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<th>On the basic G-space in equivalent K-theory</th>
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1. Introduction

Let $G$ be a compact, connected Lie group such that $\pi_\lambda(G)$ is torsion free and let $\mathcal{A}_G$ denote the category of compact, locally contractible $G$-spaces of finite covering dimension and $G$-maps. Throughout this paper all spaces will be supposed to be in $\mathcal{A}_G$ and $K^K_G$ will denote the equivariant $K$-theory defined in [5]. We use the following definition by Hodgkin [1].

**Definition.** A $G$-space $Z$ is called a basic $G$-space if the following conditions are satisfied.

(i) $K^K_G(Z)$ is projective as an $R(G)$ (= $K^K_{\operatorname{point}}$)-module.

(ii) For any $X \in \mathcal{A}_G$ the external product homomorphism

$$K^K_G(Z) \otimes K^K_G(X) \rightarrow K^K_G(Z \times X)$$

is an isomorphism.

Using the notation of [1], Snaith [6] proved that if $G$ is a torus then $\Gamma^G_\ast(-,-)$ vanishes.

In this paper we give a simple proof of Snaith's theorem ([6], Theorem 2.11) and show that if $G$ is $SU(n)$, $U(n)$, $Sp(n)$ or $G_2$ then $\Gamma^G_\ast(-,-)$ vanishes.

Consider the construction of the K"unneth formula spectral sequence [1], then we see that the above statements are equivalent to the following

**Theorem 1.1** (Snaith [6]). Let $T$ be a torus and $Z$ a $T$-space. If $K^K_T(Z)$ is projective as an $R(T)$-module then the $T$-space $Z$ is a basic $T$-space.

**Theorem 1.2.** Let $G$ denote the (special) unitary group ($SU(n)$) $U(n)$, the sympletic group $Sp(n)$ or the exceptional group $G_2$, and let $Z$ be a $G$-space. If $K^K_G(Z)$ is projective as an $R(G)$-module then the $G$-space $Z$ is a basic $G$-space.

In the following sections we denote by $\mu$ the external product homomorphism $K^K_G(X) \otimes K^K_G(Y) \rightarrow K^K_G(X \times Y)$.

2. Proof of (1.1)

**Lemma 2.1.** Let $T$ be the $n$-dimensional torus and $S$ a closed subgroup of $T$. If $K^K_T(Z)$ is projective as an $R(T)$-module for a $T$-space $Z$ then
\[ \mu : R(S) \otimes K^\#_{R(T)}(Z) \rightarrow K^\#_S(Z) \]

is isomorphic.

Proof. First we consider the following situation: Let \( T = Z_{m_1} \times \cdots \times Z_{m_{r-1}} \times S^1_r \times \cdots \times S^1_n \), \( S = Z_{m_1} \times \cdots \times Z_{m_{r-1}} \times S^1_r \times \cdots \times S^1_n \) where \( Z_{m_j} \) is a cyclic group of order \( n_{ij} \) and \( S^1 \) is the circle group, \((1 \leq j \leq r, r \leq k \leq n)\), such that \( Z_{m_i} \subset S^1_i \) and let \( Z \) be a \( T \)-space such that \( K^\#_S(Z) \) is \( R(T) \)-projective.

Let \( C(T/S) \) be the cone on \( T/S \). Then \( C(T/S) - T/S \) is isomorphic to the representation space \( V \) of the \( m_r \)-fold tensor product of the canonical 1-dimensional, non-trivial representation \( t_r \) of \( S^1_r \) since \( T/S = S^1_r/Z_{m_r} \) is isomorphic to \( S^1 \).

Consider the exact sequence for the pair \( (C(T/S) \times Z, T/S \times Z) \) then we get the diagram

\[ \begin{array}{cccc}
K^*_{R(T)}(V \times Z) & \rightarrow & K^*_{R(T)}(Z) & \rightarrow & K^*_S(Z) \\
\phi_* & \downarrow & & \downarrow & \\
& & \rightarrow & & \\
& & K^*_S(Z) & \rightarrow & 0
\end{array} \]

where the row is an exact sequence, \( \phi_* \) is the Thom isomorphism and \( j^* \phi_*(1) = 1 - t_r^{m_r} \). Since \( K^*_S(Z) \) is \( R(T) \)-projective and \( R(S^1_r) \) has no zero divisors we get a short exact sequence

\[ 0 \rightarrow K^*_S(Z) \xrightarrow{(1-t_r^{m_r})} K^*_S(Z) \rightarrow K^*_S(Z) \rightarrow 0 \]

from the above diagram.

