| Title | Whitehead double and Milnor invariants |
| :---: | :--- |
| Author(s) | Meilhan, Jean-Baptiste; Yasuhara, Akira |
| Citation | Osaka Journal of Mathematics. 2011, 48(2), p. <br> $371-381$ |
| Version Type | VoR |
| URL | https://doi.org/10.18910/7789 |
| rights |  |
| Note |  |

Osaka University Knowledge Archive : OUKA
https://ir. library.osaka-u.ac.jp/

# WHITEHEAD DOUBLE AND MILNOR INVARIANTS 

Jean-Baptiste MEILHAN and Akira YASUHARA

(Received November 24, 2009)


#### Abstract

We consider the operation of Whitehead double on a component of a link and study the behavior of Milnor invariants under this operation. We show that this operation turns a link whose Milnor invariants of length $\leq k$ are all zero into a link with vanishing Milnor invariants of length $\leq 2 k+1$, and we provide formulae for the first non-vanishing ones. As a consequence, we obtain statements relating the notions of link-homotopy and self $\Delta$-equivalence via the Whitehead double operation. By using our result, we show that a Brunnian link $L$ is link-homotopic to the unlink if and only if the link $L$ with a single component Whitehead doubled is self $\Delta$-equivalent to the unlink.


## 1. Introduction

In this paper, we consider the operation of Whitehead double, more generally of Whitehead $n$-double, on a component of a link, and we study the behavior of Milnor invariants under this operation. Milnor invariants $\bar{\mu}_{L}(I)$ of an $m$-component link $L$, where $I=i_{1} i_{2} \cdots i_{k}$ with $1 \leq i_{j} \leq m$, can be thought of as some sort of "higher order linking number" of the link. See Section 2 for a definition.

A typical example is the Whitehead link, which is a Whitehead double of the Hopf link. The linking number of the Hopf link (which coincides with Milnor invariant $\bar{\mu}(12))$ is $\pm 1$, whereas the Whitehead link has linking number 0 . On the other hand, the Whitehead link has some nontrivial higher order Milnor invariants: its Sato-Levine invariant for instance, which is equal to $-\bar{\mu}(1122)$, is $\pm 1$. Our main result, stated below, generalizes this observation.

Let $K$ be a component of a link $L$ in $S^{3}$, regarded as $h\left(\{\mathbf{0}\} \times S^{1}\right)$ for some embedding $h: D^{2} \times S^{1} \rightarrow S^{3} \backslash(L \backslash K)$, such that $K$ and $h\left((0,1) \times S^{1}\right)$ have linking number zero. Let $n$ be a (nonzero) integer. Consider in the solid torus $T=D^{2} \times S^{1}$ the knot $\mathcal{W}_{n}$ depicted in Fig. 1.1. The knot $h\left(\mathcal{W}_{n}\right)$ is called the Whitehead $n$-double of $K$, and it is denoted by $W_{n}(K)$.

Given an $m$-component link $L=K_{1} \cup \cdots \cup K_{m}$ in $S^{3}$, we denote by $W_{n}^{i}(L)$ the link $\left(L \backslash K_{i}\right) \cup W_{n}\left(K_{i}\right)$ obtained by Whitehead $n$-double on the $i^{\text {th }}$ component of $L$.

[^0]

Fig. 1.1. The knot $\mathcal{W}_{n}$ for $n<0$ and $n>0$ respectively.
Note that the case $n= \pm 1$ coincides with the usual notion of (positive or negative) Whitehead double.

Theorem 1.1. Let $L$ be an $m$-component link in $S^{3}$, and let $n(\neq 0)$ be an integer. If all Milnor invariants $\bar{\mu}_{L}(J i)$ of $L$ of length $|J i| \leq k$ are zero $(k \geq 1)$, then all Milnor invariants $\bar{\mu}_{W_{n}^{i}(L)}(I i)$ of $W_{n}^{i}(L)$ of length $|I i| \leq 2 k+1$ are zero. Moreover, if $\bar{\mu}_{L}(P i) \neq 0, \bar{\mu}_{L}(Q i) \neq 0$ with $P=p_{1} p_{2} \cdots p_{k}, Q=q_{1} q_{2} \cdots q_{k}($ possibly $P=Q)$ such that $p_{j} \neq i, q_{j} \neq i$ for all $1 \leq j \leq k$, then we have the following formulae for the first non-vanishing Milnor invariants of $W_{n}^{i}(L)$

$$
\left\{\begin{array}{l}
\bar{\mu}_{W_{i}^{i}(L)}(P i Q i)=2 n \bar{\mu}_{L}(P i) \bar{\mu}_{L}(Q i) \\
\bar{\mu}_{W_{n}^{i}(L)}(P Q i i)=-n \bar{\mu}_{L}(P i) \bar{\mu}_{L}(Q i)
\end{array}\right.
$$

