C}^{N_1} \to Y$ over f such that $\partial_w|_{w=0} s_1(z_0, w)(\mathrm{T}_0\mathbb{C}^{N_1}) \neq 0$. By assumption again, there exists a global spray $s_2 : (\mathrm{Op}\, K \times \mathbb{C}^{N_1}) \times \mathbb{C}^{N_2} \to Y$ over s_1 such that

$$\partial_w|_{w=0} s_2((z_0, 0), w)(\mathrm{T}_0\mathbb{C}^{N_2}) \not\subset \partial_w|_{w=0} s_1(z_0, w)(\mathrm{T}_0\mathbb{C}^{N_1}).$$

After repeating this process finitely many times, we obtain a global spray $s_k : \mathrm{Op}\, K \times \mathbb{C}^{N_1+\cdots+N_k} \to Y$ over f such that

$$\partial_w|_{w=0} s_k(z_0, w)(\mathrm{T}_0\mathbb{C}^{N_1+\cdots+N_k}) = \sum_{j=1}^{k} \partial_w|_{w=0} s_j((z_0, 0), w)(\mathrm{T}_0\mathbb{C}^{N_j}) = \mathrm{T}_{f(z_0)}Y.$$

Therefore, s_k is dominating on a Zariski open neighborhood $U \subset \operatorname{Op} K$ of z_0. If we repeat the above process for s_k and a point in $K \setminus U$, after finitely many times, we can obtain a dominating global spray $\operatorname{Op} K \times \mathbb{C}^N \to Y$ over f. Thus Y is an Oka manifold by Theorem 4.1. □

If we consider the space $H^0(X, f^*\mathrm{T}Y)$ of holomorphic vector fields along a holomorphic map $f : X \to Y$ as the tangent space of the space $\mathcal{O}(X, Y)$ of holomorphic maps at f, the condition in the following characterization means that we can draw an entire curve in any direction at each point of $\mathcal{O}(X, Y)$.

Corollary 4.5. *A complex manifold Y is Oka if and only if for any Stein space X, any holomorphic map $f : X \to Y$ and any $V \in H^0(X, f^*\mathrm{T}Y)$, there exists a global spray $s : X \times \mathbb{C} \to Y$ over f such that $(\partial s(\cdot, w)/\partial w)|_{w=0} = V$.*

Proof. We first prove the "only if" part. Consider $V \in H^0(X, f^*\mathrm{T}Y)$ as a holomorphic vector field on the graph $\Gamma_f \subset X \times Y$ of f tangent to the fibers of the projection $\mathrm{pr}_X : X \times Y \to X$. Since Γ_f is Stein, we can extend it to an open neighborhood of Γ_f. By using its local flow, we can construct a local spray $s_0 : U \to Y$ over f from an open neighborhood $U \subset X \times \mathbb{C}$ of $X \times \{0\}$ such that $(\partial s_0(\cdot, w)/\partial w)|_{w=0} = V$ (see the proof of Lemma 3.3). After shrinking $U \supset X \times \{0\}$ if necessary, we can extend s_0 to a continuous map $s_0 : X \times \mathbb{C} \to Y$. Then by using the Oka property (see Definition 2.5), we can obtain a global spray $s : X \times \mathbb{C} \to Y$ over f such that $(\partial s(\cdot, w)/\partial w)|_{w=0} = V$.

The converse implication follows from Corollary 4.4 since for any Stein space X and any point $x \in X$ the map

$$H^0(X, f^*\mathrm{T}Y) \to \mathrm{T}_{f(x)}Y, \quad V \mapsto V(x)$$

is surjective. □

5. Localization principle for Oka manifolds

In the proof of the localization principle for Oka manifolds (Theorem 1.8), we need the following proposition.

Proposition 5.1. *Assume that Y is a complex manifold and $U \subset Y$ is a Zariski open Oka subset. Then for any Stein space X and any holomorphic map $f : X \to Y$ there exists a global spray over f dominating on $f^{-1}(U)$.*

The proof of Proposition 5.1 is based on the proof of the implication from the Oka property to Condition Ell_1 (Proposition 3.4). To generalize it, we need the following Oka principle which is a special case of [14, Theorem 6.14.6].

Theorem 5.2 (cf. [14, Theorem 6.14.6])**.** *Let X be a Stein space, $X' \subset X$ be a closed complex subvariety, Y be a complex manifold, $A \subset Y$ be a closed complex subvariety and $f_0 : X \to Y$ be a continuous map which is holomorphic on an open neighborhood of X'. Assume that $Y \setminus A$ is Oka and $f_0^{-1}(A) \subset X'$. Then for any*

$k \in \mathbb{N}$ *there exists a homotopy* $f_t : X \to Y$ $(t \in [0,1])$ *such that the following hold for each* $t \in [0,1]$*;*

(1) f_t *is holomorphic on an open neighborhood of* X'*,*
(2) f_t *is tangent to* f_0 *to order* k *along* X'*, and*
(3) $f_1 : X \to Y$ *is holomorphic.*

Proof of Proposition 5.1. Set $A = Y \setminus U$. Since the graph $\Gamma_f \subset X \times Y$ of f is Stein, it has an open Stein neighborhood $\Omega \subset X \times Y$ by Siu's theorem (Theorem 2.4). Since Ω is Stein, Cartan's theorem A (Theorem 2.3) implies that there exist finitely many holomorphic vector fields V_k $(k = 1, \dots, N)$ on Ω tangent to the fibers of $X \times Y \to X$. For the same reason, there exist finitely many holomorphic functions $g_j \in \mathcal{O}(\Omega)$ $(j = 1, \dots, L)$ such that

$$\{z \in \Omega : g_j(z) = 0,\ j = 1, \dots, L\} = \Omega \cap (X \times A).$$

Let $\varphi_t^{j,k}$ $(t \in \mathbb{C})$ denote the local flow of $g_j V_k$. Since each of $g_j V_k$ vanishes on $\Omega \cap (X \times A)$, each flow fixes $\Omega \cap (X \times A)$ and satisfies $(\varphi_t^{j,k})^{-1}(\Omega \cap (X \times A)) \subset \Omega \cap (X \times A)$. Then

$$s_0'(x, (t_{1,1}, \dots, t_{1,N}), \dots, (t_{L,1}, \dots, t_{L,N})) = \mathrm{pr}_Y \circ \varphi_{t_{1,1}}^{1,1} \circ \dots \circ \varphi_{t_{L,N}}^{L,N}(x, f(x)) \in Y$$

defines a holomorphic map from an open neighborhood of $X \times \{0\} \subset X \times \mathbb{C}^{LN}$. Consider

$$s_0(x, t_1, \dots, t_L) = s_0'(x, g_1(x, f(x))t_1, \dots, g_L(x, f(x))t_L)$$

where we write $t_j = (t_{j,1}, \dots, t_{j,N})$. By construction, it is a holomorphic map from an open neighborhood of $X' = (X \times \{0\}) \cup (f^{-1}(A) \times \mathbb{C}^{LN})$, which satisfies

