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ON THE BASIC G -SPACE IN EQUIVARIANT K -THEORY

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1. Introduction

Let G be a compact, connected Lie group such that $\pi_1(G)$ is torsion free and let \mathcal{A}_G denote the category of compact, locally contractible G -spaces of finite covering dimension and G -maps. Throughout this paper all spaces will be supposed to be in \mathcal{A}_G and K_G^* will denote the equivariant K -theory defined in [5]. We use the following definition by Hodgkin [1].

DEFINITION. A G -space Z is called a *basic G -space* if the following conditions are satisfied.

- (i) $K_G^*(Z)$ is projective as an $R(G)$ ($=K_G^*(\text{point})$)-module.
- (ii) For any $X \in \mathcal{A}_G$ the external product homomorphism

$$K_G^*(Z) \otimes_{R(G)} K_G^*(X) \rightarrow K_G^*(Z \times X)$$

is an isomorphism.

Using the notation of [1], Snaith [6] proved that if G is a torus then $\Gamma_G^*(-, -)$ vanishes.

In this paper we give a simple proof of Snaith's theorem ([6], Theorem 2.11) and show that if G is $SU(n)$, $U(n)$, $Sp(n)$ or G_2 then $\Gamma_G^*(-, -)$ vanishes.

Consider the construction of the Künneth formula spectral sequence [1], then we see that the above statements are equivalent to the following

Theorem 1.1 (Snaith [6]). *Let T be a torus and Z a T -space. If $K_T^*(Z)$ is projective as an $R(T)$ -module then the T -space Z is a basic T -space.*

Theorem 1.2. *Let G denote the (special) unitary group ($SU(n)$) $U(n)$, the symplectic group $Sp(n)$ or the exceptional group G_2 , and let Z be a G -space. If $K_G^*(Z)$ is projective as an $R(G)$ -module then the G -space Z is a basic G -space.*

In the following sections we denote by μ the external product homomorphism $K_G^*(X) \otimes_{R(G)} K_G^*(Y) \rightarrow K_G^*(X \times Y)$.

2. Proof of (1.1)

Lemma 2.1. *Let T be the n -dimensional torus and S a closed subgroup of T . If $K_T^*(Z)$ is projective as an $R(T)$ -module for a T -space Z then*

$$\mu: R(S) \otimes_{R(T)} K_T^*(Z) \rightarrow K_S^*(Z)$$

is isomorphic.

Proof. First we consider the following situation: Let $T=Z_{m_1} \times \cdots \times Z_{m_{r-1}} \times S_r^1 \times S_{r+1}^1 \times \cdots \times S_n^1$, $S=Z_{m_1} \times \cdots \times Z_{m_{r-1}} \times Z_{m_r} \times S_{r+1}^1 \times \cdots \times S_n^1$ where Z_{m_j} is a cyclic group of order m_j and S_k^1 is the circle group, ($1 \leq j \leq r, r \leq k \leq n$), such that $Z_{m_r} \subset S_r^1$, and let Z be a T -space such that $K_T^*(Z)$ is $R(T)$ -projective.

Let $C(T/S)$ be the cone on T/S . Then $C(T/S)-T/S$ is isomorphic to the representation space V of the m_r -fold tensor product of the canonical 1-dimensional, non-trivial representation t_r of S_r^1 since $T/S=S_r^1/Z_{m_r}$ is isomorphic to S^1 .

Consider the exact sequence for the pair $(C(T/S) \times Z, T/S \times Z)$ then we get the diagram

$$\begin{array}{ccccccc} \rightarrow & K_T^*(V \times Z) & \xrightarrow{j^*} & K_T^*(Z) & \rightarrow & K_S^*(Z) & \rightarrow \\ & \uparrow \varphi_* & & & & & \\ & K_T^*(Z) & & & & & \end{array}$$

where the row is an exact sequence, φ_* is the Thom isomorphism and $j^* \varphi_*(1) = 1 - t_r^{m_r}$. Since $K_T^*(Z)$ is $R(T)$ -projective and $R(S_r^1)$ has no zero divisors we get a short exact sequence

$$0 \rightarrow K_T^*(Z) \xrightarrow{(1-t_r^{m_r}) \cdot} K_T^*(Z) \rightarrow K_S^*(Z) \rightarrow 0$$

from the above diagram.

