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Author(s)	Minami, Haruo
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Osaka University

## ON THE BASIC $G$ -SPACE IN EQUIVARIANT $K$ -THEORY

HARUO MINAMI

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### 1. Introduction

Let  $G$  be a compact, connected Lie group such that  $\pi_1(G)$  is torsion free and let  $\mathcal{A}_G$  denote the category of compact, locally contractible  $G$ -spaces of finite covering dimension and  $G$ -maps. Throughout this paper all spaces will be supposed to be in  $\mathcal{A}_G$  and  $K_G^*$  will denote the equivariant  $K$ -theory defined in [5]. We use the following definition by Hodgkin [1].

DEFINITION. A  $G$ -space  $Z$  is called a *basic  $G$ -space* if the following conditions are satisfied.

- (i)  $K_G^*(Z)$  is projective as an  $R(G)$  ( $=K_G^*(\text{point})$ )-module.
- (ii) For any  $X \in \mathcal{A}_G$  the external product homomorphism

$$K_G^*(Z) \otimes_{R(G)} K_G^*(X) \rightarrow K_G^*(Z \times X)$$

is an isomorphism.

Using the notation of [1], Snaith [6] proved that if  $G$  is a torus then  $\Gamma_G^*(-, -)$  vanishes.

In this paper we give a simple proof of Snaith's theorem ([6], Theorem 2.11) and show that if  $G$  is  $SU(n)$ ,  $U(n)$ ,  $Sp(n)$  or  $G_2$  then  $\Gamma_G^*(-, -)$  vanishes.

Consider the construction of the Künneth formula spectral sequence [1], then we see that the above statements are equivalent to the following

**Theorem 1.1** (Snaith [6]). *Let  $T$  be a torus and  $Z$  a  $T$ -space. If  $K_T^*(Z)$  is projective as an  $R(T)$ -module then the  $T$ -space  $Z$  is a basic  $T$ -space.*

**Theorem 1.2.** *Let  $G$  denote the (special) unitary group ( $SU(n)$ )  $U(n)$ , the symplectic group  $Sp(n)$  or the exceptional group  $G_2$ , and let  $Z$  be a  $G$ -space. If  $K_G^*(Z)$  is projective as an  $R(G)$ -module then the  $G$ -space  $Z$  is a basic  $G$ -space.*

In the following sections we denote by  $\mu$  the external product homomorphism  $K_G^*(X) \otimes_{R(G)} K_G^*(Y) \rightarrow K_G^*(X \times Y)$ .

### 2. Proof of (1.1)

**Lemma 2.1.** *Let  $T$  be the  $n$ -dimensional torus and  $S$  a closed subgroup of  $T$ . If  $K_T^*(Z)$  is projective as an  $R(T)$ -module for a  $T$ -space  $Z$  then*

$$\mu: R(S) \otimes_{R(T)} K_T^*(Z) \rightarrow K_S^*(Z)$$

is isomorphic.

Proof. First we consider the following situation: Let  $T=Z_{m_1} \times \cdots \times Z_{m_{r-1}} \times S_r^1 \times S_{r+1}^1 \times \cdots \times S_n^1$ ,  $S=Z_{m_1} \times \cdots \times Z_{m_{r-1}} \times Z_{m_r} \times S_{r+1}^1 \times \cdots \times S_n^1$  where  $Z_{m_j}$  is a cyclic group of order  $m_j$  and  $S_k^1$  is the circle group, ( $1 \leq j \leq r, r \leq k \leq n$ ), such that  $Z_{m_r} \subset S_r^1$ , and let  $Z$  be a  $T$ -space such that  $K_T^*(Z)$  is  $R(T)$ -projective.

Let  $C(T/S)$  be the cone on  $T/S$ . Then  $C(T/S)-T/S$  is isomorphic to the representation space  $V$  of the  $m_r$ -fold tensor product of the canonical 1-dimensional, non-trivial representation  $t_r$  of  $S_r^1$  since  $T/S=S_r^1/Z_{m_r}$  is isomorphic to  $S^1$ .