$$s_0(\cdot, 0) = f, \quad s_0^{-1}(A) = f^{-1}(A) \times \mathbb{C}^{LN}, \quad \partial_t|_{t=0} s_0(x,t)(\mathrm{T}_0\mathbb{C}^{LN}) = \mathrm{T}_{f(x)}Y$$

for all $x \in f^{-1}(U)$. After shrinking the open neighborhood if necessary, we can extend s_0 to a continuous map $s_0 : X \times \mathbb{C}^{LN} \to Y$ such that $s_0^{-1}(A) = f^{-1}(A) \times \mathbb{C}^{LN} \subset X'$. Then by Theorem 5.2, there exists a holomorphic map $s_1 : X \times \mathbb{C}^{LN} \to Y$ which agrees with s_0 to the second order along $X' = (X \times \{0\}) \cup (f^{-1}(A) \times \mathbb{C}^{LN})$. Clearly, s_1 is then a global spray over f dominating on $f^{-1}(U)$. □

By using Proposition 5.1, let us refine Theorem 1.7 and Theorem 4.1 as follows. In the case of $U = \emptyset$, the equivalence between (1) and (2) (resp. (1) and (3)) is nothing but Theorem 1.7 (resp. Theorem 4.1).

Corollary 5.3. *For a complex manifold* Y *and a Zariski open Oka subset* $U \subset Y$*, the following are equivalent:*

(1) Y *is Oka.*
(2) *For any Stein space* X *and any holomorphic map* $f : X \to Y$ *there exists a global spray over* f *dominating on* $f^{-1}(Y \setminus U)$*.*
(3) *For any compact convex set* $K \subset \mathbb{C}^n$ $(n \in \mathbb{N})$ *and any holomorphic map* $f : \mathrm{Op}\, K \to Y$ *there exists a global spray over* f *dominating on* $f^{-1}(Y \setminus U)$*.*

Proof. It suffices to prove that (3) implies (1). Let $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) be a compact convex set and $f : \operatorname{Op} K \to Y$ be a holomorphic map. By assumption, there exists a global spray $s : \operatorname{Op} K \times \mathbb{C}^N \to Y$ over f dominating on $f^{-1}(Y \setminus U)$. It follows from Proposition 5.1 that there exists a global spray $\tilde{s} : (\operatorname{Op} K \times \mathbb{C}^N) \times \mathbb{C}^L \to Y$ over s dominating on $s^{-1}(U) \supset f^{-1}(U) \times \{0\}$. This is also a global spray $\tilde{s} : \operatorname{Op} K \times \mathbb{C}^{N+L} \to Y$ over f dominating on $\operatorname{Op} K = f^{-1}(Y \setminus U) \cup f^{-1}(U)$. Therefore Y is an Oka manifold by Theorem 4.1. ☐

As an application of Proposition 5.1 and Corollary 5.3, we can prove the localization principle for Oka manifolds (Theorem 1.8).

Proof of Theorem 1.8. Note that a complex manifold Y is Oka if each compact subset of Y has an open Oka neighborhood in Y by Theorem 4.1 (or Theorem 1.1). Therefore, it suffices to prove the localization principle in the case when Y is a union of two Zariski open Oka subsets $U, V \subset Y$. Take an arbitrary holomorphic map $f : X \to Y$ from a Stein space X. By Proposition 5.1, there exists a global spray over f dominating on $f^{-1}(V) \supset f^{-1}(Y \setminus U)$. Then Y is Oka by Corollary 5.3. ☐

The localization principle (Theorem 1.8) gives many new examples of Oka manifolds. In fact, nonelliptic Oka manifolds in the next section are obtained as an application of this theorem. In the rest of this section, we give a few other examples of Oka manifolds.

It was proved by Lárusson that every smooth toric variety is Oka (see [13, Theorem 2.17]). Such a variety is Zariski locally biholomorphic to an algebraic torus $(\mathbb{C}^*)^n$. In his proof, however, the quotient construction of toric varieties was used. Now we have the localization principle, which gives the direct proof of this fact. Moreover, since an abelian complex Lie group of dimension at least two with finitely many points removed, or blown up at finitely many points, is Oka (see [14, Corollary 5.6.18 and Corollary 6.4.13]), we have the following result.

Corollary 5.4. *Let Y be a complex manifold of dimension at least two. Assume that Y is Zariski locally biholomorphic to some abelian complex Lie group. Then for any finite subset $A \subset Y$ the complement $Y \setminus A$ and the blowup $\operatorname{Bl}_A Y$ are Oka. This holds in particular for any smooth toric variety of dimension at least two.*

Since $\mathbb{P}^n$ is Zariski locally biholomorphic to $\mathbb{C}^n$, the complement of a closed algebraic subvariety of codimension at least two in $\mathbb{P}^n$ is Oka by Example 2.8 and the localization principle. In contrast to this case, the complement of a hypersurface in $\mathbb{P}^n$ is only rarely Oka (recall the Kobayashi conjecture [28, p. 127]). Hanysz [24] determined when the complements of unions of hyperplanes are Oka. It is a well-known problem whether the complement of every smooth cubic curve in $\mathbb{P}^2$ is Oka. Related to these, we prove the Oka property of the following complements.

Corollary 5.5. *(1) The complement of any quadric hypersurface in $\mathbb{P}^n$ is Oka.*
(2) The complement of any singular irreducible cubic curve in $\mathbb{P}^2$ is Oka.

Proof. (1) After a change of coordinates, we may assume that a given hypersurface is $Q = \{z_0^2 + \cdots + z_k^2 = 0\} \subset \mathbb{P}^n$ $(0 \le k \le n)$. If $k = 0$, the complement $\mathbb{P}^n \setminus Q \cong \mathbb{C}^n$ is clearly Oka. Let us consider the case of $k \ge 1$. Take arbitrary point $a = [a_0, \ldots, a_n] \in \mathbb{P}^n \setminus Q$. Since $a_j \neq 0$ for some $0 \le j \le k$, we may assume that $a_0 \neq 0$ and $a_0 + ia_k \neq 0$ by a change of coordinates. If we set $w_0 = z_0 + iz_k$ and $w_k = z_0 - iz_k$, the equation of Q becomes

$$z_0^2 + \cdots + z_k^2 = w_0 w_k + z_1^2 + \cdots + z_{k-1}^2.$$

Thus, the point $a \in \mathbb{P}^n \setminus Q$ has a Zariski open neighborhood which is biholomorphic to the complement $(\mathbb{C}^{n-1} \times \mathbb{C}) \setminus \Gamma_f$ of the graph of

$$f(x_1, \ldots, x_{n-1}) = -(x_1^2 + \cdots + x_{k-1}^2).$$

Since the complement $(\mathbb{C}^{n-1} \times \mathbb{C}) \setminus \Gamma_f \cong \mathbb{C}^{n-1} \times \mathbb{C}^*$ is Oka, the complement $\mathbb{P}^n \setminus Q$ is also Oka by the localization principle.
(2) After a change of coordinates, a given curve C can be assumed to be $\{y^2 z = x^3\}$ or $\{y^2 z = x^3 + x^2 z\}$. In both cases, the complement $\mathbb{P}^2 \setminus C$ is Zariski locally biholomorphic to the complement $(\mathbb{C} \times \mathbb{C}) \setminus \Gamma_f$ of the graph of a rational function f in one variable. By the result of Hanysz [24, Theorem 4.6], the complement $(\mathbb{C} \times \mathbb{C}) \setminus \Gamma_f$ is Oka, hence the conclusion follows from the localization principle. □

