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GLOBAL FUCHSIAN GOURSAT PROBLEM IN THE CLASS OF HOLOMORPHIC-GEVREY FUNCTIONS

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Abstract

The first and the third authors have proved the global existence of holomorphic solutions to partial differential equations with several Fuchsian variables in the sense of N.S. Madi, under some assumptions on some coefficients and on the Fuchsian characteristic polynomial. This article shows a similar global existence of solutions which are holomorphic with respect to Fuchsian variables and of projective Gevrey class with respect to non-Fuchsian variables. The proof is based on the concept of formal norms of Leray-Waelbroeck and the application of Gevrey's operator of C. Wagschal.

1. Introduction

Consider an ordinary differential equation

$$\frac{d^m u}{dt^m} + a_1(t)\frac{d^{m-1}u}{dt^{m-1}} + a_{m-1}(t)\frac{du}{dt} + a_m(t) = 0 \tag{1.1}$$

near the origin of the complex plane $\mathbb{C}$. Let the equation have a singular point $t = 0$ for the coefficients $a_l(t)$ $(1 \leq l \leq m)$. The classical Fuchs's theorem indicates that the necessary and sufficient condition for the origin $t = 0$ to be a *regular singular* point is that "$t = 0$ is a pole of a_l of multiplicity less than or equal to l" (see, for example, [5]). A very important generalization of this result to partial differential equations has been given by Baouendi and Goulaouic in 1973 ([1]). They considered partial differential operators of the form

$$t^k D_t^m + \sum_{j=0}^{m-1} \sum_{|\beta|\leq m-j} a_{j,\beta}(t,x) D_t^j D_y^\beta,$$
$$a_{j,\beta}(t,x) = t^{\max\{k-m+j,0\}}\tilde{a}_{j,\beta}(t,x), \quad \tilde{a}_{j,\beta}(0,x) = 0 \quad \text{if} \quad \beta \neq 0$$

called "Fuchsian partial differential operator with weight $m-k$", with the initial hypersurface $S = \{(t,x) \in \mathbb{C}^{n+1}; t = 0\}$. Using the concept of Fuchsian characteristic polynomial (or indicial polynomial), they gave a necessary and sufficient condition to establish results of Cauchy-Kowalevski type and of Holmgren type.

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This definition of Fuchsian partial differential operators opened a way to a large number of works ([18], [13], [4], [9], [8], [26] to name a few from the viewpoint of this article).

In [12], N.S. Madi introduced the notion of linear partial differential operators with several Fuchsian variables and gave a sufficient condition to solve the local Goursat problem in the holomorphic class. In [2], we have studied a uniqueness problem in a space of Schwartz distributions for operators with several Fuchsian variables. A global Cauchy-Kowalevski result has been shown in [3], for this type of operators.

In the present work, we study the global Goursat problem for operators with several Fuchsian variables in the class of holomorphic functions with respect to Fuchsian variables and Gevrey class with respect to the other variables. The used classes are the projective Gevrey classes, already used in the noncharacteristic Cauchy problem ([7]). To show the existence and uniqueness of the solution, we impose some conditions on the Fuchsian characteristic polynomial, which allow us to invert some operator in transforming our problem into a problem of a fixed point. We use techniques based on the concept of formal norms of Leray-Waelbroeck [10] and the application of 'Gevrey's operator' ([25]), which transforms the convergent formal series to Gevrey series, in order to introduce a family of Banach spaces where we choose some parameters to control the norms of the used operators.

We point out that H. Tahara studied intensively the characteristic Cauchy problems, in particular those for Fuchsian operators in the class of C^∞ functions ([18]–[24]). In these studies, he put the hyperbolicity condition, classically imposed in the non-characteristic case. Another work is in preparation for the study of this type of problem without the hyperbolicity.

We use the following basic notation.

$$\mathbb{N} = \{0, 1, 2, \dots\}, \quad \mathbb{N}^* = \mathbb{N} \setminus \{0\}, \quad \mathbb{Z} = \{\text{the integers}\},$$
$$\mathbb{R}_+ = \{\text{nonnegative real numbers}\}, \quad \mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}.$$

2. Definitions and results

Let $\mathscr{U}$ be an open set in $\mathbb{C}^n$ and Ω be an open neighborhood of the origin in $\mathbb{R}^q$. Let $\mathcal{C}^{\omega,\infty}(\mathscr{U} \times \Omega)$ denote the algebra of functions $f(x, y)$ holomorphic in x on $\mathscr{U}$ and of class $\mathcal{C}^\infty$ in y on Ω.

DEFINITION 2.1 ([7]). For $d \geq 1$, let $G^{(\omega,d)}(\mathscr{U} \times \Omega)$ denote the sub-algebra of functions $u \in \mathcal{C}^{\omega,\infty}(\mathscr{U} \times \Omega)$ such that for any $h > 0$, there exists a constant $C_h > 0$ satisfying

$$\forall \delta \in \mathbb{N}^q, \quad \sup_{\mathscr{U} \times \Omega} \left| D_y^\delta u(x, y) \right| \leq C_h h^{|\delta|} |\delta|!^d. \tag{2.1}$$

Let $\mathcal{G}^{(\omega,d)}(\mathscr{U} \times \Omega)$ denote the set of functions u such that $u \in G^{(\omega,d)}(\mathscr{U}' \times \Omega')$ for any relatively compact $\mathscr{U}' \Subset \mathscr{U}$ and $\Omega' \Subset \Omega$, where $A \Subset B$ means that the closure $\overline{A}$ is compact and $\overline{A} \subset B$. This space is called a *projective Gevrey class* with index d in the second set of variables.

REMARK 2.1. Let $d = 1$. From the inequality (2.1), we have that for any $h > 0$, there exists a positive constant C_h such that

$$\forall x \in \mathscr{U}, \quad \forall \delta \in \mathbb{N}^q, \quad \sup_{y \in \Omega} \left| D_y^\delta u(x,y) \right| \le C_h h^{|\delta|} |\delta|!,$$

which implies that $u(x, \cdot)$ can be extended entirely to $\mathbb{C}^q$. Thus, $\mathcal{G}^{(\omega,1)}(\mathscr{U} \times \Omega)$ is the set of all analytic functions on $\mathscr{U} \times \mathbb{C}^q$.

Let $d \ge 1$ and let us consider an operator $\mathcal{P}$ on $\mathbb{C}^n \times \Omega$ defined by

$$\mathcal{P} = \sum_{|\alpha|+d|\beta| \le m} a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta, \quad \text{where} \quad a_{\alpha,\beta} \in \mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega). \tag{2.2}$$

DEFINITION 2.2. (1) For multiindices $\alpha, \beta \in \mathbb{Z}^n$, we write $\alpha \le \beta$ and say that α is smaller than or equal to β, if $\alpha_i \le \beta_i$ $(\forall i)$.
(2) We say that the weight of monomial $a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta$ with respect to x is smaller than or equal to $\tau \in \mathbb{Z}^n$, if there exists a function $\overline{a}_{\alpha,\beta}(x,y) \in \mathcal{C}^{\omega,\infty}(\mathbb{C}^n \times \Omega)$ such that

$$a_{\alpha,\beta}(x,y) = x^{\alpha-\tau} \overline{a}_{\alpha,\beta}(x,y) \quad \text{with} \quad \alpha - \tau \in \mathbb{N}^n,$$

which is denoted by

$$a_{\alpha,\beta}(x,y) = \mathcal{O}(x^{\alpha-\tau}).$$

DEFINITION 2.3 ([12]). A partial differential operator $\mathcal{P}$ defined by (2.2) is called of Fuchs type (or Fuchsian) with weight $\mu \in \mathbb{N}^n$ with respect to x, if all weights of monomials $a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta$ with respect to x are smaller than or equal to μ, and if for any $\beta \ne 0$, the following holds.
(i) The weight of the monomial $a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta$, with respect to x are strictly smaller than μ.
(ii) If $\mu_i \ne 0$, the weight with respect to x_i of monomial $a_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta$, is strictly smaller than μ_i.

From this definition, there exists $\widetilde{a}_{\alpha,0} \in \mathcal{C}^{\omega,\infty}(\mathbb{C}^n \times \Omega)$ such that

$$a_{\alpha,0}(x,y) = x^{[\alpha-\mu]_+} \widetilde{a}_{\alpha,0}(x,y), \tag{2.3}$$

where $[s]_+ = \max\{s, 0\}$ for $s \in \mathbb{R}$, and $[\alpha]_+ = ([\alpha_1]_+, \dots, [\alpha_n]_+)$ for $\alpha \in \mathbb{R}^n$.

DEFINITION 2.4. Let $\mathcal{P}$ be a Fuchsian operator defined by (2.2). Its Fuchsian characteristic polynomial $\widetilde{Q}$ is defined by

$$\widetilde{Q}(k,y) = \sum_{\substack{|\alpha|\le m\\ \alpha\ge\mu}} \widetilde{a}_{\alpha,0}(0,y)\mathcal{C}_{\alpha-\mu}(k), \quad k\in\mathbb{N}^n, \quad y\in\Omega,$$

where $\mathcal{C}_0(h)=1$, $\mathcal{C}_p(h)=\prod_{l=0}^{p-1}(h-l)$ for $(p,h)\in\mathbb{N}^*\times\mathbb{C}$, and $\mathcal{C}_\alpha(k)=\prod_{i=1}^n \mathcal{C}_{\alpha_i}(k_i)$ for $(\alpha,k)\in\mathbb{N}^n\times\mathbb{C}^n$.

Since we consider the existence of global solutions, it would be natural to assume that *some* coefficients of the operator are polynomials in y. The order of $a(x,y)\in\mathcal{C}^\omega(\mathbb{C}^n)[y]$ with respect to y is denoted by $\mathrm{ord}_y\, a$. By definition, $\mathrm{ord}_y\, a<0$ means $a\equiv 0$.

In this paper, we assume

$\mathcal{H}_0$) For all $\alpha\ge\mu$, we have the following.

— If $|\alpha|<m$, $\widetilde{a}_{\alpha,0}(0,y)=\widetilde{a}_\alpha\in\mathbb{C}$.

— If $|\alpha|=m$, $\widetilde{a}_{\alpha,0}(x,y)=\widetilde{a}_\alpha\in\mathbb{C}$.

Especially, $\widetilde{Q}(k,y)=\widetilde{Q}(k)$ do not depend on y.

$\mathcal{H}_1$) For all (α,β) such that "$\alpha\ge\mu$ with $\beta\ne 0$ or $|\alpha|\le|\mu|$" and $m-1<|\alpha|+d|\beta|\le m$, the coefficients $a_{\alpha,\beta}(x,y)$ are polynomials in y with $\mathrm{ord}_y\, a_{\alpha,\beta}<|\beta|$, whose coefficients are entire functions of x.

$\mathcal{H}_2$) For all (α,β) such that $\alpha\not\ge\mu$ and $|\alpha|>|\mu|$, we have the following.

— If $m-|\mu|-1<|\alpha|+d|\beta|\le m-|\mu|$, then $a_{\alpha,\beta}(x,y)$ is a polynomial in y such that $\mathrm{ord}_y\, a_{\alpha,\beta}<|\beta|$.

— If $|\alpha|+d|\beta|>m-|\mu|$, then $a_{\alpha,\beta}\equiv 0$.