Apply the functor \( \otimes_{R(T)} K^*_S(Z) \) to the exact sequence obtained by putting \( Z = a \) point in the above short exact sequence then we also have an exact sequence

\[ 0 \rightarrow K^*_{R(T)}(Z) \xrightarrow{(1-t_r^{m_r})} K^*_{R(T)}(Z) \rightarrow K^*_S(Z) \rightarrow 0 \]

Here consider the commutative diagram

\[ \begin{array}{cccc}
0 & \rightarrow & K^*_{R(T)}(Z) & \xrightarrow{f} & K^*_{R(T)}(Z) & \rightarrow & K^*_S(Z) & \rightarrow & 0 \\
\| & & \| & \downarrow & \| & \downarrow & & & \mu \\
0 & \rightarrow & K^*_{R(T)}(Z) & \xrightarrow{f} & K^*_{R(T)}(Z) & \rightarrow & R(S) \otimes_{R(T)} K^*_S(Z) & \rightarrow & 0
\end{array} \]

where the rows are exact and \( f = (1-t_r^{m_r}) \cdot \). Then we see that \( \mu : R(S) \otimes R(T) K^*_S(Z) \rightarrow K^*_S(Z) \) is an isomorphism by the five lemma.

In the general case we may consider that \( T = S^1_1 \times \cdots \times S^1_r \times H \), \( S = Z_{m_1} \times \cdots \times Z_{m_j} \times H \) and \( Z_{m_j} \subset S^1_j \), \((1 \leq j \leq l)\), where \( H \) is a torus, by Proof of [1], Lemma 7.1 or [6], Lemma 2.3.
Put \( S_k = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k} \times S_{k+1}^1 \times \cdots \times S_3^1 \times H \) for \( 0 \leq k \leq l \). By the preceding discussion we have an isomorphism

\[
R(S_k) \otimes_{R(S_{k-1})} K_{S_{k-1}}^*(\mathbb{Z}) \to K_{S_k}^*(\mathbb{Z})
\]

for \( 1 \leq k \leq l \) inductively. This completes the proof of Lemma 2.1.

**Proof of (1.1).** \( K_{\mathbb{R}}^*(\mathbb{Z}) \otimes_{R(T)} K_{\mathbb{R}}^*(_,\mathbb{Z}) \) is a cohomology theory since \( K_{\mathbb{R}}^*(\mathbb{Z}) \) is \( R(T) \)-projective and \( K_{\mathbb{R}}^*(\mathbb{Z} \times _,\mathbb{Z}) \) is so. Using the Segal's spectral sequence [5] and the natural transformation \( \mu : K_{\mathbb{R}}^*(\mathbb{Z}) \otimes_{R(T)} K_{\mathbb{R}}^*(_,\mathbb{Z}) \to K_{T}^*(\mathbb{Z} \times _,\mathbb{Z}) \), compare these cohomology theories. Then Lemma 2.1 shows that \( \mu \) induces an isomorphism of the \( E_r \)-terms of these spectral sequences. Therefore this concludes (1.1).

**3. Proof of (1.2)**

Let \( T \) be a maximal torus of \( G \). According to [6], §3 it suffices to show that

\[
\mu_G = \mu : R(T) \otimes_{R(G)} R(T) \to K_{\mathbb{R}}^*(G/T)
\]

is an isomorphism

for a proof of (1.2). However, from Proof of [6], Theorem 3.6 we see that

\[
\mu_G \text{ is a monomorphism for any compact, connected Lie group } G \text{ such that } \pi_1(G) \text{ is free.}
\]

Therefore it suffices to prove that \( \mu_G \) is an epimorphism.

Now, since \( R(T) \) is a projective \( R(G) \)-module [4], we see by (1.1) that

\[
\text{(3.3) If (3.1) is true then the } T \text{-space } G/T \text{ is a basic } T \text{-space.}
\]

(1) **Proof for \( U(n) \).** This follows from [5], Proposition (3.9) (See [6], Corollary 3.7).

(2) **Proof for \( SU(n) \).** Let \( T \) be a maximal torus of \( U(n) \) and put \( ST = T \cap SU(n) \). Then \( ST \) is a maximal torus of \( SU(n) \) and \( SU(n)/ST \cong U(n)/T \) as \( T \)-spaces.

By (1) and (3.3), \( U(n)/T \) is a basic \( T \)-space and so

\[
K_{\mathbb{R}}^*(U(n)/T) \cong K_{\mathbb{R}}^*(ST \times U(n)/T)
\]

\[
\cong R(ST) \otimes_{R(T)} K_{\mathbb{R}}^*(U(n)/T)
\]

\[
\cong R(ST) \otimes_{R(U(n))} R(T).
\]

Hence we get the following commutative diagram
where $i: SU(n) \to U(n)$ is the inclusion of $SU(n)$, and this shows that $\mu$ is surjective for $G=SU(n)$.

