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A GENERALIZATION OF HALL QUASIFIELDS

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1. Introduction

Let $Q=Q(+, \circ)$ be a right quasifield which satisfies the following conditions:

(1.1) Q is a two dimensional left vector space over its kernel K with a basis $\{1, \lambda\}$.

(1.2) There exist two mappings r and s from $K^* = K - \{0\}$ into K such that every element $\xi = a + b\lambda$ of Q not in K satisfies the equation $\xi^2 - r(b)\xi - s(b) = 0$.

(1.3) Each element of K commutes with all the elements of Q .

Several examples of such Q are known. For example, the Hall quasifields satisfy the conditions above, where r and s are constant functions and the quadratic polynomial $x^2 - rx - s$ is irreducible over K . Moreover, the quasifields which correspond to the spread sets constructed by Narayana Rao and Satyanarayana [3] also satisfy the conditions above, where $r(x) = 3x^{-1}$, $s(x) = 2x^{-2}$ and $K = GF(5^{2n-1})$.

The purpose of this paper is to study the quasifields satisfying the conditions (1.1)-(1.3). In §2 we prove the following theorem which gives a condition for $Q(+, \circ)$ to be a quasifield.

Theorem 1. *Let K be a field and let r and s be mappings from K^* into K such that (i) $x^2 - r(u)x - s(u)$ is irreducible over K for each $u \in K^*$ and (ii) $v^2 - r(x)v - s(x) = wx$ has a unique solution in K^* for each $v \in K$, $w \in K^*$. Let $Q = \{x + y\lambda \mid x, y \in K\}$ be a left vector space over K . If a multiplication \circ on Q is defined by*

$$(z + t\lambda) \circ (x + y\lambda) = \begin{cases} zx - ty^{-1}F(x, y) + (zy - tx + t r(y))\lambda & \text{if } y \neq 0, \\ zx + tx\lambda & \text{if } y = 0, \end{cases}$$

where $F(x, y) = x^2 - r(y)x - s(y)$, then $Q(+, \circ)$ is a quasifield which satisfies (1.1)-(1.3).

Let $K = GF(q)$ and let Φ_K be the set of the ordered pairs (r, s) such that r and s satisfy (i) and (ii) of Theorem 1. The spread set $\Sigma_{r,s}$ which corre-

sponds to $(r, s) \in \Phi_K$ is defined as follows: $\Sigma_{r,s} = \{M(x, y) \mid x, y \in K\}$, where $M(x, 0) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ for $x \in K$ and $M(x, y) = \begin{pmatrix} x & y \\ f(x, y) & g(x, y) \end{pmatrix}$ for $x \in K$ and $y \in K^\#$. Here $f(x, y) = -y^{-1}(x^2 - r(y)x - s(y))$ and $g(x, y) = -x + r(y)$.

Let $\pi_{r,s}$ be the translation plane constructed by $\Sigma_{r,s}$ and set $L(x, y) = \{(v, vM(x, y)) \mid v \in K \times K\}$ for $x, y \in K$, $L(\infty) = \{(0, 0, v) \mid v \in K \times K\}$. Let G be the linear translation complement of $\pi_{r,s}$ and set $\Delta = \{L(x, 0) \mid x \in K\} \cup \{L(\infty)\}$ and $\Omega = \{L(x, y) \mid x \in K, y \in K^\#\}$. In §3 we prove the following theorem.

Theorem 2. *If $G_{L(\infty), L(0,0)}$ is transitive on Ω , then $r(x) = ax^n$ and $s(x) = bx^{2n}$ for some $a, b \in K$ and n with $0 \leq n \leq q-2$.*

The Hall planes and the planes of Narayana Rao and Satyanarayana satisfy the condition of this theorem. But an element of Φ_K is not always represented in this form (Remark 3.6.).

Throughout the paper notations are standard and taken from [1] and [2]. All sets and groups are finite except in §2.