Remark 1.2. In the case of a 2 -component link, the formulae given in Theorem 1.1 for the first nonvanishing Milnor invariants of $W_{n}^{i}(L)$ provide, as an immediate corollary, a generalization of a result of Shibuya and the second author [14] as follows: Let $L=K_{1} \cup K_{2}$ in $S^{3}$. Let $n \neq 0$ be an integer, and let $W_{n}(L)$ be obtained by Whitehead $n$-double on a component of $L$. Then the Sato-Levine invariant $\beta_{2}$ of $W_{n}(L)$ satisfies

$$
\beta_{2}\left(W_{n}(L)\right)=n\left(l k\left(K_{1}, K_{2}\right)\right)^{2} .
$$

(Note that the Sato-Levine invariant of $W_{n}(L)$ is well-defined, as Theorem 1.1 ensures that the link has zero linking number.)

Recall that two links are link-homotopic if they are related by a sequence of ambient isotopies and self crossing changes, which are crossing changes involving two strands of the same component, see the left-hand side of Fig. 1.2. In particular, a link is called link-homotopically trivial if it is link-homotopic to the unlink. It has long been known that Milnor invariants with no repeating indices are invariants of link-homotopy [5]. Like crossing change, the $\Delta$-move is an unknotting operation [6]. Here we consider


Fig. 1.2. A crossing change and a $\Delta$-move.
the notion of self $\Delta$-move for links, which is a local move as illustrated in the righthand side of Fig. 1.2 involving three strands of the same component. Two links are self $\Delta$-equivalent if they are related by a finite sequence of ambient isotopies and self $\Delta$-moves. Self $\Delta$-equivalence is a generalized link-homotopy, i.e., self $\Delta$-equivalence implies link-homotopy. The self $\Delta$-equivalence was introduced by Shibuya [10, 11], and was subsequently studied by various authors $[2,7,8,9,13,14,16]$. A link is self $\Delta$-trivial if it is self $\Delta$-equivalent to the unlink.

The following is a consequence of our main result.
Corollary 1.3. Let $L$ be an m-component link in $S^{3}$ which is not link-homotopically trivial. Then, for any $n(\neq 0)$ and $i(1 \leq i \leq m), W_{n}^{i}(L)$ is not self $\Delta$-trivial.

Recall now that a link $L$ is Brunnian if all proper sublinks of $L$ are trivial. The next result shows that the converse of Corollary 1.3 also holds for Brunnian links.

Theorem 1.4. Let $L$ be an m-component Brunnian link in $S^{3}$. Let $n(\neq 0)$ and $i(1 \leq i \leq m)$ be integers. Then $L$ is link-homotopically trivial if and only if $W_{n}^{i}(L)$ is self $\Delta$-trivial.

Observe that an $m$-component Brunnian link always has vanishing Milnor invariants of length $\leq m-1$ since these are Milnor invariants of sublinks of a Brunnian link, which are trivial links. So Theorem 1.1 implies that all Milnor invariants of $W_{n}^{i}(L)$ of length $\leq 2 m-1$ are zero for any choice of $1 \leq i \leq m$ and $n(\neq 0)$. In other words, for $m$-component Brunnian links, Whitehead doubling kills all Milnor invariants of length $\leq$ $2 m-1$. It follows from a more general result (stated and proved in Section 4) that an additional Whitehead doubling, on either the same or another component of the link, actually kills all Milnor invariants, as the resulting link is always a boundary link, see Corollary 4.2.

The rest of the paper is organized as follows. In Section 2 we recall the definition of Milnor invariants and prove Theorem 1.1. In Section 3 we prove the two statements relating Whitehead doubling and self $\Delta$-equivalence, namely Corollary 1.3 and Theorem 1.4. In Section 4 we consider more general satellite constructions, involving a knot which is null-homologous in the solid torus. When applied twice to a Brunnian link, such a construction always yields a boundary link.

## 2. Milnor invariants

J. Milnor defined in [4,5] a family of invariants of oriented, ordered links in $S^{3}$, known as Milnor's $\bar{\mu}$-invariants.

Given an $m$-component link $L$ in $S^{3}$, denote by $\pi(L)$ the fundamental group of $S^{3} \backslash L$, and by $\pi_{q}(L)$ the $q^{t h}$ subgroup of the lower central series of $\pi(L)$. We have a presentation of $\pi(L) / \pi_{q}(L)$ with $m$ generators, given by a meridian $\alpha_{i}$ of the $i^{\text {th }}$ component of $L$. So for $1 \leq i \leq m$, the longitude $l_{i}$ of the $i^{\text {th }}$ component of $L$ is expressed modulo $\pi_{q}(L)$ as a word in the $\alpha_{i}$ 's (abusing notations, we still denote this word by $l_{i}$ ).

