Tanabe, Hiroki
Osaka Math. J.
**12** (1960), 145-166.

# *Remarks on the Equations of Evolution in a Banach Space*

By Hiroki TANABE

§ 0. **Introduction.** The contents of this paper consist of a slight extension of the previous paper [5] and some supplements to it. As in [5], we consider a certain type of the equations of evolution in a Banach space $\mathfrak{X}$:

$$dx(t)/dt = (A(t)+B(t))x(t)+f(t) \tag{0. 1}$$

and the associated homogeneous equation

$$dx(t)/dt = (A(t)+B(t))x(t). \tag{0. 1'}$$

Here, $A(t)$ and $B(t)$ satisfy all the assumptions in [5] only replacing $\|\exp(tA(s))\|\leqq 1$ by $\|\exp(tA(s))\|\leqq M$, where $M$ is a positive constant which is independent of $t$ or $s$ and generally greater than one. A necessary and sufficient condition that a closed operator generates a semi-group of bounded operators satisfying such an inequality was given by R. S. Phillips [3]. In [5], we assumed $M=1$ so that we were assured of the uniqueness of the solution of (0.1) by Theorem 1 of T. Kato [1]. But, we shall show the uniqueness in this paper without making such an assumption by examining the property of the fundamental solution $U(t, s)$ constructed in [5] a little closely. Note that it was unnecessary to assume $M=1$ in constructing $U(t, s)$ in [5].

In [5], we constructed the fundamental solution $U(t, s)$ first for the equation with $B(t)\equiv 0$, and then for the equation with $B(t)\not\equiv 0$ by a perturbation method. In this paper, we shall construct $U(t, s)$ directly even when $B(t)\not\equiv 0$ without using a perturbation method and show the further differentiability of the solution of (0.1) under the assumption that $A(t)$, $B(t)$ and $f(t)$ are sufficiently smooth, which was done only when $B(t)\equiv 0$ and $f(t)\equiv 0$ in [5]. In § 5, we shall give some remarks on generalized solutions in the sense of Solomiak [4], and finally in § 6, the case will be considered in which $\mathfrak{D}(A(t))$ changes smoothly with $t$ in the sense of T. Kato [2].

## § 1. The uniqueness of the solution and some remarks.

Throughout this paper except in § 6, we assume that the operators

$A(t)$ and $B(t)$ satisfy the hypotheses in [5] with one replacement mentioned in § 0. To make sure, we write them down here.

**Hypotheses** 1.1°. $A(t)$ is defined for $a \leqq t \leqq b$ and is an infinitesimal generator of a semi-group of bounded operators with the norm not exceeding some positive constant $M$ independent of $t$.

1.2° 1) The domain $\mathfrak{D}$ of $A(t)$ is independent of $t$, 2) the bounded operator $B(t, s) = [I - A(t)][I - A(s)]^{-1}$ is uniformly bounded for $a \leqq s$, $t \leqq b$. 3) $B(t, s)$ satisfies Lipschitz condition in $t$ for every $s$ in the uniform operator topology.

1.3°. $B(t, s)$ is strongly continuously differentiable in $t$ for every $s$.

1.4°. For each $s$ and $t$ with $a \leqq s \leqq b$ and $t > 0$, $(d/dt) \exp(tA(s))$ is a bounded operator and there exist positive constants $C$ and $t_0$ such that

$$\|(d/dt) \exp(tA(s))\| = \|A(s) \exp(tA(s))\| \leqq C/t \tag{1.1}$$

for any $s$ and $0 < t \leqq t_0$.

2.1°. $B(t)$ is closed and defined for $a \leqq t \leqq b$ and has a domain (which may be dependent on $t$) containing the domain $\mathfrak{D}$ of $A(t)$.

2.2°. A bounded operotor $B(t)A(s)^{-1}$ is continuous in $a \leqq t \leqq b$ for every $s$ in the uniform operator topology.

2.3°. There exist positive constants $C_1, C_2$, $\rho \leqq 1$ and $\lambda \leqq 1$ such that

$$\|B(t) \exp(\tau A(s))\| \leqq C_1 \tau^{-(1-\rho)}, \tag{1.2}$$

$$\|(B(t') - B(t)) \exp(\tau A(s))\| \leqq C_2 |t' - t|^{\lambda} \tau^{-(1-\rho)} \tag{1.3}$$

for $a \leqq t$, $t'$, $s \leqq b$ and $\tau > 0$.

As in [5], we shall write $A(t)$ instead of $A(t) - I$. Then, there exist positive constants $L$ and $N$ such that for every $t$, $s$ and $r$,

$$\|A(t)A(s)^{-1}\| \leqq L, \quad \|(A(t) - A(r))A(s)^{-1}\| \leqq N|t - r| \tag{1.4}$$

Throughout this paper, the constant $K$ will be used to denote positive constants depending only on the constants appearing in the above Hypotheses and (1.4), and it is not necessarily equal in every occurrence. And we use $K_\alpha, K_\beta, \cdots$ to denote constants depending also on $\alpha, \beta, \cdots$ besides the constants mentioned above.

First, we consider the equation with $B(t) \equiv 0$:

$$dx(t)/dt = A(t)x(t) + f(t). \tag{1.5}$$

Let an operator-valued function $V(t, s)$ be such that

i) it is strongly continuous in $a \leqq s \leqq t \leqq b$,

ii) $V(s, s) = I$ for each $s$,

iii) for $x \in \mathfrak{D}$, $V(t, s)x$ is strongly continuously differentiable in $s$ and satisfies

$$-(\partial/\partial s)V(t, s)x = V(t, s)A(s)x, \quad a \leqq s < t \leqq b. \tag{1.6}$$

Then for any solution $x(t)$ of the homogeneous equation of (1.5), we have

$$0 = \int_s^t \frac{\partial}{\partial \sigma}(V(t, \sigma)x(\sigma))d\sigma = x(t) - V(t, s)x(s),$$

which implies the uniqueness of the solution of (1.5). Hence, in order to prove the uniqueness for (1.5), we have only to show the existence of such an operator $V(t, s)$.

Let us write $V(t, s)$ in the form

$$V(t, s) = \exp((t-s)A(s)) + \int_s^t Q(t, \tau)\exp((\tau - s)A(s))d\tau. \tag{1.7}$$

By formal calculation,

$$\frac{\partial}{\partial s}V(t, s) = \frac{\partial}{\partial s}\exp((t-s)A(s)) - Q(t, s) + \int_s^t Q(t, \tau)\frac{\partial}{\partial s}\exp((t-s)A(s))d\tau$$

$$\begin{aligned} V(t, s)A(s) &= \exp((t-s)A(s))A(s) + \int_s^t Q(t, \tau)\exp((\tau - s)A(s))A(s)d\tau \\ &= \frac{\partial}{\partial t}\exp((t-s)A(s)) + \int_s^t Q(t, \tau)\frac{\partial}{\partial \tau}\exp((\tau - s)A(s))d\tau. \end{aligned}$$

Hence, putting $Q_1(t, s) = (\partial/\partial t + \partial/\partial s)\exp((t-s)A(s))$, we obtain

$$\frac{\partial}{\partial s}V(t, s) + V(t, s)A(s) = Q_1(t, s) - Q(t, s) + \int_s^t Q(t, \tau)Q_1(\tau, s)d\tau.$$

So, let us determine $Q(t, s)$ as the solution of

$$Q(t, s) - \int_s^t Q(t, \tau)Q_1(\tau, s)d\tau = Q_1(t, s) \tag{1.8}$$

By Lemma 1.4. in [5], we can solve the above integral equation by a successive approximation method:

$$Q(t, s) = \textstyle\sum_{m=1}^{\infty} Q_m(t, s), \tag{1.9}$$

where $Q_m(t, s) = \int_s^t Q_{m-1}(t, \tau)Q_1(\tau, s)d\tau$, $m = 2, 3, \cdots$.