DEFINITION 2.5 ([12]). We say that $\mathcal{P}$ satisfies the condition (A), if there exists a constant $C>0$ such that:

$$\text{(A)}\qquad \forall k\in\mathbb{N}^n, \qquad \left|\widetilde{Q}(k)\right|\ge C(|k|+1)^{m-|\mu|}.$$

The Goursat problem associated with $\mathcal{P}$ is to find a function u satisfying the following equations.

$$\begin{cases}\mathcal{P}u=f,\\ u-w=\mathcal{O}(x^\mu).\end{cases} \tag{2.4}$$

Theorem 2.1. *Let $d\ge 1$. Under the hypotheses $\mathcal{H}_0$)–$\mathcal{H}_2$), if $\mathcal{P}$ is a partial differential operator of Fuchs type with weight μ with respect to x, and if it satisfies the condition* (A), *then for any f and w in $\mathcal{G}^{(\omega,d)}(\mathbb{C}^n\times\Omega)$, the problem* (2.4) *admits a unique solution $u\in\mathcal{G}^{(\omega,d)}(\mathbb{C}^n\times\Omega)$.*

REMARK 2.2. According to Remark 2.1, if $d = 1$ then the result of this theorem is included in the main theorem given in [3].

3. Transformation of the problem

In order to give a proof of the theorem, we first transform the problem.

If u is a solution of (2.4), there exists a function $v \in \mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$ such that $u(x,y) - w(x,y) = x^\mu v(x,y)$. By changing the unknown function from u to v, we get the equivalence of our problem with the following differential equation.

$$\mathcal{P}_1 v(x,y) = g(x,y), \tag{3.1}$$

where

$$\begin{cases} \mathcal{P}_1 v(x,y) = \mathcal{P}\big(x^\mu v\big)(x,y) \\ \text{and} \\ g(x,y) = f(x,y) - \mathcal{P}w(x,y). \end{cases}$$

Proposition 3.1 ([3]). *If $\mathcal{P}$ is a Fuchsian operator defined by* (2.2), *then*

$$\mathcal{P}_1[v(x,y)] = Q[v(x,y)] + \sum_{\substack{|\alpha|+d|\beta|\le m \\ \alpha \ge \mu}} x^{\alpha-\mu} \left(\sum_{i=1}^n x_i \tilde{a}^{(i)}_{\alpha,\beta}(x,y) \right) D_x^\alpha D_y^\beta [x^\mu v(x,y)]$$
$$+ \sum_{\substack{|\alpha|+d|\beta|\le m \\ \alpha \not\ge \mu}} x^{[\alpha-\mu]_+} \tilde{a}_{\alpha,\beta}(x,y) D_x^\alpha D_y^\beta [x^\mu v(x,y)],$$

where $\tilde{a}^{(i)}_{\alpha,\beta}, \tilde{a}_{\alpha,\beta} \in \mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$, and

$$Q(k) = \widetilde{Q}(k) C_\mu(k+\mu),$$
$$Q = Q(xD_x) = \sum_{\substack{|\alpha|\le m \\ \alpha\ge\mu}} \tilde{a}_\alpha \mathcal{C}_{\alpha-\mu}(xD_x)\mathcal{C}_\mu(xD_x+\mu).$$

For $R > 0$, we set

$$B_R = \left\{ x \in \mathbb{C}^n;\ \max_{1\le i\le n} |x_i| < R \right\}$$

and

$$G^{(\omega,d)}_{\text{loc}}(B_R \times \Omega) = \left\{ u \in \mathcal{C}^{\omega,\infty}(B_R \times \Omega); \forall r \in]0,R[,\ u \in G^{(\omega,d)}(B_r \times \Omega) \right\}.$$

Proposition 3.2. *If the operator* $\mathcal{P}$ *satisfies the condition* (A), *then the operator* Q *is an automorphism of the Fréchet space* $G^{(\omega,d)}_{\mathrm{loc}}(B_R \times \Omega)$*, and for* $u = \sum_{k\in\mathbb{N}^n} D^k_x u(0,y)x^k/k! \in G^{(\omega,d)}_{\mathrm{loc}}(B_R \times \Omega)$*, we have*

$$Q^{-1}u = \sum_{k\in\mathbb{N}^n} \frac{D^k_x u(0,y)}{Q(k)} \frac{x^k}{k!}. \tag{3.2}$$

Proof. STEP 1. Let us prove that Q is an endomorphism in $G^{(\omega,d)}_{\mathrm{loc}}(B_R \times \Omega)$.

Take an arbitrary r such that $0 < r < R$, and take s as $r < s < R$. There exists a constant $C_1 \colon 1 < C_1 < s/r$.

If $u \in G^{(\omega,d)}_{\mathrm{loc}}(B_R \times \Omega)$, then for every $h > 0$, there exists a constant $C^0_h \geq 0$ such that

$$\forall \delta \in \mathbb{N}^q, \quad \sup_{B_s\times\Omega} \left| D^\delta_y u(x,y)\right| \leq C^0_h h^{|\delta|}|\delta|!^d.$$

By Cauchy's formula in the ball B_s, we have

$$\forall k \in \mathbb{N}^n, \quad \left| D^k_x D^\delta_y u(0,y)\right| \leq \frac{k!}{s^{|k|}} C^0_h h^{|\delta|}|\delta|!^d. \tag{3.3}$$

It is well-known that

$$\lim_{|k|\to+\infty} \frac{(|k|+|\mu|)^m}{C_1^{|k|}} = 0,$$

which implies that there exists a positive constant $A_0 > 0$ such that

$$\forall k \in \mathbb{N}^n, \quad (|k|+|\mu|)^m \leq A_0 C_1^{|k|}. \tag{3.4}$$

By using the expression of Fuchsian characteristic polynomial, we obtain

$$\begin{aligned}\forall k \in \mathbb{N}^n, \quad |Q(k)| &\leq \sum_{\substack{|\alpha|\leq m\\ \alpha\geq\mu}} |\widetilde{a}_\alpha| \prod_{i=1}^{n} \prod_{l=0}^{\alpha_i-\mu_i-1} (k_i - l) \prod_{l=0}^{\mu_i-1} (k_i+\mu_i-l)\\ &\leq \sum_{\substack{|\alpha|\leq m\\ \alpha\geq\mu}} |\widetilde{a}_\alpha| (|k|+|\mu|)^{|\alpha|}.\end{aligned}$$

From (3.4), we have

$$\forall k \in \mathbb{N}^n, \quad |Q(k)| \leq \sum_{\substack{|\alpha|\leq m\\ \alpha\geq\mu}} |\widetilde{a}_\alpha| A_0 C_1^{|k|} = C_2 C_1^{|k|}, \tag{3.5}$$

where $C_2 = \sum_{|\alpha|\leq m, \alpha\geq\mu} |\widetilde{a}_\alpha| A_0$.

On the other hand, we have

$$D_y^\delta(Qu(x,y)) = Q\left(D_y^\delta u(x,y)\right) = \sum_{k\in\mathbb{N}^n} Q(k) D_x^k D_y^\delta u(0,y)\frac{x^k}{k!}, \quad \forall\delta\in\mathbb{N}^q.$$

By the estimate (3.3) and (3.5), we have

$$\tag{3.6} \begin{aligned} &\forall\delta\in\mathbb{N}^q, \quad \forall(x,y)\in B_r\times\Omega,\\ &\left|D_y^\delta Qu(x,y)\right| \le \sum_{k\in\mathbb{N}^n} |Q(k)|\frac{|x^k|}{s^{|k|}} C_h^0 h^{|\delta|}|\delta|!^d \le \sum_{k\in\mathbb{N}^n}|Q(k)|\left(\frac{r}{s}\right)^{|k|} C_h^0 h^{|\delta|}|\delta|!^d\\ &\qquad\qquad\qquad \le \sum_{k\in\mathbb{N}^n} C_2 C_1^{|k|}\left(\frac{r}{s}\right)^{|k|} C_h^0 h^{|\delta|}|\delta|!^d\\ &\qquad\qquad\qquad \le C_3 C_h^0 h^{|\delta|}|\delta|!^d, \end{aligned}$$

with another constant C_3, because $C_1 r/s < 1$. By substituting this result in (3.6), we get

$$\forall\delta\in\mathbb{N}^q, \quad \forall(x,y)\in B_r\times\Omega, \quad \left|D_y^\delta Qu(x,y)\right| \le C_h h^{|\delta|}|\delta|!^d$$

with $C_h = C_3 C_h^0$.

Therefore, Q is an endomorphism in $G_{\mathrm{loc}}^{(\omega,d)}(B_R\times\Omega)$.

STEP 2. Let us show that Q is bijective in $G_{\mathrm{loc}}^{(\omega,d)}(B_R\times\Omega)$.

If $g\in G_{\mathrm{loc}}^{w,d}(B_R\times\Omega)$, then we have

$$g(x,y) = \sum_{k\in\mathbb{N}^n}\frac{D_x^k g(0,y)}{k!}x^k,$$

and for every $s\in\,]0,R[$ and $h>0$, there exists a positive constant $C_h^{(0)}>0$ such that

$$\forall\delta\in\mathbb{N}^q, \quad \sup_{B_s\times\Omega}\left|D_y^\delta g(x,y)\right| \le C_h^{(0)} h^{|\delta|}|\delta|!^d.$$

According to the definition of the operator Q and the condition (A), the following series

$$u(x,y) = \sum_{k\in\mathbb{N}^n}\frac{D_x^k g(0,y)}{Q(k)}\frac{x^k}{k!}$$

is a unique solution of equation $Qu = g$. Thus, it is enough to prove that $u\in G_{\mathrm{loc}}^{w,d}(B_R\times\Omega)$.

From Cauchy's formula, for all $s\in\,]0,R[$ and $h>0$, we have

$$\tag{3.7} \forall(k,\delta)\in\mathbb{N}^n\times\mathbb{N}^q, \quad \forall y\in\Omega, \quad \left|D_x^k D_y^\delta g(0,y)\right| \le C_h^{(0)} h^{|\delta|}|\delta|!^d\frac{k!}{s^{|k|}}.$$

For an arbitrary fixed $r \in]0, R[$, we take s as $r < s < R$. By using the result (3.7) and the expression of the solution u, we have $\forall h > 0$, $\exists C_h^{(0)} > 0$ such that

$$
\begin{aligned}
\forall \delta \in \mathbb{N}^n, \quad \forall y \in \Omega, \quad \left| D_y^\delta u(x, y) \right| &\leq \sum_{k \in \mathbb{N}^n} \frac{\left| D_x^k D_y^\delta g(0, y) \right|}{|Q(k)|} \frac{|x^k|}{k!} \\
&\leq \sum_{k \in \mathbb{N}^n} \frac{C_h^{(0)} h^{|\delta|} |\delta|!^d}{|Q(k)|} \frac{k!}{s^{|k|}} \frac{|x^k|}{k!}.
\end{aligned} \tag{3.8}
$$

Since there exists a positive constant C satisfying

$$\forall k \in \mathbb{N}^n, \quad |Q(k)| > C,$$

we get from (3.8)

$$\forall \delta \in \mathbb{N}^n, \quad \forall y \in \Omega, \quad \left| D_y^\delta u(x, y) \right| \leq \frac{1}{C} C_h^{(0)} h^{|\delta|} |\delta|!^d \sum_{k \in \mathbb{N}^n} \frac{|x^k|}{s^{|k|}}$$

which implies that $\forall h > 0$, $\exists C_h > 0$ such that

$$\forall \delta \in \mathbb{N}^q, \quad \sup_{B_r \times \Omega} \left| D_y^\delta u(x, y) \right| \leq C_h h^{|\delta|} |\delta|!^d,$$

where $C_h = (1/C) C_h^{(0)} \sup_{x \in B_r} \sum_{k \in \mathbb{N}^n} |x^k| / s^{|k|} < \infty$. Thus, we have $u \in G_{\text{loc}}^{(\omega,d)}(B_R \times \Omega)$. $\square$

By this proposition, we easily get the following corollary.