Consider the exact sequence for the pair  $(C(T/S) \times Z, T/S \times Z)$  then we get the diagram

$$\begin{array}{ccccccc} \rightarrow & K_T^*(V \times Z) & \xrightarrow{j^*} & K_T^*(Z) & \rightarrow & K_S^*(Z) & \rightarrow \\ & \uparrow \varphi_* & & & & & \\ & K_T^*(Z) & & & & & \end{array}$$

where the row is an exact sequence,  $\varphi_*$  is the Thom isomorphism and  $j^* \varphi_*(1) = 1 - t_r^{m_r}$ . Since  $K_T^*(Z)$  is  $R(T)$ -projective and  $R(S_r^1)$  has no zero divisors we get a short exact sequence

$$0 \rightarrow K_T^*(Z) \xrightarrow{(1-t_r^{m_r}) \cdot} K_T^*(Z) \rightarrow K_S^*(Z) \rightarrow 0$$

from the above diagram.

Apply the functor  $\otimes_{R(T)} K_T^*(Z)$  to the exact sequence obtained by putting  $Z=a$  point in the above short exact sequence then we also have an exact sequence

$$0 \rightarrow K_T^*(Z) \xrightarrow{(1-t_r^{m_r}) \cdot} K_T^*(Z) \rightarrow R(S) \otimes_{R(T)} K_T^*(Z) \rightarrow 0$$

Here consider the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & K_T^*(Z) & \xrightarrow{f} & K_T^*(Z) & \longrightarrow & K_S^*(Z) & \longrightarrow 0 \\ & \parallel & & \parallel & & \uparrow \mu & \\ 0 \rightarrow & K_T^*(Z) & \xrightarrow{f} & K_T^*(Z) & \rightarrow & R(S) \otimes_{R(T)} K_T^*(Z) & \rightarrow 0 \end{array}$$

where the rows are exact and  $f=(1-t_r^{m_r}) \cdot$ . Then we see that  $\mu: R(S) \otimes_{R(T)} K_T^*(Z) \rightarrow K_S^*(Z)$  is an isomorphism by the five lemma.

In the general case we may consider that  $T=S_1^1 \times \cdots \times S_l^1 \times H$ ,  $S=Z_{m_1} \times \cdots \times Z_{m_l} \times H$  and  $Z_{m_j} \subset S_j^1$ , ( $1 \leq j \leq l$ ), where  $H$  is a torus, by Proof of [1], Lemma 7.1 or [6], Lemma 2.3.

Put  $S_k = Z_{m_1} \times \cdots \times Z_{m_k} \times S_{k+1}^1 \times \cdots \times S_l^1 \times H$  for  $0 \leq k \leq l$ . By the preceding discussion we have an isomorphism

$$R(S_k) \otimes_{R(S_{k-1})} K_{S_{k-1}}^*(Z) \rightarrow K_{S_k}^*(Z)$$

for  $1 \leq k \leq l$  inductively. This completes the proof of Lemma 2.1.

*Proof of (1.1).*  $K_T^*(Z) \otimes_{R(T)} K_T^*(-)$  is a cohomology theory since  $K_T^*(Z)$  is  $R(T)$ -projective and  $K_T^*(Z \times -)$  is so. Using the Segal's spectral sequence [5] and the natural transformation  $\mu: K_T^*(Z) \otimes_{R(T)} K_T^*(-) \rightarrow K_T^*(Z \times -)$ , compare these cohomology theories. Then Lemma 2.1 shows that  $\mu$  induces an isomorphism of the  $E_2$ -terms of these spectral sequences. Therefore this concludes (1.1).

### 3. Proof of (1.2)

Let  $T$  be a maximal torus of  $G$ . According to [6], §3 it suffices to show that

$$(3.1) \quad \mu_G = \mu: R(T) \otimes_{R(G)} R(T) \rightarrow K_T^*(G/T) \text{ is an isomorphism}$$

for a proof of (1.2). However, from Proof of [6], Theorem 3.6 we see that

$$(3.2) \quad \mu_G \text{ is a monomorphism for any compact, connected Lie group } G \text{ such that } \pi_1(G) \text{ is free.}$$

Therefore it suffices to prove that  $\mu_G$  is an epimorphism.

Now, since  $R(T)$  is a projective  $R(G)$ -module [4], we see by (1.1) that

$$(3.3) \quad \text{If (3.1) is true then the } T\text{-space } G/T \text{ is a basic } T\text{-space.}$$

(1) *Proof for  $U(n)$ .* This follows from [5], Proposition (3.9) (See [6], Corollary 3.7).