It is easy to see that $V(t, s)$ obtained by inserting the above $Q(t, s)$ in (1.7) really satisfies i), ii) and iii). Hence, the uniqueness in question is obtained. Incidentally, $V(t, s) = U(t, s)$ is also proved. But, we can also prove this fact directly by showing

$$\int_s^t Q_m(t, \tau) \exp((\tau - s)A(s))d\tau = \int_s^t \exp((t-\tau)A(\tau))R_m(\tau, s)d\tau$$

for each $m$ by induction. As to the meaning of $R_m(t, s)$, see [5].

Next, we consider the equation with $B(t) \not\equiv 0$. The fundamental solution of (0.1′) was constructed by perturbation method in [5]: $U(t, s) = \sum_{m=0}^{\infty} U_m(t, s)$, where $U_0(t, s)$ is the fundamental solution of (1.5) and

$$U_m(t, s) = \int_s^t U_0(t, \sigma)B(\sigma)U_{m-1}(\sigma, s)d\sigma = \int_s^t U_{m-1}(t, \sigma)B(\sigma)U_0(\sigma, s)d\sigma . \tag{1.10}$$

Let $x$ be any element in $\mathfrak{D}$. $-(\partial/\partial s)U_0(t, s)x = U_0(t, s)A(s)x$ was shown above. For general $m$,

$$\begin{aligned} \frac{\partial}{\partial s} U_m(t, s)x &= \frac{\partial}{\partial s}\int_s^t U_{m-1}(t, \sigma)B(\sigma)U_0(\sigma, s)xd\sigma \\ &= -U_{m-1}(t, s)B(s)x - \int_s^t U_{m-1}(t, \sigma)B(\sigma)U_0(\sigma, s)A(s)xd\sigma \\ &= -U_{m-1}(t, s)B(s)x - U_m(t, s)A(s)x , \end{aligned}$$

where we used the easily verified fact that $(\partial/\partial s)B(\sigma)U_0(\sigma, s)x = B(\sigma)(\partial/\partial s)U_0(\sigma, s)x$. Hence,

$$\begin{aligned} \frac{\partial}{\partial s} U(t, s)x &= \frac{\partial}{\partial s}\sum_{m=0}^{\infty} U_m(t, s)x \\ &= -U_0(t, s)A(s)x - \sum_{m=1}^{\infty} (U_{m-1}(t, s)B(s)x + U_m(t, s)A(s)x) \\ &= -U(t, s)A(s)x - U(t, s)B(s)x . \end{aligned}$$

Consequently, the uniqueness for (0.1) is also proved.

**Theorem 1.1.** *Under Hypotheses* 1.1°–2.3°, *the solution of* (0.1) *is uniquely determined in* $s \leqq t \leqq b$ *by the initial condition at* $t = s$ *and the right member* $f(t)$. *The fundamental solution* $U(t, s)$ *whose existence was shown in* [5] *satisfies*

$$-\frac{\partial}{\partial s} U(t, s)x = U(t, s)A(s)x + U(t, s)B(s)x , \tag{1.11}$$

*for any* $x \in \mathfrak{D}$ *in* $a \leqq s < t \leqq b$.

Next, we also assume that $(\partial/\partial t)B(t, s)$ is Hölder continuous in $t$:

$$\begin{aligned} \|(\partial/\partial t)B(t, s) - (\partial/\partial \tau)B(\tau, s)\| &= \|(A'(t) - A'(\tau))A(s)^{-1}\| \\ &\leqq H|t-\tau|^{\alpha}, \quad H > 0,\ 0 < \alpha \leqq 1 , \end{aligned} \tag{1.12}$$

and we will show the boundedness of $(\partial/\partial s)U(t, s)$, $s<t$, for the equation (1.5) ($B(t)\equiv 0$) under this additional assumption. We denote by $K'$ constants which depend also on $H$ and $\alpha$ besides the constants appearing in Hypotheses 1.1°-2.3°.

**Lemma 1.1.** *The following inequalities hold*:

$$\left\|\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)R_1(t, s)\right\| \leqq K'(t-s)^{\alpha-1}, \tag{1.13}$$

$$\left\|\frac{\partial}{\partial s}R_m(t, s)\right\| \leqq K'K^{m-2}\frac{(t-s)^{m-2}}{(m-2)!} \text{ for } m\geqq 2.$$

Proof. The first one is a direct consequence of

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)R_1(t, s) \tag{1.14}$$
$$= (A'(t)-A'(s))\exp((t-s)A(s))+(A(t)-A(s))\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\exp((t-s)A(s))$$

and the assumption (1.12) and the uniform boundedness of the second term on the right member which can be easily shown. Further, taking into consideration that

$$\left\|R_1(t, \sigma)-R_1(t, s)\right\| = \left\|\int_s^\sigma \frac{\partial}{\partial \xi}R_1(t, \xi)d\xi\right\| \leqq K\int_s^\sigma \frac{d\xi}{t-\xi} = K\log\frac{t-s}{t-\sigma}$$

for $s<\sigma<t$, we may write

$$\frac{\partial}{\partial s}R_2(t, s) = \int_s^t (R_1(t, \sigma)-R_1(t, s))\frac{\partial}{\partial s}R_1(\sigma, s)d\sigma \tag{1.15}$$
$$+R_1(t, s)\int_s^t\left(\frac{\partial}{\partial \sigma}+\frac{\partial}{\partial s}\right)R_1(\sigma, s)d\sigma - R_1(t, s)R_1(t, s).$$

Noting that $\log(1+x)\leqq x$ for $x\geqq 0$, we see that the norm of the first term is bounded by

$$K\int_s^t \frac{1}{\sigma-s}\log\frac{t-s}{t-\sigma}d\sigma = K\int_s^{(t+s)/2}\frac{1}{t-\sigma}d\sigma+\frac{K}{t-s}\int_{(t+s)/2}^t \log\frac{t-s}{t-\sigma}d\sigma \leqq K.$$

Clearly, the remaining terms of the right member of (1.15) are uniformly bounded. Hence, we find that $(\partial/\partial s)R_2(t, s)$ is uniformly bounded. For general $m>2$, we have

$$\frac{\partial}{\partial s}R_m(t, s) = \int_s^t R_{m-2}(t, \sigma)\frac{\partial}{\partial s}R_2(\sigma, s)d\sigma.$$

If (1.13) has been proved for $m-1$, we have

$$\left\|\frac{\partial}{\partial s}R_m(t,s)\right\| \leqq \int_s^t \frac{K^{m-2}(t-\sigma)^{m-3}}{(m-3)!}K'd\sigma = K'K^{m-2}\frac{(t-s)^{m-2}}{(m-2)!}.$$

Thus, the lemma is proved.

**Theorem 1.2.** *Under the assumption* (1.12), $(\partial/\partial s)U(t,s)$ *is bounded for* $t>s$ *and it satisfies*

$$\left\|\frac{\partial}{\partial s}U(t,s)\right\| \leqq \frac{K'}{t-s}. \tag{1.16}$$

Proof. Let us estimate the derivative of each term of the right member of

$$U(t,s) = \exp((t-s)A(s)) + \int_s^t \exp((t-\tau)A(\tau))R_1(\tau,s)d\tau$$
$$+\int_s^t \exp((t-\tau)A(\tau))\sum_{m=2}^{\infty}R_m(\tau,s)d\tau.$$

Estimating as in the preceding lemma, we can show that the derivative of the second term is uniformly bounded. That of the last term is seen to be bounded by $K'(t-s)$ in norm by Lemma 1.1. (1.16) is a direct consequence of these facts.

The above theorem implies

**Theorem 1.3.** $U(t,s)^*$ *is weakly differentiable in* $s$ *and satisfies in* $a\leqq s\leqq t$

$$-\frac{\partial}{\partial s}U(t,s)^* = A(s)^*U(t,s)^*,\ U(t,t)^* = I$$

$$\left\|\frac{\partial}{\partial s}U(t,s)^*\right\| \leqq \frac{K'}{t-s}.$$

If $\rho+\lambda>1$ we can obtain the same results for the equation (0.1′) in which $B(t)\not\equiv 0$.