Corollary 3.1. *Q is an automorphism of $\mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$.*

This corollary allows us to transform the problem along the same steps used in the holomorphic case ([3]), which gives the equivalence between the problem (2.4) and the following equation in $\mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$.

$$u(x, y) = (\mathcal{A} + \mathcal{B}) u(x, y) + g(x, y), \tag{3.9}$$

where

$$\mathcal{Q} = Q^{-1}$$

and

$$\mathcal{B}v(x,y) = \sum_{\substack{|\alpha|+d|\beta|\le m\\ \alpha\ge\mu}} x^{(\alpha-\mu)}\left(\sum_{i=1}^{n} x_i \tilde{a}^{(i)}_{\alpha,\beta}(x,y)\right) D_x^{\alpha} D_y^{\beta} x^{\mu} \mathcal{Q}v(x,y),$$

$$\mathcal{A}v(x,y) = \sum_{\substack{|\alpha|+d|\beta|\le m\\ \alpha\ngeq\mu}} x^{[\alpha-\mu]_+} \tilde{a}_{\alpha,\beta}(x,y) D_x^{\alpha} D_y^{\beta} x^{\mu} \mathcal{Q}v(x,y).$$

Actually, we will first solve the equation (3.9) in another space $G^{(\omega,d)}(\Omega_T)$ for an arbitrary T, which gives a solution in $\mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$.

4. Holomorphic-Gevrey formal series

We recall some definitions and properties given by C. Wagschal in [25]. Let $\mathbb{R}_+[[x]]$ denote the set of formal power series in x whose coefficients are all non-negative.

Consider a formal series $\Phi \in \mathbb{R}_+[[x,y]]$, where we can write

$$\Phi(x,y) = \sum_{\delta\in\mathbb{N}^q} \phi_\delta(x)\frac{y^\delta}{\delta!}, \quad \phi_\delta \in \mathbb{R}_+[[x]]. \tag{4.1}$$

We assume that there exists an open neighborhood $\mathscr{U}$ of the origin in $\mathbb{C}^n$ such that all the series ϕ_δ converge in $\mathscr{U}$.

DEFINITION 4.1 (Wagschal [25]). (1) For $u = \sum_{\alpha\in\mathbb{N}^n} u_\alpha x^\alpha \in \mathbb{C}[[x]]$ and $\phi = \sum_{\alpha\in\mathbb{N}^n} \phi_\alpha x^\alpha \in \mathbb{R}_+[[x]]$, we write $u \ll \phi$ if $|u_\alpha| \le \phi_\alpha$ for every $\alpha \in \mathbb{N}^n$. This is the usual majorant series in x.
(2) If $\Psi(x,y) = \sum_{\delta\in\mathbb{N}^q} \psi_\delta(x) y^\delta/(\delta!)$ is another formal series and has the same properties as Φ in (4.1), then we write $\Phi \ll \Psi$ if

$$\forall\delta \in \mathbb{N}^q, \quad \phi_\delta \ll \psi_\delta \quad \text{(in the sense of majorant series in } x\text{)}.$$

(3) For a function $u \in \mathcal{C}^{\omega,\infty}(\mathscr{U} \times \Omega)$, we write

$$u \lll \Phi,$$

if

$$\forall\delta \in \mathbb{N}^q,\ \forall y \in \Omega,\ D_y^\delta u(x,y) \ll \phi_\delta(x) \quad \text{(in the sense of majorant series in } x\text{)}. \tag{4.2}$$

In other words,

$$u \lll \Phi \Longleftrightarrow \forall (k,\delta) \in \mathbb{N}^n \times \mathbb{N}^q, \quad \forall y \in \Omega, \quad \left|D_x^k D_y^\delta u(0,y)\right| \le D_x^k \phi_\delta(0). \tag{4.3}$$

We have the following proposition.

Proposition 4.1. *If* $u, v \in \mathcal{C}^{\omega,\infty}(\mathscr{U} \times \Omega)$, *then we have the following.*
1. $u \lll \Phi$ *and* $\Phi \ll \Psi \Longrightarrow u \lll \Psi$.
2. *For* $\lambda, \mu \in \mathbb{C}$, *we have*

$$u \lll \Phi \quad \textit{and} \quad v \lll \Psi \Longrightarrow \lambda u + \mu v \lll |\lambda|\Phi + |\mu|\Psi \quad \textit{and} \quad uv \lll \Phi\Psi.$$

3.

$$u \lll \Phi \Longrightarrow \forall(k, \delta) \in \mathbb{N}^n \times \mathbb{N}^q, \quad D_x^k D_y^\delta u \lll D_x^k D_y^\delta \Phi. \tag{4.4}$$

The following definition is a simple extension of that by C. Wagschal ([25]).

DEFINITION 4.2. For a real $d \geq 1$ and a formal series $\Psi = \sum_{\delta \in \mathbb{N}^q} \psi_\delta y^\delta$, where $\psi_\delta \in \mathbb{R}_+[[x]]$, we set

$$\Psi^d = \sum_{\delta \in \mathbb{N}^q} \psi_\delta |\delta|!^{d-1} y^\delta.$$

Lemma 4.1. *If* Φ *and* Ψ *are in* $\mathbb{R}_+[[x, y]]$, *then we have*:
a. $\Phi \ll \Psi \Longrightarrow \Phi^d \ll \Psi^d$.
b. $\Phi^d \Psi^d \ll (\Phi\Psi)^d$.

Proof. a. It is enough to use Definitions 4.1 and 4.2.
b. Let us consider

$$\Phi(y) = \sum_{\delta \in \mathbb{N}^q} \phi_\delta y^\delta \quad \text{and} \quad \Psi(y) = \sum_{\delta \in \mathbb{N}^q} \psi_\delta y^\delta$$

with $\phi_\delta, \psi_\delta \in \mathbb{R}_+[[x]]$, we have

$$\Phi^d(y)\Psi^d(y) = \sum_{\delta \in \mathbb{N}^q} \left(\sum_{0 \leq \nu \leq \delta} \phi_{\delta-\nu} |\delta - \nu|!^{d-1} \psi_\nu |\nu|!^{d-1} \right) y^\delta.$$

For all $\delta \in \mathbb{N}^q$ and $0 \leq \nu \leq \delta$, we have $|\delta - \nu|!^{d-1}|\nu|!^{d-1} = (|\delta - \nu|)!\,|\nu|!)^{d-1} \leq |\delta|!^{d-1}$, and hence

$$\sum_{0 \leq \nu \leq \delta} \phi_{\delta-\nu} |\delta - \nu|!^{d-1} \psi_\nu |\nu|!^{d-1} \ll \left(\sum_{0 \leq \nu \leq \delta} \phi_{\delta-\nu} \psi_\nu \right) |\delta|!^{d-1}.$$

Thus, from Definitions 4.1 and 4.2, we obtain

$$\Phi^d \Psi^d \ll \sum_{\delta \in \mathbb{N}^q} \left(\sum_{0 \leq \nu \leq \delta} \phi_{\delta-\nu} \psi_\nu \right) |\delta|!^{d-1} y^\delta = (\Phi\Psi)^d. \qquad \square$$

Let τ and ξ be one-dimensional variables. We set

$$\varphi_R^a(\xi) = \frac{e^{a\xi}}{R-\xi} \in \mathbb{R}_+[[\xi]].$$

Lemma 4.2. *Let $p,q \in \mathbb{N}$ and $\nu = \lfloor dp \rfloor$, which is the smallest integer larger than or equal to dp. Then, we have*:

1. $D^p\varphi_R^a(\xi) \ll a^{-q}\ D^{p+q}\varphi_R^a(\xi)$.
2. $D^p\varphi_R^a(\xi) \ll R^{-q}(p!/(p+q)!)D^{p+q}\varphi_R^a(\xi)$.
3. $D^p[(D^q\varphi_R^a)^d(\xi)] \ll R^{\nu-p}(D^{q+\nu}\varphi_R^a)^d(\xi)$.

Proof. 1 and 2 are proved in holomorphic case ([3]).

3. We know that for every ξ such that $|\xi| < R$, we have

$$\varphi_R^a(\xi) = \frac{e^{a\xi}}{R-\xi} = \frac{1}{R}\sum_{n=0}^{+\infty}\frac{a^n}{n!}\xi^n\sum_{n=0}^{+\infty}\frac{1}{R^n}\xi^n = \frac{1}{R}\sum_{n=0}^{+\infty}\left(\sum_{j=0}^{n}\frac{a^j}{j!}\frac{1}{R^{n-j}}\right)\xi^n.$$

Hence, by differentiating this function q times, we get

$$D^q\varphi_R^a(\xi) = \frac{1}{R}\sum_{n=0}^{+\infty}\left(\sum_{j=0}^{n+q}\frac{a^j}{j!}\frac{1}{R^{n+q-j}}\right)\frac{\xi^n}{n!}(n+q)!.$$

Then, from Definition 4.2, we have

$$(D^q\varphi_R^a)^d(\xi) = \frac{1}{R}\sum_{n=0}^{+\infty}\left(\sum_{j=0}^{n+q}\frac{a^j}{j!}\frac{1}{R^{n+q-j}}\right)\frac{\xi^n}{n!}(n+q)!\,(n!)^{d-1},$$

which gives

$$D^p[(D^q\varphi_R^a)^d(\xi)] = \frac{1}{R}\sum_{n=0}^{+\infty}\left(\sum_{j=0}^{n+q+p}\frac{a^j}{j!}\frac{1}{R^{n+q+p-j}}\right)\frac{\xi^n}{n!}(n+q+p)!\,((n+p)!)^{d-1}. \tag{4.5}$$

Similarly, we have

$$(D^{q+\nu}\varphi_R^a)^d(\xi) = \frac{1}{R}\sum_{n=0}^{+\infty}\left(\sum_{j=0}^{n+q+\nu}\frac{a^j}{j!}\frac{1}{R^{n+q+\nu-j}}\right)\frac{\xi^n}{n!}(n+q+\nu)!\,(n!)^{d-1}. \tag{4.6}$$

Since

$$\left(\frac{(n+p)!}{n!}\right)^{d-1} \leq (n+p)^{p(d-1)}$$

and

$$\forall \nu \geq p, \quad \frac{(n+q+p)!}{(n+q+\nu)!} = \frac{1}{(n+q+p+1)\cdots(n+q+p+(\nu-p))} \leq \frac{1}{(n+p)^{\nu-p}},$$

we obtain for all $\nu \geq dp$,

$$\frac{(n+q+p)!}{(n+q+\nu)!}\left(\frac{(n+p)!}{n!}\right)^{d-1} \leq \frac{(n+p)^{p(d-1)}}{(n+p)^{\nu-p}} = \frac{1}{(n+p)^{\nu-dp}} \leq 1. \tag{4.7}$$

If $\nu \geq p$, then we have

$$\begin{aligned} \sum_{j=0}^{n+q+p} \frac{a^j}{j!} \frac{1}{R^{n+q+p-j}} &= R^{\nu-p} \sum_{j=0}^{n+q+p} \frac{a^j}{j!} \frac{1}{R^{n+q+\nu-j}} \\ &\leq R^{\nu-p} \sum_{j=0}^{n+q+\nu} \frac{a^j}{j!} \frac{1}{R^{(n+q+\nu-j)}}, \end{aligned} \tag{4.8}$$

and therefore, by using the estimates (4.7) and (4.8), from (4.5) and (4.6), we obtain our lemma. □

For $R > 0$, we set $\Omega_R = \mathscr{U}_R \times \Omega$, where

$$\mathscr{U}_R = \left\{ x \in \mathbb{C}^n; \sum_{i=1}^{n} |x_i| < R \right\}.$$

5. Banach spaces $\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R)$

Take an integer $s > m+1$ and set $s' = s-1$. For $a, R \in \mathbb{R}_+^*$, we set

$$\Phi_R^a(\tau,\xi) = \sum_{p\in\mathbb{N}} \tau^p R^{s'p} \frac{D^{sp}\varphi_R^a(\xi)}{(sp)!} \in R_+[[\tau,\xi]],$$

which converges in the set $\{(\tau,\xi) \in \mathbb{C}\times\mathbb{R};\ R^{s'/s}|\tau|^{1/s} + |\xi| < R\}$.