(2) *Proof for  $SU(n)$ .* Let  $T$  be a maximal torus of  $U(n)$  and put  $ST = T \cap SU(n)$ . Then  $ST$  is a maximal torus of  $SU(n)$  and  $SU(n)/ST \cong U(n)/T$  as  $T$ -spaces.

By (1) and (3.3),  $U(n)/T$  is a basic  $T$ -space and so

$$\begin{aligned} K_{ST}^*(U(n)/T) &\cong K_T^*(T/ST \times U(n)/T) \\ &\cong R(ST) \otimes_{R(T)} K_T^*(U(n)/T) \\ &\cong R(ST) \otimes_{R(U(n))} R(T). \end{aligned}$$

Hence we get the following commutative diagram

$$\begin{array}{ccc}
 K_{ST}^*(SU(n)/ST) & \xleftarrow{\cong} & K_{ST}^*(U(n)/T) \\
 \mu \uparrow & & \uparrow \cong \\
 R(ST) \otimes_{R(SU(n))} R(ST) & \xleftarrow{1 \otimes i^*} & R(ST) \otimes_{R(U(n))} R(T)
 \end{array}$$

where  $i: SU(n) \rightarrow U(n)$  is the inclusion of  $SU(n)$ , and this shows that  $\mu$  is surjective for  $G = SU(n)$ .

(3) *Proof for  $Sp(n)$ .* We regard  $Sp(n)$  as a closed subgroup of  $U(2n)$  by the canonical embedding. Then  $Sp(1) = SU(2)$  and so the proof for  $Sp(1)$  follows from (2). We shall prove the case of (3) by induction on  $n$ .

Suppose  $Sp(k)$  satisfies (3.1) for  $1 \leq k \leq n-1$ . Then (3.1) is true for  $Sp(n-1) \times Sp(1)$ . Because

$$\begin{aligned}
 K_{T_1 \times T_2}^*(Sp(n-1) \times Sp(1)/T_1 \times T_2) &\cong K_{T_1}^*(Sp(n-1)/T_1) \otimes K_{T_2}^*(Sp(1)/T_2) \\
 &\cong R(T_1 \times T_2) \otimes_{R(Sp(n-1) \times Sp(1))} R(T_1 \times T_2)
 \end{aligned}$$

where  $T_1$  and  $T_2$  are maximal tori of  $Sp(n-1)$  and  $Sp(1)$  respectively, by the inductive hypothesis and [3]. Therefore, by [6], Theorem 3.6  $Sp(n-1) \times Sp(1)/T$  is a basic  $Sp(n-1) \times Sp(1)$ -space and so

$$R(T) \otimes_{R(Sp(n-1) \times Sp(1))} K_{Sp(n-1) \times Sp(1)}^*(Sp(n)/T) \cong K_T^*(Sp(n)/T)$$

where  $T$  is the standard maximal torus of  $Sp(n)$ . Hence it suffices to show that

$$R(T) \otimes_{R(Sp(n))} R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(Sp(n)/Sp(n-1) \times Sp(1))$$

is an isomorphism, because of  $K_T^*(Sp(n)/Sp(n-1) \times Sp(1)) \cong K_{Sp(n-1) \times Sp(1)}^*(Sp(n)/T)$ .

Put  $R(T) = Z[t_1, \dots, t_n; t_1^{-1}, \dots, t_n^{-1}]$ , then  $R(Sp(n)) = Z[\sigma_1, \dots, \sigma_n]$  as a subring where  $\sigma_k$  is the  $k$ -th elementary symmetric function in the  $n$  variables  $t_1 + t_1^{-1}, \dots, t_n + t_n^{-1}$  ([2], §13, Theorem 6.1).

Define the ring homomorphism  $\phi: R(Sp(n))[\theta] \rightarrow R(Sp(n-1) \times Sp(1))$  by the restriction  $R(Sp(n)) \rightarrow R(Sp(n-1) \times Sp(1))$  and the correspondence  $\theta \mapsto t_n + t_n^{-1}$ . Then we have

**Lemma 3.1.**  $R(Sp(n))[\theta]/(\sum_{j=0}^n (-1)^j \sigma_j \theta^{n-j}) \cong R(Sp(n-1) \times Sp(1))$ .

*Proof.* By the definition of  $\phi$ ,  $\phi$  is surjective obviously.