§2. **The construction of the fundamental solution for (1.1′) by the second method.** In this section, we construct the fundamental solution $U(t,s)$ for (0.1′) by a more direct method without using a perturbation theory as in [5]. Those two fundamental solutions are identical to each other due to the uniqueness theorem proved in the preceding section. As before, we determine an operator $R(t,s)$ so that

$$U(t,s) = \exp((t-s)A(s)) + \int_s^t \exp((t-\tau)A(\tau))R(\tau,s)d\tau \tag{2.1}$$

should be the fundamental solution of (0.1′). Let us determine $R(t,s)$

as the solution of

$$(2.2)\qquad R(t, s)-\int_s^t (A(t)+B(t)-A(\tau)) \exp((t-\tau)A(\tau))R(\tau, s)d\tau$$
$$= (A(t)+B(t)-A(s)) \exp((t-s)A(s)).$$

We put $R_1(t, s)=(A(t)+B(t)-A(s)) \exp((t-s)A(s))$.

**Lemma 2.1.** *$R_1(t, s)$ is strongly continuous in $s$ and $t$ in $a\leqq s<t\leqq b$ and satisfies*

$$(2.3)\qquad ||R_1(t, s)||\leqq K(t-s)^{\rho-1}.$$

Proof. By Lemma 1.1 in [5], we have only to show the continuity of $B(t)\exp(t-s)A(s))$, but it is clear.

The integral equation (2.2) can be solved by a successive approximation method:

$$(2.4)\qquad R(t, s)=\sum_{m=1}^{\infty} R_m(t, s),$$

where $R_m(t, s)=\int_s^t R_1(t, \sigma)R_{m-1}(\sigma, s)d\sigma$, $m=2, 3, \cdots$.

**Lemma 2.2.** *The series* (2.4) *converges uniformly in $a\leqq s<t\leqq b$ in the uniform operator topology in the wider sense, and the sum is strongly continuous in the same region and satisfies*

$$(2.5)\qquad ||R(t, s)||\leqq K(t-s)^{\rho-1}$$

Proof. The lemma follows immediately from

$$||R(t, s)||\leqq \frac{(K\Gamma(\rho))^m(t-s)^{m\rho-1}}{\Gamma(m\rho)}, \quad m=2, 3, \cdots,$$

which is easily seen by induction.

**Lemma 2.3.** *For $s<\tau<t$, we have*

$$(2.6)\qquad ||R(t, s)-R(\tau, s)||$$
$$\leqq K_\alpha\left\{\frac{t-\tau}{(t-s)(\tau-s)^{1-\rho}}+\frac{(t-\tau)^\rho}{(\tau-s)^{1-\rho}}+\frac{(t-\tau)^\lambda}{(\tau-s)^{1-\rho}}+\frac{(t-\tau)^\alpha}{(\tau-s)^{1-2\rho+\alpha}}\right\}$$

*where $\alpha>0$ is an arbitrary constant less than $\rho$.*

Proof. We begin with the estimation of

$$R_1(t, s)-R_1(\tau, s)=(A(t)-A(s)) \exp((t-s)A(s))-(A(\tau)-A(s)) \exp((\tau-s)A(s))$$
$$+B(t) \exp((t-s)A(s))-B(\tau) \exp((\tau-s)A(s)).$$

In [5], the norm of the first difference of the right member was proved to be bounded by $K(t-\tau)(t-s)^{-1}$. By hypotheses,

$$||(B(t)-B(\tau))\exp((t-s)A(s))|| \leqq C'(t-\tau)^{\lambda}(t-s)^{\rho-1}.$$

$$\begin{aligned}
&||B(\tau)\{\exp((t-s)A(s))-\exp((\tau-s)A(s))\}|| = \\
&= \left\| B(\tau)\int_{\tau-s}^{t-s} A(s)\exp(\sigma A(s))d\sigma \right\| \\
&\leqq \int_{\tau-s}^{t-s}\left\| B(\tau)\exp\left(\frac{\sigma}{2}A(s)\right)A(s)\exp\left(\frac{\sigma}{2}A(s)\right)\right\| d\sigma = \\
&\leqq K\int_{\tau-s}^{t-s}\sigma^{\rho-2}d\sigma = K\{(\tau-s)^{\rho-1}-(t-s)^{\rho-1}\} \\
&= K(\tau-s)^{\rho-1}\left\{1-\left(\frac{\tau-s}{t-s}\right)^{1-\rho}\right\} \leqq K(\tau-s)^{\rho-1}\left(1-\frac{\tau-s}{t-s}\right) \\
&= K\frac{t-\tau}{t-s}(\tau-s)^{\rho-1},
\end{aligned}$$

where we used that $(\tau-s)/(t-s)<1$. Thus, we obtain

$$||R_1(t,s)-R_1(\tau,s)|| \leqq K\{(t-\tau)^{\lambda}(t-s)^{\rho-1}+(t-\tau)(t-s)^{-1}(\tau-s)^{\rho-1}\}. \tag{2.7}$$

Noting that $R(t,s)=R_1(t,s)+\int_s^t R_1(t,\sigma)R(\sigma,s)d\sigma$, we have only to estimate

$$\int_s^t R_1(t,\sigma)R(\sigma,s)d\sigma-\int_s^{\tau} R_1(\tau,\sigma)R(\sigma,s)d\sigma = \int_{\tau}^t R_1(t,\sigma)R(\sigma,s)d\sigma$$
$$+\int_s^{\tau}(R_1(t,\sigma)-R_1(\tau,\sigma))R(\sigma,s)d\sigma,$$

in order to complete the proof. As for the first term of the right member,

$$\begin{aligned}
\left\|\int_{\tau}^t R_1(t,\sigma)R(\sigma,s)d\sigma\right\| &\leqq K\int_{\tau}^t (t-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma \\
&\leqq K(\tau-s)^{\rho-1}\int_{\tau}^t (t-\sigma)^{\rho-1}d\sigma \leqq K(t-\tau)^{\rho}(\tau-s)^{\rho-1}.
\end{aligned}$$

As for the second term, using (2.7) we have

$$\begin{aligned}
&\left\|\int_s^{\tau}(R_1(t,\sigma)-R_1(\tau,\sigma))R(\sigma,s)d\sigma\right\| \\
&\leqq K\int_s^{\tau}\left\{\frac{(t-\tau)^{\lambda}}{(t-\sigma)^{1-\rho}}+\frac{t-\tau}{(t-\sigma)(\tau-\sigma)^{1-\rho}}\right\}(\sigma-s)^{\rho-1}d\sigma \\
&\leqq K(t-\tau)^{\lambda}\int_s^{\tau}(\tau-\sigma)^{\rho-1}(\sigma-s)^{\rho-1}d\sigma+K_{\alpha}(t-\tau)^{\alpha}\int_s^{\tau}(\tau-\sigma)^{-\alpha+\rho-1}(\sigma-s)^{\rho-1}d\sigma \\
&\leqq K_{\alpha}\{(t-\tau)^{\lambda}(\tau-s)^{2\rho-1}+(t-\tau)^{\alpha}(\tau-s)^{2\rho-\alpha-1}\}.
\end{aligned}$$

Summing up the above results, we obtain (2.6).