This function satisfies the following estimates.

Lemma 5.1 (P. Pongérard [16]). *For every $\eta > 1$ and $R > 0$, we have*:

1. $(\eta R/(\eta R - (\tau+\xi)))\Phi_R^a(\tau,\xi) \ll (\eta/(\eta-1))\Phi_R^a(\tau,\xi)$.
2. $1/(R-(\tau+\xi)) \ll \Phi_R^a(\tau,\xi)$.

From Lemma 4.1, we have the following corollary.

Corollary 5.1. *a*) *For any $\eta > 1$, $(\eta R/(\eta R - (\tau+\xi))\Phi_R^a(\tau,\xi))^d \ll (\eta/(\eta-1))(\Phi_R^a)^d(\tau,\xi)$.*

b) $(1/(R-(\tau+\xi)))^d \ll (\Phi_R^a)^d(\tau,\xi)$.

For all $x=(x_1,\dots,x_n)\in\mathbb{C}^n$ and $y=(y_1,\dots,y_q)\in\mathbb{R}^q$, we set

$$\sigma = x_1+\cdots+x_n,\quad \xi = y_1+\cdots+y_q,\quad \tau=\rho\sigma,$$
$$\Phi^a_{\rho,R}(x,y)=\Phi^a_{\rho R}(\rho\sigma,\xi)=\sum_{p\in\mathbb{N}}\tau^p(\rho R)^{s'p}\frac{D^{sp}\varphi^a_{\rho R}(\xi)}{(sp)!},$$

where ρ is another positive parameter.

By Definition 4.2, we obtain

$$\begin{aligned}\left(\Phi^a_{\rho,R}\right)^d(x,y)&=\sum_{p\in\mathbb{N}}\tau^p(\rho R)^{s'p}\frac{\left(D^{sp}\varphi^a_{\rho R}\right)^d(\xi)}{(sp)!}=\sum_{n\in\mathbb{N}}D^n_\xi\Phi^a_{\rho,R}(x,0)n!^{d-1}\frac{\xi^n}{n!}\\&=\sum_{\delta\in\mathbb{N}^q}D^{|\delta|}_\xi\Phi^a_{\rho,R}(x,0)|\delta|!^{d-1}\frac{y^\delta}{\delta!},\end{aligned}\tag{5.1}$$

where the series $D^{|\delta|}_\xi\Phi^a_{\rho,R}(x,0)|\delta|!^{(d-1)}$ converges in $\mathscr{U}_R$.

From Definition 4.1, we have the following proposition.

Proposition 5.1. *Let $u\in\mathcal{C}^{\omega,\infty}(\Omega_R)$. We have $u(x,y)\lll\left(\Phi^a_{\rho,R}\right)^d(x,y)$ if and only if*

$$\forall\delta\in\mathbb{N}^q,\quad\forall y\in\Omega,\quad D^\delta_y u(x,y)\ll D^{|\delta|}_\xi\Phi^a_{\rho,R}(x,0)|\delta|!^{(d-1)}$$

(*in the sense of majorant series in x*).

We set

$$\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R)=\left\{u\in\mathcal{C}^{\omega,\infty}(\Omega_R);\exists C\geq 0: u(x,y)\lll C\left(\Phi^a_{\rho,R}\right)^d(x,y)\right\}.$$

Proposition 5.2 (Wagschal [25]). *The space $\mathcal{G}^{(\omega,d)}_{R,a,\rho.}(\Omega_R)$ is a Banach space with the norm defined by*

$$\|u\|=\min\left\{C\geq 0; u(x,y)\lll C\left(\Phi^a_{\rho,R}\right)^d(x,y)\right\}$$

Lemma 5.2. *If $R_1>0$ and $f\in G^{(\omega,d)}(B_{R_1}\times\Omega)$, then for every $R\in\,]0,R_1[$ and $\rho>0$, there exists a constant $C>0$ such that*

$$f(x,y)\lll C\left(\frac{\rho R}{\rho R-(\tau+\xi)}\right)^d.$$

Proof. Let $R_1 > 0$ and $f \in G^{(\omega,d)}(B_{R_1} \times \Omega)$. For every $h > 0$, there exists a constant C_h such that

$$\forall \delta \in \mathbb{N}^q, \quad \forall y \in \Omega, \quad \sup_{x \in B_{R_1}} \left| D_y^\delta f(x,y) \right| \le C_h h^{|\delta|} |\delta|!^d. \tag{5.2}$$

By applying Cauchy's formula in the ball B_R with $R \in]0, R_1[$ for the holomorphic function

$$x \mapsto D_y^\delta f(x,y),$$

we obtain as a formal series in x

$$\forall \delta \in \mathbb{N}^q, \quad \forall y \in \Omega, \quad D_y^\delta f(x,y) \ll C_h h^{|\delta|} |\delta|!^d \frac{R}{R-\sigma} = C_h h^{|\delta|} |\delta|!^d \frac{\rho R}{\rho R - \tau}. \tag{5.3}$$

We know that

$$\frac{\rho R}{\rho R - \tau} \gg 1,$$

and hence, by recursion we obtain

$$(\rho R)^{|\delta|} \frac{1}{|\delta|!} D^{|\delta|} \left[\frac{\rho R}{\rho R - \tau} \right] = \left(\frac{\rho R}{\rho R - \tau} \right)^{|\delta|+1} \gg \frac{\rho R}{\rho R - \tau}.$$

By substituting this result in (5.3), we will have

$$\forall \delta \in \mathbb{N}^q, \quad \forall y \in \Omega, \quad D_y^\delta f(x,y) \ll C_h h^{|\delta|} |\delta|!^{d-1} (\rho R)^{|\delta|} D^{|\delta|} \left[\frac{\rho R}{\rho R - \tau} \right]. \tag{5.4}$$

Thus, for h smaller than $1/(\rho R)$, we obtain

$$\forall \delta \in \mathbb{N}^q, \quad \forall y \in \Omega, \quad D_y^\delta f(x,y) \ll C_h D^{|\delta|} \left[\frac{\rho R}{\rho R - \tau} \right] |\delta|!^{d-1}. \tag{5.5}$$

On the other hand, for all (τ, ξ) with $|\tau| + |\xi| < \rho R$, we have

$$\frac{\rho R}{\rho R - (\tau + \xi)} = \sum_{n \in \mathbb{N}} D^n \left[\frac{\rho R}{\rho R - \tau} \right] \frac{\xi^n}{n!}.$$

From Definition 4.2, we have

$$\begin{aligned} \left(\frac{\rho R}{\rho R - (\tau + \xi)} \right)^d &= \sum_{n \in \mathbb{N}} D^n \left[\frac{\rho R}{\rho R - \tau} \right] n!^{d-1} \frac{\xi^n}{n!} \\ &= \sum_{\delta \in \mathbb{N}^q} D^{|\delta|} \left[\frac{\rho R}{\rho R - \tau} \right] |\delta|!^{d-1} \frac{y^\delta}{\delta!}. \end{aligned} \tag{5.6}$$

By Definition 4.1 and the equality (5.6) and the estimate (5.5), we obtain

$$\forall R \in \left]0, R_1\right[, \quad f(x, y) \lll C\left(\frac{\rho R}{\rho R - (\tau + \xi)}\right)^d. \tag{5.7}$$ □

By using the second property of Corollary 5.1 and Lemma 5.2, we have the following.

Corollary 5.2. *If $f \in \mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$, then for all $R, \rho, a \in \mathbb{R}_+^*$ and $\Omega' \Subset \Omega$, we have*

$$f \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega'_R), \quad \textit{where} \quad \Omega'_R = \mathcal{U}_R \times \Omega'.$$

Proposition 5.3. *Let $p \in \mathbb{N}$ and a function F be defined by*

$$F(x, y) = \sum_{|\gamma| \le p} f_\gamma(x) y^\gamma$$

where f_γ are entire functions on $\mathbb{C}^n$. Then, for every $R > 0$ there exists a positive constant $C(R)$ independent of the parameter ρ such that for every $\rho > 0$ with $\rho R \ge$ $\max\{\operatorname{diam}(\Omega), 1\}$, *we have the following.*

$$F(x, y) \lll C(R)(\rho R)^p \left(\frac{\rho R}{\rho R - (\tau + \xi)}\right)^d. \tag{5.8}$$

Proof. By using Cauchy's formula in the polydisk B_R for functions f_γ, we obtain

$$\forall k \in \mathbb{N}^n, \quad \left|D_x^k f_\gamma(0)\right| \le M_\gamma(R) \frac{k!}{R^{|k|}},$$

which gives, for every $(k, \delta) \in \mathbb{N}^{n+q}$ and $y \in \Omega$,

$$\begin{aligned} \left|D_y^\delta D_x^k F(0, y)\right| &= \left| \sum_{|\gamma| \le p,\, \gamma \ge \delta} D_x^k f_\gamma(0) \frac{\gamma!}{(\gamma - \delta)!} y^{\gamma - \delta} \right| \\ &\le \frac{k!}{R^{|k|}} (\rho R)^{p - |\delta|} |\delta|! \sum_{|\gamma| \le p,\, \gamma \ge \delta} \frac{M_\gamma(R) |\gamma|!}{|\delta|!\,(|\gamma| - |\delta|)!} \\ &\le C(R)(\rho R)^p \frac{|\delta|!}{(\rho R)^{|\delta|}} \frac{k!}{R^{|k|}}, \end{aligned}$$

where $M(R) = \sup_{|\gamma| \le p} M_\gamma(R)$ and $C(R) = M(R) \sum_{|\gamma| \le p,\, \gamma \ge \delta} |\gamma|!/(|\delta|!(|\gamma| - |\delta|)!)$. Thus, by Definition 4.1, we get

$$F(x, y) \lll C(R)(\rho R)^p \frac{R}{R - \sigma} \frac{\rho R}{\rho R - \xi}. \tag{5.9}$$

Since we have $|\gamma|!^{d-1} \geq 1$ for all $d \geq 1$, we have

$$\frac{\rho R}{\rho R-\tau}\,\frac{\rho R}{\rho R-\xi} \ll \frac{\rho R}{\rho R-(\tau+\xi)} \ll \left(\frac{\rho R}{\rho R-(\tau+\xi)}\right)^{d}.$$

Therefore, from (5.9), we obtain our result. □

6. Solution of the equation (3.9) in $\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R)$

Since we assumed that the coefficients belong to $\mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$, for every $\Omega' \Subset \Omega$, they belong to $G^{(\omega,d)}(\Omega'_R)$ with $\forall R > 0$ and to $\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega'_R)$ with $\forall R, \rho, a \in \mathbb{R}^*_+$. We may assume that they belong to $G^{(\omega,d)}(\Omega_R)$ and $\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R)$ by changing the notation without loss of generality.