If  $\phi(f(\theta)) = 0$  for  $f(\theta) \in R(Sp(n))$  then  $(\theta - (t_n + t_n^{-1}))$  divides  $f(\theta)$ . By symmetry,  $(\theta - (t_j + t_j^{-1}))$  divides  $f(\theta)$  for  $1 \leq j \leq n$ . Hence  $\sum_{j=0}^n (-1)^j \sigma_j \theta^{n-j}$  divides  $f(\theta)$ . This shows Lemma 3.1.

The following lemma completes the proof for  $Sp(n)$  by the preceding discussion.

**Lemma 3.2.**  $\mu: R(T) \otimes_{R(Sp(n))} R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(Sp(n)/Sp(n-1) \times Sp(1))$  is an isomorphism for any  $n \geq 2$ .

Proof.  $Sp(n)/Sp(n-1) \times Sp(1)$  is homeomorphic to the projective space of dimension  $n-1$  over the quaternion number field. By the canonical embedding  $P^{n-2}(\mathbf{Q}) \subset P^{n-1}(\mathbf{Q})$  we have an equivariant embedding  $i: Sp(n-1)/Sp(n-2) \times Sp(1) \subset Sp(n)/Sp(n-1) \times Sp(1)$ .

For simplicity we write  $P^{n-1}(\mathbf{Q})$  for  $Sp(n)/Sp(n-1) \times Sp(1)$ . Then we have

$$(a) \quad \mu': R(T) \otimes_{R(Sp(n-1))} R(Sp(n-2) \times Sp(1)) \xrightarrow{\cong} K_T^*(P^{n-2}(\mathbf{Q}))$$

by the inductive hypothesis and

$$(b) \quad \mu: R(T) \otimes_{R(Sp(n))} R(Sp(n-1) \times Sp(1)) \rightarrow K_T^*(P^{n-1}(\mathbf{Q}))$$
 is a monomorphism

by the analogous argument to the proof for (3.2). Moreover the  $T$ -space  $P^{n-1}(\mathbf{Q}) - P^{n-2}(\mathbf{Q})$  is isomorphic to the representation space  $W$  of  $t_1 t_n^{-1} \oplus \dots \oplus t_{n-1} t_n^{-1} \oplus t_1^{-1} t_n^{-1} \oplus \dots \oplus t_{n-1}^{-1} t_n^{-1}$ .

Consider the exact sequence for the pair  $(P^{n-1}(\mathbf{Q}), P^{n-2}(\mathbf{Q}))$ , then by Lemma 3.1, (a) and (b) we obtain the diagram

$$\begin{array}{ccccccc}
 0 \rightarrow & K_T^*(W) & \xrightarrow{j^*} & K_T^*(P^{n-1}(\mathbf{Q})) & \xrightarrow{i^*} & K_T^*(P^{n-2}(\mathbf{Q})) & \rightarrow 0 \\
 & \varphi_* \uparrow & & \mu \uparrow & & \cong \uparrow \mu' & \\
 & R(T) & R(T)[\theta]/(\sum_{j=0}^n (-1)^j \sigma_j \theta^{n-j}) & & R(T)[\theta']/(\sum_{j=0}^{n-1} (-1)^j \sigma'_j \theta'^{n-j-1}) & & \\
 & & & \uparrow & & & \\
 & & & 0 & & & 
 \end{array}$$

where the row is an exact sequence,  $\varphi_*$  is the Thom isomorphism and the definition of  $\theta'$  and  $\sigma'_j$ , ( $0 \leq j \leq n-1$ ), are similar to that of  $\theta$  and  $\sigma_j$ . In this diagram we see that  $i^*$  is surjective from the fact that  $i^*(\mu(\theta)) = \mu'(\theta')$ , and furthermore we can easily check that  $j^* \varphi_*(1) = (t_n^{-1})^{n-1} \sum_{j=0}^{n-1} (-1)^j \sigma'_j \mu(\theta)^{n-j-1}$ . Therefore we see that  $\mu$  is surjective. q.e.d.

This completes the induction.