The Magnus expansion $E\left(l_{i}\right)$ of $l_{i}$ is the formal power series in non-commuting variables $X_{1}, \ldots, X_{m}$ obtained by substituting $1+X_{j}$ for $\alpha_{j}$ and $1-X_{j}+X_{j}^{2}-X_{j}^{3}+\cdots$ for $\alpha_{j}^{-1}, 1 \leq j \leq m$.

Let $I=i_{1} i_{2} \cdots i_{k-1} j$ be a multi-index (i.e., a sequence of possibly repeating indices) among $\{1, \ldots, m\}$. Denote by $\mu_{L}(I)$ the coefficient of $X_{i_{1}} \cdots X_{i_{k-1}}$ in the Magnus expansion $E\left(l_{j}\right)$. Milnor invariant $\bar{\mu}_{L}(I)$ is the residue class of $\mu_{L}(I)$ modulo the greatest common divisor of all $\mu_{L}(J)$ such that $J$ is obtained from $I$ by removing at least one index, and permutating the remaining indices cyclically. We call $|I|=k$ the length of Milnor invariant $\bar{\mu}_{L}(I)$.

The indeterminacy comes from the choice of the meridians $\alpha_{i}$ or, equivalently, from the indeterminacy of representing the link as the closure of a string link [3].

Proof of Theorem 1.1. Without loss of generality, we may suppose that $i=m$. We give the proof of the case $n<0$. The case $n>0$ is strictly similar and we omit it.

We denote by $\alpha_{1}, \ldots, \alpha_{m-1}, \alpha_{m}$ and $a$ meridians of $K_{1}, \ldots, K_{m-1}, K_{m}$ and $W_{n}\left(K_{m}\right)$ respectively, such that $\alpha_{1}, \ldots, \alpha_{m}$ generate $\pi(L) / \pi_{q}(L)$ and $\alpha_{1}, \ldots, \alpha_{m-1}$, a generate $\pi\left(W_{n}^{m}(L)\right) / \pi_{q}\left(W_{n}^{m}(L)\right)$.

The Magnus expansion of the longitude $l_{m} \in \pi(L) / \pi_{q}(L)$ of $K_{m}$, written as a word in $\alpha_{1}, \ldots, \alpha_{m}$, has the form

$$
E\left(l_{m}\right)=1+\sum \mu_{L}\left(i_{1} \cdots i_{r}, m\right) X_{i_{1}} \cdots X_{i_{r}}=1+f\left(X_{1}, \ldots, X_{m}\right),
$$

where $E\left(\alpha_{i}\right)=1+X_{i}$ for all $1 \leq i \leq m$.
Now consider the Whitehead $n$-double of $K_{m}$, and consider $2 n+1$ elements $a_{0}, a_{1}, \ldots$, $a_{2 n}$ of $S^{3} \backslash W_{n}^{m}(L)$ as represented in Fig. 2.1. Let $\phi\left(l_{m}\right)=l$, where $\phi: \pi(L) / \pi_{q}(L) \rightarrow$ $\pi\left(W_{n}^{m}(L)\right) / \pi_{q}\left(W_{n}^{m}(L)\right)$ is the natural map that maps $\alpha_{i}$ to itself $(1 \leq i \leq m-1)$ and maps $\alpha_{m}$ to $a_{2 n}^{-1} a$. (Abusing notation, we still denote by $a_{i}, 0 \leq i \leq 2 n$, the corresponding elements in $\pi\left(W_{n}^{m}(L)\right) / \pi_{q}\left(W_{n}^{m}(L)\right)$.)


Fig. 2.1. The Whitehead $n$-double of $K_{m}$ for $n<0$.
It follows from repeated uses of Wirtinger relations that

$$
\begin{cases}a_{0}=l^{-1} a l, & \text { for all } \quad r \geq 1, \\ a_{2 r}=R^{r} a R^{-r}, & \\ a_{2 r+1}=R^{r} a R^{-(r+1)}, & \text { for all } \quad r \geq 0\end{cases}
$$

where $R=a l^{-1} a^{-1} l$. In particular we have that

$$
\phi\left(\alpha_{m}\right)=a_{2 n}^{-1} a=R^{n} a^{-1} R^{-n} a .
$$

Let $E(a)=1+X$ denote the Magnus expansion of $a$. Observe that

$$
\begin{aligned}
E(R)=E\left(a l^{-1} a^{-1} l\right) & =(1+X) E\left(l^{-1}\right)(1-X) E(l)+\mathcal{O}_{X}(2) \\
& =1+X-E\left(l^{-1}\right) X E(l)+\mathcal{O}_{X}(2),
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(R^{-1}\right)=E\left(l^{-1} a l a^{-1}\right) & =E\left(l^{-1}\right)(1+X) E(l)(1-X)+\mathcal{O}_{X}(2) \\
& =1-X+E\left(l^{-1}\right) X E(l)+\mathcal{O}_{X}(2),
\end{aligned}
$$

where $\mathcal{O}_{X}(2)$ denotes terms which contain $X$ at least 2 times. So we have

$$
\begin{aligned}
E\left(\phi\left(\alpha_{m}\right)\right)= & \left(1+X-E\left(l^{-1}\right) X E(l)\right)^{n}(1-X) \\
& \times\left(1-X+E\left(l^{-1}\right) X E(l)\right)^{n}(1+X)+\mathcal{O}_{X}(2) \\
= & \left(1+n X-n E\left(l^{-1}\right) X E(l)\right)(1-X) \\
& \times\left(1-n X+n E\left(l^{-1}\right) X E(l)\right)(1+X)+\mathcal{O}_{X}(2) \\
= & 1+\mathcal{O}_{X}(2) .
\end{aligned}
$$