Next, we consider the Oka property of a blowup of $\mathbb{C}^n$. In fact, a blowup of $\mathbb{C}^n$ is in general not Oka. More precisely, there exists a discrete set $D \subset \mathbb{C}^n$ $(n > 1)$ such that the blowup $\mathrm{Bl}_D \mathbb{C}^n$ is not Oka [32, Example A.3]. Let us first recall the following result of Forstnerič.

Proposition 5.6 (cf. [14, Proposition 6.4.12])**.** *Assume that $n > 1$ and $D \subset \mathbb{C}^n$ is a tame discrete set (cf. Example 2.8). Then the blowup $\mathrm{Bl}_D \mathbb{C}^n$ is Oka.*

It seems that Proposition 5.6 also holds for any tame closed complex submanifold $A \subset \mathbb{C}^n$ of codimension at least two. Here we give the following generalization of Proposition 5.6 under the stronger tameness assumption. Note that in the case of $\dim A = 0$ it is nothing but Proposition 5.6 by the result of Rosay and Rudin [40, Theorem 3.9].

Corollary 5.7. *Let $A \subset \mathbb{C}^n$ be a closed complex submanifold of pure dimension $k \le n - 2$. Assume that there exists a holomorphic automorphism φ of $\mathbb{C}^n$ such that $\varphi(A) \subset \mathbb{C}^{k+1} \times \{0\} \subset \mathbb{C}^n$. Then the blowup $\mathrm{Bl}_A \mathbb{C}^n$ is Oka.*

Proof. We may assume that A is a smooth hypersurface in $\mathbb{C}^{k+1} \times \{0\} \subset \mathbb{C}^n$. Since the second Cousin problem on $\mathbb{C}^n$ is solvable (cf. [14, §5.2]), there exists a holomorphic function $f \in \mathcal{O}(\mathbb{C}^{k+1})$ such that $A = \{z \in \mathbb{C}^{k+1} : f(z) = 0\}$ and $\mathrm{d}f \neq 0$ on A. Then the blowup $\mathrm{Bl}_A \mathbb{C}^n$ can be described as follows:

$$\mathrm{Bl}_A \mathbb{C}^n = \{(z, z', w) \in \mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1} \times \mathbb{P}^{n-k-1} : w = [f(z), z'_1, \ldots, z'_{n-k-1}]\}.$$

Note that this is covered by the Zariski open subset

$$\{(z, z', w) \in \mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1} \times \mathbb{C}^{n-k-1} : z' = f(z)w\} \cong \mathbb{C}^n$$

and Zariski open subsets of the same form

$$\begin{aligned}&\{(z, z', w) \in \mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1} \times \mathbb{C}^{n-k-1} : f(z) = z'_1 w_1,\ (z'_j)_{j=2}^{n-k-1} = z'_1 (w_j)_{j=2}^{n-k-1}\}\\ \cong\ &\{(z, z'_1, w_1) \in \mathbb{C}^{k+1} \times \mathbb{C} \times \mathbb{C} : f(z) = z'_1 w_1\} \times \mathbb{C}^{n-k-2}.\end{aligned}$$

Since the hypersurface $\{(z, z'_1, w_1) \in \mathbb{C}^{k+1} \times \mathbb{C} \times \mathbb{C} : f(z) = z'_1 w_1\}$ is Oka by the result of Kaliman and Kutzschebauch [26, Theorem 2], the blowup $\mathrm{Bl}_A \mathbb{C}^n$ is also Oka by the localization principle. □

6. Oka complements of countable sets and nonelliptic Oka manifolds

The goal of this section is to obtain an example of a nonelliptic Oka manifold. To this end, we consider the complements of closed countable sets $S \subset \mathbb{C}^n$ which are *not necessarily discrete*. That is, we do not assume that the derived set S' (the set of accumulation points of S) is empty. For such closed countable sets, we define tameness in the same manner as in Example 2.8.

Definition 6.1. A closed countable set $S \subset \mathbb{C}^n$ is *tame* if there exists a holomorphic automorphism φ of $\mathbb{C}^n$ such that the closure of $\varphi(S)$ in the projective space $\mathbb{P}^n \supset \mathbb{C}^n$ does not contain the hyperplane at infinity $\mathbb{P}^n \setminus \mathbb{C}^n$.

For example, every compact countable set $S \subset \mathbb{C}^n$ is tame. Our main result in this section is the following generalization of the result of Forstnerič and Prezelj for tame discrete sets (Example 2.8). Theorem 1.9 is proved as its application (see Corollary 6.4).

Theorem 6.2. *For any tame closed countable set $S \subset \mathbb{C}^n$ $(n > 1)$ with a discrete derived set, the complement $\mathbb{C}^n \setminus S$ is Oka.*

In the proof of Theorem 6.2, we need the following approximation theorem. Let $\overline{\mathbb{B}^n(a, r)}$ denote the closed ball in $\mathbb{C}^n$ of radius $r > 0$ centered at $a \in \mathbb{C}^n$.

Lemma 6.3. *For any discrete sequence $\{a_j\}_j$ in $\mathbb{C}^n$ there exists a sequence $\{r_j\}_j$ of positive numbers such that*

(1) the closed balls $\overline{\mathbb{B}^n(a_j, r_j)}$ are mutually disjoint, and

(2) for any holomorphic functions $f_j : \mathrm{Op}\,\overline{\mathbb{B}^n(a_j, r_j)} \to \mathbb{C}$ and any sequence $\{\varepsilon_j\}_j$ of positive numbers there exists a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ such that $\sup_{\overline{\mathbb{B}^n(a_j, r_j)}} |f - f_j| \leq \varepsilon_j$ for all j.