By using the same technique used for the decomposition of operators $\mathcal{A}$ and $\mathcal{B}$ in the holomorphic case ([3]), we obtain the following expressions.

$$\mathcal{B}[u(x,y)] = \sum_{\substack{|\alpha|+d|\beta|\leq m\\ \alpha\geq\mu}} \sum_{i=1}^{n} \tilde{a}^{(i)}_{\alpha,\beta}(x,y)\mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y), \tag{6.1}$$

$$\begin{aligned}\mathcal{A}[u(x,y)] = &\sum_{\substack{|\alpha|+d|\beta|\leq m\\ \alpha\ngeq\mu \text{ and } |\alpha|<|\mu|}} a_{\alpha,\beta}(x,y)\mathcal{A}_{\alpha,\beta}u(x,y)\\ &+ \sum_{\substack{|\alpha|+d|\beta|\leq m\\ \alpha\ngeq\mu \text{ and } |\alpha|=|\mu|}} \sum_{i=1}^{n} x_i a^{(i)}_{\alpha,\beta}(x,y)\mathcal{A}_{\alpha,\beta}u(x,y)\\ &+ \sum_{\substack{|\alpha|+d|\beta|\leq m\\ \alpha\ngeq\mu \text{ and } |\alpha|>|\mu|}} \sum_{i=1}^{n} \tilde{b}^{i}_{\alpha,\beta}(x,y)\mathcal{B}^{i}_{(\alpha,\beta)}u(x,y)\end{aligned} \tag{6.2}$$

with

$$\mathcal{A}_{\alpha,\beta}u(x,y) = \sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \frac{x^{k+\mu-\alpha}}{\widetilde{Q}(k)\mathcal{C}_\mu(k+\mu)}\,\frac{1}{k!}\,\frac{(k+\mu)!}{(k+\mu-\alpha)!}D^\beta_y D^k_x u(0,y) \tag{6.3}$$

and

$$\mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y) = \sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \frac{x^{k+e_i}}{\widetilde{Q}(k)\mathcal{C}_\mu(k+\mu)}\,\frac{1}{k!}\,\frac{(k+\mu)!}{(k+\mu-\alpha)!}D^\beta_y D^k_x u(0,y), \tag{6.4}$$

where $e_i = (0,\ldots,\overset{i}{\hat{1}},\ldots,0)$.

REMARK 6.1. From the hypotheses $\mathcal{H}_1$) and $\mathcal{H}_2$), we obtain that for all (α,β) with $m-1<|\alpha|+d|\beta|\leq m$, the functions $\tilde{a}^{(i)}_{\alpha,\beta}(x,y)$, $a^{(i)}_{\alpha,\beta}(x,y)$ and $a_{\alpha,\beta}(x,y)$ are polynomials in y of order smaller than $|\beta|$ whose coefficients are entire functions in x. It is the same for functions $\tilde{b}^{i}_{\alpha,\beta}(x,y)$ with $m-|\mu|-1<|\alpha|+d|\beta|\leq m-|\mu|$.

Proposition 6.1. *Under the hypotheses* $\mathcal{H}_0$)–$\mathcal{H}_2$), *if the operator* $\mathcal{P}$ *satisfies the condition* (A), *then for every* $R>0$, *there exists* $\rho_0>0$ *such that the following holds.*

$\forall\rho\geq\rho_0$, $\exists a_\rho>0$ *such that for every* $a\geq a_\rho$, *the problem* (3.9) *admits a unique solution* u *in* $\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R)$.

Proof. Let $u(x,y)=\sum_{k\in\mathbb{N}^n}\big(D^k_x u(0,y)/k!\big)x^k\in\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R)$. We have

$$u(x,y)\lll\|u\|\big(\Phi^a_{\rho,R}\big)^d(x,y),$$

and by the property (4.4) given in Proposition 4.1, we have

$$D^\beta_y D^k_x u(x,y)\lll\|u\|D^\beta_y D^k_x\big(\Phi^a_{\rho,R}\big)^d(x,y) \tag{6.5}$$

which implies

$$D^\beta_y D^k_x u(0,y)\lll\|u\|D^\beta_y D^k_x\Big[\big(\Phi^a_{\rho,R}\big)^d(x,y)\Big]_{|x=0}. \tag{6.6}$$

By using the construction of the formal series $\big(\Phi^a_{\rho,R}\big)^d$, we obtain

$$\begin{aligned}
&D^\beta_y D^k_x\Big[\big(\Phi^a_{\rho,R}\big)^d(x,y)\Big]_{|x=0}\\
&=D^\beta_y D^k_x\left[\sum_{p=0}^{+\infty}(\rho(x_1+\cdots+x_n))^p(\rho R)^{s'p}\frac{(D^{sp}\varphi_{\rho R})^d(y_1+\cdots+y_q)}{(sp)!}\right]_{|x=0}\\
&=\sum_{p=|k|}^{+\infty}\frac{p!}{(p-|k|)!}\rho^p(x_1+\cdots+x_n)^{p-|k|}(\rho R)^{s'p}\frac{D^{|\beta|}_\xi[(D^{sp}\varphi_{\rho R})^d(\xi)]}{(sp)!}\Big|_{x=0}\\
&=|k|!\,\rho^{|k|}(\rho R)^{s'|k|}\frac{D^{|\beta|}_\xi[(D^{s|k|}\varphi_{\rho R})^d(\xi)]}{(s|k|)!}.
\end{aligned}$$

By the third estimate of Lemma 4.2, we get

$$D^\beta_y D^k_x\Big[\big(\Phi^a_{\rho,R}\big)^d(x,y)\Big]_{|x=0}\ll|k|!\,\rho^{|k|}(\rho R)^{s'|k|}(\rho R)^{\nu-|\beta|}\frac{(D^{s|k|+\nu}\varphi_{\rho R})^d(\xi)}{(s|k|)!},$$

where $\nu=\lfloor d|\beta|\rfloor$. By substituting this into (6.6), we obtain

$$D^\beta_y D^k_x u(0,y)\lll\|u\|\rho^{|k|}|k|!\,(\rho R)^{s'|k|}(\rho R)^{\nu-|\beta|}\frac{(D^{s|k|+\nu}\varphi_{\rho R})^d(\xi)}{(s|k|)!}. \tag{6.7}$$

Thus, from (6.4), we obtain

$$\begin{aligned}\mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y) \lll \sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \frac{x^{k+e_i}}{|\widetilde{Q}(k)|\mathcal{C}_\mu(k+\mu)}\frac{1}{k!}\frac{(k+\mu)!}{(k+\mu-\alpha)!}\|u\|\rho^{|k|}|k|!\\ \times(\rho R)^{s'|k|}(\rho R)^{\nu-|\beta|}\frac{(D^{s|k|+\nu}\varphi_{\rho R})^d(\xi)}{(s|k|)!}.\end{aligned}\tag{6.8}$$

Lemma 6.1. *Let* $\alpha \geq \mu$ *and* $\nu = \lfloor d|\beta|\rfloor$.
1) *If* $|\alpha|+\nu < m$, *then for all* $u \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R)$, *there exists* $C_{\alpha,\beta}(R,\rho) > 0$ *independent of* a *such that*

$$\mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y) \lll C_{\alpha,\beta}(R,\rho)a^{-1}\|u\|\left(\Phi^a_{\rho,R}\right)^d(x,y).$$

2) *For all* $|\alpha|+\nu = m$, *there exists a positive constant* C_0 *independent of* R, a *and* ρ *such that*

$$\forall u\in\mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R),\quad \mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y)\lll C_0R^{1-|\beta|}\rho^{-|\beta|}\|u\|\left(\Phi^a_{\rho,R}\right)^d(x,y).$$

Proof. 1) By applying the first and the second estimates of Lemma 4.2 and then the majoration a) of Lemma 4.1, we obtain

$$\begin{aligned}(D^{s|k|+\nu}\varphi_{\rho R})^d(\xi) &\ll a^{-1}(D^{s|k|+\nu+1}\varphi_{\rho R})^d(\xi)\\ &\ll a^{-1}(\rho R)^{s-\nu-1}\frac{(s|k|+\nu+1)!}{\{s(|k|+1)\}!}(D^{s(|k|+1)}\varphi_{\rho R})^d(\xi).\end{aligned}$$

By substituting this into (6.8), we obtain

$$\begin{aligned}\mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y) \lll \sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \frac{x^{k+e_i}}{|\widetilde{Q}(k)|\mathcal{C}_\mu(k+\mu)}\frac{1}{k!}\frac{(k+\mu)!}{(k+\mu-\alpha)!}\|u\|\rho^{|k|}|k|!\,(\rho R)^{s'|k|}(\rho R)^{\nu-|\beta|}\\ \times a^{-1}(\rho R)^{s-\nu-1}\frac{(s|k|+\nu+1)!}{(s|k|)!}\frac{(D^{s(|k|+1)}\varphi_{\rho R})^d(\xi)}{\{s(|k|+1)\}!}.\end{aligned}\tag{6.9}$$

It is easy to show that for all $k, k' \in \mathbb{N}^n$ and $i, j\in\mathbb{N}$ such that $j \geq i$ and $k \geq K'$, we have

$$\mathcal{C}_\mu(k+\mu) = \frac{(k+\mu)!}{k!} > 1,\quad \frac{k!}{k'!}\leq\frac{|k|!}{|k'|!}\quad\text{and}\quad\frac{j!}{(j-i)!}\leq j^i.\tag{6.10}$$

By using these properties, we obtain for $\alpha \geq \mu$ and $|\alpha| + \nu < m$

$$\begin{aligned}\mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y) \lll \|u\|\rho^{-1}a^{-1} &\sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \frac{1}{|\widetilde{Q}(k)|}\frac{|k|!}{(|k|-(|\alpha|-|\mu|))!}\rho^{|k|+1}(\rho R)^{s'|k|}x_i\frac{|k|!\,x^k}{k!}\\ &\times(\rho R)^{-|\beta|}(\rho R)^{s-1}\frac{(s|k|+\nu+1)!}{(s|k|)!}\frac{(D^{s(|k|+1)}\varphi_{\rho R})^d(\xi)}{\{s(|k|+1)\}!}\\ \lll \|u\|\rho^{-1}a^{-1} &\sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \frac{1}{|\widetilde{Q}(k)|}|k|^{|\alpha|-|\mu|}\rho^{|k|+1}(\rho R)^{s'(|k|+1)}x_i\frac{|k|!\,x^k}{k!}\\ &\times(\rho R)^{-|\beta|}(s|k|+\nu+1)^{\nu+1}\frac{(D^{s(|k|+1)}\varphi_{\rho R})^d(\xi)}{\{s(|k|+1)\}!}\\ \lll \|u\|\rho^{-1}a^{-1}(\rho R)^{-|\beta|} &\sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \rho^{|k|+1}R^{s'|k|}x_i\frac{|k|!x^k}{k!}\\ &\times\frac{(s|k|+m)^{m-|\mu|}}{|\widetilde{Q}(k)|}\frac{(D^{s(|k|+1)}\varphi_{\rho R})^d(\xi)}{\{s(|k|+1)\}!}.\end{aligned}$$