This implies that

$$
\begin{aligned}
E(l) & =1+f\left(X_{1}, \ldots, X_{m-1}, \mathcal{O}_{X}(2)\right) \\
& =1+f_{1}\left(X_{1}, \ldots, X_{m-1}\right)+f_{2}\left(X_{1}, \ldots, X_{m-1}, X\right)
\end{aligned}
$$

where

$$
f_{1}\left(X_{1}, \ldots, X_{m-1}\right)=f\left(X_{1}, \ldots, X_{m-1}, 0\right) \in \mathcal{O}(k)
$$

and

$$
f_{2}\left(X_{1}, \ldots, X_{m-1}, X\right)=f\left(X_{1}, \ldots, X_{m-1}, \mathcal{O}_{X}(2)\right)-f_{1}\left(X_{1}, \ldots, X_{m-1}\right) \in \mathcal{O}(k+1)
$$

and $\mathcal{O}(u)$ denotes terms of degree at least $u$ (the degree of a monomial in the $X_{j}$ is simply defined by the sum of the powers). Similarly we have

$$
\begin{aligned}
E\left(l^{-1}\right) & =1+g\left(X_{1}, \ldots, X_{m-1}, \mathcal{O}_{X}(2)\right) \\
& =1+g_{1}\left(X_{1}, \ldots, X_{m-1}\right)+g_{2}\left(X_{1}, \ldots, X_{m-1}, X\right),
\end{aligned}
$$

where $g_{1}\left(X_{1}, \ldots, X_{m-1}\right) \in \mathcal{O}(k)$ and $g_{2}\left(X_{1}, \ldots, X_{m-1}, X\right) \in \mathcal{O}(k+1)$.
Let $f_{1}, f_{2}, g_{1}, g_{2}$ denote $f_{1}\left(X_{1}, \ldots, X_{m-1}\right), f_{2}\left(X_{1}, \ldots, X_{m-1}, X\right), g_{1}\left(X_{1}, \ldots, X_{m-1}\right)$, $g_{2}\left(X_{1}, \ldots, X_{m-1}, X\right)$ respectively, and set $f=f_{1}+f_{2}$ and $g=g_{1}+g_{2}$. Set $E\left(a^{-1}\right)=$ $1-X+X^{2}-X^{3}+\cdots=1+Y$. Note that $(1+f)(1+g)=(1+g)(1+f)=1$ and $(1+X)(1+Y)=(1+Y)(1+X)=1$, hence $f+g=-f g=-g f \in \mathcal{O}(2 k)$ and $X+Y=-X Y=-Y X$. One can check, by induction, that

$$
\left\{\begin{array}{l}
E\left(R^{n}\right)=1+n(g Y-X f+X g Y+g Y f)+\mathcal{O}(2 k+2), \\
E\left(R^{-n}\right)=1+n(X f-g Y+X f Y+g X f)+\mathcal{O}(2 k+2), \\
E\left(\left(a^{-1} R\right)^{n}\right)=(1+Y)^{n}+(1+Y)^{n} f-f(1+Y)^{n}+n(g Y f-f g Y)+\mathcal{O}(2 k+2) .
\end{array}\right.
$$

Since the preferred longitude $L_{m}$ of $W_{n}^{m}\left(K_{m}\right)$ is presented in $\pi\left(W_{n}^{m}(L)\right) / \pi_{q}\left(W_{n}^{m}(L)\right)$ by the word

$$
L_{m}=l a^{-1} a_{2}^{-1} \cdots a_{2 n-2}^{-1} l^{-1} a_{2 n-1}^{-1} a_{2 n-3}^{-1} a_{3}^{-1} a_{1}^{-1} a^{2 n}=l\left(a^{-1} R\right)^{n} R^{-n} l^{-1} R^{n} a^{n},
$$

we have

$$
\begin{aligned}
E\left(L_{m}\right)= & (1+f)\left[(1+Y)^{n}+(1+Y)^{n} f-f(1+Y)^{n}+n(g Y f-f g Y)\right] \\
& \times[1+n(X f-g Y+X f Y+g X f)](1+g) \\
& \times[1+n(g Y-X f+X g Y+g Y f)](1+X)^{n} \\
= & {\left[(1+Y)^{n}+n\left(2 f X f-f^{2} X-X f^{2}\right)\right](1+X)^{n}+\mathcal{O}(2 k+2) } \\
= & 1+n(2 f X f-f f X-X f f)+\mathcal{O}(2 k+2) .
\end{aligned}
$$

Because $f \in \mathcal{O}(k)$, the first non-trivial terms in the Magnus expansion $E\left(L_{m}\right)$ are of degree $2 k+1$. It follows that all Milnor invariants $\bar{\mu}_{W_{n}^{m}(L)}(\operatorname{Im})$ of length $|\operatorname{Im}| \leq$ $2 k+1$ of $W_{n}^{m}(L)$ are zero.