Proof. We write $\overline{\mathbb{B}}_R = \overline{\mathbb{B}^n(0, R)}$. Take an increasing sequence $\{R_l\}_{l \in \mathbb{N}}$ of positive numbers such that $\lim_{l \to \infty} R_l = \infty$ and $\{a_j\}_j \cap \bigcup_{l \in \mathbb{N}} \partial \overline{\mathbb{B}}_{R_l} = \emptyset$. We define $\overline{\mathbb{B}}_{R_0} = \emptyset$ for convenience. After renumbering $\{a_j\}_j$ if necessary, we may assume that there exists an increasing sequence $0 = k_0 < k_1 < k_2 < \cdots$ of integers such that $\{a_j\}_{j=k_l+1}^{k_{l+1}} \subset \overline{\mathbb{B}}_{R_{l+1}} \setminus \overline{\mathbb{B}}_{R_l}$ for all $l \geq 0$. Since $\overline{\mathbb{B}}_{R_l} \cap \{a_j\}_{j=k_l+1}^{k_{l+1}} = \emptyset$ for each l, there exists a sequence $\{r_j\}_j$ of small positive numbers such that

- the closed balls $\overline{\mathbb{B}^n(a_j, r_j)}$ are mutually disjoint,
- $\bigcup_{j=k_l+1}^{k_{l+1}} \overline{\mathbb{B}^n(a_j, r_j)} \subset \overline{\mathbb{B}}_{R_{l+1}} \setminus \overline{\mathbb{B}}_{R_l}$ for all $l \geq 0$, and
- $\overline{\mathbb{B}}_{R_l} \cup \bigcup_{j=k_l+1}^{k_{l+1}} \overline{\mathbb{B}^n(a_j, r_j)}$ is polynomially convex for each $l \geq 0$.

Let us verify the condition (2). Let $g_0 : \emptyset \to \mathbb{C}$ be the unique holomorphic function on $\emptyset$. Assume inductively that a holomorphic function $g_l : \operatorname{Op} \overline{\mathbb{B}}_{R_l} \to \mathbb{C}$ has been chosen for some $l \geq 0$. By Theorem 2.6, there exists a holomorphic function $g_{l+1} : \operatorname{Op} \overline{\mathbb{B}}_{R_{l+1}} \to \mathbb{C}$ such that

- $\sup_{\overline{\mathbb{B}}_{R_l}} |g_{l+1} - g_l| \leq \min\{\varepsilon_j\}_{j=1}^{k_l} / 2^{l+1}$, and
- $\sup_{\overline{\mathbb{B}^n(a_j, r_j)}} |g_{l+1} - f_j| \leq \varepsilon_j / 2$ for all $j = k_l + 1, \ldots, k_{l+1}$.

Then the limit $f = \lim_{l \to \infty} g_l$ exists uniformly on compacts in $\mathbb{C}^n$ and has the desired property. □

Proof of Theorem 6.2. By the localization principle for Oka manifolds (Theorem 1.8), it suffices to prove that for any fixed point $z_0 \in \mathbb{C}^n \setminus S$ there exists a Zariski open Oka neighborhood of z_0. Since S is tame, there exist a holomorphic coordinate system $z = (z', z_n)$ on $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ and a constant $C > 0$ such that $z_0 = 0$ and

$$S \subset \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z_n| \leq C(1 + \|z'\|)\}$$

by definition. Furthermore, since S is countable, we may also assume that $S \subset (\mathbb{C}^{n-1} \setminus \{0\}) \times \mathbb{C}$ and $\operatorname{pr}'|_S : S \to \mathbb{C}^{n-1}$ is injective where $\operatorname{pr}' : \mathbb{C}^n \to \mathbb{C}^{n-1}$ denotes the projection $z \mapsto z'$. Note that the restriction of pr' to $\{(z', z_n) : |z_n| \leq C(1 + \|z'\|)\}$ is proper. Thus $\operatorname{pr}'(S')$ is discrete in $\mathbb{C}^{n-1}$. Let us enumerate $\operatorname{pr}'(S') = \{a_j\}_j$. Take a sequence $\{r_j\}_j$ of positive numbers which satisfies the conditions (1) and (2) in Lemma 6.3. Note that $\operatorname{pr}'(S) \setminus \bigcup_j \overline{\mathbb{B}^{n-1}(a_j, r_j)} \subset \mathbb{C}^{n-1}$ is discrete. By Theorem 2.6, there exists a holomorphic function $g : \mathbb{C}^{n-1} \to \mathbb{C}$ such that

- $g(0) = 1$, and
- $g(z') = -z_n$ for all $(z', z_n) \in S' \cup (S \setminus \operatorname{pr}'^{-1}(\bigcup_j \overline{\mathbb{B}^{n-1}(a_j, r_j)}))$.

By the condition (2) in Lemma 6.3, there exists a holomorphic function $f : \mathbb{C}^{n-1} \to \mathbb{C}$ such that on each closed ball $\overline{\mathbb{B}^{n-1}(a_j, r_j)}$ the real part of f satisfies

$$\Re f \leq \log \frac{1}{j \cdot \sup_{z' \in \overline{\mathbb{B}^{n-1}(a_j, r_j)}} (C(1 + \|z'\|) + |g(z')|)}.$$

It follows that for all $(z', z_n) \in S \cap \mathrm{pr}'^{-1}(\overline{\mathbb{B}^{n-1}(a_j, r_j)})$,

$$\left| e^{f(z')}(z_n + g(z')) \right| \le e^{\Re f(z')}(C(1 + \|z'\|) + |g(z')|) \le \frac{1}{j}.$$

Consider the holomorphic automorphism φ of $\mathbb{C}^n$ defined by

$$\varphi(z', z_n) = \left(z', e^{f(z')}(z_n + g(z')) \right).$$

Note that $\varphi(z_0) = (0, e^{f(0)}) \in \mathbb{C}^{n-1} \times \mathbb{C}^*$, $\varphi(S') \subset \mathbb{C}^{n-1} \times \{0\}$ and the discrete set $D = \varphi(S) \cap (\mathbb{C}^{n-1} \times \mathbb{C}^*)$ in $\mathbb{C}^{n-1} \times \mathbb{C}^*$ is contained in $\bigcup_j (\overline{\mathbb{B}^{n-1}(a_j, r_j)} \times \overline{\mathbb{B}^1(0, 1/j)})$. Thus $\mathrm{pr}_n(D)$ is discrete in $\mathbb{C}^*$ where $\mathrm{pr}_n : \mathbb{C}^n \to \mathbb{C}$ is the n-th projection, and hence $\mathrm{pr}_n((\mathrm{id}_{\mathbb{C}^{n-1}} \times \exp)^{-1}(D)) = \exp^{-1}(\mathrm{pr}_n(D))$ is discrete in $\mathbb{C}$. This implies that $(\mathrm{id}_{\mathbb{C}^{n-1}} \times \exp)^{-1}(D)$ is a tame discrete set in $\mathbb{C}^n$ by the result of Rosay and Rudin [40, Theorem 3.9]. Therefore $(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus \varphi(S)$ is a Zariski open Oka neighborhood of $\varphi(z_0)$ because its universal covering $\mathbb{C}^n \setminus (\mathrm{id}_{\mathbb{C}^{n-1}} \times \exp)^{-1}(D)$ is Oka (see Proposition 2.9). It follows that the preimage $\varphi^{-1}(\mathbb{C}^{n-1} \times \mathbb{C}^*) \setminus S \subset \mathbb{C}^n \setminus S$ is a Zariski open Oka neighborhood of z_0. □

As we mentioned in the introduction, we prove the following stronger result than Theorem 1.9.