We denote by $W(t, s)$ the second term of the right member of (2.1):

$$W(t, s) = \int_s^t \exp((t-\tau)A(\tau))R(\tau, s)d\tau .$$

By Lemma 2.3 and the uniform boundedness of $(\partial/\partial t+\partial/\partial s)\exp((t-s)A(s))$ which was proved in [5], we obtain as in [5]:

$$\begin{aligned} \frac{\partial}{\partial t} U(t, s) &= A(s)\exp((t-s)A(s)) \\ &+\int_s^t A(\tau)\exp((t-\tau)A(\tau))(R(\tau, s)-R(t, s))d\tau \\ &+\int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\exp((t-\tau)A(\tau))R(t, s)d\tau \\ &+\exp((t-s)A(s))R(t, s). \end{aligned} \tag{2.8}$$

The following inequalities are easily obtained using Lemmas 2.2 and 2.3:

$$\begin{aligned} &\|W(t, s)\| \leqq K(t-s)^{\rho}, \\ &\left\|\int_s^t A(\tau)\exp((t-\tau)A(\tau))(R(\tau, s)-R(t, s))d\tau\right\| \leqq K(t-s)^{\rho-1} \\ &\left\|\int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\exp((t-\tau)A(\tau))R(t, s)d\tau\right\| \leqq K(t-s)^{\rho}, \\ &\|\exp((t-s)A(s))R(t, s)\| \leqq K(t-s)^{\rho-1}. \end{aligned} \tag{2.9}$$

Hence, we have

$$\|(\partial/\partial t)W(t, s)\| \leqq K(t-s)^{\rho-1}, \tag{2.10}$$

$$\|(\partial/\partial t)U(t, s)\| \leqq K(t-s)^{-1}. \tag{2.11}$$

The similar results hold for $A(t)U(t, s)$ and it is easily verified as in [5] that $U(t, s)$ constructed above is a fundamental solution of (0.1′).

**Theorem 2.1.** *There exists a unique fundamental solution* $U(t, s)$ *of* (0.1′), *which satisfies*

$$\left\|\frac{\partial}{\partial t} U(t, s)\right\| \leqq \frac{K}{t-s}, \quad \|A(t)U(t, s)\| \leqq \frac{K}{t-s}, \quad \|B(t)U(t, s)\| \leqq \frac{K}{(t-s)^{1-\rho}}.$$

## § 3. Hölder continuity of the derivative of the solution.

In this section, we consider about the Hölder continuity of the derivative of a solution of (0.1). For the sake of simplicity, we assume that $B(t)\equiv 0$ throughout this section. Hence, (2.5) and (2.6) reduce to

$$\|R(t, s)\| \leqq K \exp (K(t-s)). \tag{3.1}$$

$$\|R(t, s)-R(\tau, s)\| \leqq K\left(\frac{t-\tau}{t-s}+(t-\tau) \log \frac{t-s}{t-\tau}\right). \tag{3.2}$$

We begin with the Hölder continuity of $(\partial/\partial t)U(t, s)$. The result about the first and last terms of the right member of (2.8) are easily obtained, i.e.

$$\|A(s) \exp ((t-s)A(s))-A(s) \exp ((\tau-s)A(s))\| \leqq K \frac{t-\tau}{(t-s)(\tau-s)}, \tag{3.3}$$

$$\begin{aligned} &\|\exp ((t-s)A(s))R(t, s)-\exp ((\tau-s)A(s))R(\tau, s)\| \\ &\leqq K\left(\frac{t-\tau}{t-s}+(t-\tau) \log \frac{t-s}{t-\tau}+\log \frac{t-s}{\tau-s}\right). \end{aligned} \tag{3.4}$$

As for the second term,

$$\begin{aligned} &\int_s^t A(\sigma) \exp ((t-\sigma)A(\sigma))(R(\sigma, s)-R(t, s))d\sigma \\ &-\int_s^\tau A(\sigma) \exp ((\tau-\sigma)A(\sigma))(R(\sigma, s)-R(\tau,s))d\sigma \\ &=\int_\tau^t A(\sigma) \exp ((t-\sigma)A(\sigma))(R(\sigma, s)-R(t, s))d\sigma \\ &-\int_s^\tau A(\sigma) \exp ((t-\sigma)A(\sigma))(R(t, s)-R(\tau, s))d\sigma \\ &+\int_s^\tau \{A(\sigma) \exp ((t-\sigma)A(\sigma))-A(\sigma) \exp ((\tau-\sigma) A(\sigma))\} \\ &(R(\sigma, s)-R(\tau, s))d\sigma . \end{aligned}$$

It is easy to see that

$$\left\|\int_\tau^t A(\sigma)\exp((t-\sigma)A(\sigma))(R(\sigma, s)-R(t, s))d\sigma\right\| \leqq K\left\{\frac{t-\tau}{t-s}+(t-\tau)\left(\log\frac{t-s}{t-\tau}+1\right)\right\},$$

$$\begin{aligned} &\left\|\int_s^\tau A(\sigma) \exp ((t-\sigma)A(\sigma))(R(t, s)-R(\tau, s))d\sigma\right\| \\ &\leqq K\left(\frac{t-\tau}{t-s}+(t-\tau) \log \frac{t-s}{t-\tau}\right)\log\frac{t-s}{t-\tau}. \end{aligned}$$

Using $\log x \leqq x^\mu/\mu$ for $0<\mu<1$ and $x>1$, we get

$$\begin{aligned} &\left\|\int_s^\tau \{A(\sigma) \exp ((t-\sigma)A(\sigma))-A(\sigma) \exp ((\tau-\sigma)A(\sigma))\} (R(\sigma, s)-R(\tau, s))d\sigma\right\| \\ &\leqq K\int_s^\tau \frac{t-\tau}{(t-\sigma)(\tau-\sigma)}\left(\frac{\tau-\sigma}{\tau-s}+(\tau-\sigma) \log \frac{\tau-s}{\tau-\sigma}\right)d\sigma \\ &\leqq K\frac{t-\tau}{\tau-s}\log\frac{t-s}{t-\tau}+K_\mu(t-\tau)^{1-\mu}(\tau-s)^\mu , \end{aligned}$$

with an arbitrary constant $\mu$ such that $0<\mu<1$.

Finally, we estimate the difference of the third term of (2.8):

$$\int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \sigma}\right)\exp((t-\sigma)A(\sigma))R(t, s)d\sigma$$

$$-\int_s^\tau\left(\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \sigma}\right)\exp((\tau-\sigma)A(\sigma))R(\tau, s)d\sigma$$

$$(3.5)\quad =\int_\tau^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \sigma}\right)\exp((t-\sigma)A(\sigma))R(t, s)d\sigma$$

$$+\int_s^\tau\left\{\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \sigma}\right)\exp((t-\sigma)A(\sigma))-\left(\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \sigma}\right)\exp((\tau-\sigma)A(\sigma))\right\}d\sigma\cdot R(t, s)$$

$$+\int_s^\tau\left(\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \sigma}\right)\exp((\tau-\sigma)A(\sigma))d\sigma(R(t, s)-R(\tau, s)).$$

Using the easily verified fact that $(\partial/\partial r+\partial/\partial\sigma)\exp((r-\sigma)A(\sigma))$ has a strongly continuous first derivative in $r$ satisfying

$$\left\|\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial \sigma}\right)\exp((r-\sigma)A(\sigma))\right\|\leqq\frac{K}{t-\sigma},$$

we see that the norm of the second term of the right member of (3.5) is bounded by $K\{-(t-\tau)\log(t-\tau)+(t-\tau)\log(t-s)+(t-\tau)\}$. Estimating remaining terms in (3.5) in a obvious way, we find that the norm of the left member of (3.5) is bounded by $K(t-\tau)\{1+\log(t-s)(t-\tau)^{-1}\}$. Summing up, we obtain

$$(3.6)\qquad \left\|\frac{\partial}{\partial t}W(t, s)-\frac{\partial}{\partial t}W(\tau, s)\right\|\leqq K_\mu(t-\tau)^{1-\mu}(t-s)^\mu(\tau-s)^{-1}$$
$$\text{for } s<\tau<t,$$

where $\mu$ is an arbitrary constant such that $0<\mu<1$. Thus, we are led to

**Theorem 3.1.** *The first derivative in t of the fundamental solution $U(t, s)$ of (0.1′) with $B(t)\equiv 0$ satisfies in $a\leqq s<t\leqq b$*

$$(3.7)\qquad \left\|\frac{\partial}{\partial t}U(t, s)-\frac{\partial}{\partial \tau}U(\tau, s)\right\|\leqq K\frac{t-\tau}{(t-s)(\tau-s)}+K_\mu\frac{(t-\tau)^{1-\mu}(t-s)^\mu}{\tau-s},$$

*where $\mu$ is an arbitrary constant such that* $0<\mu<1$.