From the condition (A), there exists a positive constant C_0 such that

$$\frac{(s|k|+m)^{m-|\mu|}}{|\widetilde{Q}(k)|} \leq C_0.$$

From this estimate and the properties

$$\{k\in\mathbb{N}^n; k+\mu-\alpha\geq 0\} \subset \{k\in\mathbb{N}^n; |k|+|\mu|-|\alpha|\geq 0\} \tag{6.11}$$

and

$$\sum_{|k|=p,\, k\in\mathbb{N}^n}\frac{|k|!}{k!}x^k = (x_1+x_2+\cdots+x_n)^p = \sigma^p, \quad x_i \ll \sigma, \tag{6.12}$$

we obtain

$$\begin{aligned}&\mathcal{B}^{(i)}_{(\alpha,\beta)}u(x,y)\\ &\lll \|u\|C_0a^{-1}\rho^{-1}(\rho R)^{-|\beta|}\sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}}\rho^{|k|+1}R^{s'(|k|+1)}x_i\frac{|k|!x^k}{k!}\frac{(D^{s(|k|+1)}\varphi_{\rho R})^d(\xi)}{\{s(|k|+1)\}!}\\ &\lll \|u\|C_0a^{-1}\rho^{-1}(\rho R)^{-|\beta|}\sum_{p+|\mu|-|\alpha|\geq 0}\rho^{p+1}R^{s'(p+1)}\frac{(D^{s(p+1)}\varphi_{\rho R})^d(\xi)}{\{s(p+1)\}!}x_i\sum_{\substack{|k|=p\\ k\in\mathbb{N}^n}}\frac{|k|!x^k}{k!}\\ &\lll \|u\|C_0a^{-1}\rho^{-1}(\rho R)^{-|\beta|}\sum_{p\in\mathbb{N}}\rho^{p+1}\sigma^{p+1}R^{s'(p+1)}\frac{(D^{s(p+1)}\varphi_{\rho R})^d(\xi)}{\{s(p+1)\}!}\end{aligned}$$

$$\lll \|u\| C_0 a^{-1} \rho^{-1} (\rho R)^{-|\beta|} (\Phi^a_{\rho,R})^d (x,y).$$

2) Let $\alpha \geq \mu$ and $\beta \neq 0$ such that $|\alpha| + \nu = m$.

By applying the second estimate of Lemma 4.2 and the first majoration of Lemma 4.1, we obtain

$$\mathcal{B}^{(i)}_{(\alpha,\beta)} u(x,y) \lll \|u\| \sum_{\substack{k+\mu-\alpha\geq 0\\ k\in\mathbb{N}^n}} \frac{1}{\left|\widetilde{Q}(k)\right|} \frac{|k|!}{(|k|+|\mu|-|\alpha|)!} \frac{|k|!}{k!} x_i x^k \rho^{|k|}$$
$$\times (\rho R)^{s'|k|} (\rho R)^{\nu-|\beta|} (\rho R)^{s-\nu} \frac{(s|k|+\nu)!}{(s|k|)!} \frac{(D^{s(|k|+1)}\varphi_{\rho R})^d(\xi)}{\{s(|k|+1)\}!}.$$

By using the condition (A), (6.11) and (6.12), we obtain

$$\mathcal{B}^{(i)}_{(\alpha,\beta)} u(x,y) \lll C_0 \rho^{-1} (\rho R)^{1-|\beta|} \|u\|$$
$$(6.13) \qquad \times \sum_{p\geq|\alpha|-|\mu|} x_i \left(\sum_{\substack{|k|=p\\ k\in\mathbb{N}^n}} \frac{|k|!}{k!} x^k \right) \rho^{p+1} (\rho R)^{s'(p+1)} \frac{(D^{s(p+1)}\varphi_{\rho R})^d(\xi)}{\{s(p+1)\}!}$$
$$\lll C_0 \rho^{-1} \|u\| (\rho R)^{1-|\beta|} \left(\Phi^a_{\rho,R}\right)^d (x,y). \qquad \square$$

Thus, by using the same techniques presented in the proofs of lemmas given in the holomorphic case ([3]), we obtain the following results.

Lemma 6.2. *Let $\alpha \ngeq \mu$ such that $|\alpha| < |\mu|$ and $\beta \in \mathbb{N}^q$.*

1) *If $|\alpha| + \nu < m$, then there exists $C_{(\alpha,\beta)}(R,\rho) > 0$, independent of a, such that*

$$\forall u \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R), \quad \mathcal{A}_{(\alpha,\beta)} u(x,y) \lll C_{(\alpha,\beta)}(R,\rho) a^{-1} \|u\| \left(\Phi^a_{\rho,R}\right)^d (x,y).$$

2) *If $|\alpha| + \nu = m$, then there exists $C_{(\alpha,\beta)}(R)$ which depends only on R such that*

$$\forall u \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R), \quad \mathcal{A}_{(\alpha,\beta)} u(x,y) \lll C_{(\alpha,\beta)}(R) \|u\| R^{1-|\beta|} \rho^{-|\beta|} \left(\Phi^a_{\rho,R}\right)^d (x,y).$$

Lemma 6.3. *Let $\alpha \ngeq \mu$ such that $|\alpha| = |\mu|$ and $\beta \in \mathbb{N}^q$.*

1) *If $|\alpha| + \nu < m$, then there exists $C_{(\alpha,\beta)}(R,\rho) > 0$, independent of a, such that*

$$\forall u \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R), \quad x_i \mathcal{A}_{(\alpha,\beta)} u(x,y) \lll C_{(\alpha,\beta)}(R,\rho) a^{-1} \|u\| \left(\Phi^a_{\rho,R}\right)^d (x,y).$$

2) *If $|\alpha| + \nu = m$, then there exists $C_0 > 0$, independent of a, ρ and R, such that*

$$\forall u \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R), \quad x_i \mathcal{A}_{(\alpha,\beta)} u(x,y) \lll C_0 R^{1-|\beta|} \|u\| \rho^{-|\beta|} \left(\Phi^a_{\rho,R}\right)^d (x,y).$$

Lemma 6.4. *Let $\alpha \ngeq \mu$ such that $|\alpha| > |\mu|$ and $\beta \in \mathbb{N}^q$.*

1) *If* $|\alpha| + \nu < m - |\mu|$, *then there exists* $C_{(\alpha,\beta)}(R,\rho) > 0$, *independent of* a, *such that*

$$\forall u \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R), \quad \mathcal{B}^{(i)}_{(\alpha,\beta)} u(x,y) \lll C_{\alpha,\beta}(R,\rho) a^{-1} \|u\| \left(\Phi^a_{\rho,R}\right)^d (x,y).$$

2) *If* $|\alpha| + \nu = m - |\mu|$, *then there exists* $C_0 > 0$, *independent of* a, ρ *and* R, *such that*

$$\forall u \in \mathcal{G}^{(\omega,d)}_{R,a,\rho}(\Omega_R), \quad \mathcal{B}^{(i)}_{(\alpha,\beta)} u(x,y) \lll C_0 \|u\| \rho^{-|\beta|} R^{1-|\beta|} \left(\Phi^a_{\rho,R}\right)^d (x,y).$$

7. End of the proof of Proposition 6.1

In this section, we take an arbitrary parameter $\eta > 1$ and fix it.

Since the coefficients of the operator $\mathcal{P}$ belong to $\mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$, we can use Lemma 5.2. For every $R > 0$ and $\rho > 0$ satisfying $\eta R \rho > \max\{\text{diam}(\Omega), 1\}$, there exists some constants $\widetilde{M}^i_{\alpha,\beta}(\rho,R)$, $M_{\alpha,\beta}(\rho,R)$, $M^i_{\alpha,\beta}(\rho,R)$, and $C^i_{\alpha,\beta}(\rho,R)$ depending only on ρ and R such that the following holds.

For $|\alpha| + d|\beta| \leq m-1$,

$$\text{if} \quad \alpha \geq \mu, \quad \text{then} \quad \tilde{a}^{(i)}_{\alpha,\beta}(x,y) \lll \widetilde{M}^i_{\alpha,\beta}(\rho,R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau+\xi)}\right)^d, \tag{7.1}$$

if $\alpha \not\geq \mu$ and $|\alpha| < |\mu|$, then

$$a_{\alpha,\beta}(x,y) \lll M_{\alpha,\beta}(\rho,R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau+\xi)}\right)^d, \tag{7.2}$$

if $\alpha \not\geq \mu$ and $|\alpha| = |\mu|$, then

$$a^{(i)}_{\alpha,\beta}(x,y) \lll M^i_{\alpha,\beta}(\rho,R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau+\xi)}\right)^d. \tag{7.3}$$

For $|\alpha| + d|\beta| \leq m - |\mu| - 1$, if $\alpha \not\geq \mu$ and $|\alpha| > |\mu|$, then

$$\tilde{b}^i_{\alpha,\beta}(x,y) \lll C^i_{\alpha,\beta}(\rho,R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau+\xi)}\right)^d. \tag{7.4}$$

By applying the first estimate of Lemmas 6.1–6.4, we obtain

$$\begin{aligned} &\sum_{\substack{|\alpha|+d|\beta|\leq m-1 \\ \alpha\geq\mu}} \sum_{i=1}^n \tilde{a}^{(i)}_{\alpha,\beta}(x,y) \mathcal{B}^{(i)}_{\alpha,\beta} u(x,y) \\ &\lll M_1(\rho,R) a^{-1} \|u\| \left(\frac{\eta\rho R}{\eta\rho R - (\tau+\xi)}\right)^d \left(\Phi^a_{\rho,R}\right)^d (x,y), \end{aligned} \tag{7.5}$$

$$
\begin{aligned}
(7.6)\quad & \sum_{\substack{|\alpha|+d|\beta|\leq m-1 \\ \alpha\not\geq\mu \text{ and } |\alpha|<|\mu|}} a_{\alpha,\beta}(x,y)\mathcal{A}_{\alpha,\beta}u(x,y) \\
& \lll M_2(\rho,R)a^{-1}\|u\| \left(\frac{\eta\rho R}{\eta\rho R-(\tau+\xi)}\right)^d \left(\Phi^a_{\rho,R}\right)^d(x,y),
\end{aligned}
$$

$$
\begin{aligned}
(7.7)\quad & \sum_{\substack{|\alpha|+d|\beta|\leq m-1 \\ \alpha\not\geq\mu \text{ and } |\alpha|=|\mu|}} \sum_{i=1}^{n} a^{(i)}_{\alpha,\beta}(x,y)x_i\mathcal{A}_{\alpha,\beta}u(x,y) \\
& \lll M_3(\rho,R)a^{-1}\|u\| \left(\frac{\eta\rho R}{\eta\rho R-(\tau+\xi)}\right)^d \left(\Phi^a_{\rho,R}\right)^d(x,y),
\end{aligned}
$$