Corollary 6.4. *For any $n \ge 3$, the complement $\mathbb{C}^n \setminus ((\overline{\mathbb{N}^{-1}})^2 \times \{0\}^{n-2})$ is Oka but not weakly subelliptic.*

In order to prove Corollary 6.4, we need to improve the result of Andrist, Shcherbina and Wold [2, Theorem 1.1] as follows. The proof is based on their idea and Gromov's method of composed sprays (cf. [14, §6.3]).

Lemma 6.5. *Assume that Y is a Stein manifold of dimension at least three and $K \subset Y$ is a compact $\mathcal{O}(Y)$-convex set with an infinite derived set. Then the complement $Y \setminus K$ is not weakly subelliptic.*

Proof. To reach a contradiction, we assume that $Y \setminus K$ is weakly subelliptic. Take a relatively compact open neighborhood $U \subset Y$ of K. By assumption, there exists a family of global sprays $s_j : E_j \to Y$ $(j = 1, \ldots, k)$ over the identity map $\mathrm{id}_{Y \setminus K}$ dominating on ∂U. By the Hartogs extension theorem for holomorphic vector bundles and sprays [2, Theorem 1.2 and Theorem 4.1], we can extend E_j $(j = 1, \ldots, k)$ to holomorphic vector bundles $\tilde{p}_j : \widetilde{E}_j \to Y \setminus A$ $(j = 1, \ldots, k)$, where $A \subset K$ is a finite set, and s_j $(j = 1, \ldots, k)$ to holomorphic maps $\tilde{s}_j : \widetilde{E}_j \to Y$ $(j = 1, \ldots, k)$. Note that $\tilde{s}_j(0_y) = y$ for each $y \in Y \setminus A$ by the identity theorem.

Let $B \subset Y \setminus A$ denote the closed complex subvariety of points $y \in Y \setminus A$ such that $\sum_{j=1}^k (\mathrm{d}\tilde{s}_j)_{0_y}(\widetilde{E}_{j,y}) \ne \mathrm{T}_y Y$ and set $S = A \cup B$. Since U is relatively compact and $B \cap \partial U = \emptyset$, the intersection $B \cap U \subset U \setminus A$ must be a discrete set. Hence K is not contained in S and thus we may take a point $y_0 \in \partial K \setminus S$. Let us consider the fiber product and the associated maps

$$E = \left\{ (e_1, \dots, e_k) \in \prod_{j=1}^{k} \left(\widetilde{E}_j \setminus \left(\tilde{p}_j^{-1}(S) \cup \tilde{s}_j^{-1}(S) \right) \right) : \begin{array}{c} \tilde{s}_j(e_j) = \tilde{p}_{j+1}(e_{j+1}), \\ j = 1, \dots, k-1 \end{array} \right\},$$

$$p : E \to Y \setminus S,\ (e_1, \dots, e_k) \mapsto \tilde{p}_1(e_1), \quad s : E \to Y \setminus S,\ (e_1, \dots, e_k) \mapsto \tilde{s}_k(e_k).$$

Note that $\mathrm{d}s_{(0_y,\dots,0_y)}(\mathrm{T}_{(0_y,\dots,0_y)}p^{-1}(y)) = \sum_{j=1}^{k} (\mathrm{d}\tilde{s}_j)_{0_y}(\widetilde{E}_{j,y}) = \mathrm{T}_y Y$ for each $y \in Y \setminus S$ (cf. [14, Lemma 6.3.6]). Therefore the fiber preserving map $E \to (Y \setminus S)^2,\ e \mapsto (p(e), s(e))$ restricts to a holomorphic submersion from a neighborhood of the zero section $\{(0_y, \dots, 0_y) : y \in Y \setminus S\}$ onto a neighborhood of the diagonal $\{(y, y) : y \in Y \setminus S\}$. Thus there exists an open neighborhood $V \subset Y \setminus S$ of y_0 such that for each $y \in V$ there exists $e \in E$ such that $p(e) = y$ and $s(e) = y_0$. This contradicts to $y_0 \in \partial K$ and $s(p^{-1}(Y \setminus K)) \subset Y \setminus K$. □

Proof of Corollary 6.4. Since $\overline{\mathbb{N}^{-1}} \subset \mathbb{C}$ is polynomially convex, $(\overline{\mathbb{N}^{-1}})^2 \times \{0\}^{n-2} \subset \mathbb{C}^n$ is also polynomially convex. Clearly the derived set of $(\overline{\mathbb{N}^{-1}})^2 \times \{0\}^{n-2} \subset \mathbb{C}^n$ is infinite. Thus Lemma 6.5 implies that its complement is not weakly subelliptic.

In order to prove that the complement is Oka, we use the localization principle for Oka manifolds (Theorem 1.8). Set $U_j = \mathbb{C}^{j-1} \times \mathbb{C}^* \times \mathbb{C}^{n-j}$ $(j = 1, \dots, n)$ and $S = (\overline{\mathbb{N}^{-1}})^2 \times \{0\}^{n-2}$. Note that $\mathbb{C}^n \setminus S = \bigcup_{j=1}^{n} (U_j \setminus S)$ and each $U_j \setminus S$ is Zariski open in $\mathbb{C}^n \setminus S$. By the localization principle, it suffices to show that each $U_j \setminus S$ is Oka. For $j \geq 3$, $U_j \cap S = \emptyset$ and thus $U_j \setminus S = U_j$ is Oka. Consider the exponential map $\pi = \exp \times \mathrm{id}_{\mathbb{C}^{n-1}} : \mathbb{C}^n \to U_1$. Since π is a covering map, $U_1 \setminus S$ is Oka if and only if $\mathbb{C}^n \setminus \pi^{-1}(S)$ is Oka by Proposition 2.9. By definition, $\pi^{-1}(S) = \exp^{-1}(\mathbb{N}^{-1}) \times (\overline{\mathbb{N}^{-1}}) \times \{0\}^{n-2}$. Note that $\pi^{-1}(S)$ is tame and its derived set $\pi^{-1}(S)' = \exp^{-1}(\mathbb{N}^{-1}) \times \{0\}^{n-1}$ is discrete. Therefore Theorem 6.2 implies that $\mathbb{C}^n \setminus \pi^{-1}(S)$ is Oka and thus $U_1 \setminus S \cong U_2 \setminus S$ are Oka. □

In the same manner as in the proof of Theorem 6.2, we can also prove the following theorem for blowups.

Theorem 6.6 ([31, Theorem 1.3])**.** *For any tame closed countable set $S \subset \mathbb{C}^n$ $(n > 1)$ with a discrete derived set S', the blowup $\mathrm{Bl}_{S \setminus S'}(\mathbb{C}^n \setminus S')$ is Oka.*

It is a fundamental problem whether any point in an Oka manifold of dimension at least two has an Oka complement and an Oka blowup (see Corollary 5.4). As an application of Theorem 6.2 and Theorem 6.6, we solve this problem for Hopf manifolds (in particular, we solve [13, Problem 2.42]). Recall that a complex manifold Y is called a *Hopf manifold* if it is compact and universally covered by $\mathbb{C}^{\dim Y} \setminus \{0\}$. The latter condition and Proposition 2.9 imply that every Hopf manifold is Oka and the dimension of a Hopf manifold must be greater than 1.