2. Proof of Theorem 1

Let Q be a set with two binary operations $+$, \circ satisfying the assumption of Theorem 1. Since Q is a left vector space, the following holds.

Lemma 2.1. *$(Q, +)$ is an abelian group.*

Lemma 2.2. *Let $a, b, c, d \in K$ and assume $a + b\lambda \neq 0$ and $c + d\lambda \neq 0$. Then the equation $(a + b\lambda)(x + y\lambda) = c + d\lambda$ has a unique solution for $x + y\lambda$ in $Q^\# = Q - \{0\}$.*

Proof. (2.1) is equivalent to

$$ax - by^{-1}(x^2 - r(y)x - s(y)) = c, \quad (2.2)$$

$$b r(y) + ay - bx = d \text{ if } y \neq 0$$

or
$$ax = c, bx = d \text{ if } y = 0. \quad (2.3)$$

By the second equation of (2.2),

$$b r(y) = bx - ay + d \quad (2.4)$$

Substituting this into the first equation of (2.2), we have

$$y^{-1}(dx + b s(y)) = c. \quad (2.5)$$

Hence $b s(y) + dx = cy$. By this and the second equation of (2.2),

$$d^2 - bd r(y) - b^2 s(y) = (ad - bc) y. \quad (2.6)$$

Therefore (2.2) is equivalent to (2.4) and (2.6) when $b \neq 0$.

Assume $b=0$ and $d=0$. Then $a \neq 0$ and $c \neq 0$. Hence (2.1) has no solution in $Q-K$ and has a unique solution $a^{-1}c+0\lambda$ in $K^\#$

Assume $b=0$ and $d \neq 0$. Then $a \neq 0$. By (2.3), (2.1) has no solution in K and by (2.2) it has a unique solution $a^{-1}c+a^{-1}d\lambda$ in $Q-K$.

Assume $b \neq 0$ and $ad-bc=0$. Then (2.6) is equivalent to $(b^{-1}d)^2-r(y)(b^{-1}d)-s(y)=0$. By the assumption (i) of Theorem 1, (2.1) has no solution in $Q-K$. Therefore it has a unique solution $b^{-1}d+0\lambda$ in $K^\#$.

Assume $b \neq 0$ and $ad-bc \neq 0$. Then (2.1) has no solution in K by (2.3). Since $b \neq 0$, (2.6) is equivalent to $(b^{-1}d)^2-r(y)(b^{-1}d)-s(y)=b^{-2}(ad-bc)y$ and hence (2.6) has a unique solution y' in $K^\#$ by the assumption (ii) of Theorem 1. Let x' be the unique solution of $br(y')=bx-ay'+d$. Then $x'+y'\lambda$ is a unique solution of (2.1).

Lemma 2.3. *Let $a, b, c, d \in K$ and assume $a+b\lambda \neq 0, c+d\lambda \neq 0$. Then the equation*

$$(x+y\lambda)(a+b\lambda) = c+d\lambda \tag{2.7}$$

has a unique solution for $x+y\lambda$ in $Q^\#$.

Proof. If $b=0$, (2.7) has a unique solution $a^{-1}c+a^{-1}d\lambda$. Assume $b \neq 0$. Then (2.7) is equivalent to linear equations

$$\begin{aligned} xa-yb^{-1}(a^2-r(b)a-s(b)) &= c, \\ xb+y(r(b)-a) &= d. \end{aligned} \tag{2.8}$$

Since $a(r(b)-a)-b(-b^{-1}(a^2-r(b)a-s(b)))=-s(b) \neq 0$ by the assumption (i) of Theorem 1, (2.8) has a unique solution $(x, y) \neq (0, 0)$. Thus (2.7) has a unique solution in $Q^\#$.

Proof of Theorem 1.