We proceed to the solution of the inhomogeneous equation (0.1) with a right member $f(t)$ satisfying

$$(3.8)\qquad \|f\|_0+\|f\|_\gamma=\sup_{a\leqq t\leqq b}\|f(t)\|+\sup_{a\leqq s,t\leqq b}\frac{\|f(t)-f(s)\|}{|t-s|^\gamma}<\infty,$$
$$0<\gamma<1.$$

First, we note that we can estimate the difference of each term of the right member of

$$(3.9)\quad \frac{\partial}{\partial t}\int_s^t \exp((t-\sigma)A(\sigma))f(\sigma)d\sigma = \int_s^t A(\sigma)\exp((t-\sigma)A(\sigma))(f(\sigma)-f(t))d\sigma$$
$$+\int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial\sigma}\right)\exp((t-\sigma)A(\sigma))f(t)d\sigma+\exp((t-s)A(s))f(t)$$

just in the same way as we did for $(\partial/\partial t)\int_s^t \exp((t-\sigma)A(\sigma))R(\sigma, s)d\sigma$, using (2.8) instead of (3.2), and the result is

$$\left\|\frac{\partial}{\partial t}\int_s^t \exp((t-\sigma)A(\sigma))f(\sigma)d\sigma-\frac{\partial}{\partial\tau}\int_s^\tau \exp((\tau-\sigma)A(\sigma))f(\sigma)d\sigma\right\|$$
$$\leqq K\left\{(t-\tau)\left(1+\log\frac{t-s}{t-\tau}\right)+\log\frac{t-s}{\tau-s}\right\}\|f\|_0+K_{\gamma'}\|f\|_\gamma(t-\tau)^{\gamma'},$$

where $\gamma'$ is an arbitrary positive number less than $\gamma$. It is easy to see that

$$\left\|\frac{\partial}{\partial t}\int_s^t W(t, \sigma)f(\sigma)d\sigma-\frac{\partial}{\partial\tau}\int_s^\tau W(\tau, \sigma)f(\sigma)d\sigma\right\|\leqq K(t-\tau)\|f\|_0+K_\mu(t-\tau)^{1-\mu}\|f\|_\gamma,$$

with $0<\mu<1$. Hence, we obtain

**Theorem 3.2.** *For $f(t)$ satisfying* (3.8), *we have*

$$\left\|\frac{\partial}{\partial t}\int_s^t U(t, \sigma)f(\sigma)d\sigma-\frac{\partial}{\partial\tau}\int_s^\tau U(\tau, \sigma)f(\sigma)d\sigma\right\|$$
$$\leqq K(t-\tau)\left(\frac{1}{\tau-s}+\log\frac{t-s}{t-\tau}\right)\|f\|_0+K_{\gamma'}(t-\tau)^{\gamma'}\|f\|_\gamma,$$

*where $\gamma'$ is an arbitrary constant such that $0<\gamma'<\gamma$.*

§ **4. Higher derivatives of the solution.** In this section, we assume also that

4.1°. $A(t)A(s)^{-1}$ is twice and $B(t)A(s)^{-1}$ is once continuously differentiable in $t$ for each $s$.

4.2°. $B'(t)A(s)^{-1}=(\partial/\partial t)(B(t)A(s)^{-1})$ is Hölder-continuous in $t$ in the uniform operator topology:

$$(4.1)\qquad \|B'(t)A(s)^{-1}-B'(\tau)A(s)^{-1}\|\leqq C_3|t-\tau|^\lambda.$$

4.3°. $B'(t)$ satisfies the following inequalities:

$$(4.2)\qquad \|B'(t)\exp(\tau A(s))\|\leqq C_4\tau^{\rho-1}$$

(4.3) $$\|(B'(t)-B'(r))\exp(\tau A(s))\| \leqq C_5|t-r|^{\lambda}\tau^{\rho-1}, \quad 0<\lambda<1,$$

where $\rho$ is the same constant as in (1.2).

In this section, we denote by $K$ constants which depend on the constants appearing in (4.1)-(4.3) besides the ones in Hypotheses 1.1°-2.3° and (1.4). The meaning of $K_\alpha$ etc. would be clear.

**Lemma 4.1.** *For $a \leqq s < t \leqq b$ and $m=1, 2, \cdots$,*

(4.4) $$\left\|\frac{\partial}{\partial t}R_1(t, s)\right\| \leqq K(t-s)^{\rho-2}, \quad \left\|\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)R_m(t, s)\right\| \leqq K_m(t-s)^{m\rho-1}$$

Proof. The first inequality follows immediately from

$$\frac{\partial}{\partial t}R_1(t, s) = (A(t)+B(t)-A(s))A(s)\exp((t-s)A(s)) \\ +(A'(t)+B'(t))\exp((t-s)A(s)).$$

The second one for $m=1$ is a direct consequence of

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)R_1(t, s) = (A(t)+B(t)-A(s))\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)\exp((t-s)A(s)) \\ +(A'(t)+B'(t)-A'(s))\exp((t-s)A(s)).$$

For general $m$, it can be proved by induction noting

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)R_m(t, s) = \int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)R_{m-1}(t, \tau)R_1(\tau, s)d\tau \\ +\int_s^t R_{m-1}(t, \tau)\left(\frac{\partial}{\partial \tau}+\frac{\partial}{\partial s}\right)R_1(\tau, s)d\tau.$$

**Lemma 4.2.** *For $a \leqq s < t \leqq b$, we have*

(4.5) $$\left\|\frac{\partial}{\partial t}\int_s^t R_m(t, \sigma)d\sigma\right\| \leqq K_m(t-s)^{m\rho-1}.$$

Proof. For sufficiently small positive $h$, we have

$$\frac{\partial}{\partial t}\int_s^{t-h} R_m(t, \sigma)d\sigma = R_m(t, t-h)+\int_s^{t-h}\frac{\partial}{\partial t}R_m(t, \sigma)d\sigma \\ = \int_s^{t-h}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \sigma}\right)R_m(t, \sigma)d\sigma+R_m(t, s).$$

Letting $h \downarrow 0$, we get

(4.6) $$\frac{\partial}{\partial t}\int_s^t R_m(t, \sigma)d\sigma = \int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \sigma}\right)R_m(t, \sigma)d\sigma+R_m(t, s).$$

The inequality (4.5) follows immediately from (4.6) and the preceding lemma.

**Lemma 4.3.** *For $a \leqq s < t \leqq b$, and $m=1, 2, \cdots$, we have*

$$\left\| \frac{\partial}{\partial t} R_m(t, s) \right\| \leqq K_m (t-s)^{m\rho-2}, \quad \left\| \frac{\partial}{\partial t} R(t, s) \right\| \leqq K(t-s)^{\rho-2}. \tag{4.7}$$

Proof. The above inequality can be proved by induction using

$$\frac{\partial}{\partial t} R_m(t, s) = \int_s^t \frac{\partial}{\partial t} R_{m-1}(t, \sigma)(R_1(\sigma, s) - R_1(t, s)) d\sigma$$

$$+ \frac{\partial}{\partial t} \int_s^t R_{m-1}(t, \sigma) d\sigma R_1(t, s) \quad \text{for } m \text{ not very large,}$$

$$\frac{\partial}{\partial t} R_m(t, s) = \int_s^t \frac{\partial}{\partial t} R_{m-1}(t, \sigma) R_1(\sigma, s) d\sigma \quad \text{for } m \text{ sufficiently large.}$$