(3) Proof for $Sp(n)$. We regard $Sp(n)$ as a closed subgroup of $U(2n)$ by the canonical embedding. Then $Sp(1)=SU(2)$ and so the proof for $Sp(1)$ follows from (2). We shall prove the case of (3) by induction on $n$.

Suppose $Sp(k)$ satisfies (3.1) for $1 \leq k \leq n-1$. Then (3.1) is true for $Sp(n-1) \times Sp(1)$. Because

$$K^\ast_{T_1 \times T_2}(Sp(n-1) \times Sp(1)/T_1 \times T_2) \cong K^\ast_{T_1}(Sp(n-1)/T_1) \otimes K^\ast_{T_2}(Sp(1)/T_2)$$

$$\cong R(T_1 \times T_2) \underset{R(Sp(n-1) \times Sp(1))}{\otimes} R(T_1 \times T_2)$$

where $T_1$ and $T_2$ are maximal tori of $Sp(n-1)$ and $Sp(1)$ respectively, by the inductive hypothesis and [3]. Therefore, by [6], Theorem 3.6 $Sp(n-1) \times Sp(1)/T$ is a basic $Sp(n-1) \times Sp(1)$-space and so

$$R(T) \underset{R(Sp(n-1) \times Sp(1))}{\otimes} K^\ast_{Sp(n-1) \times Sp(1)}(Sp(n)/T) \cong K^\ast_{T}(Sp(n)/T)$$

where $T$ is the standard maximal torus of $Sp(n)$. Hence it suffices to show that

$$R(T) \underset{R(Sp(n-1) \times Sp(1))}{\otimes} K^\ast_{Sp(n-1) \times Sp(1)}(Sp(n)/Sp(n-1) \times Sp(1))$$

is an isomorphism, because of $K^\ast_{Sp(n)/Sp(n-1) \times Sp(1)}(Sp(n-1) \times Sp(1)) \cong K^\ast_{Sp(n-1) \times Sp(1)}(Sp(n-1) \times Sp(1))$.

Put $R(T)=\mathbb{Z}[t_1, \ldots, t_n; t_1^{-1}, \ldots, t_n^{-1}]$, then $R(Sp(n))=\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ as a subring where $\sigma_k$ is the $k$-th elementary symmetric function in the $n$ variables $t_1+t_1^{-1}, \ldots, t_n+t_n^{-1}$ ([2], §13, Theorem 6.1).

Define the ring homomorphism $\phi: R(Sp(n))[\theta] \to R(Sp(n-1) \times Sp(1))$ by the restriction $R(Sp(n)) \to R(Sp(n-1) \times Sp(1))$ and the correspondence $\theta \mapsto t_n+t_n^{-1}$. Then we have

**Lemma 3.1.** $R(Sp(n))[\theta]/(\sum_{j=0}^{n}(-1)^j \sigma_j \theta^{-j}) \cong R(Sp(n-1) \times Sp(1))$.

**Proof.** By the definition of $\phi$, $\phi$ is surjective obviously.

If $\phi(f(\theta))=0$ for $f(\theta) \in R(Sp(n))$ then $(\theta-(t_n+t_n^{-1}))$ divides $f(\theta)$. By symmetry, $(\theta-(t_j+t_j^{-1}))$ divides $f(\theta)$ for $1 \leq j \leq n$. Hence $\sum_{j=0}^{n}(-1)^j \sigma_j \theta^{-j}$ divides $f(\theta)$. This shows Lemma 3.1.

The following lemma completes the proof for $Sp(n)$ by the preceding discussion.
Lemma 3.2. $\mu: R(T) \otimes R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(Sp(n)/Sp(n-1) \times Sp(1))$ is an isomorphism for any $n \geq 2$.

Proof. $Sp(n)/Sp(n-1) \times Sp(1)$ is homeomorphic to the projective space of dimension $n-1$ over the quaternion number field. By the canonical embedding $P^{n-2}(Q) \subset P^{n-1}(Q)$ we have an equivariant embedding $i: Sp(n-1)/Sp(n-2) \times Sp(1) \subset Sp(n)/Sp(n-1) \times Sp(1)$.

For simplicity we write $P^{n-1}(Q)$ for $Sp(n)/Sp(n-1) \times Sp(1)$. Then we have

(a) $\mu': R(T) \otimes R(Sp(n-2) \times Sp(1)) \rightarrow K_T^*(P^{n-1}(Q))$

by the inductive hypothesis and

(b) $\mu: R(T) \otimes R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(P^{n-1}(Q))$ is a monomorphism

by the analogous argument to the proof for (3.2). Moreover the $T$-space $P^{n-2}(Q)$ is isomorphic to the representation space $W$ of $t_1 t_{n-1} \oplus \cdots \oplus t_1 t_{n-1}$.