(4) *Proof for  $G_2$ .*  $G_2$  contains  $SU(3)$  as a closed subgroup of maximal rank and the homogeneous space  $G_2/SU(3)$  is homeomorphic to the unit sphere  $S^6$ .

Let  $T$  denote a maximal torus of  $SU(3)$  and put  $R(T) = Z[t_1, t_2, t_3; t_1^{-1}, t_2^{-1}, t_3^{-1}]/(t_1 t_2 t_3 - 1)$ . Moreover we denote the representation space of  $t_1 \oplus t_2 \oplus t_3$  by  $W$  and the unit sphere in  $\mathbf{R} \oplus W$  by  $S(\mathbf{R} \oplus W)$  where  $\mathbf{R}$  is the real number field.

Then we see easily that

**Lemma 3.3.**  $G_2/SU(3)$  is homeomorphic to  $S(\mathbf{R} \oplus W)$  as  $T$ -spaces.

The following lemma completes the proof for  $G_2$  by the same reason as for  $Sp(n)$ .

**Lemma 3.4.**  $\mu: R(T) \otimes_{R(\mathbb{G}_2)} R(SU(3)) \rightarrow K_{\mathbb{Z}}^*(G_2/SU(3))$  is an epimorphism.

Proof. Consider the exact sequence for the pair consisting of the unit ball  $D(W)$  and the unit sphere  $S(W)$  in  $W$ , then we have the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_{\mathbb{Z}}^*(W) & \xrightarrow{j^*} & K_{\mathbb{Z}}^*(D(W)) & \xrightarrow{i^*} & K_{\mathbb{Z}}^*(S(W)) \rightarrow 0 \\ & & \uparrow \varphi_* & & \parallel & & \\ & & R(T) & & R(T) & & \end{array}$$

where the row is exact and  $\varphi_*$  is the Thom isomorphism, and then we get

$$K_{\mathbb{Z}}^*(S(W)) = R(T)/(\lambda_2 - \lambda_1)$$

since  $j^* \varphi_*(1) = \lambda_2 - \lambda_1$  where  $\lambda_1$  and  $\lambda_2$  are the ring generators of  $R(SU(3))$  as in [2], §13, Theorem 3.1.

Next we divide  $S(\mathbf{R} \oplus W)$  into two closed  $T$ -subspaces  $D^\pm$  as follows: Put  $D^\pm = \{(r, z_1, z_2, z_3) \in S(\mathbf{R} \oplus W); r \geq 0 \text{ or } r \leq 0\}$  and then  $D^+ \cup D^- = S(\mathbf{R} \oplus W)$  and  $D^+ \cap D^- = S(W)$ . Consider the diagram obtained by the Mayer-Vietoris exact sequence for the triple  $(S(\mathbf{R} \oplus W); D^+, D^-)$  then we obtain the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & K_{\mathbb{Z}}^*(S(\mathbf{R} \oplus W)) & \xrightarrow{(j_+^*, j_-^*)} & K_{\mathbb{Z}}^*(D^+) \oplus K_{\mathbb{Z}}^*(D^-) & \xrightarrow{i_+^* - i_-^*} & K_{\mathbb{Z}}^*(S(W)) \rightarrow 0 \\ & & \uparrow \mu & & \parallel & & \parallel \\ & & R(T) \otimes_{R(\mathbb{G}_2)} R(SU(3)) & & R(T) \oplus R(T) & & R(T)/(\lambda_2 - \lambda_1) \end{array}$$

where the row is exact and  $j_\pm: D^\pm \rightarrow S(\mathbf{R} \oplus W)$  and  $i_\pm: S(W) \rightarrow D^\pm$  are the inclusion maps. Then we see that  $K_{\mathbb{Z}}^*(S(\mathbf{R} \oplus W))$  is isomorphic to the submodule of  $R(T) \oplus R(T)$  over  $R(T)$  generated by  $(1, 1)$  and  $(\lambda_2 - \lambda_1, 0)$ , and  $\mu$  satisfies  $(j_+^*, j_-^*)\mu(1 \otimes 1) = (1, 1)$  and  $(j_+^*, j_-^*)\mu(1 \otimes \lambda_1) = (\lambda_1, \lambda_2)$ . This shows that  $\mu$  is surjective.

OSAKA CITY UNIVERSITY

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