**Lemma 4.4.** *For $a \leqq s < b$, we have with any $\alpha$, $0 < \alpha < 1$,*

$$\left\| \frac{\partial}{\partial t} R_1(t, s) - \frac{\partial}{\partial \tau} R_1(\tau, s) \right\| \leqq K \left\{ \frac{(t-\tau)^\lambda}{(t-s)^{1-\rho}} + \frac{t-\tau}{(t-s)(\tau-s)^{2-\rho}} \right\}, \tag{4.8}$$

$$\left\| \frac{\partial}{\partial t} R(t, s) - \frac{\partial}{\partial \tau} R(\tau, s) \right\| \tag{4.9}$$

$$\leqq K_\alpha \left\{ \frac{(t-\tau)^\lambda}{(\tau-s)^{1-\rho}} + \frac{t-\tau}{(t-s)(\tau-s)^{2-\rho}} + \frac{(t-\tau)^\alpha}{(\tau-s)^{2+\alpha-2\rho}} \right\},$$

$$\left\| \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) R_1(t, s) - \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial s} \right) R_1(\tau, s) \right\| \tag{4.10}$$

$$\leqq K \left\{ \frac{(t-\tau)^\lambda}{(t-s)^{1-\rho}} + \frac{t-\tau}{(t-s)(\tau-s)^{1-\rho}} \right\},$$

$$\left\| \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) R(t, s) - \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial s} \right) R(\tau, s) \right\| \tag{4.11}$$

$$\leqq K_\alpha \left\{ \frac{(t-\tau)^\lambda}{(t-s)^{1-\rho}} + \frac{t-\tau}{(t-s)(\tau-s)^{1-\rho}} + \frac{(t-\tau)^\rho}{(\tau-s)^{1-\rho}} + \frac{(t-\tau)^\alpha}{(\tau-s)^{1+\alpha-2\rho}} \right\}.$$

Proof. (4.8) and (4.10) are easily obtained. (4.9) follows from

$$\frac{\partial}{\partial t} R_m(t, s) - \frac{\partial}{\partial \tau} R_m(\tau, s) = \int_\tau^t \frac{\partial}{\partial t} R_{m-1}(t, \sigma)(R_1(\sigma, s) - R_1(t, s)) d\sigma$$

$$+ \int_s^\tau \left( \frac{\partial}{\partial t} R_{m-1}(t, \sigma) - \frac{\partial}{\partial \tau} R_{m-1}(\tau, \sigma) \right) (R_1(\sigma, s) - R_1(\tau, s)) d\sigma$$

$$+ \left( \frac{\partial}{\partial t} \int_s^t R_{m-1}(t, \sigma) d\sigma - \frac{\partial}{\partial \tau} \int_s^\tau R_{m-1}(\tau, \sigma) d\sigma \right) R_1(\tau, s)$$

$$+ \left( \frac{\partial}{\partial t} \int_s^t R_{m-1}(t, \sigma) d\sigma \right) (R_1(t, s) - R_1(\tau, s)),$$

for not very large $m$, and

$$\frac{\partial}{\partial t} R_m(t, s) - \frac{\partial}{\partial \tau} R_m(\tau, s) = \int_s^\tau \left( \frac{\partial}{\partial t} R_{m-1}(t, \sigma) - \frac{\partial}{\partial \tau} R_{m-1}(\tau, \sigma) \right) R_1(\sigma, s) d\sigma$$

$$+ \int_\tau^t \frac{\partial}{\partial t} R_{m-1}(t, \sigma) R_1(\sigma, s) d\sigma, \text{ for sufficiently large } m.$$

(4.11) can be shown similary.

Under the assumption 4.1°–4.3°, $(\partial/\partial t)U(t, s)$ can be expressed as follows:

$$\frac{\partial}{\partial t} U(t, s) = A(s) \exp((t-s)A(s)) + \int_s^{s_1} A(\tau) \exp((t-\tau)A(\tau)) R(\tau, s) d\tau$$

$$(4.12) \quad + \int_{s_1}^t \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) \exp((t-\tau)A(\tau)) R(\tau, s) d\tau + \int_{s_1}^t \exp((t-\tau)A(\tau)) \frac{\partial}{\partial t} R(\tau, s) d\tau$$

$$+ \exp((t-s_1)A(s_1)) R(s_1, s), \quad (s < s_1 < t).$$

The derivatives in $t$ of the first, second and last terms of the right member can be obtained easily. As for those of the remaining terms, we can write them down in a form similar to (2.8) noticing Lemma 4.4. We can estimate all those derivatives using the above lemmas. Thus, we obtain

**Theorem 4.1.** *Under the assumptions* 4.1°–4.3°; *the fundamental solution $U(t, s)$ has a second derivative in $t$ satisfying*

$$(4.13) \qquad \left\| \frac{\partial^2}{\partial t^2} U(t, s) \right\| \leq \frac{K}{(t-s)^2}.$$

Next, we consider the differentiability of the solution of the inhomogeneous equation. Let us assume that

4.4°. $f(t)$ has a Hölder continuous first derivative in $a \leq t \leq b$.

**Lemma 4.5.** *The following inequalities hold*:

$$(4.14) \quad \left\| \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) W(t, s) \right\| \leq K(t-s)^\rho, \quad \left\| \frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) W(t, s) \right\| \leq K(t-s)^{\rho-1}.$$

Proof. The first one is a direct consequence of

$$\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) W(t, s) = \int_s^t \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) \exp((t-\tau)A(\tau)) R(\tau, s) d\tau$$

$$+ \int_s^t \exp((t-\tau)A(\tau)) \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial s} \right) R(\tau, s) d\tau.$$

Using Lemma 4.4, we can express $(\partial/\partial t)(\partial/\partial t + \partial/\partial s) W(t, s)$ explicitly and obtain the second inequality of (4.14).

Now, we show the existence of the second derivatives of both terms of the right member of

$$\int_s^t U(t, \sigma)f(\sigma)d\sigma = \int_s^t \exp((t-\sigma)A(\sigma))f(\sigma)d\sigma + \int_s^t W(t, \sigma)f(\sigma)\, d\sigma .$$

As for the first term, we can express the derivative explicitly as we did for $\int_s^t \exp((t-\sigma)A(\sigma))R(\sigma, s)d\sigma$. But, in the present case we need not divide the interval $[s, t]$ into two parts as we did for the latter, so the computation is a little simpler this time. As for the second term, we rewrite its first derivative in the following form:

$$\begin{aligned}\frac{\partial}{\partial t}\int_s^t W(t, \sigma)f(\sigma)d\sigma &= \int_s^t\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial\sigma}\right)W(t, \sigma)f(\sigma)d\sigma + W(t, s)f(s)\\ &\quad + \int_s^t W(t, \sigma)\frac{df(\sigma)}{d\sigma}\, d\sigma .\end{aligned}$$

Hence, by Lemma 4.6 we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int_s^t W(t, \sigma)f(\sigma)d\sigma &= \int_s^t \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial\sigma}\right)W(t, \sigma)f(\sigma)d\sigma + \frac{\partial}{\partial t}\, W(t, s)f(s)\\ &\quad + \int_s^t \frac{\partial}{\partial t}\, W(t, \sigma)\, \frac{df(\sigma)}{d\sigma}\, d\sigma .\end{aligned}$$

**Theorem 4.2.** *Under the assumption* 4.1°–4.4°, *the solution $x(t)$ of the inhomogeneous equation* (0.1) *is twice continuously differentiable. Furthermore, for each $t$, $dx(t)/dt \in \mathfrak{D}$ and $A(t)dx(t)/dt$ is continuous.*

Proof. The first half of the theorem has already been proved. The last half is a direct consequence of

$$\begin{aligned}A(t)\,\frac{x(t+h)-x(t)}{h} &= \frac{1}{h}\left\{\frac{d}{dt}\,x(t+h) - \frac{d}{dt}\,x(t)\right\}\\ &\quad - \frac{1}{h}\,(A(t+h)-A(t))x(t+h) - \frac{1}{h}\,(f(t+h)-f(t))\end{aligned}$$

and the closedness of $A(t)$.

