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REFLECTING RECOLLEMENTS

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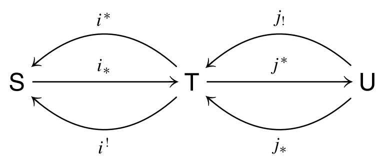
Abstract

A recollement describes one triangulated category T as "glued together" from two others, S and U. The definition is not symmetrical in S and U, but this note shows how S and U can be interchanged when T has a Serre functor.

A recollement of triangulated categories S, T, U is a diagram of triangulated functors

(1)

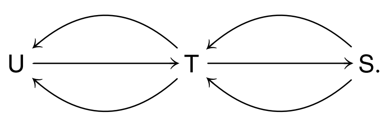


satisfying a number of conditions given in Remark 2 below.

Recollements are important in algebraic geometry and representation theory, see for instance [1], [3], [4]. They were introduced and developed in [1], and as indicated by the terminology, one thinks of T as being "glued together" from S and U. Indeed, in the canonical example of a recollement, T is a derived category of sheaves on a space, and S and U are derived categories of sheaves on a closed subset and its open complement, respectively. Other examples of a more algebraic nature can be found in [3].

The recollement (1) is not symmetrical in S and U: There are only two arrows pointing to the right, but four pointing to the left. So there is no particular reason to think that it should be possible to interchange S and U, that is, use (1) to construct another recollement of the form

(2)



Nevertheless, that is precisely what this note does in Theorem 7 below, under the assumption that T has a Serre functor; see Remark 3 for the definition.

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In fact, it will be showed that there are two different ways to get recollements of the form (2), one involving the four upper functors from (1) and another involving the four lower functors.

SETUP 1. Let k denote a field and assume that the category T of the recollement (1) is a skeletally small k-linear triangulated category with finite dimensional Hom-sets and split idempotents.

Let me start with two remarks explaining the formalism of recollements and Serre functors.

REMARK 2 (Recollements, cf. [1, Section 1.4]). The recollement (1) is defined by the following properties.

(i) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, and (j^*, j_*) are pairs of adjoint functors.

(ii) $j^* i_* = 0$.

(iii) i_*, $j_!$, and j_* are fully faithful.

(iv) Each object X in T determines distinguished triangles

(a) $i_* i^! X \to X \to j_* j^* X \to$ and

(b) $j_! j^* X \to X \to i_* i^* X \to$

where the arrows into and out of X are counit and unit morphisms of the relevant adjunctions.

REMARK 3 (Serre functors, cf. [5, Section I.1]). Let $(-)^\vee$ denote the functor $\mathrm{Hom}_k(-, k)$. A right Serre functor T for T is an endofunctor for which there are natural isomorphisms

$$\mathsf{T}(X, Y) \cong \mathsf{T}(Y, TX)^\vee,$$

and a left Serre functor $\tilde{T}$ is an endofunctor for which there are natural isomorphisms

$$\mathsf{T}(X, Y) \cong \mathsf{T}(\tilde{T}Y, X)^\vee.$$

A Serre functor is an essentially surjective right Serre functor.

A right Serre functor is fully faithful, and hence a Serre functor is an autoequivalence.

If there is a right Serre functor T and a left Serre functor $\tilde{T}$, then T is in fact a Serre functor and $\tilde{T}$ is a quasi-inverse of F.

SETUP 4. Assume that T has a Serre functor T with quasi-inverse $\tilde{T}$.

It is now possible to prove that the categories S and U in (1) can be interchanged. First, however, two propositions which may be of independent interest.

Proposition 5. *The category* S *has a Serre functor* S *with quasi-inverse* $\tilde{S}$:

$$S = i^! T i_* \quad \textit{and} \quad \tilde{S} = i^* \tilde{T} i_*.$$

The category U *has a Serre functor* U *with quasi-inverse* $\tilde{U}$:

$$U = j^* T j_! \quad \textit{and} \quad \tilde{U} = j^* \tilde{T} j_*.$$

Proof. By Remark 3, it is enough to show that S and $\tilde{S}$ are, respectively, a right and a left Serre functor for S, and similarly for U and $\tilde{U}$. This can be done directly,