Apply the functor $\otimes_{R(T)} K_T^*(Z)$ to the exact sequence obtained by putting $Z=a$ point in the above short exact sequence then we also have an exact sequence

$$0 \rightarrow K_T^*(Z) \xrightarrow{(1-t_r^{m_r}) \cdot} K_T^*(Z) \rightarrow R(S) \otimes_{R(T)} K_T^*(Z) \rightarrow 0$$

Here consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_T^*(Z) & \xrightarrow{f} & K_T^*(Z) & \longrightarrow & K_S^*(Z) \longrightarrow 0 \\ & & \parallel & & \parallel & & \uparrow \mu \\ 0 & \rightarrow & K_T^*(Z) & \xrightarrow{f} & K_T^*(Z) & \rightarrow & R(S) \otimes_{R(T)} K_T^*(Z) \rightarrow 0 \end{array}$$

where the rows are exact and $f=(1-t_r^{m_r}) \cdot$. Then we see that $\mu: R(S) \otimes_{R(T)} K_T^*(Z) \rightarrow K_S^*(Z)$ is an isomorphism by the five lemma.

In the general case we may consider that $T=S_1^1 \times \cdots \times S_l^1 \times H$, $S=Z_{m_1} \times \cdots \times Z_{m_l} \times H$ and $Z_{m_j} \subset S_j^1$, ($1 \leq j \leq l$), where H is a torus, by Proof of [1], Lemma 7.1 or [6], Lemma 2.3.

Put $S_k = Z_{m_1} \times \cdots \times Z_{m_k} \times S_{k+1}^1 \times \cdots \times S_l^1 \times H$ for $0 \leq k \leq l$. By the preceding discussion we have an isomorphism

$$R(S_k) \otimes_{R(S_{k-1})} K_{S_{k-1}}^*(Z) \rightarrow K_{S_k}^*(Z)$$

for $1 \leq k \leq l$ inductively. This completes the proof of Lemma 2.1.

Proof of (1.1). $K_T^*(Z) \otimes_{R(T)} K_T^*(-)$ is a cohomology theory since $K_T^*(Z)$ is $R(T)$ -projective and $K_T^*(Z \times -)$ is so. Using the Segal's spectral sequence [5] and the natural transformation $\mu: K_T^*(Z) \otimes_{R(T)} K_T^*(-) \rightarrow K_T^*(Z \times -)$, compare these cohomology theories. Then Lemma 2.1 shows that μ induces an isomorphism of the E_2 -terms of these spectral sequences. Therefore this concludes (1.1).

3. Proof of (1.2)

Let T be a maximal torus of G . According to [6], §3 it suffices to show that

$$(3.1) \quad \mu_G = \mu: R(T) \otimes_{R(G)} R(T) \rightarrow K_T^*(G/T) \text{ is an isomorphism}$$

for a proof of (1.2). However, from Proof of [6], Theorem 3.6 we see that

$$(3.2) \quad \mu_G \text{ is a monomorphism for any compact, connected Lie group } G \text{ such that } \pi_1(G) \text{ is free.}$$

Therefore it suffices to prove that μ_G is an epimorphism.

Now, since $R(T)$ is a projective $R(G)$ -module [4], we see by (1.1) that

$$(3.3) \quad \text{If (3.1) is true then the } T\text{-space } G/T \text{ is a basic } T\text{-space.}$$

(1) *Proof for $U(n)$.* This follows from [5], Proposition (3.9) (See [6], Corollary 3.7).