Moreover, we actually have

$$
E\left(L_{m}\right)=1+n\left(2 f_{1} X f_{1}-f_{1} f_{1} X-X f_{1} f_{1}\right)+\mathcal{O}(2 k+2)
$$

So if $\bar{\mu}_{L}(P m) \neq 0, \bar{\mu}_{L}(Q m) \neq 0$ for some multi-indices $P=p_{1} \cdots p_{k}, Q=q_{1} \cdots q_{k}$ ( $P \neq Q$ ) with $p_{j} \neq m, q_{j} \neq m$ for all $1 \leq j \leq k$, then

$$
f_{1}=\bar{\mu}_{L}(P m) X_{p_{1}} \cdots X_{p_{k}}+\bar{\mu}_{L}(Q m) X_{q_{1}} \cdots X_{q_{k}}+\mathcal{O}(k),
$$

and it follows from the above formula that

$$
\begin{aligned}
E\left(L_{m}\right)= & 1+2 n \bar{\mu}_{L}(P m) \bar{\mu}_{L}(P m) X_{p_{1}} \cdots X_{p_{k}} X X_{p_{1}} \cdots X_{p_{k}} \\
& +2 n \bar{\mu}_{L}(P m) \bar{\mu}_{L}(Q m) X_{p_{1}} \cdots X_{p_{k}} X X_{q_{1}} \cdots X_{q_{k}} \\
& +2 n \bar{\mu}_{L}(Q m) \bar{\mu}_{L}(P m) X_{q_{1}} \cdots X_{q_{k}} X X_{p_{1}} \cdots X_{p_{k}} \\
& +2 n \bar{\mu}_{L}(Q m) \bar{\mu}_{L}(Q m) X_{q_{1}} \cdots X_{q_{k}} X X_{q_{1}} \cdots X_{q_{k}} \\
& -n \bar{\mu}_{L}(P m) \bar{\mu}_{L}(P m) X_{p_{1}} \cdots X_{p_{k}} X_{p_{1}} \cdots X_{p_{k}} X \\
& -n \bar{\mu}_{L}(P m) \bar{\mu}_{L}(Q m) X_{p_{1}} \cdots X_{p_{k}} X_{q_{1}} \cdots X_{q_{k}} X \\
& -n \bar{\mu}_{L}(Q m) \bar{\mu}_{L}(P m) X_{q_{1}} \cdots X_{q_{k}} X_{p_{1}} \cdots X_{p_{k}} X \\
& -n \bar{\mu}_{L}(Q m) \bar{\mu}_{L}(Q m) X_{q_{1}} \cdots X_{q_{k}} X_{q_{1}} \cdots X_{q_{k}} X \\
& -n \bar{\mu}_{L}(P m) \bar{\mu}_{L}(P m) X X_{p_{1}} \cdots X_{p_{k}} X_{p_{1}} \cdots X_{p_{k}} \\
& -n \bar{\mu}_{L}(P m) \bar{\mu}_{L}(Q m) X X_{p_{1}} \cdots X_{p_{k}} X_{q_{1}} \cdots X_{q_{k}} \\
& -n \bar{\mu}_{L}(Q m) \bar{\mu}_{L}(P m) X X_{q_{1}} \cdots X_{q_{k}} X_{p_{1}} \cdots X_{p_{k}} \\
& -n \bar{\mu}_{L}(Q m) \bar{\mu}_{L}(Q m) X X_{q_{1}} \cdots X_{q_{k}} X_{q_{1}} \cdots X_{q_{k}}+\mathcal{O}(2 k+1)
\end{aligned}
$$

which implies the desired formulae for the first nonvanishing Milnor invariants of $W_{n}^{m}(L)$.

REmARK 2.1. One may wonder what happens when we consider, in the definition of a Whitehead $n$-double, an odd number of half-twists in place of $n$ full twists. For a link $L$, denote by $W_{\text {odd }}^{i}(L)$ any link obtained by such a satellite construction with an odd number of half-twists on the $i^{\text {th }}$ component of $L$. Then we can prove the following: If all Milnor invariants of $L$ with length $\leq k$ vanish, then for any multiindex $I i$ with $|I i| \leq k+1, \bar{\mu}_{W_{\text {odd }}^{i}(L)}(I i)=2^{r_{i}+1} \bar{\mu}_{L}(I i)$, where $r_{i}$ is the number of times that the index $i$ appears in $I$.