It follows immediately from the definition that $Q(+, \circ)$ satisfies the following.

$$\xi 1 = 1\xi = \xi \text{ for all } \xi \in Q. \tag{2.9}$$

$$(\xi+\eta)\mu = \xi\mu+\eta\mu \text{ for all } \xi, \eta, \mu \in Q. \tag{2.10}$$

$$\xi 0 = 0 \text{ for all } \xi \in Q. \tag{2.11}$$

By Lemmas 2.1-2.3 and (2.9)-(2.11), Q is a weak quasifield. Since Q is a finite dimensional vector space over K , it is a quasifield by Theorem 7.3 of [1]. Thus we have the theorem.

Suppose $|K| < \infty$. The spread set $\Sigma_{r,s} = \{M(x, y) | x, y \in K\}$ which

corresponds to the above quasifield is defined as follows: Let $K=GF(q)$ and let $M(x, y)=\begin{pmatrix} x & y \\ f(x, y) & g(x, y) \end{pmatrix}$ be a 2×2 matrix over K . Define $M(x, y) \in \Sigma_{r,s}$ if and only if $\lambda \circ (x+y\lambda) = f(x, y) + g(x, y)\lambda$. Then we have $M(x, 0) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ for $x \in K$ and $M(x, y) = \begin{pmatrix} x & y \\ f(x, y) & g(x, y) \end{pmatrix}$, where $f(x, y) = -y^{-1}(x^2 - r(y)x - s(y))$ and $g(x, y) = -x + r(y)$ for $x \in K$ and $y \in K^*$.

3. The proof of Theorem 2

Throughout this section let p be a prime and $K=GF(q)$, $q=p^m$. We use the following notations.

$$K^2 = \{k^2 \mid k \in K\}$$

Φ_K the set of ordered pairs (r, s) of r and s which satisfy the conditions (i) and (ii) of Theorem 1

$M_2(K)$ the set of 2×2 matrices over K

$tr(M)$ the trace of a matrix M of $M_2(K)$

$\det(M)$ the determinant of a matrix M of $M_2(K)$

Let $(r, s) \in \Phi_K$ and $\Sigma_{r,s}$ the corresponding spread set defined in the last paragraph of §2. Let $\pi_{r,s}$ be the translation plane of order q^2 constructed from $\Sigma_{r,s}$.

Lemma 3.1. (i) *Let $M(x, y) \in \Sigma_{r,s}$. If $y \neq 0$, then $r(y) = tr(M(x, y))$ and $s(y) = -\det(M(x, y))$.*

(ii) *Let $P, M \in M_2(K)$ with $\det(P) \neq 0$ and set $P^{-1}MP = \begin{pmatrix} * & y \\ * & * \end{pmatrix}$. Assume $y \neq 0$. Then $P^{-1}MP \in \Sigma_{r,s}$ if and only if $r(y) = tr(P^{-1}MP)$ and $s(y) = -\det(P^{-1}MP)$.*

Proof. By an easy computation we have (i).

The "only if" part of (ii) is an immediate consequence of (i). Assume $r(y) = tr(P^{-1}MP)$ and $s(y) = -\det(P^{-1}MP)$ and set $P^{-1}MP = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$. Since $tr(P^{-1}MP) = tr(M)$ and $\det(P^{-1}MP) = \det(M)$, we have

$$r(y) = tr(M) = x + u \tag{3.1}$$

and

$$s(y) = -\det(M) = -xu + yz. \tag{3.2}$$

By (3.1), $u = -x + r(y)$. Substituting this into (3.2) gives $s(y) = x^2 - r(y)x + yz$. As $y \neq 0$, $z = -y^{-1}(x^2 - r(y)x - s(y))$. Hence $\begin{pmatrix} x & y \\ z & u \end{pmatrix} = \begin{pmatrix} x & y \\ f(x, y) & g(x, y) \end{pmatrix} \in \Sigma_{r,s}$ by what we have mentioned in the last paragraph of §2.