(2) *Proof for $SU(n)$.* Let T be a maximal torus of $U(n)$ and put $ST = T \cap SU(n)$. Then ST is a maximal torus of $SU(n)$ and $SU(n)/ST \cong U(n)/T$ as T -spaces.

By (1) and (3.3), $U(n)/T$ is a basic T -space and so

$$\begin{aligned} K_{ST}^*(U(n)/T) &\cong K_T^*(T/ST \times U(n)/T) \\ &\cong R(ST) \otimes_{R(T)} K_T^*(U(n)/T) \\ &\cong R(ST) \otimes_{R(U(n))} R(T). \end{aligned}$$

Hence we get the following commutative diagram

$$\begin{array}{ccc}
 K_{ST}^*(SU(n)/ST) & \xleftarrow{\cong} & K_{ST}^*(U(n)/T) \\
 \mu \uparrow & & \uparrow \cong \\
 R(ST) \otimes_{R(SU(n))} R(ST) & \xleftarrow{1 \otimes i^*} & R(ST) \otimes_{R(U(n))} R(T)
 \end{array}$$

where $i: SU(n) \rightarrow U(n)$ is the inclusion of $SU(n)$, and this shows that μ is surjective for $G = SU(n)$.

(3) *Proof for $Sp(n)$.* We regard $Sp(n)$ as a closed subgroup of $U(2n)$ by the canonical embedding. Then $Sp(1) = SU(2)$ and so the proof for $Sp(1)$ follows from (2). We shall prove the case of (3) by induction on n .

Suppose $Sp(k)$ satisfies (3.1) for $1 \leq k \leq n-1$. Then (3.1) is true for $Sp(n-1) \times Sp(1)$. Because

$$\begin{aligned}
 K_{T_1 \times T_2}^*(Sp(n-1) \times Sp(1)/T_1 \times T_2) &\cong K_{T_1}^*(Sp(n-1)/T_1) \otimes K_{T_2}^*(Sp(1)/T_2) \\
 &\cong R(T_1 \times T_2) \otimes_{R(Sp(n-1) \times Sp(1))} R(T_1 \times T_2)
 \end{aligned}$$

where T_1 and T_2 are maximal tori of $Sp(n-1)$ and $Sp(1)$ respectively, by the inductive hypothesis and [3]. Therefore, by [6], Theorem 3.6 $Sp(n-1) \times Sp(1)/T$ is a basic $Sp(n-1) \times Sp(1)$ -space and so

$$R(T) \otimes_{R(Sp(n-1) \times Sp(1))} K_{Sp(n-1) \times Sp(1)}^*(Sp(n)/T) \cong K_T^*(Sp(n)/T)$$

where T is the standard maximal torus of $Sp(n)$. Hence it suffices to show that

$$R(T) \otimes_{R(Sp(n))} R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(Sp(n)/Sp(n-1) \times Sp(1))$$

is an isomorphism, because of $K_T^*(Sp(n)/Sp(n-1) \times Sp(1)) \cong K_{Sp(n-1) \times Sp(1)}^*(Sp(n)/T)$.

Put $R(T) = Z[t_1, \dots, t_n; t_1^{-1}, \dots, t_n^{-1}]$, then $R(Sp(n)) = Z[\sigma_1, \dots, \sigma_n]$ as a subring where σ_k is the k -th elementary symmetric function in the n variables $t_1 + t_1^{-1}, \dots, t_n + t_n^{-1}$ ([2], §13, Theorem 6.1).

Define the ring homomorphism $\phi: R(Sp(n))[\theta] \rightarrow R(Sp(n-1) \times Sp(1))$ by the restriction $R(Sp(n)) \rightarrow R(Sp(n-1) \times Sp(1))$ and the correspondence $\theta \mapsto t_n + t_n^{-1}$. Then we have

Lemma 3.1. $R(Sp(n))[\theta]/(\sum_{j=0}^n (-1)^j \sigma_j \theta^{n-j}) \cong R(Sp(n-1) \times Sp(1))$.