## 3. On self $\Delta$-equivalence

In this section we provide the proofs for Corollary 1.3 and Theorem 1.4.
We need the following additional notation. Given a multi-index $I$, we denote by $r(I)$ the maximum number of times that any index appears in $I$. For example, $r(1123)=2$ and $r(1233212)=3$.

Proof of Corollary 1.3. Let $L$ be an $m$-component link which is not linkhomotopically trivial. Then by [4] there exists some multi-index $I=i_{1} \cdots i_{p}$ with $r(I)=1$ such that $\bar{\mu}_{L}(I) \neq 0$ and $\bar{\mu}_{L}(J)=0$ for all multi-index $J$ with length $|J|<$ $|I|$ and $r(J)=1$.

Let $n(\neq 0)$ and $i(1 \leq i \leq m)$ be integers. If $I$ does not contain $i$, then $\bar{\mu}_{W_{n}^{i}(L)}(I)=\bar{\mu}_{L}(I) \neq 0$. So $W_{n}^{i}(L)$ is not link-homotopically trivial. Hence $W_{n}^{i}(L)$ is not self $\Delta$-trivial. Suppose that $I$ contains $i$. By "cyclic symmetry" ([5, Theorem 6]), we may assume that $i_{p}=i$. By Theorem 1.1, the link $W_{n}^{i}(L)$ thus satisfies $\bar{\mu}_{W_{n}^{i}(L)}(M) \neq 0$ for some multi-index $M$ with $r(M) \leq 2$. Since Milnor invariants with $r \leq 2$ are self $\Delta$-equivalence invariants [1], $W_{n}^{i}(L)$ is not self $\Delta$-trivial.

Proof of Theorem 1.4. Let $L$ be an $m$-component Brunnian link. Let $n(\neq 0)$ and $i$ $(1 \leq i \leq m)$ be integers. By Corollary 1.3 we already know that $L$ is link-homotopically trivial if $W_{n}^{i}(L)$ is self $\Delta$-trivial. Let us prove that the converse is also true.

The link $L$ being Brunnian, $\bar{\mu}_{L}(I)=0$ if $I$ does not contain an index in $\{1, \ldots, m\}$. Moreover, if $L$ is link-homotopically trivial, then $\bar{\mu}_{L}(I)=0$ for any $I$ with $r(I)=1$. In particular $\bar{\mu}_{L}(I)=0$ for all $|I| \leq m$, and by Theorem 1.1 the link $W_{n}^{i}(L)$ thus satisfies $\bar{\mu}_{W_{n}^{i}(L)}(I)=0$ for all $|I| \leq 2 m+1$. This implies that $\bar{\mu}_{W_{n}^{i}(L)}(I)=0$ for any multi-index $I$ with $r(I) \leq 2$. By [16, Corollary 1.5], we have that $W_{n}^{i}(L)$ is self $\Delta$-trivial.

## 4. From Brunnian links to boundary links

4.1. Boundary links from satellite construction. In this section we consider a more general satellite construction.

Let $L=K_{1} \cup \cdots \cup K_{m}$ be an $m$-component link in $S^{3}$, and let $h_{i}: D^{2} \times S^{1} \rightarrow$ $S^{3}$ be an embedding such that $h_{i}\left(\{\mathbf{0}\} \times S^{1}\right)$ is the $i^{\text {th }}$ component $K_{i}$ of $L$ (as in the introduction, we assume that $K_{i}$ and $h\left((0,1) \times S^{1}\right)$ have linking number zero). Now, instead of the knot $\mathcal{W}_{n}$ depicted in Fig. 1.1, consider in the solid torus $T=D^{2} \times S^{1}$ a fixed knot $\mathcal{K}$ which is null-homologous in $T$. Denote by $W_{\mathcal{K}}^{i}(L)$ the $\operatorname{link}\left(L \backslash K_{i}\right) \cup$ $h_{i}(\mathcal{K})$. We have the following result.

Theorem 4.1. Let $L=K_{1} \cup \cdots \cup K_{m}$ be an m-component link in $S^{3}$, and let $\mathcal{K}$, $\mathcal{K}^{\prime}$ be two null-homologous knots in the solid torus $T$. Then
(i) If $L \backslash K_{i}$ is a boundary link, then $W_{\mathcal{K}}^{i}\left(W_{\mathcal{K}^{\prime}}^{i}(L)\right)$ is a boundary link.
(ii) If $L \backslash\left(K_{i} \cup K_{j}\right)$ is a boundary link and $K_{i} \cup K_{j}$ is null-homotopic in $S^{3} \backslash(L \backslash$
$\left.\left(K_{i} \cup K_{j}\right)\right)$, then $W_{\mathcal{K}}^{i}\left(W_{\mathcal{K}^{\prime}}^{j}(L)\right)$ is a boundary link.