Corollary 6.7. *For any Hopf manifold Y and any finite set $S \subset Y$, the complement $Y \setminus S$ and the blowup $\mathrm{Bl}_S Y$ are Oka.*

Proof. Set $n = \dim Y > 1$. By Kodaira's argument [29, Theorem 30] (see also [25]), there exists a finite (unramified) covering map $(\mathbb{C}^n \setminus \{0\})/\langle\varphi\rangle \to Y$ where φ is a holomorphic contraction, i.e. φ is a holomorphic automorphism of $\mathbb{C}^n$ such that $\varphi(0) = 0$ and $\varphi^j \to 0$ uniformly on compacts as $j \to \infty$. Thus we may assume that $Y = (\mathbb{C}^n \setminus \{0\})/\langle\varphi\rangle$ from the beginning. Let $\pi : \mathbb{C}^n \setminus \{0\} \to Y$ denote the quotient map. Since φ is a holomorphic contraction, there exists a holomorphic coordinate system on $\mathbb{C}^n$ in which φ is lower triangular (cf. [14, p. 117]). In this coordinate system, the discrete set $\pi^{-1}(S) \setminus \overline{\mathbb{B}^n(0,1)} \subset \mathbb{C}^n$ projects to a discrete set in the first coordinate, and hence it is tame by the result of Rosay and Rudin [40, Theorem 3.9]. Note that $(\pi^{-1}(S) \cap \overline{\mathbb{B}^n(0,1)}) \cup \{0\}$ is compact. Thus $\pi^{-1}(S) \cup \{0\} \subset \mathbb{C}^n$ is tame and $(\pi^{-1}(S) \cup \{0\})' = \{0\}$. Therefore Theorem 6.2 (resp. Theorem 6.6) implies the Oka property of the complement $(\mathbb{C}^n \setminus \{0\}) \setminus \pi^{-1}(S)$ (resp. the blowup $\mathrm{Bl}_{\pi^{-1}(S)}(\mathbb{C}^n \setminus \{0\})$) and hence the complement $Y \setminus S$ (resp. the blowup $\mathrm{Bl}_S Y$) is Oka by Proposition 2.9. □

7. Further developments

In this last section, we present the results obtained in [34, 36] which generalize and improve the above results.

7.1. **Elliptic characterization and unification of Oka maps.** The definition of the Oka property of a complex manifold (Definition 2.5) can be generalized to the relative setting as follows.

Definition 7.1. A holomorphic submersion $\pi : Y \to B$ (between complex spaces) enjoys *the Oka property* if the following holds: For any Stein space X, any closed complex subvariety $X' \subset X$, any compact $\mathcal{O}(X)$-convex set $K \subset X$, any compact Hausdorff spaces $P_0 \subset P$, any family of holomorphic maps $F : P \times X \to B$ and any family of continuous maps $f_0 : P \times X \to Y$ such that

(1) $\pi \circ f_0 = F$, and
(2) $f_0|_{P_0 \times X}$, $f_0|_{P \times X'}$ and $f_0|_{P \times \mathrm{Op}\, K}$ are families of holomorphic maps,

there exists a homotopy $f_t : P \times X \to Y$ $(t \in [0,1])$ such that the following hold for each $t \in [0,1]$;

(a) $\pi \circ f_t = F$,
(b) $f_t = f_0$ on $(P_0 \times X) \cup (P \times X')$,
(c) $f_t|_{P \times \mathrm{Op}\, K}$ is a family of holomorphic maps which approximates f_0 uniformly on $P \times K$, and
(d) $f_1 : P \times X \to Y$ is a family of holomorphic maps.

If, in addition, $f_0|_{P \times \mathrm{Op}\, X'}$ is a family of holomorphic maps, the homotopy f_t can be chosen tangent to f_0 to a given finite order along X'.

As in the case of complex manifolds, there are mainly two types of Oka principles for holomorphic submersions. One is the Gromov type Oka principle [9, 12, 23] which states that every subelliptic submersion enjoys the Oka property (see Theorem 1.3, and see [14, Definition 6.1.2] for the definition of a subelliptic submersion). The other is the Forstnerič type Oka principle [11, 12] which shows the Oka property of a holomorphic fiber bundle whose fiber enjoys CAP (see Theorem 1.1). These Oka principles are known to be mutually independent. Namely, there exists a subelliptic submersion which is not locally trivial at any base point (e.g. a complete family of complex tori [38, Theorem 16]), and there exists a non-subelliptic holomorphic fiber bundle with a CAP fiber (Corollary 6.4). Moreover, there is a holomorphic submersion which enjoys the Oka property but is neither subelliptic nor locally trivial at any base point as follows. In fact, the Oka property of this example is proved by using the localization principle for Oka manifolds (Theorem 1.8).

Proposition 7.2 ([34, Proposition 5.10]). *Let $\{a_j\}_{j\in\mathbb{N}}$ be a dense sequence in $\mathbb{C}$ and set*

$$S = \{(a_j, j, 0)\}_{j\in\mathbb{N}} \times \left(\overline{\mathbb{N}^{-1}}\right)^2 \subset \mathbb{C}^5.$$

Then the submersion $\pi : \mathbb{C}^5 \setminus S \to \mathbb{C},\ (z_1, \dots, z_5) \mapsto z_1$ enjoys the Oka property but for any nonempty open set $U \subset \mathbb{C}$ the restriction $\pi^{-1}(U) \to U$ is neither trivial nor subelliptic.

Thus it is natural to ask whether there exists a characterization of the Oka property of a holomorphic submersion which implies the Oka principles by Gromov and Forstnerič. In order to solve this problem, let us generalize the definition of convex ellipticity (Definition 3.5) to holomorphic submersions.

Definition 7.3. A holomorphic submersion $\pi : Y \to B$ is *convexly elliptic* if for any point $b \in B$ there exists an open neighborhood $U \subset B$ of b such that for any compact convex set $K \subset \mathbb{C}^n$ ($n \in \mathbb{N}$) and any holomorphic map $f : \mathrm{Op}\, K \to \pi^{-1}(U)$ there exists a *dominating global π-spray* $s : \mathrm{Op}\, K \times \mathbb{C}^N \to Y$ over f, i.e. a holomorphic map $s : \mathrm{Op}\, K \times \mathbb{C}^N \to Y$ such that

(1) $\pi \circ s(z, w) = \pi \circ f(z)$, $s(z, 0) = f(z)$ for all $(z, w) \in \mathrm{Op}\, K \times \mathbb{C}^N$, and
(2) $s(z, \cdot) : \mathbb{C}^N \to \pi^{-1}(\pi \circ f(z))$ is a submersion at 0 for each $z \in K$.