$$
\begin{aligned}
(7.8)\quad & \sum_{\substack{|\alpha|+d|\beta|\leq m-|\mu|-1 \\ \alpha\not\geq\mu \text{ and } |\alpha|>|\mu|}} \tilde{b}^{i_\alpha}_{\alpha,\beta}(x,y)\mathcal{B}^{i_\alpha}_{\alpha,\beta}u(x,y) \\
& \lll M_4(\rho,R)a^{-1}\|u\| \left(\frac{\eta\rho R}{\eta\rho R-(\tau+\xi)}\right)^d \left(\Phi^a_{\rho,R}\right)^d(x,y),
\end{aligned}
$$

with

$$M_1(\rho,R)=C_m \max\left\{\sum_{i=1}^{n}\widetilde{M}^i_{\alpha,\beta}(\rho,R)C_{(\alpha,\beta)}(R,\rho);\, |\alpha|+d|\beta|\leq m-1,\ \alpha\geq\mu\right\},$$

$$M_2(\rho,R)=C_m \max\left\{C_{(\alpha,\beta)}(R,\rho)M_{\alpha,\beta}(\rho,R);\, |\alpha|+d|\beta|\leq m-1,\ \alpha\not\geq\mu,\ |\alpha|<|\mu|\right\},$$

$$M_3(\rho,R)=C_m \max\left\{\sum_{i=1}^{n}M^i_{\alpha,\beta}(\rho,R)C_{(\alpha,\beta)}(R,\rho);\, |\alpha|+d|\beta|\leq m-1,\ \alpha\not\geq\mu,\ |\alpha|=|\mu|\right\},$$

$$M_4(\rho,R)=C_m \max\left\{C^i_{\alpha,\beta}(\rho,R)C_{(\alpha,\beta)}(R,\rho);\, |\alpha|+d|\beta|\leq m-|\mu|-1,\ \alpha\not\geq\mu,\ |\alpha|>|\mu|\right\},$$

where C_m is the number of (α,β) satisfying $|\alpha|+d|\beta|\leq m$.

From the second majoration of Lemma 4.1 and the first of Corollary 5.1, we obtain

$$(7.9)\quad \sum_{\substack{|\alpha|+d|\beta|\leq m-1 \\ \alpha\geq\mu}} \sum_{i=1}^{n} \tilde{a}^{(i)}_{\alpha,\beta}(x,y)\mathcal{B}^{(i)}_{\alpha,\beta}u(x,y) \lll M'_1(\rho,R)a^{-1}\|u\|\left(\Phi^a_{\rho,R}\right)^d(x,y),$$

$$(7.10)\quad \sum_{\substack{|\alpha|+d|\beta|\leq m-1 \\ \alpha\not\geq\mu \text{ and } |\alpha|<|\mu|}} a_{\alpha,\beta}(x,y)\mathcal{A}_{\alpha,\beta}u(x,y) \lll M'_2(\rho,R)a^{-1}\|u\|\left(\Phi^a_{\rho,R}\right)^d(x,y),$$

$$(7.11)\quad \sum_{\substack{|\alpha|+d|\beta|\leq m-1 \\ \alpha\not\geq\mu \text{ and } |\alpha|=|\mu|}} \sum_{i=1}^{n} a^{(i)}_{\alpha,\beta}(x,y)x_i\mathcal{A}_{\alpha,\beta}u(x,y) \lll M'_3(\rho,R)a^{-1}\|u\|\left(\Phi^a_{\rho,R}\right)^d(x,y),$$

$$(7.12)\quad \sum_{\substack{|\alpha|+d|\beta|\leq m-|\mu|-1 \\ \alpha\not\geq\mu \text{ and } |\alpha|>|\mu|}} \tilde{b}^{i}_{\alpha,\beta}(x,y)\mathcal{B}^{i}_{\alpha,\beta}u(x,y) \lll M'_4(R)a^{-1}\|u\|(\Phi^a_{\rho,R})^d(x,y),$$

where $M_l'(\rho, R) = (\eta/(\eta - 1))M_l(\rho, R)$, $l \in \{1, 2, 3, 4\}$.

According to Remark 6.1 and the estimate (5.8) of Proposition 5.3, there exists constants $A^{(i)}_{\alpha,\beta}(R)$, $b_{\alpha,\beta}(R)$, $C^{(i)}_{\alpha,\beta}(R)$, $d^{(i_\alpha)}_{\alpha,\beta}(R)$ independent of ρ and a, such that the following holds. For $m - 1 < |\alpha| + d|\beta| \le m$, we have

$$\forall \alpha \geq \mu : \tilde{a}^{(i)}_{\alpha,\beta}(x, y) \lll (\eta\rho R)^{|\beta|-1} A^{(i)}_{\alpha,\beta}(R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d, \tag{7.13}$$

$$\forall \alpha \ngeq \mu; |\alpha| < |\mu| : a_{\alpha,\beta}(x, y) \lll (\eta\rho R)^{|\beta|-1} b_{\alpha,\beta}(R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d, \tag{7.14}$$

$$\forall \alpha \ngeq \mu; |\alpha| = |\mu| : a^{(i)}_{\alpha,\beta}(x, y) \lll (\eta\rho R)^{|\beta|-1} C^{(i)}_{\alpha,\beta}(R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d, \tag{7.15}$$

and for $m - |\mu| - 1 < |\alpha| + d|\beta| \le m - |\mu|$, we have

$$\forall \alpha \ngeq \mu; |\alpha| > |\mu| : \tilde{b}^{i}_{\alpha,\beta}(x, y) \lll (\eta\rho R)^{|\beta|-1} d^{i}_{\alpha,\beta}(R) \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d. \tag{7.16}$$

By these estimates and the second estimates of Lemmas 6.1–6.4, we have

$$\begin{aligned} &\sum_{\substack{m-1<|\alpha|+d|\beta|\le m \\ \alpha\ge\mu}} \sum_{i=1}^{n} \tilde{a}^{(i)}_{\alpha,\beta}(x, y) \mathcal{B}^{(i)}_{\alpha,\beta} u(x, y) \\ &\lll C_1(R)\rho^{-1}\|u\| \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d \left(\Phi^a_{\rho,R}\right)^d (x, y), \end{aligned} \tag{7.17}$$

$$\begin{aligned} &\sum_{\substack{m-1<|\alpha|+d|\beta|\le m \\ \alpha\ngeq\mu \text{ and } |\alpha|<|\mu|}} a_{\alpha,\beta}(x, y) \mathcal{A}_{\alpha,\beta} u(x, y) \\ &\lll C_2(R)\rho^{-1}\|u\| \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d \left(\Phi^a_{\rho,R}\right)^d (x, y), \end{aligned} \tag{7.18}$$

$$\begin{aligned} &\sum_{\substack{m-1<|\alpha|+d|\beta|\le m \\ \alpha\ngeq\mu \text{ and } |\alpha|=|\mu|}} \sum_{i=1}^{n} a^{(i)}_{\alpha,\beta}(x, y) x_i \mathcal{A}_{\alpha,\beta} u(x, y) \\ &\lll C_3(R)\rho^{-1}\|u\| \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d \left(\Phi^a_{\rho,R}\right)^d (x, y), \end{aligned} \tag{7.19}$$

$$\begin{aligned} &\sum_{\substack{m-|\mu|-1<|\alpha|+d|\beta|\le m-|\mu| \\ \alpha\ngeq\mu \text{ and } |\alpha|>|\mu|}} \tilde{b}^{i}_{\alpha,\beta}(x, y) \mathcal{B}^{i}_{\alpha,\beta} u(x, y) \\ &\lll C_4(R)\rho^{-1}\|u\| \left(\frac{\eta\rho R}{\eta\rho R - (\tau + \xi)} \right)^d \left(\Phi^a_{\rho,R}\right)^d (x, y), \end{aligned} \tag{7.20}$$

where

$$
\begin{aligned}
C_1(R) &= C_m C_0 \max\left\{\eta^{|\beta|-1}\sum_{i=1}^{n} A^{(i)}_{\alpha,\beta}(R);\, m-1<|\alpha|+d|\beta|\le m,\ \alpha\ge\mu\right\},\\
C_2(R) &= C_m \max\left\{\eta^{|\beta|-1} b_{\alpha,\beta}(R) C_{\alpha,\beta}(R);\, m-1<|\alpha|+d|\beta|\le m,\ \alpha\not\ge\mu,\ |\alpha|<|\mu|\right\},\\
C_3(R) &= C_m C_0 \max\left\{\eta^{|\beta|-1}\sum_{i=1}^{n} C^{(i)}_{\alpha,\beta}(R);\, m-1<|\alpha|+d|\beta|\le m,\ \alpha\not\ge\mu,\ |\alpha|=|\mu|\right\},\\
C_4(R) &= C_m C_0 \max\Big\{\eta^{|\beta|-1} d^{i}_{\alpha,\beta}(R);\\
&\qquad\qquad m-|\mu|-1<|\alpha|+d|\beta|\le m-|\mu|,\ \alpha\not\ge\mu,\ |\alpha|>|\mu|\Big\}.
\end{aligned}
$$

Hence, again by the second majoration of Lemma 4.1 and the first of Corollary 5.1, we obtain

$$
\sum_{\substack{m-1<|\alpha|+d|\beta|\le m\\ \alpha\ge\mu}}\ \sum_{i=1}^{n} \tilde{a}^{(i)}_{\alpha,\beta}(x,y)\mathcal{B}^{(i)}_{\alpha,\beta}u(x,y) \lll C_1(R)\rho^{-1}\|u\|\frac{\eta}{\eta-1}\left(\Phi^a_{\rho,R}\right)^d(x,y), \tag{7.21}
$$

$$
\sum_{\substack{m-1<|\alpha|+d|\beta|\le m\\ \alpha\not\ge\mu \text{ and } |\alpha|<|\mu|}} a_{\alpha,\beta}(x,y)\mathcal{A}_{\alpha,\beta}u(x,y) \lll C_2(R)\rho^{-1}\|u\|\frac{\eta}{\eta-1}\left(\Phi^a_{\rho,R}\right)^d(x,y), \tag{7.22}
$$

$$
\sum_{\substack{m-1<|\alpha|+d|\beta|\le m\\ \alpha\not\ge\mu \text{ and } |\alpha|=|\mu|}}\ \sum_{i=1}^{n} a^{(i)}_{\alpha,\beta}(x,y)x_i\mathcal{A}_{\alpha,\beta}u(x,y) \lll C_3(R)\rho^{-1}\|u\|\frac{\eta}{\eta-1}\left(\Phi^a_{\rho,R}\right)^d(x,y), \tag{7.23}
$$

$$
\sum_{\substack{m-|\mu|-1<|\alpha|+d|\beta|\le m-|\mu|\\ \alpha\not\ge\mu \text{ and } |\alpha|>|\mu|}} \tilde{b}^{(i_\alpha)}_{\alpha,\beta}(x,y)\mathcal{B}^{(i_\alpha)}_{\alpha,\beta}u(x,y) \lll C_4(R)\rho^{-1}\|u\|\frac{\eta}{\eta-1}\left(\Phi^a_{\rho,R}\right)^d(x,y). \tag{7.24}
$$

Thus, by using the estimates (7.9)–(7.12), (7.21)–(7.24), and the expressions of the operators $\mathcal{A}$ and $\mathcal{B}$ given in (6.2) and (6.1), respectively, we obtain that if $R\eta\rho > \max\{\operatorname{diam}(\Omega), 1\}$ and $a>1$, then

$$
(\mathcal{B}+\mathcal{A})[u(x,y)] \lll \|u\|\left(\frac{C(R)}{\rho}+\frac{M(R,\rho)}{a}\right)\left(\Phi^a_{\rho,R}\right)^d(x,y)
$$

where

$$
C(R)=\frac{\eta}{\eta-1}(C_1(R)+C_2(R)+C_3(R)+C_4(R))
$$

and

$$
M(R,\rho)=M_1'(R,\rho)+M_2'(R,\rho)+M_3'(R,\rho)+M_4'(R,\rho).
$$

Hence, from the definition of the norm $\|\cdot\|$, we obtain

$$\|\mathcal{B}+\mathcal{A}\| \leq \frac{C(R)}{\rho} + \frac{M(R,\rho)}{a}.$$

If we take parameters ρ and a such that

$$\rho > 2C(R) \quad \text{and} \quad a > 2M(R,\rho),$$

then we get $\|\mathcal{B}+\mathcal{A}\| < 1$ and therefore the operator $(I-(\mathcal{B}+\mathcal{A}))$ is invertible in the Banach space $\mathcal{G}^{(\omega,d)}_{R,a,\ \rho}(\Omega_R)$, which shows the existence and uniqueness of the solution of the problem (3.9) in this space. $\square$

8. Construction of global solutions

From now on, the norm $\|\cdot\|_i$ denotes the norm of $\mathcal{G}^{(\omega,d)}_{R,a_i,\rho_i}(\Omega_R)$.