Fig. 4.1. The boundary links $W_{-4,2}^{1,1}(B)$ and $W_{-4,2}^{1,2}(B)$.


Fig. 4.2. The link $L_{i}$.
Note that in particular a Brunnian link $L$ always satisfies the conditions in (i) and (ii). It follows that a link obtained from a Brunnian link by taking twice Whitehead double (on either the same or another component of the link) kills all Milnor invariants.

Corollary 4.2. Let $L$ be an $m$-component Brunnian link in $S^{3}$. Let $p, q(p q \neq 0)$ and $i, j \in\{1, \ldots, m\}$ (possibly equal) be integers. Then the link $W_{p, q}^{i, j}(L)$, obtained by respectively Whitehead p-double and Whitehead $q$-double on the $i^{\text {th }}$ and $j^{\text {th }}$ components of $L$, is a boundary link.

Fig. 4.1 below illustrates this result in the case of the Borromean rings $B$.
4.2. Proof of Theorem 4.1. Before proving Theorem 4.1, we will introduce the notion of band presentation of a link.

Let $L_{i}=\gamma_{i 0} \cup \gamma_{i 1} \cup \gamma_{i 2} \cup \cdots \cup \gamma_{i p_{i}}$ be a link as illustrated in Fig. 4.2. Let $L_{1} \cup$ $\cdots \cup L_{m}$ be a split union of the links $L_{1}, \ldots, L_{m}$, and let $\Delta=\bigcup \Delta_{i j}$ be a disjoint union of disks $\Delta_{i j}\left(1 \leq i \leq m ; 1 \leq j \leq p_{i}\right)$ such that $\partial \Delta_{i j}=\gamma_{i j}$ and $\Delta_{i j} \cap\left(\bigcup_{k} \gamma_{k 0}\right)=$ $\Delta_{i j} \cap \gamma_{i 0}$ consists of a single point. It is known [15] that an $m$-component link $L$ in a 3-manifold $M$ which is null-homotopic in $M$ can be expressed as a band sum of $L_{1} \cup \cdots \cup L_{m}$, which is contained in a 3-ball in $M$, along mutually disjoint bands $b_{i j}$ $\left(1 \leq i \leq m ; 1 \leq j \leq p_{i}\right)$, disjoint from int $\Delta$, such that $b_{i j}$ connect $\gamma_{i j}$ and $\left(\bigcup_{k} \gamma_{k 0}\right) .{ }^{1}$ This presentation is called a band presentation of $L$, and $L_{1} \cup \cdots \cup L_{m}$ is called the base link.

[^1]Proof of Theorem 4.1. (i) We may suppose that $i=m$ without loss of generality. Since $K_{1} \cup \cdots \cup K_{m-1}$ is a boundary link, it bounds a disjoint union of surfaces $E=E_{1} \cup \cdots \cup E_{m-1}$. Denote by $W_{\mathcal{K}^{\prime}}\left(K_{m}\right)$ the $m^{\text {th }}$ component of $W_{\mathcal{K}^{\prime}}^{m}(L)$. Since $W_{\mathcal{K}^{\prime}}\left(K_{m}\right)$ is null-homologous in $h_{m}\left(D^{2} \times S^{1}\right)$, it is null-homotopic in $S^{3} \backslash\left(L \backslash K_{m}\right)$. Hence there is a band presentation of $W_{\mathcal{K}^{\prime}}\left(K_{m}\right)$ such that the base link is disjoint from $E$ and such that the intersections of each band and $E$ are ribbon singularities. So $W_{\mathcal{K}^{\prime}}\left(K_{m}\right) \cap E$ is a union of copies of $S^{0}$, which are the endpoints of these ribbon singularities. By tubing the surfaces $E_{i}$ suitably at these endpoints, we obtain a union of mutually disjoint surfaces $F_{1}, \ldots, F_{m-1}$ such that $F_{i}=\partial K_{i}$ and $F_{i} \cap W_{\mathcal{K}^{\prime}}\left(K_{m}\right)=\emptyset$ for all $1 \leq i \leq m-1$. Since the $m^{\text {th }}$ component of $W_{\mathcal{K}}^{m}\left(W_{\mathcal{K}^{\prime}}^{m}(L)\right)$ bounds a Seifert surface $F_{m}$ in a regular neighborhood of $W_{\mathcal{K}^{\prime}}\left(K_{m}\right)$, it follows that the components of $W_{\mathcal{K}}^{m}\left(W_{\mathcal{K}^{\prime}}^{m}(L)\right)$ bound $m$ mutually disjoint Seifert surfaces $F_{1}, \ldots, F_{m}$.
(ii) We may suppose that $i=m-1$ and $j=m$ without loss of generality. $K_{1} \cup$ $\cdots \cup K_{m-2}$ being a boundary link, it bounds a disjoint union of surfaces $E=E_{1} \cup$ $\cdots \cup E_{m-2}$. Since $K_{m-1} \cup K_{m}$ is null-homotopic in $S^{3} \backslash\left(K_{1} \cup \cdots \cup K_{m-2}\right)$, there is a band presentation of $K_{m-1} \cup K_{m}$ such that the base link is disjoint from $E$ and such that the intersections of each band and $E$ are ribbon singularities. By tubing the surfaces $E_{i}$ suitably at the endpoints of theses singularities, we obtain a union of mutually disjoint surfaces $F_{1}, \ldots, F_{m-2}$ such that $F_{i}=\partial K_{i}$ and $F_{i} \cap\left(K_{m-1} \cup K_{m}\right)=\emptyset$ for all $1 \leq i \leq m-2$. Since the $(m-1)^{\text {th }}$ and $m^{\text {th }}$ components of $W_{\mathcal{K}}^{m-1}\left(W_{\mathcal{K}^{\prime}}^{m}(L)\right)$ bound a disjoint union $F_{m-1} \cup F_{m}$ of Seifert surfaces in a regular neighborhood of $K_{m-1} \cup K_{m}$, it follows that the components of $W_{\mathcal{K}}^{m-1}\left(W_{\mathcal{K}^{\prime}}^{m}(L)\right)$ bound $m$ mutually disjoint Seifert surfaces $F_{1}, \ldots, F_{m}$.