Without using the Oka principles by Gromov and Forstnerič, we can easily prove that every subelliptic submersion is convexly elliptic (see Remark 3.6) and that every holomorphic fiber bundle with a CAP fiber is convexly elliptic (see the proof of Theorem 4.1). Thus the following Oka principle unifies them.

Theorem 7.4 ([34, Theorem 1.3 and Remark 5.4]). *A holomorphic submersion enjoys the Oka property if and only if it is convexly elliptic.*

Since a complex manifold Y is an Oka manifold if and only if the constant submersion $Y \to *$ enjoys the Oka property, Theorem 7.4 generalizes our previous characterization of Oka manifolds (Theorem 4.1). Recall that the previous characterization gave an affirmative answer to Gromov's conjecture [23, §1.4.E″] which essentially states that the Oka property and Gromov's Condition Ell_1 are equivalent for manifolds (Conjecture 1.6). Therefore Theorem 7.4 can be considered as an affirmative answer to the relative version of Gromov's conjecture.

The proof of Theorem 7.4 consists of the proofs of the Oka principles by Gromov and Forstnerič, and the following approximation theorem (see Theorem 4.3). Here, a Cartan pair (A, B) is said to be *very special* if there exists a holomorphic coordinate system on $\mathrm{Op}\, B$ in which B is convex and $A \cap B$ is the intersection of B and a closed half space.

Theorem 7.5 ([34, Theorem 3.12]). *Let $\pi : Z \to X$ be a holomorphic submersion onto a complex manifold X, (A, B) be a very special Cartan pair in X, $P_0 \subset P$ be compact Hausdorff spaces and $f_0 : Q \times \mathrm{Op}(A \cup B) \to Z$ be a family of continuous sections of π where $Q = P \times [0, 1]$, $Q_0 = (P \times \{0\}) \cup (P_0 \times [0, 1])$ such that $f_0|_{Q_0 \times \mathrm{Op}(A \cup B)}$ and $f_0|_{Q \times \mathrm{Op}\, A}$ are families of holomorphic sections. Assume that for each $q \in Q$ there exists a dominating global π-spray over $f_0(q, \cdot)|_{\mathrm{Op}(A \cap B)}$. Then there exists a homotopy $f_t : Q \times \mathrm{Op}(A \cup B) \to Z$ $(t \in [0, 1])$ of families of continuous sections such that the following hold for each $t \in [0, 1]$;*

1. *$f_t = f_0$ on $Q_0 \times \mathrm{Op}(A \cup B)$,*
2. *$f_t|_{Q \times \mathrm{Op}\, A}$ is a family of holomorphic sections which approximates f_0 uniformly on $Q \times A$, and*
3. *$f_1 : Q \times \mathrm{Op}(A \cup B) \to Z$ is a family of holomorphic sections.*

Theorem 7.4 has various applications (cf. [34, §5]). The following corollary generalizes Proposition 2.9.

Corollary 7.6 ([34, Corollary 5.6]). *Let $\pi_1 : Y_1 \to B$, $\pi_2 : Y_2 \to B$ and $\pi : Y_1 \to Y_2$ be holomorphic submersions. Assume that π is a surjective* Oka map *(i.e. a holomorphic Serre fibration with the Oka property) such that $\pi_2 \circ \pi = \pi_1$. Then π_1 enjoys the Oka property if and only if π_2 enjoys the Oka property.*

In the same way as in Section 5, Theorem 7.4 also implies the localization principle for the Oka property of a holomorphic submersion.

Corollary 7.7 ([34, Corollary 5.13]). *Let $\pi : Y \to B$ be a holomorphic submersion. Assume that each point of Y has a Zariski open neighborhood $U \subset Y$ such that the restriction $\pi|_U : U \to B$ enjoys the Oka property. Then π enjoys the Oka property.*

As another important application, Theorem 7.4 implies the Oka property of the complement of a holomorphically convex set; see the next subsection for this subject.

7.2. **Oka properties of complements of holomorphically convex sets.** A complex manifold Y is said to enjoy the *density property* if the Lie algebra generated by all $\mathbb{C}$-complete holomorphic vector fields on Y is dense in the Lie algebra of all holomorphic vector fields on Y with respect to the compact-open topology. The most typical examples are complex Euclidean spaces of dimension at least two[5]. This is a consequence of the classical Andersén–Lempert theory [1] (see [14, §4.10] and [27]). For a Stein manifold, it is known that the density property implies the Oka property (cf. [14, Proposition 5.6.23]).

As we have seen in the previous sections, it is important to understand when a closed set in an Oka manifold has an Oka complement. The only examples which have Oka complements so far were closed sets of positive codimension (closed complex subvarieties of codimension at least two [8, 17, 23, 32, 34, 35], closed complex hypersurfaces [24, 32] and closed countable sets of codimension at least two [31, 46]). The main theorem of our paper [36] gives whole new examples of arbitrary codimension.

Theorem 7.8 ([36, Theorem 1.2])**.** *For any Stein manifold Y with the density property and any compact $\mathcal{O}(Y)$-convex set $K \subset Y$, the complement $Y \setminus K$ is Oka.*

This theorem is motivated by the well-known long-standing problem whether the complement of a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$ is Oka (cf. [16, Problem], [10, Problem 3.11]). Forstnerič and Ritter studied such a complement and proved that it enjoys a restricted version of the Oka property [18, Theorem 1] (see also [15, Remark 1.3]). Since the complex Euclidean space $\mathbb{C}^n$ $(n > 1)$ enjoys the density property, Theorem 7.8 gives the following positive answer to this problem.

Corollary 7.9. *For any compact polynomially convex set $K \subset \mathbb{C}^n$ $(n > 1)$, the complement $\mathbb{C}^n \setminus K$ is Oka.*

As another application of Theorem 7.8, we can obtain new examples of nonelliptic Oka manifolds. Lemma 6.5 and Theorem 7.8 imply the following. Note that Corollary 6.4 is a special case of this corollary.

Corollary 7.10. *Assume that Y is a Stein manifold of dimension at least three with the density property and $K \subset Y$ is a compact $\mathcal{O}(Y)$-convex set with an infinite derived set. Then the complement $Y \setminus K$ is Oka but not weakly subelliptic.*

In fact, the relative version of Theorem 7.8 was also obtained in [36]. To state this theorem, let us recall the following notions. Here, the $\mathcal{O}(Y)$-hull $\widehat{S}_{\mathcal{O}(Y)}$ of a (not necessarily compact) closed set $S \subset Y$ is defined by $\widehat{S}_{\mathcal{O}(Y)} = \bigcup_{j\in\mathbb{N}} \widehat{(S_j)}_{\mathcal{O}(Y)}$ where $S = \bigcup_{j\in\mathbb{N}} S_j$ is an exhaustion of S by compact sets (see Definition 2.1).