Proof. By the definition of ϕ , ϕ is surjective obviously.

If $\phi(f(\theta)) = 0$ for $f(\theta) \in R(Sp(n))$ then $(\theta - (t_n + t_n^{-1}))$ divides $f(\theta)$. By symmetry, $(\theta - (t_j + t_j^{-1}))$ divides $f(\theta)$ for $1 \leq j \leq n$. Hence $\sum_{j=0}^n (-1)^j \sigma_j \theta^{n-j}$ divides $f(\theta)$. This shows Lemma 3.1.

The following lemma completes the proof for $Sp(n)$ by the preceding discussion.

Lemma 3.2. $\mu: R(T) \otimes_{R(Sp(n))} R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(Sp(n)/Sp(n-1) \times Sp(1))$ is an isomorphism for any $n \geq 2$.

Proof. $Sp(n)/Sp(n-1) \times Sp(1)$ is homeomorphic to the projective space of dimension $n-1$ over the quaternion number field. By the canonical embedding $P^{n-2}(\mathbb{Q}) \subset P^{n-1}(\mathbb{Q})$ we have an equivariant embedding $i: Sp(n-1)/Sp(n-2) \times Sp(1) \subset Sp(n)/Sp(n-1) \times Sp(1)$.

For simplicity we write $P^{n-1}(\mathbb{Q})$ for $Sp(n)/Sp(n-1) \times Sp(1)$. Then we have

$$(a) \quad \mu': R(T) \otimes_{R(Sp(n-1))} R(Sp(n-2) \times Sp(1)) \xrightarrow{\cong} K_T^*(P^{n-2}(\mathbb{Q}))$$

by the inductive hypothesis and

$$(b) \quad \mu: R(T) \otimes_{R(Sp(n))} R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(P^{n-1}(\mathbb{Q}))$$
 is a monomorphism

by the analogous argument to the proof for (3.2). Moreover the T -space $P^{n-1}(\mathbb{Q}) - P^{n-2}(\mathbb{Q})$ is isomorphic to the representation space W of $t_1 t_n^{-1} \oplus \dots \oplus t_{n-1} t_n^{-1} \oplus t_1^{-1} t_n^{-1} \oplus \dots \oplus t_{n-1}^{-1} t_n^{-1}$.

Consider the exact sequence for the pair $(P^{n-1}(\mathbb{Q}), P^{n-2}(\mathbb{Q}))$, then by Lemma 3.1, (a) and (b) we obtain the diagram

$$\begin{array}{ccccccc}
 0 \rightarrow & K_T^*(W) & \xrightarrow{j^*} & K_T^*(P^{n-1}(\mathbb{Q})) & \xrightarrow{i^*} & K_T^*(P^{n-2}(\mathbb{Q})) & \rightarrow 0 \\
 & \uparrow \varphi_* & & \uparrow \mu & & \cong \uparrow \mu' & \\
 & R(T) & R(T)[\theta]/(\sum_{j=0}^n (-1)^j \sigma_j \theta^{n-j}) & & R(T)[\theta']/(\sum_{j=0}^{n-1} (-1)^j \sigma'_j \theta'^{n-j-1}) & & \\
 & & \uparrow & & & & \\
 & & 0 & & & &
 \end{array}$$

where the row is an exact sequence, φ_* is the Thom isomorphism and the definition of θ' and σ'_j , ($0 \leq j \leq n-1$), are similar to that of θ and σ_j . In this diagram we see that i^* is surjective from the fact that $i^*(\mu(\theta)) = \mu'(\theta')$, and furthermore we can easily check that $j^* \varphi_*(1) = (t_n^{-1})^{n-1} \sum_{j=0}^{n-1} (-1)^j \sigma'_j \mu(\theta)^{n-j-1}$. Therefore we see that μ is surjective. q.e.d.

This completes the induction.