## References

[1] T. Fleming and A. Yasuhara: Milnor's invariants and self $C_{k}$-equivalence, Proc. Amer. Math. Soc. 137 (2009), 761-770.
[2] T. Fleming and A. Yasuhara: Milnor numbers and the self delta classification of 2-string links; in Intelligence of Low Dimensional Topology 2006, Ser. Knots Everything 40, World Sci. Publ., Hackensack, NJ, 27-34, 2007.
[3] N. Habegger and X.-S. Lin: The classification of links up to link-homotopy, J. Amer. Math. Soc. 3 (1990), 389-419.
[4] J. Milnor: Link groups, Ann. of Math. (2) 59 (1954), 177-195.
[5] J. Milnor: Isotopy of links, Algebraic geometry and topology; in A Symposium in Honor of S. Lefschetz, Princeton Univ. Press, Princeton, NJ, 280-306, 1957.
[6] H. Murakami and Y. Nakanishi: On a certain move generating link-homology, Math. Ann. 284 (1989), 75-89.
[7] Y. Nakanishi and Y. Ohyama: Delta link homotopy for two component links, III, J. Math. Soc. Japan 55 (2003), 641-654.
[8] Y. Nakanishi and T. Shibuya: Link homotopy and quasi self delta-equivalence for links, J. Knot Theory Ramifications 9 (2000), 683-691.
[9] Y. Nakanishi, T. Shibuya and A. Yasuhara: Self delta-equivalence of cobordant links, Proc. Amer. Math. Soc. 134 (2006), 2465-2472.
[10] T. Shibuya: Two self-\#-equivalences of links in solid tori, Mem. Osaka Inst. Tech. Ser. A 35 (1990), 13-24.
[11] T. Shibuya: Self $\Delta$-equivalence of ribbon links, Osaka J. Math. 33 (1996), 751-760.
[12] T. Shibuya and A. Yasuhara: Self $C_{k}$-move, quasi self $C_{k}$-move and the Conway potential function for links, J. Knot Theory Ramifications 13 (2004), 877-893.
[13] T. Shibuya and A. Yasuhara: Boundary links are self delta-equivalent to trivial links, Math. Proc. Cambridge Philos. Soc. 143 (2007), 449-458.
[14] T. Shibuya and A. Yasuhara: Self $\Delta$-equivalence of links in solid tori in $S^{3}$, Kobe J. Math. 25 (2008), 59-64.
[15] S. Suzuki: Local knots of 2-spheres in 4-manifolds, Proc. Japan Acad. 45 (1969), 34-38.
[16] A. Yasuhara: Self delta-equivalence for links whose Milnor's isotopy invariants vanish, Trans. Amer. Math. Soc. 361 (2009), 4721-4749.

Jean-Baptiste Meilhan<br>Institut Fourier Université Grenoble 1<br>100 rue des Maths - BP 74<br>38402 St Martin d'Hères<br>France<br>e-mail: jean-baptiste.meilhan@ujf-grenoble.fr<br>Akira Yasuhara<br>Tokyo Gakugei University<br>Department of Mathematics<br>Koganeishi, Tokyo 184-8501<br>Japan<br>e-mail: yasuhara@u-gakugei.ac.jp


[^0]:    1991 Mathematics Subject Classification. Primary 57M25; Secondary 57M27.
    The second author is partially supported by a Grant-in-Aid for Scientific Research (C) (\#20540065) of the Japan Society for the Promotion of Science.

[^1]:    ${ }^{1}$ The result is given in [15] for knots in $S^{3}$, but it can be easily extended to the link case.