Lemma 3.2. (i) *The equation $v^2 - r(x)v - s(x) = wx$ has a unique solution*

in $K^\#$ for any $v \in K$, $w \in K^\#$ if and only if $(x-y)v^2 - (xr(y) - yr(x))v - (xs(y) - ys(x)) \neq 0$ for any $v \in K$ and $x, y \in K^\#, x \neq y$.

(ii) Assume $p > 2$. Then $(r, s) \in \Phi_K$ if and only if the following two conditions are satisfied.

(a) $(r(y))^2 + 4s(y) \notin K^2$ for any $y \in K^\#$.

(b) $(xr(y) - yr(x))^2 + 4(x-y)(xs(y) - ys(x)) \notin K^2$ for any $x, y \in K^\#, x \neq y$.

Proof. Assume $(x-y)v^2 - (xr(y) - yr(x))v - (xs(y) - ys(x)) = 0$ for some $V \in K$ and $x, y \in K^\#, x \neq y$. Thdn $x(v^2 - r(y)v - s(y)) = y(v^2 - r(x)v - s(x))$. Hence $(v^2 - r(y)v - s(y))/y = (v^2 - r(x)v - s(x))/x$. Put $w = (v^2 - r(x)v - s(x))/x$. Then $w \neq 0$ as $v^2 - r(x)v - s(x) \neq 0$ by the assumption (i) of Theorem 1 and the equation $v^2 - r(\xi)v - s(\xi) = w\xi$ has at least two solutions for ξ .

Conversely, assume $v^2 - r(x)v - s(x) = wx$ and $v^2 - r(y)v - s(y) = wy$ for some $x, y \in K^\#, x \neq y$. Then $wxy = y(v^2 - r(x)v - s(x)) = x(v^2 - r(y)v - s(y))$. This gives $(x-y)v^2 - (xr(y) - yr(x))v - (xs(y) - ys(x)) = 0$. Therefore (i) holds.

Assume $p > 2$. Then it is well known that a quadratic equation $ax^2 + bx + c = 0$ over K has no solution in K if and only if $b^2 - 4ac \notin K^2$. Hence (ii) follows immediately from (i).

Lemma 3.3. Assume $|K| > 3$ and let $P, Q \in M_2(K)$. If $P + xQ \in \Sigma_{r,s}$ for any $x \in K$, then either (i) $Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $P \in \Sigma_{r,s}$ or (ii) P and Q are scalar matrices.

Proof. Set $\Sigma = \Sigma_{r,s}$, $P = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$, $i, j, k, l \in K$ and $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in K$. Then $P + xQ = \begin{pmatrix} i+ax & j+bx \\ k+cx & l+dx \end{pmatrix}$.

Assume $j = b = 0$. Then $P + xQ \in \Sigma$ if and only if $P + xQ$ is a scalar matrix. Hence $i + ax = l + dx$ and $k + cx = 0$. Since x is arbitrary, it follows that $i = l$, $a = d$ and $k = c = 0$. Thus (ii) holds when $j = b = 0$.

Assume $j \neq 0$ and $b = 0$. By Lemma 3.1, $P + xQ \in \Sigma$ if and only if $r(j) = tr(P + xQ)$ and $s(j) = -\det(P + xQ)$. Hence

$$r(j) = i + l + (a + d)x \tag{3.3}$$

and
$$s(j) = -adx^2 + (jc - al - id)x + jk - il. \tag{3.4}$$

Since (3.3) and (3.4) hold for all $x \in K$, we have

$$r(j) = i + l, \quad a + d = 0 \tag{3.5}$$

and
$$s(j) = jk - il, \quad jc - al - id = 0, \quad ad = 0 \tag{3.6}$$

Hence $a = d = 0$ so $jc = 0$. As $j \neq 0$, $c = 0$. Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus (i) holds when $j \neq 0$ and $b = 0$.