**Theorem 4.3.** *In addition to the assumption of the preceding theorem, if we assume that for each $t$, $f(t) \in \mathfrak{D}$ and for some $r$, $A(r)f(t)$ is continuous in $t$, then the solution $x(t)$ of the inhomogeneous equation* (0.1) *satisfies that $A(t)x(t) \in \mathfrak{D}$ for each $t$ and $A(t)^2x(t)$ is continuous in $t$.*

Proof. Under the assumption of the theorem, $A(t)f(t) = A(t)A(r)^{-1}$

$A(r)f(t)$ is also continuous in $t$ and the right member of $A(t)x(t)=dx(t)/dt-f(t)$ belongs to $\mathfrak{D}$ by the preceding theorem. By applying the above theorem to $U(t, s)$, we obtain

**Theorem 4.4.** *Under the assumption* 4.1°–4.3°, *the range of* $(\partial/\partial t)U(t, s)(=A(t)U(t, s))$ *is contained in* $\mathfrak{D}$ *for each* $s$ *and* $t$ *and* $A(t)(\partial/\partial t)U(t, s)(=A(t)^2U(t, s))$ *is strongly continuous in* $t$ *in* $s<t\leqq b$. *Furthermore, it satisfies*

$$\|A(t)^2U(t, s)\|=\|A(t)(\partial/\partial t)U(t, s)\|\leqq\frac{K}{(t-s)^2}.$$

§ 5. **Generalized solution.** We denote by $\mathfrak{B}_p[a, b]$ the set of all strongly measurable $\mathfrak{X}$-valued functions whose norms are $L^p$-summable in $[a, b]$.

After Solomiak [4], we give the following definition.

Definition. $x(t)$ is said to be a generalized solution of (0.1) of class $\mathfrak{B}_p(1<p<\infty)$ if it satisfies the following conditions:

1. For $a<t<b$, $x(t)$ is absolutely continuous and belongs to $\mathfrak{D}$.
2. $x(t)$ is strongly differentiable almost everywhere.
3. $dx(t)/dt$ and $A(t)x(t)$ belongs to $\mathfrak{B}_p[a, b]$.

For a given $f(t)$ in $\mathfrak{B}_p[a, b]$, we put

$$\omega(\sigma\,; f)=\left(\int_a^{b-\sigma}\|f(t+\sigma)-f(t)\|^p dt\right)^{1/p}$$

**Theorem 5.1.** *If* $f(t)\in\mathfrak{B}_p[a, b]$ *and if* $\int_0^c\frac{\omega(\sigma\,; f)}{\sigma}d\sigma<\infty$ *for any finite number* $c>0$, *then* $\int_s^t U(t, \sigma)f(\sigma)d\sigma$ *gives the solution of* (0.1) *of class* $\mathfrak{B}_p$ *in* $s<t<b$ *with vanishing initial value for any* $s$.

Proof. We have only to estimate each term of the right member of (3.9) and $\int_s^t\frac{\partial}{\partial t}W(t, \sigma)f(\sigma)d\sigma$ for a smooth $f(t)$. We begin with the first term of (3.9). Let $\varphi(t)$ be any element in $\mathfrak{B}'_{p'}[a, b]$ where $1/p+1/p'=1$ and $\mathfrak{X}$ is replaced by its conjugate space $\mathfrak{X}'$.

$$\int_s^b\left(\int_s^t A(\sigma)\exp\,((t-\sigma)A(\sigma))(f(\sigma)-f(t))d\sigma,\ \varphi(t)\right)dt$$
$$=\int_s^b\int_0^{t-s}\left\{A(t-\sigma)\exp\,(\sigma A(t-\sigma))(f(t-\sigma)-f(t)),\ \varphi(t)\right\}d\sigma dt$$
$$=\int_0^{b-s}\int_{\sigma+s}^b\left\{A(t-\sigma)\exp\,(\sigma A(t-\sigma))(f(t-\sigma)-f(t)),\ \varphi(t)\right\}dt d\sigma.$$

Hence, the absolute value of the left member is not larger than

$$\int_0^{b-s}\int_{\sigma+s}^{b}\frac{C}{\sigma}\|f(t-\sigma)-f)t)\|\ \|\varphi(t)\|\,dtd\sigma$$

$$\leqq\int_0^{b-s}\frac{C}{\sigma}\Big(\int_{\sigma+s}^{b}\|f(t-\sigma)-f(t)\|^{p}dt\Big)^{1/p}\Big(\int_{\sigma+s}^{b}\|\varphi(t)\|^{p'}dt\Big)^{1/p'}d\sigma$$

$$\leqq C\int_0^{b}\frac{\omega(\sigma\,;f)}{\sigma}\,d\sigma\Big(\int_s^{b}\|\varphi(t)\|^{p'}dt\Big)^{1/p'}.$$

Thus, we obtain

$$\Big(\int_s^b\Big\|\int_s^t A(\sigma)\exp\,((t-\sigma)A(\sigma))(f(\sigma)-f(t))d\sigma\Big\|^p dt\Big)^{1/p}$$

$$\leqq C\int_0^b\frac{\omega(\sigma\,;f)}{\sigma}\,d\sigma\,.$$

In a similar way, we get

$$\Big(\int_s^b\Big\|\int_s^t\frac{\partial}{\partial t}W(t,\sigma)f(\sigma)d\sigma\Big\|^p dt\Big)^{1/p}\leqq\frac{K(b-s)^\rho}{\rho}\Big(\int_s^b\|f(t)\|^p dt\Big)^{1/p}.$$

The estimation of the remaining terms is trivially obtained.

## § 6. The equation in which $A(t)$ has a variable domain.

Let us weaken slightly the assumption of the independence of $\mathfrak{D}(A(t))$ of $t$ and assume that $\mathfrak{D}(A(t))$ changes smoothly with $t$ in the sense of T. Kato [2]. Namely, we assume that $A(t)$ satisfies Hypotheses 1.1° and 1.4° in §1 and the following slightly weaker Hypotheses than 1.2° and 1.3°.

**Hypoteses** 6.1°. There exists a bounded operator-valued function $R(t)$ with a bounded inverse and with a Hölder continuous first derivative $\dot{R}(t)$ in $a\leqq t\leqq b$ such that the operator $\bar{A}(t)=R(t)A(t)R(t)^{-1}$ has a domain $\mathfrak{D}$ independent of $t$.

6.2°. The bounded operator $\bar{B}(t,s)=[I-\bar{A}(t)][I-\bar{A}(s)]^{-1}$ is uniformly bounded and satisfies Lipschitz condition in $t$ for each $s$ in the uniform operator topology.

6.3°. $\bar{B}(t,s)$ is strongly continuously differentiable in $t$ for each $s$.

Next, we assume that $B(t)$ satisfies

6.4°. $\mathfrak{D}(B(t))$ contains $\mathfrak{D}(A(t))$ for each $t$.

6.5°. $B(t)R(t)^{-1}R(s)A(s)^{-1}$ is continuous in $a\leqq t\leqq b$ for each $s$ in the uniform operator topology.

6.6°. There exist positive constants $C_3, C_4$, $\rho\leqq 1$ and $\lambda\leqq 1$ such that

$$\|B(t)R(t)^{-1}R(s)\exp\,(\tau A(s))\|\leqq C_3\tau^{-(1-\rho)}\,, \tag{6.1}$$

$$\|(B(t')R'(t')^{-1}-B(t)R(t)^{-1})R(s)\exp\,(\tau A(s))\|\leqq C_4|t'-t|^{\lambda}\tau^{-(1-\rho)}\,, \tag{6.2}$$

for $a \leqq t,\ t',\ s \leqq b$ and $\tau > 0$.