(4) *Proof for G_2 .* G_2 contains $SU(3)$ as a closed subgroup of maximal rank and the homogeneous space $G_2/SU(3)$ is homeomorphic to the unit sphere S^6 .

Let T denote a maximal torus of $SU(3)$ and put $R(T) = Z[t_1, t_2, t_3; t_1^{-1}, t_2^{-1}, t_3^{-1}]/(t_1 t_2 t_3 - 1)$. Moreover we denote the representation space of $t_1 \oplus t_2 \oplus t_3$ by W and the unit sphere in $\mathbb{R} \oplus W$ by $S(\mathbb{R} \oplus W)$ where \mathbb{R} is the real number field.

Then we see easily that

Lemma 3.3. $G_2/SU(3)$ is homeomorphic to $S(\mathbb{R} \oplus W)$ as T -spaces.

The following lemma completes the proof for G_2 by the same reason as for $Sp(n)$.

Lemma 3.4. $\mu: R(T) \otimes_{R(\mathbb{G}_2)} R(SU(3)) \rightarrow K_{\mathbb{Z}}^*(G_2/SU(3))$ is an epimorphism.

Proof. Consider the exact sequence for the pair consisting of the unit ball $D(W)$ and the unit sphere $S(W)$ in W , then we have the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_{\mathbb{Z}}^*(W) & \xrightarrow{j^*} & K_{\mathbb{Z}}^*(D(W)) & \xrightarrow{i^*} & K_{\mathbb{Z}}^*(S(W)) \rightarrow 0 \\ & & \uparrow \varphi_* & & \parallel & & \\ & & R(T) & & R(T) & & \end{array}$$

where the row is exact and φ_* is the Thom isomorphism, and then we get

$$K_{\mathbb{Z}}^*(S(W)) = R(T)/(\lambda_2 - \lambda_1)$$

since $j^* \varphi_*(1) = \lambda_2 - \lambda_1$ where λ_1 and λ_2 are the ring generators of $R(SU(3))$ as in [2], §13, Theorem 3.1.

Next we divide $S(\mathbf{R} \oplus W)$ into two closed T -subspaces D^\pm as follows: Put $D^\pm = \{(r, z_1, z_2, z_3) \in S(\mathbf{R} \oplus W); r \geq 0 \text{ or } r \leq 0\}$ and then $D^+ \cup D^- = S(\mathbf{R} \oplus W)$ and $D^+ \cap D^- = S(W)$. Consider the diagram obtained by the Mayer-Vietoris exact sequence for the triple $(S(\mathbf{R} \oplus W); D^+, D^-)$ then we obtain the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_{\mathbb{Z}}^*(S(\mathbf{R} \oplus W)) & \xrightarrow{(j_+^*, j_-^*)} & K_{\mathbb{Z}}^*(D^+) \oplus K_{\mathbb{Z}}^*(D^-) & \xrightarrow{i_+^* - i_-^*} & K_{\mathbb{Z}}^*(S(W)) \rightarrow 0 \\ & & \uparrow \mu & & \parallel & & \parallel \\ & & R(T) \otimes_{R(\mathbb{G}_2)} R(SU(3)) & & R(T) \oplus R(T) & & R(T)/(\lambda_2 - \lambda_1) \end{array}$$

where the row is exact and $j_\pm: D^\pm \rightarrow S(\mathbf{R} \oplus W)$ and $i_\pm: S(W) \rightarrow D^\pm$ are the inclusion maps. Then we see that $K_{\mathbb{Z}}^*(S(\mathbf{R} \oplus W))$ is isomorphic to the submodule of $R(T) \oplus R(T)$ over $R(T)$ generated by $(1, 1)$ and $(\lambda_2 - \lambda_1, 0)$, and μ satisfies $(j_+^*, j_-^*)\mu(1 \otimes 1) = (1, 1)$ and $(j_+^*, j_-^*)\mu(1 \otimes \lambda_1) = (\lambda_1, \lambda_2)$. This shows that μ is surjective.

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