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## SPECTRAL ASYMPTOTICS FOR DIRICHLET ELLIPTIC OPERATORS WITH NON-SMOOTH COEFFICIENTS

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### Abstract

We consider a  $2m$ -th-order elliptic operator of divergence form in a domain  $\Omega$  of  $\mathbb{R}^n$ , assuming that the coefficients are Hölder continuous of exponent  $r \in (0, 1]$ . For the self-adjoint operator associated with the Dirichlet boundary condition we improve the asymptotic formula of the spectral function  $e(\tau^{2m}, x, y)$  for  $x = y$  to obtain the remainder estimate  $O(\tau^{n-\theta} + \text{dist}(x, \partial\Omega)^{-1}\tau^{n-1})$  with any  $\theta \in (0, r)$ , using the  $L^p$  theory of elliptic operators of divergence form. We also show that the spectral function is in  $C^{m-1, 1-\varepsilon}$  with respect to  $(x, y)$  for any small  $\varepsilon > 0$ . These results extend those for the whole space  $\mathbb{R}^n$  obtained by Miyazaki [19] to the case of a domain.

### Introduction

Let us consider a  $2m$ -th-order elliptic operator of divergence form

$$(0.1) \quad Au(x) = \sum_{|\alpha| \leq m, |\beta| \leq m} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x))$$

with  $L^\infty(\mathbb{R}^n)$  coefficients in  $\mathbb{R}^n$  and assume that the leading coefficients are in  $C^{0,r}(\mathbb{R}^n)$  for some  $r \in (0, 1]$ . Here we use the notation

$$D = (D_1, \dots, D_n), \quad D_j = -i \frac{\partial}{\partial x_j} \quad (j = 1, \dots, n), \quad i = \sqrt{-1}.$$

Let  $\Omega$  be a domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ ,  $A_{L^2(\Omega)}$  the self-adjoint realization associated with the Dirichlet boundary condition in  $\Omega$ , and  $e_\Omega(\tau, x, y)$  the spectral function of  $A_{L^2(\Omega)}$ .

We are interested in obtaining a better estimate for the remainder term of the asymptotic formula of  $e_\Omega(\tau, x, x)$  when the smoothness index  $r$  of the leading coefficients is given. For simplicity of notation we consider  $e_\Omega(\tau^{2m}, x, x)$  instead of  $e_\Omega(\tau, x, x)$  when

we give its asymptotic formulas. In [19] we showed that  $e_{\mathbb{R}^n}(\tau, x, y)$  is in  $C^{m-1, 1-\varepsilon}$  with respect to  $(x, y)$  for any small  $\varepsilon > 0$  and that the asymptotic formula

$$(0.2) \quad e_{\mathbb{R}^n}(\tau^{2m}, x, x) = c_A(x)\tau^n + O(\tau^{n-\theta}) \quad \text{as } \tau \rightarrow \infty$$

holds with any  $\theta \in (0, r)$  if  $\Omega = \mathbb{R}^n$ , where

$$c_A(x) = (2\pi)^{-n} \int_{\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} < 1} d\xi,$$

and  $O$ -estimate is uniform with respect to  $x$ . Formula (0.2) is based on the theorem of  $L^p$  resolvents of elliptic operators of divergence form in  $\mathbb{R}^n$  [18, Main Theorem] and the asymptotic formula for spectral functions of pseudodifferential operators due to Zielinski [30]. Now that we have established the  $L^p$  theory of elliptic operators under the Dirichlet boundary condition in [20, 21, 22], it is natural to try to extend the results for  $\mathbb{R}^n$  to the case  $\Omega \neq \mathbb{R}^n$ . Accordingly, the purpose of this paper is to show that  $e_{\Omega}(\tau, x, y)$  is in  $C^{m-1, 1-\varepsilon}$  with respect to  $(x, y)$  for any small  $\varepsilon > 0$  and to derive the asymptotic formula

$$(0.3) \quad e_{\Omega}(\tau^{2m}, x, x) = c_A(x)\tau^n + O(\tau^{n-\theta} + \text{dist}(x, \partial\Omega)^{-1}\tau^{n-1}) \quad \text{as } \tau \rightarrow \infty$$

with any  $\theta \in (0, r)$ .