$$\begin{aligned} \mathsf{S}(Y, SX)^\vee &= \mathsf{S}(Y, i^! T i_* X)^\vee && \text{by definition} \\ &\cong \mathsf{T}(i_* Y, T i_* X)^\vee && i_* \text{ left-adjoint of } i^! \\ &\cong \mathsf{T}(i_* X, i_* Y) && T \text{ right Serre functor} \\ &\cong \mathsf{S}(X, Y) && i_* \text{ fully faithful} \end{aligned}$$

and

$$\begin{aligned} \mathsf{S}(\tilde{S} Y, X)^\vee &= \mathsf{S}(i^* \tilde{T} i_* Y, X)^\vee && \text{by definition} \\ &\cong \mathsf{T}(\tilde{T} i_* Y, i_* X)^\vee && i_* \text{ right-adjoint of } i^* \\ &\cong \mathsf{T}(i_* X, i_* Y) && \tilde{T} \text{ left Serre functor} \\ &\cong \mathsf{S}(X, Y) && i_* \text{ fully faithful.} \end{aligned}$$

Similar computations work for U and $\tilde{U}$. □

Proposition 6. *The functors* i^* *and* $j_!$ *have left-adjoint functors given by*

$$i_! = \tilde{T} i_* S = \tilde{T} i_* i^! T i_* \quad \textit{and} \quad j^? = \tilde{U} j^* T = j^* \tilde{T} j_* j^* T.$$

The functors $i^!$ *and* j_* *have right-adjoint functors given by*

$$i_? = T i_* \tilde{S} = T i_* i^* \tilde{T} i_* \quad \textit{and} \quad j^! = U j^* \tilde{T} = j^* T j_! j^* \tilde{T}.$$

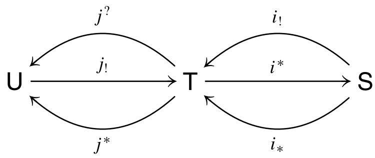
Proof. This can be proved directly, for instance

$$\begin{aligned} \mathsf{T}(i_! X, Y) &= \mathsf{T}(\tilde{T} i_* S X, Y) && \text{by definition} \\ &\cong \mathsf{T}(Y, i_* S X)^\vee && \tilde{T} \text{ left Serre functor} \\ &\cong \mathsf{S}(i^* Y, S X)^\vee && i^* \text{ left-adjoint of } i_* \\ &\cong \mathsf{S}(X, i^* Y), && S \text{ right Serre functor,} \end{aligned}$$

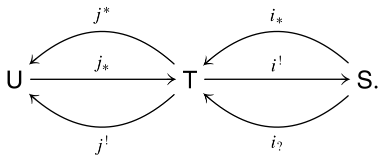
and similarly for the other cases. □

This permits the proof of the main result of this note.

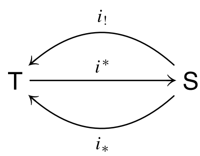
Theorem 7. *There are recollements*



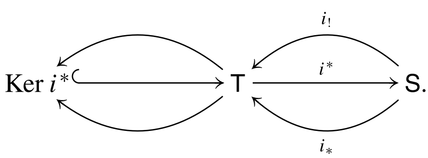
and



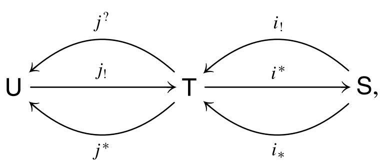
Proof. Proposition 6 implies that there is



where $(i_!, i^*)$ and (i^*, i_*) are pairs of adjoint functors. The functor i_* is fully faithful, and it follows from [4, Proposition 2.7] or [2, Proposition 1.14] that there is a recollement



It is standard recollement theory that $\operatorname{Ker} i^* = \operatorname{Ess.Im} j_!$, see [3, Theorem 1] or [2, Remark 1.5 (8)], and $j_!$ can be used to replace $\operatorname{Ess.Im} j_!$ with U, so the first recollement of the theorem,



follows. The functors from T to U must be $j^?$ and j^* since, by the definition of recollements, they are the left- and the right-adjoint of the functor $j_!$ from U to T.

The second recollement of the theorem can be obtained by the dual procedure. □

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