Consider the exact sequence for the pair $(P^{n-1}(Q), P^{n-2}(Q))$, then by Lemma 3.1, (a) and (b) we obtain the diagram

$$
\begin{array}{cccc}
0 & \rightarrow & K_T^*(W) & \rightarrow K_T^*(P^{n-1}(Q)) & \rightarrow K_T^*(P^{n-2}(Q)) & \rightarrow 0 \\
& & \varphi * & \mu & \cong & \mu' \\
R(T) & \rightarrow & R(T)[\theta]/(\sum_{j=0}^{n-1} (-1)^j \sigma_j \theta^{n-j}) & \rightarrow & R(T)[\theta]/(\sum_{j=0}^{n-1} (-1)^j \sigma_j \theta^{n-j-1}) & \rightarrow 0
\end{array}
$$

where the row is an exact sequence, $\varphi *$ is the Thom isomorphism and the definition of $\theta'$ and $\sigma_j'$, $(0 \leq j \leq n-1)$, are similar to that of $\theta$ and $\sigma_j$. In this diagram we see that $i^*$ is surjective from the fact that $i^*(\mu(\theta)) = \mu'(\theta')$, and furthermore we can easily check that $j^* \varphi_*(1) = (t_{n-1})^{n-1} \sum_{j=0}^{n-1} (-1)^j \sigma_j \mu(\theta)^{n-j-1}$. Therefore we see that $\mu$ is surjective. q.e.d.

This completes the induction.

(4) Proof for $G_2$. $G_2$ contains $SU(3)$ as a closed subgroup of maximal rank and the homogeneous space $G_2/SU(3)$ is homeomorphic to the unit sphere $S^9$.

Let $T$ denote a maximal torus of $SU(3)$ and put $R(T) = Z[t_1, t_2, t_3; t_1^{-1}, t_2^{-1}, t_3^{-1}](t_1 t_2 t_3 - 1)$. Moreover we denote the representation space of $t_1 t_2 t_3$ by $W$ and the unit sphere in $R \oplus W$ by $S(R \oplus W)$ where $R$ is the real number field. Then we see easily that

Lemma 3.3. $G_2/SU(3)$ is homeomorphic to $S(R \oplus W)$ as $T$-spaces.

The following lemma completes the proof for $G_2$ by the same reason as for $Sp(n)$.
Lemma 3.4. \( \mu : R(T) \otimes R(SU(3)) \to K^*_T(G_2/SU(3)) \) is an epimorphism.

Proof. Consider the exact sequence for the pair consisting of the unit ball \( D(W) \) and the unit sphere \( S(W) \) in \( W \), then we have the diagram

\[
0 \to K^*_T(W) \xrightarrow{j^*} K^*_T(D(W)) \xrightarrow{i^*} K^*_T(S(W)) \to 0
\]

where the row is exact and \( \varphi_* \) is the Thom isomorphism, and then we get

\[ K^*_T(S(W)) = R(T)/(\lambda_2 - \lambda_1) \]

since \( j^* \varphi_*(1) = \lambda_2 - \lambda_1 \) where \( \lambda_1 \) and \( \lambda_2 \) are the ring generators of \( R(SU(3)) \) as in [2], §13, Theorem 3.1.

Next we divide \( S(R \oplus W) \) into two closed \( T \)-subspaces \( D^\pm \) as follows: Put \( D^\pm = \{(r, z_1, z_2, z_3) \in S(R \oplus W); r \geq 0 \) or \( r \leq 0 \} \) and then \( D^+ \cup D^- = S(R \oplus W) \) and \( D^+ \cap D^- = S(W) \). Consider the diagram obtained by the Mayer-Vietoris exact sequence for the triple \( (S(R \oplus W); D^+, D^-) \) then we obtain the diagram

\[
0 \to K^*_T(S(R \oplus W)) \xrightarrow{(j^*_+, j^*_-)} K^*_T(D^+) \oplus K^*_T(D^-) \xrightarrow{i^*_+ - i^*_-} K^*_T(S(W)) \to 0
\]

where the row is exact and \( j^\pm \): \( D^\pm \to S(R \oplus W) \) and \( i^\pm \): \( S(W) \to D^\pm \) are the inclusion maps. Then we see that \( K^*_T(S(R \oplus W)) \) is isomorphic to the submodule of \( R(T) \oplus R(T) \) over \( R(T) \) generated by \((1,1)\) and \((\lambda_2 - \lambda_1, 0)\), and \( \mu \) satisfies \((j^*_+, j^*_-)\mu(1 \otimes 1) = (1, 1)\) and \((j^*, j^*)\mu(1 \otimes \lambda_1) = (\lambda_1, \lambda_2)\). This shows that \( \mu \) is surjective.

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References