Evidently, $R(t)$ maps $\mathfrak{D}(A(t))$ onto $\mathfrak{D}$ in a one-to-one fashion. If we transform $x(t)$ into $\bar{x}(t) = R(t)x(t)$ in (0.1′), we obtain

$$d\bar{x}(t)/dt = \bar{A}(t)\bar{x}(t) + \bar{B}(t)\bar{x}(t), \tag{6.3}$$

where

$$\bar{B}(t) = R(t)B(t)R(t)^{-1} + \dot{R}(t)R(t)^{-1}. \tag{6.4}$$

$\bar{B}(t)$ is closed because it is the sum of a closed operator and a bounded one. By $\exp(t\bar{A}(s)) = R(s)\exp(tA(s))R(s)^{-1}$ and the uniform boundedness of $R(t)$, $\dot{R}(t)$ and $R(t)^{-1}$, it is easily seen that Hypotheses 1.1°~2.3° are all satisfied by $\bar{A}(t)$ and $\bar{B}(t)$ instead of $A(t)$ and $B(t)$ respectively replacing the constants $M$, $C$ etc. by other suitable ones if necessary. Hence, there is a unique fundamental solution $\bar{U}(t, s)$ of (6.3):

$$\begin{cases} \dfrac{\partial}{\partial t}\bar{U}(t, s) = (\bar{A}(t) + \bar{B}(t))\bar{U}(t, s), \\ \bar{U}(s, s) = I, \end{cases} \tag{6.5}$$

satisfying

$$\begin{aligned} &\|(\partial/\partial t)\bar{U}(t, s)\| \leqq K(t-s)^{-1}, \\ &\|\bar{A}(t)\bar{U}(t, s)\| \leqq K(t-s)^{-1}, \\ &\|\bar{B}(t)\bar{U}(t, s)\| \leqq K(t-s)^{\rho-1}, \end{aligned} \tag{6.6}$$

$$\partial(\bar{U}(t, s)x)/\partial s = -\bar{U}(t, s)(\bar{A}(s) + \bar{B}(s))x \quad \text{for any } x \in \mathfrak{D}. \tag{6.7}$$

$\bar{U}(t, s)$ has the following form

$$\bar{U}(t, s) = \exp((t-s)\bar{A}(s)) + \bar{W}(t, s), \tag{6.8}$$

where $\bar{W}(t, s)$ satisfies

$$\begin{aligned} &\|\bar{W}(t, s)\| \leqq K(t-s)^{\rho}, \quad \|(\partial/\partial t)\bar{W}(t, s)\| \leqq K(t-s)^{\rho-1}, \\ &\|\bar{A}(t)\bar{W}(t, s)\| \leqq K(t-s)^{\rho-1}, \quad \|\bar{B}(t)\bar{W}(t, s)\| \leqq K(t-s)^{2\rho-1}. \end{aligned}$$

The fundamental solution $U(t, s)$ of the original solution (0.1′) is easily seen to be given by

$$U(t, s) = R(t)^{-1}\bar{U}(t, s)R(s). \tag{6.9}$$

Let $x$ by any element in $\mathfrak{D}(A(s))$. Then,

$$\begin{aligned} &h^{-1}\{U(t, s+h) - U(t, s)\}x \\ &\quad = R(t)^{-1}h^{-1}\{\bar{U}(t, s+h)R(s+h) - \bar{U}(t, s)R(s)\}x \\ &\quad = R(t)^{-1}\bar{U}(t, s+h)h^{-1}\{R(s+h) - R(s)\}x + R(t)^{-1}h^{-1}\{\bar{U}(t, s+h) - \bar{U}(t, s)\}R(s)x. \end{aligned}$$

By $R(s)x \in \mathfrak{D}$, (6.7) and (6.4), we get

$$\begin{aligned}&\lim_{h\to 0} h^{-1}\{U(t, s+h)-U(t, s)\}x \\ &= R(t)^{-1}\bar{U}(t, s)\dot{R}(s)x - R(t)^{-1}\bar{U}(t, s)(\bar{A}(s)+\bar{B}(s))R(s)x \\ &= -U(t, s)(A(s)+B(s))x .\end{aligned}$$

Thus, we obtain

**Theorem 6.1.** *Under Hypotheses* 1.1°, 1.4° *and* 6.1°~6.6°, *there exists a unique fundamental solution* $U(t, s)$ *of* (0.1′), *defined in* $a \leqq s \leqq t \leqq b$ *with the following properties*:

(6.10) $U(t, s)$ *is strongly continuous in* $a \leqq s \leqq t \leqq b$,

(6.11) $\begin{cases} (\partial/\partial t)U(t, s),\ A(t)U(t, s) \text{ and } B(t)U(t, s) \text{ are bounded for} \\ a \leqq s < t \leqq b \text{ and they satisfy} \\ \left\| \dfrac{\partial}{\partial t} U(t, s) \right\| \leqq \dfrac{K}{t-s},\ \|A(t)U(t, s)\| \leqq \dfrac{K}{t-s},\ \|B(t)U(t, s)\| \leqq \dfrac{K}{(t-s)^{1-\rho}}, \end{cases}$

(6.12) $\begin{cases} \text{if we set } U(t, s) = R(t)^{-1}R(s)\exp((t-s)A(s)) + W(t, s), \\ \text{then } W(t, s) \text{ satisfies} \\ \|W(t, s)\| \leqq K(t-s)^{\rho},\ \|(\partial/\partial t)W(t, s)\| \leqq K(t-s)^{\rho-1}, \\ \text{and} \\ \|B(t)W(t, s)\| \leqq K(t-s)^{2\rho-1}. \end{cases}$

(6.13) $\begin{cases} \text{if } x \in \mathfrak{D}(A(s)), \text{ then } U(t, r)x \text{ is strongly differentiable} \\ \text{in } r \text{ at } r = s \text{ and} \\ \dfrac{\partial}{\partial s} U(t, s)x = -U(t, s)(A(s)+B(s))x . \end{cases}$

**Theorem 6.2.** *If* $f(t)$ *satisfies one of the following conditions*:

(6.14) $f(t)$ *is Hölder continuous*,

*or*

(6.15) $f(t) \in \mathfrak{D}(A(t))$, *and* $A(t)f(t)$ *and* $f(t)$ *are strongly cotinuous*,

*then the solution* $x(t)$ *of the inhomogeneous equation* (0.1) *corresponding to the initial data* $x(s)=x$ *is given by*

$$x(t) = U(t, s)x + \int_s^t U(t, \sigma)f(\sigma)d\sigma . \tag{6.16}$$

Proof. In fact, the replacement $\bar{x}(t) = R(t)x(t)$ and $\bar{f}(t) = R(t)f(t)$ transforms (0.1) into

$$(6.17) \qquad d\bar{x}(t)/dt = (\bar{A}(t)+\bar{B}(t))\bar{x}(t)+\bar{f}(t).$$

As $\bar{f}(t)$ satisfies one of the following conditions: $\bar{f}(t)$ is Hölder continuous; or $\bar{f}(t)\in\mathfrak{D}$, and $\bar{A}(t)\bar{f}(t)$ and $\bar{f}(t)$ are strongly continuous, in accordance with the assumption about $f(t)$, $\bar{x}(t)$ is given by

$$\bar{x}(t) = \bar{U}(t, s)R(s)x+\int_s^t \bar{U}(t, \sigma)\bar{f}(\sigma)d\sigma.$$

If we return to the original notation, we obtain (6.16).

For the solution of the equation considered in this section, we could easily deduce those results similar to the ones proved in the previous sections under Hypotheses 1.1°~2.3°.

(Received March 13, 1960)

### References

[1] T. Kato: Integration of the equation of evolution in a Banach space, J. Math. Soc. Japan **5** (1953), 208–234.

[2] T. Kato: On linear differential equations in Banach spaces, Comm. Pure Appl. Math. **9** (1956), 479–486.

[3] R. S. Phillips: Perturbation theory for semi-groups of linear operators, Trans. Amer. Math. Soc. **74** (1954), 199–221.

[4] M. Z. Solomiak: The application of the semi-group theory to the study of differential equations in Banach spaces, Doklady Acad. Nauk, SSSR **122** (1958), 766–769.

[5] H. Tanabe: A class of the equations of evolution in a Banach space, Osaka Math. J. **11** (1959), 121–145.