Assume $b \neq 0$. Set $x = -b^{-1}j$. Then $j + bx = 0$ and so $P + xQ$ is a scalar

matrix. Hence $k - cb^{-1}j = 0$ and $i - ab^{-1}j = l - db^{-1}j$. Setting $w = -b^{-1}j$, we have $j = -bw$, $k = -cw$ and $l = i + aw - dw$. Putting $y = j + bx$ gives $P + xQ = \begin{pmatrix} ab^{-1}y + aw + i & y \\ b^{-1}cy & b^{-1}dy + aw + i \end{pmatrix}$. By Lemma 3.1, we have $r(y) = b^{-1}(a+d)y + 2(aw+i)$ and $s(y) = -b^{-2}(ad-bc)y^2 - b^{-1}(a+d)(aw+i)y - (aw+i)^2$. In particular

$$xr(y) - yr(x) = 2(i+aw)(x-y)$$

$$\text{and } xs(y) - ys(x) = (x-y)(b^{-2}(ad-bc)xy - (aw+i)^2). \quad (3.7)$$

If $p=2$, then $xr(y) - yr(x) = 0$ by (3.7). Hence we have a contradiction by Lemma 3.2 (i).

If $p > 2$, then $(xr(y) - yr(x))^2 + 4(x-y)(xs(y) - ys(x)) = 4(x-y)^2 b^{-2}(ad-bc)xy \notin K^2$ by Lemma 3.2 (ii). Let x be any element of $K^2 - \{0\}$ and let y be any element of $K - K^2$. Then clearly $4(x-y)^2 b^{-2}xy \notin K^2$. Hence $ad-bc$ must be an element of K^2 . From this $x'y' \notin K^2$ for any $x', y' \in K^\sharp$, $x' \neq y'$. In particular $K^2 = \{0, 1\}$, which implies $K = GF(3)$. This contradicts the assumption.

Set $L(x, y) = \{(v, vM(x, y)) \mid v \in K \times K\}$ for $x, y \in K$, $L(\infty) = \{(0, 0, v) \mid v \in K \times K\}$ and $\Delta = \{L(x, 0) \mid x \in K\} \cup \{L(\infty)\}$, $\Omega = \{L(x, y) \mid x \in K, y \in K^\sharp\}$. Then $\Delta \cup \Omega$ is the set of lines of $\pi_{r,s}$ through $(0, 0, 0, 0)$. Let G be the linear translation complement of $\pi_{r,s}$ and set $H = G_{L(\infty), L(0,0)}$, the stabilizer of the lines $L(\infty)$ and $L(0, 0)$ in G . Let $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a nonsingular 4×4 matrix, where $A, B, C, D \in M_2(K)$. Then the following criterion is well known: σ is an element of G if and only if the following conditions are satisfied.

(1) If C is nonsingular, then $C^{-1}D \in \Sigma_{r,s}$. (In this case $L(\infty)\sigma = L(u, v)$, where $C^{-1}D = M(u, v) \in \Sigma_{r,s}$.)

(2) If C is singular, then $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and D is nonsingular. (In this case $L(\infty)\sigma = L(\infty)$.)

(3) If $A + M(x, y)C$ is nonsingular, then $(A + M(x, y)C)^{-1}(B + M(x, y)D) \in \Sigma_{r,s}$. (In this case $L(x, y)\sigma = L(u, v)$, where $(A + M(x, y)C)^{-1}(B + M(x, y)D) = M(u, v) \in \Sigma_{r,s}$.)

(4) If $A + M(x, y)C$ is singular, then $A + M(x, y)C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. (In this case $L(x, y)\sigma = L(\infty)$.)

Lemma 3.4. *Assume either r or s is not a constant function. Let A, B, C and D be elements of $M_2(K)$ and set $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. If $\sigma \in H$, then the following hold.*

(i) $B = C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, $D = kA$ for some $a, d, k \in K^\sharp$ and $c \in K$.

(ii) $r(a^{-1}dky) = kr(y)$, $s(a^{-1}dky) = k^2s(y)$. Moreover $L(x, y) = L(k(x + a^{-1}cy), ka^{-1}dy)$ for all $x, y \in K, y \neq 0$.