Definition 7.11. Let $\pi : Y \to B$ be a holomorphic submersion.

(1) π enjoys the *(fibered) density property* if the Lie algebra generated by all $\mathbb{C}$-complete holomorphic vector fields on Y tangent to the fibers of π is dense in the Lie algebra of all holomorphic vector fields on Y tangent to the fibers of π with respect to the compact-open topology.

[5]In fact, there are no 1-dimensional Stein manifolds with the density property.

(2) A subset $S \subset Y$ is called a *family of compact holomorphically convex sets* if the restriction $\pi|_S : S \to B$ is proper and each point of B admits an open neighborhood $U \subset B$ such that $S \cap \pi^{-1}(U) \subset \pi^{-1}(U)$ is $\mathcal{O}(\pi^{-1}(U))$-convex (i.e. it coincides with its $\mathcal{O}(\pi^{-1}(U))$-hull).

For example, if Y is a Stein manifold with the density property and B is a Stein space, then the trivial bundle $B \times Y \to B$ enjoys the density property by [44, Lemma 3.5]. The following is the relative version of Theorem 7.8.

Theorem 7.12 ([36, Theorem 4.2])**.** *Let $\pi : Y \to B$ be a holomorphic submersion between complex spaces and $S \subset Y$ be a family of compact holomorphically convex sets. Assume that each point of B admits an open neighborhood $U \subset B$ such that $\pi^{-1}(U)$ is Stein and the restriction $\pi^{-1}(U) \to U$ enjoys the density property. Then the restriction $\pi|_{Y \setminus S} : Y \setminus S \to B$ enjoys the Oka property.*

By using this relative version, the following generalization of Corollary 7.9 was proved.

Theorem 7.13 ([36, Theorem 1.6])**.** *Let $S \subset \mathbb{C}^n$ $(n > 1)$ be a closed polynomially convex set. Assume that for some $C > 0$ there exists a holomorphic automorphism φ of $\mathbb{C}^n$ such that*

$$\varphi(S) \subset \left\{(z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : \|w\| \leq C(1 + \|z\|)\right\}.$$

Then the complement $\mathbb{C}^n \setminus S$ is Oka.

It is also a well-known problem whether the complement of a totally real affine subspace $\mathbb{R}^k \subset \mathbb{C}^n$ is Oka for $n > 1$ and $1 \leq k \leq n$ [19, Problem 1.5]. As in the case of compact polynomially convex sets, Forstnerič and Wold proved that for $n > 1$ and $1 \leq k \leq n - 1$ the complement $\mathbb{C}^n \setminus \mathbb{R}^k$ enjoys a restricted version of the Oka property [19, Theorem 1.3] (see also [37]). Observe that

$$\mathbb{C}^n \setminus \mathbb{R}^{2k} \cong \mathbb{C}^n \setminus \left\{(z, 0, \ldots, 0, \bar{z}) : z \in \mathbb{C}^k\right\},$$
$$\mathbb{C}^n \setminus \mathbb{R}^{2l+1} \cong \mathbb{C}^n \setminus \left(\mathbb{R} \times \left\{(z, 0, \ldots, 0, \bar{z}) : z \in \mathbb{C}^l\right\}\right),$$

and the sets

$$\left\{(z, 0, \ldots, 0, \bar{z}) : z \in \mathbb{C}^k\right\}, \quad \mathbb{R} \times \left\{(z, 0, \ldots, 0, \bar{z}) : z \in \mathbb{C}^l\right\}$$

are contained in $\{(z, w) \in \mathbb{C}^{n-2} \times \mathbb{C}^2 : \|w\| \leq 1 + \|z\|\}$ if $\max\{k, l + 1\} \leq n - 2$. Note that the only exceptions are $\mathbb{C}^2 \setminus \mathbb{R}$, $\mathbb{C}^2 \setminus \mathbb{R}^2$ and $\mathbb{C}^3 \setminus \mathbb{R}^3$. Since every closed set S in $\mathbb{R}^n \subset \mathbb{C}^n$ is polynomially convex (cf. [43, p. 3]), the above observation and Theorem 7.13 imply the following corollary.

Corollary 7.14. *If $n > 1$, then for any compact set K in $\mathbb{R}^n \subset \mathbb{C}^n$ the complement $\mathbb{C}^n \setminus K$ is Oka. If in addition $(n, k) \neq (2, 1), (2, 2), (3, 3)$, then for any closed set S in $\mathbb{R}^k \subset \mathbb{C}^n$ the complement $\mathbb{C}^n \setminus S$ is Oka. In particular, the complement $\mathbb{C}^n \setminus \mathbb{R}^k$ of a totally real affine subspace is Oka if $n > 1$ and $(n, k) \neq (2, 1), (2, 2), (3, 3)$.*

In the above results, we only consider the complements of holomorphically convex sets. It can be easily seen that a domain in a complex manifold with a strongly pseudoconvex boundary point cannot be Oka (cf. [18, Example 5]). Thus it is natural to ask whether the Oka property of the complement of a closed set $S \subset \mathbb{C}^n$ implies polynomial convexity of S. In fact, we can obtain the following negative answer to this question as an application of Theorem 7.13.

Corollary 7.15 ([36, Corollary 1.8])**.** *For any rectifiable simple closed curve $C \subset \mathbb{C}^n$ $(n \geq 3)$, the complement $\mathbb{C}^n \setminus C$ is Oka. In particular, the complement $\mathbb{C}^n \setminus S^1$ of the unit circle S^1 in a complex line $\mathbb{C} \subset \mathbb{C}^n$ $(n \geq 3)$ is Oka.*

Since every rectifiable arc in $\mathbb{C}^n$ is polynomially convex (cf. [43, Corollary 3.1.2]), Corollary 7.9 also implies the following.

Corollary 7.16. *For any rectifiable arc $C \subset \mathbb{C}^n$ $(n > 1)$, the complement $\mathbb{C}^n \setminus C$ is Oka.*

Recall that for a continuous function $f : \mathbb{D} \to \mathbb{C}$, the projection $(\mathbb{D}\times\mathbb{C})\setminus\Gamma_f \to \mathbb{D}$ from the complement of the graph Γ_f of f enjoys the Oka property if and only if f is holomorphic (cf. [14, Corollary 7.4.10]). By Theorem 7.12, however, there exists a non-holomorphic real analytic map $f : \mathbb{D} \to \mathbb{C}^n$ $(n > 1)$ such that the projection $(\mathbb{D} \times \mathbb{C}^n) \setminus \Gamma_f \to \mathbb{D}$ enjoys the Oka property (e.g. $f(z) = (\bar{z}, 0, \ldots, 0)$). This phenomenon leads us to the following question.

Question 7.17. Assume that $f : \mathbb{D} \to \mathbb{C}^n$ $(n > 1)$ is a continuous map. Is there a characterization of the Oka property of the projection $(\mathbb{D} \times \mathbb{C}) \setminus \Gamma_f \to \mathbb{D}$ by some function-theoretic property of f?

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