Lemma 8.1. *If* $R>0$, $a_1>0$, $a_2>0$, $\rho_1 \geq \rho_2 > 0$, *then we have a continuous embedding from* $\mathcal{G}^{(\omega,d)}_{R,a_1,\rho_1}(\Omega_R)$ *into* $\mathcal{G}^{(\omega,d)}_{R,a_2,\rho_2}(\Omega_R)$.

Proof. If $u \in \mathcal{G}^{(\omega,d)}_{R,a_1,\rho_1}(\Omega_R)$, then

$$\forall \delta \in \mathbb{N}^q, \quad \forall y \in \Omega, \quad D^\delta_y u(x,y) \ll \|u\|_1 D^{|\delta|}_\xi \Phi^{a_1}_{\rho_1,R}(x,0)|\delta|!^{d-1}. \tag{8.1}$$

By using the construction of formal series $\Phi^a_{\rho,R}$, we obtain

$$\begin{aligned} D^{|\delta|}_\xi \Phi^{a_1}_{\rho_1,R}(x,0) &= \sum_{p\in\mathbb{N}} \tau^p(\rho_1 R)^{s'p}\frac{(|\delta|+sp)!}{(sp)!}\sum_{l=0}^{|\delta|+sp}\frac{a_1^l}{l!}\frac{1}{(\rho_1 R)^{sp+|\delta|+1-l}} \\ &= \sum_{p\in\mathbb{N}}\left(\frac{\sigma}{R}\right)^p\frac{(|\delta|+sp)!}{(sp)!}\frac{1}{(\rho_1 R)^{|\delta|+1}}\sum_{l=0}^{|\delta|+sp}\frac{(R\rho_1 a_1)^l}{l!}. \end{aligned} \tag{8.2}$$

Since we have

$$\forall a>0, \quad \forall \rho>0, \quad 1 \leq \sum_{l=0}^{|\delta|+sp}\frac{(R\rho a)^l}{l!} \leq e^{R\rho a},$$

and

$$\frac{1}{(\rho_1 R)^{|\delta|+1}} \leq \frac{1}{(\rho_2 R)^{|\delta|+1}} \quad \text{for} \quad \rho_1 \geq \rho_2 > 0,$$

we obtain the following estimates from (8.2).

$$D^{|\delta|}_\xi \Phi^{a_1}_{\rho_1,R}(x,0) \ll \sum_{p\in\mathbb{N}}\left(\frac{\sigma}{R}\right)^p\frac{(|\delta|+sp)!}{(sp)!}\frac{1}{(\rho_2 R)^{|\delta|+1}}\left(\sum_{l=0}^{|\delta|+sp}\frac{(R\rho_1 a_1)^l}{l!}\right)\sum_{l=0}^{|\delta|+sp}\frac{(R\rho_2 a)^l}{l!}$$

$$\ll e^{\rho_1 R a_1} D_\xi^{|\delta|} \Phi_{\rho_2,R}^{a_2}(x,0).$$

By substituting this estimate into (8.1), we obtain

$$\forall \delta \in \mathbb{N}^q, \quad \forall y \in \Omega, \quad D_y^\delta u(x,y) \ll \|u\|_1 e^{\rho_1 R a_1} D_\xi^{|\delta|} \Phi_{\rho_2,R}^{a_2}(x,0) |\delta|!^{d-1}.$$

Thus, we obtain our embedding and

$$\|u\|_2 \leq \|u\|_1 e^{\rho_1 R a_1}. \qquad \square$$

This lemma allows us to use the same techniques by Gourdin and Mechab in [7], for unique global construction of the solution.

By using Proposition 6.1, we obtain the following result for each fixed $R > 0$.

Proposition 8.1. *There exists* $n_0 \in \mathbb{N}^*$ *and an increasing sequence of positive numbers* $(a_n)_{n \geq n_0}$ *such that the problem* (3.9) *admits a unique solution* u_{n_0} *satisfying*

$$u_{n_0} \in \bigcap_{n \geq n_0} \mathcal{G}_{R,a_n,n}^{(\omega,d)}(\Omega_R).$$

Proof. By Proposition 6.1, there exists $n_0 \in \mathbb{N}^*$ such that

$$\forall n \geq n_0,\ \exists a_n > 0 \text{: the problem (3.9) admits a unique solution } u_n \in \mathcal{G}_{R,a_n,n}^{(\omega,d)}(\Omega_R).$$

We also have a unique solution $u_{n+1} \in \mathcal{G}_{R,a_{n+1},n+1}^{(\omega,d)}(\Omega_R)$.

By Lemma 8.1, we have

$$\mathcal{G}_{R,a_{n+1},n+1}^{(\omega,d)}(\Omega_R) \subset \mathcal{G}_{R,a_n,n}^{(\omega,d)}(\Omega_R),$$

which implies that the solutions u_n and u_{n+1} belong to the same space $\mathcal{G}_{R,a_n,n}^{(\omega,d)}(\Omega_R)$, where there is the uniqueness of solution. Hence,

$$u_n = u_{n_0}, \quad \text{on } \Omega_R \text{ for } n \geq n_0. \qquad \square$$

9. Existence of solutions in $G^{(\omega,d)}(\Omega_{R/2^s})$

Proposition 9.1.

$$u_{n_0} \in G^{(\omega,d)}(\Omega_{R/2^s}).$$

Proof. By Proposition 8.1, we have

$$\forall n \geq n_0, \quad \forall \delta \in \mathbb{N}^q, \quad \forall y \in \Omega, \quad D_y^\delta u_{n_0}(x,y) \ll \|u_{n_0}\|_n D_\xi^{|\delta|} \Phi_{n,R}^{a_n}(x,0) |\delta|!^{d-1}. \tag{9.1}$$

Since the function $\Phi_{n,R}^{a_n}$ is analytic in the open set

$$\left\{(\tau,\xi)\in\mathbb{C}^2;\,(nR)^{s'/s}|\tau|^{1/s}+|\xi|<nR\right\},$$

which contains the polydisk

$$\Theta_R=\left\{(\tau,\xi)\in\mathbb{C}\times\mathbb{R};\,|\tau|\le\frac{nR}{2^s}\text{ and }|\xi|\le\frac{nR}{2}\right\}.$$

By using Cauchy's formula for this polydisk, we obtain the existence of constant $C_n=C_n(a_n,n,R)>0$ satisfying

$$\Phi_{n,R}^{a_n}(x,y)\ll M_n\frac{nR/2^s}{(nR/2^s)-\tau}\,\frac{nR/2}{(nR/2)-\xi},$$

which implies

$$D_\xi^{|\delta|}\Phi_{n,R}^{a_n}(x,0)\ll M_n\frac{nR/2^s}{(nR/2^s)-\tau}\,\frac{|\delta|!}{(nR/2)^{|\delta|}}\quad\text{for}\quad|\tau|=|n\sigma|<\frac{nR}{2^s}.\tag{9.2}$$

By substituting this result in (9.1), we get

$$\forall n\ge n_0,\quad\forall\delta\in\mathbb{N}^q,\quad\forall y\in\Omega,\quad D_y^\delta u_{n_0}(x,y)\ll\|u_{n_0}\|_n M_n\frac{R/2^s}{(R/2^s)-\sigma}\,\frac{|\delta|!}{(nR/2)^{|\delta|}}|\delta|!^{\,d-1},\tag{9.3}$$

which gives

$$\forall n\ge n_0,\quad\forall\delta\in\mathbb{N}^q,\quad\forall y\in\Omega,\quad\left|D_y^\delta u_{n_0}(x,y)\right|\le C_n\frac{1}{(nR/2)^{|\delta|}}|\delta|!^{\,d},\tag{9.4}$$

where

$$C_n=M_n\sup_{x\in\mathscr{U}_{R/2^s}}\left|\frac{R/2^s}{(R/2^s)-\sigma}\right|\|u_{n_0}\|_n.$$

Hence, we obtain that for all $h>0$, there exists an integer $n>\max\{n_0,2/(hR)\}$ such that

$$\forall\delta\in\mathbb{N}^q,\quad\forall(x,y)\in\mathscr{U}_{R/2^s}\times\Omega,\quad\left|D_y^\delta u_{n_0}(x,y)\right|\le C_h h^{|\delta|}|\delta|!^{\,d}.\tag{9.5}$$

Thus, we have $u_{n_0}\in G^{(\omega,d)}(\Omega_{R/2^s})$. □

10. Uniqueness of solution in $G^{(\omega,d)}(\Omega_T)$

Let $T>0$ and $u_1,u_2\in G^{(\omega,d)}(\Omega_T)$ be two solutions of the problem (3.9). From Corollary 5.2, there exists $R>0$ such that

$$u_1,u_2\in\mathcal{G}_{R,a,\rho}^{(\omega,d)}(\Omega_R),$$

where there is the uniqueness. Thus, we have

$$\forall y \in \Omega, \quad u_1(\,\cdot\,, y) = u_2(\,\cdot\,, y), \quad \text{on} \quad \mathscr{U}_R.$$

Since $\mathscr{U}_R$ is an connected open set in $\mathscr{U}_T$, and since u_1, u_2 are analytic on $\mathscr{U}_T$, by the analytic continuation, we get

$$u_1(\,\cdot\,, y) = u_2(\,\cdot\,, y), \quad \text{on} \quad \mathscr{U}_T.$$

11. End of the proof of the main theorem

In this final section, we finish the proof of Theorem 2.1.

Let $g \in \mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$ be a function introduced from function f in Section 3. For an arbitrary $R > 0$ and an arbitrary open set $\Omega' \Subset \Omega$, we have $g \in G^{(\omega,d)}(B_R \times \Omega')$.

Applying the results up to Section 10 for Ω' instead of Ω, we have proved that for all $R > 0$, the problem (3.9) admits a unique solution u_R in $G^{(\omega,d)}(\Omega'_R)$. Hence, by the uniqueness, we have a unique solution $u \in \mathcal{G}^{(\omega,d)}(\mathbb{C}^n \times \Omega)$ of the problem (3.9).

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