Proof. Since σ fixes $L(\infty)$ and $L(0, 0)$, $B=D=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $\sigma=\begin{pmatrix} A & O \\ O & D \end{pmatrix}$ and $A^{-1}MD\in\Sigma$ for any $M\in\Sigma$. In particular $A^{-1}M(x, 0)D=xA^{-1}D\in\Sigma$ for each $x\in K$. If $K=GF(3)$, $s(1)=s(-1)\neq 0$ and $(r(1)+r(-1))^2=(r(1))^2=(r(-1))^2=-(s(\pm 1))-1$ by Lemma 3.2 (ii). Hence r and s are constant functions. Applying Lemma 3.3 we have $A^{-1}D=k$ for some $k\in K^\sharp$. Hence $A^{-1}MD=kA^{-1}MA\in\Sigma$ for any $M\in\Sigma$. Put $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $x\in K$, $y\in K^\sharp$. Then $kA^{-1}M(x, y)A=M(u, v)$ for some $u\in K$ and $v\in K^\sharp$. Set $M(x, y)=\begin{pmatrix} x & y \\ f & g \end{pmatrix}$ and $M(u, v)=\begin{pmatrix} u & v \\ f' & g' \end{pmatrix}$. Since $\begin{pmatrix} kx & ky \\ kf & kg \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & v \\ f' & g' \end{pmatrix}$, we have

$$bkx+dky = av+bg', \tag{3.8}$$

$$bkf+dkg = cv+dg' \tag{3.9}$$

and
$$akx+cky = au+bf'. \tag{3.10}$$

Hence $d(av)-b(cv)=d(bkx+dky)-b(bkf+dkg)$ by (3.8) and (3.9). From this we have

$$(ad-bc)v=k((b^2y^{-1})x^2+(2bd-b^2y^{-1}r(y))x+(d^2y-b^2y^{-1}s(y)-bdr(y))). \tag{3.11}$$

On the other hand, by Lemma 3.1 (i), we have

$$\begin{aligned} kr(y) &= r(v) \\ \text{and} \quad k^2s(y) &= s(v) \end{aligned} \tag{3.12}$$

We argue $b=0$. Suppose $b\neq 0$ and set $\Psi_y=\{v|r(v)=kr(y)\}$ for $y\in K^\sharp$. By (3.11), $|\Psi_y|\geq(q+1)/2$ if $p>2$ and $|\Psi_y|\geq q/2$ if $p=2$ for any $y\in K^\sharp$. Thus $|\Psi_y|>|K^\sharp|/2$ for any $y\in K^\sharp$ so we have $\Psi_y\cap\Psi_z\neq\phi$ for all $y, z\in K^\sharp$. This implies that r is a constant function. Similarly s is also a constant function. This contradicts the assumption. Therefore $b=0$.

From (3.8) and (3.10), $v=a^{-1}dky$ and $u=kx+a^{-1}cky$. Hence $r(a^{-1}dky)=kr(y)$, $s(a^{-1}dky)=k^2s(y)$ by (3.12) and $L(x, y)\sigma=L(u, v)=L(kx+a^{-1}cky, a^{-1}dky)$. Thus lemma holds.

Lemma 3.5. Set $\Omega_y=\{L(x, y)|x\in K\}$ for $y\in K^\sharp$ and $H_1=\left\{\begin{pmatrix} A & O \\ O & A \end{pmatrix} \mid A=\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, a\in K^\sharp, c\in K\right\}$. Then $H_1\subset H$. Moreover H_1 acts on Ω_y and is transitive on Ω_y for each $y\in K^\sharp$.

Proof. Let $\sigma=\begin{pmatrix} A & O \\ O & A \end{pmatrix}\in H_1$. Since $A^{-1}M(x, y)A=\begin{pmatrix} x+a^{-1}cy & y \\ * & * \end{pmatrix}$, $A^{-1}M(x, y)A\in\Sigma$ by Lemma 3.1 (ii) so $\sigma\in H$. Moreover $L(x, y)\sigma=L(x+a^{-1}cy, y)$. Since $a\in K^\sharp$ and $c\in K$ are arbitrary, we have the lemma.

Proof of Theorem 2.

Any mapping from K^\sharp into K can be uniquely written in the form $\sum_{i=0}^{q-2} c_i x^i$, $c_i \in K$, $0 \leq i \leq q-2$. Set $r(y) = \sum_{i=0}^{q-2} c_i y^i$ and $s(y) = \sum_{i=0}^{q-2} d_i y^i$. We may assume that r or s is not a constant function. By Lemma 3.4 (ii), $L(0, 1)\sigma = L(a^{-1}ck, a^{-1}dk)$, where $\sigma = \begin{pmatrix} A & O \\ O & kA \end{pmatrix}$, $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. By Lemma 3.5, H is transitive on Ω if and only if $K^\sharp = \{a^{-1}dk | \begin{pmatrix} A & O \\ O & kA \end{pmatrix} \in H, A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\}$. Set $h = a^{-1}dk$. Then, by Lemma 3.4,

$$r(hy) = ad^{-1}hr(y) \quad (3.13)$$

and
$$s(hy) = (ad^{-1})^2 h^2 s(y) \quad (3.14)$$

Suppose H is transitive on Ω . Then, for any $h \in K^\sharp$, there exist a and d in K^\sharp which satisfy (3.13) and (3.14) simultaneously. Hence $\sum_{i=0}^{q-1} c_i h^i y^i = \sum_{i=0}^{q-2} c_i ad^{-1} h y^i$ and $\sum_{i=0}^{q-2} d_i h^i y^i = \sum_{i=0}^{q-2} d_i (ad^{-1})^2 h^2 y^i$. Therefore $c_i h^i = c_i ad^{-1} h$ and $d_i h^i = d_i (ad^{-1})^2 h^2$ for all i with $0 \leq i \leq q-2$. If $c_m \neq 0$ and $c_n \neq 0$ for some m, n with $0 \leq m, n \leq q-2$, then $h^{m-1} = h^{n-1} = ad^{-1}$ and so $h^{m-n} = 1$ for any $h \in K^\sharp$. Thus $m = n$, so that we have $r(y) = c_n y^n$. By a similar argument above, we have $s(y) = d_t y^t$ for some t with $0 \leq t \leq q-2$.

Since $(r(hy))^2 / (r(y))^2 = s(hy) / s(y)$ by (3.13) and (3.14), $c_n^2 h^{2n} y^{2n} / c_n^2 y^{2n} = d_t h^t y^t / d_t y^t$. From this $h^{2n-t} = 1$ for any $h \in K^\sharp$. Thus $t \equiv 2n \pmod{q-1}$.

REMARK 3.6. An element of Φ_K is not always represented in the form $(r(x), s(x))$, $r(x) = ax^n$, $s(x) = bx^{2n}$. We list some of such examples below, which were obtained by a computer search using Lemma 3.2.

- (i) $K = GF(7)$, $r(x) = 4x^5 + 6x^4$, $s(x) = 6x^5 + 3x^4 + 6x^3 + 4x^2 + 3$.
- (ii) $K = GF(11)$, $r(x) = 5x^9 + 6x^7 + 9x^6 + 2$, $s(x) = 3x^9 + 5x^8 + 6x^7 + 9x^6 + 4x^5 + 10x^4 + 9x^3 + 2x^2 + 9$.
- (iii) $K = GF(11)$, $r(x) = 2x^9 + 6x^8 + 4x^7 + 3x^6 + 8$, $s(x) = 5x^9 + x^8 + 8x^6 + 10x^5 + x^4 + 2x^3 + 10x^2 + 10$.

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