To contrast with known results we set  $\delta(x) = \min\{1, \text{dist}(x, \partial\Omega)\}$  and note that (0.3) remains unchanged if we replace  $\text{dist}(x, \partial\Omega)$  by  $\delta(x)$ . In [10, 11, 17, 26] the asymptotic formula for  $e_{\Omega}(\tau^{2m}, x, x)$  was obtained with the remainder term of the form  $O(\delta(x)^{-\theta} \tau^{n-\theta})$ , where one can take any  $\theta \in (0, r/(r+3))$  in [10],  $\theta \in (0, r/(r+2))$  in [11, 26], and  $\theta \in (0, r/(r+1))$  in [17]. Our remainder estimate makes the range of  $\theta$  wider. In addition,  $O(\tau^{n-\theta} + \delta(x)^{-1}\tau^{n-1})$  is better than  $O(\delta(x)^{-\theta} \tau^{n-\theta})$ , since  $\delta(x)^{-\theta} \tau^{n-\theta} = \tau^{n-\theta^2}$  and  $\delta(x)^{-1}\tau^{n-1} = \tau^{n-\theta}$  if we choose  $x \in \Omega$  so that  $\delta(x) = \tau^{\theta-1}$ . Hence our estimate improves those in [10, 11, 17, 26]. Moreover, it appears that (0.3) splits the remainder term into two parts: one depending on the smoothness of the coefficients and one influenced by the boundary. When the coefficients are in  $C^\infty$ , it was proved independently by Brüning [4] and Tsujimoto [27] that (0.3) holds with  $\theta = 1$  (see also [13]).

In this paper, we derive (0.3) with any  $\theta \in (0, r)$  for a given  $r \in (0, 1]$  as a corollary of the proposition stating that if  $A_{L^2(\mathbb{R}^n)}$  satisfies (0.2) with some  $\theta \in (0, 1]$  then  $A_{L^2(\Omega)}$  satisfies (0.3) with the same  $\theta$ . In order to prove this proposition we follow the spirit of Hörmander [5] and Brüning [4]. We first estimate the difference between the resolvent kernel for  $A_{L^2(\Omega)}$  and that for  $A_{L^2(\mathbb{R}^n)}$ , then show that the kernel of  $\exp(-zA_{L^2(\Omega)}^{1/(2m)}) - \exp(-zA_{L^2(\mathbb{R}^n)}^{1/(2m)})$ , which is defined for  $\text{Re } z > 0$ , is analytically continued to some disk with center 0, and finally apply a Fourier Tauberian theorem.

We would like to emphasize that our results can be obtained without assuming  $2m > n$ . In most papers the assumption  $2m > n$  was essential, since the resolvent kernel has

singularities on the diagonal when  $2m \leq n$ . Otherwise, extra assumptions were needed such as  $D(A_{L^2(\Omega)}^k) \subset H^{2mk,2}(\Omega)$  for some  $k$  with  $2mk > n$ . Such additional assumptions are, however, not required with the help of the  $L^p$  theory for the Dirichlet problem in a domain. Instead of the regularity such as  $D(A_{L^2(\Omega)}^k) \subset H^{2mk,2}(\Omega)$ , which is impossible in the case of non-smooth coefficients, the  $L^p$  theory leads us to  $D(A_{L^2(\Omega)}^k) \subset C^{m-1,1-\varepsilon}(\Omega)$  for a small  $\varepsilon > 0$  if  $k$  is large enough. The idea of using the  $L^p$  theory for the case of non-smooth coefficients goes back to Beals [2], who considered elliptic operators of non-divergence form.

When  $\Omega$  is bounded, the spectrum of  $A_{L^2(\Omega)}$  consists only of eigenvalues with finite multiplicities accumulating only at  $\infty$ . Let  $N_\Omega(\tau)$  denote the number of the eigenvalues of  $A_{L^2(\Omega)}$  not exceeding  $\tau$ . The asymptotic behavior of  $N_\Omega(\tau)$  is related to that of the spectral function, for  $N_\Omega(\tau)$  is obtained by integrating  $e_\Omega(\tau, x, x)$  with respect to  $x$  over  $\Omega$ . Thanks to the min-max principle, the investigation for  $N_\Omega(\tau)$  has always been ahead of that for  $e_\Omega(\tau, x, x)$ . Improving the results in [10, 11, 12, 14, 16, 26], Zielinski [29] obtained the asymptotic formula

$$(0.4) \quad N_\Omega(\tau^{2m}) = c_{A,\Omega} \tau^n + O(\tau^{n-\theta}) \quad \text{as } \tau \rightarrow \infty$$

with any  $\theta \in (0, r)$  for a general boundary problem when  $2m > n$  (see also [28, 30]), where  $c_{A,\Omega} = \int_\Omega c_A(x) dx$ . In some special cases, including the case  $n = 1$ , Miyazaki [15, 16] showed that (0.4) holds with  $\theta = r$ . Formula (0.4) can be derived by combining (0.3) with the estimate  $|e_\Omega(\tau^{2m}, x, y)| \leq C\tau^n$ . Accordingly, we could say that the investigation for  $e_\Omega(\tau, x, x)$  has caught up with that for  $N_\Omega(\tau)$  as long as we treat the Dirichlet boundary condition, a domain with smooth boundary and the remainder term  $O(\tau^{n-\theta})$  with  $\theta < 1$ .

For the case of  $C^\infty$  coefficients we refer to [6, 7, 23], where the two-term asymptotic formula for  $N_\Omega(\tau)$  is also considered. It is known that  $\theta = 1$  is the best possible in (0.4) for the case of  $C^\infty$  coefficients. It is remarkable that (0.4) with  $\theta = 1$  was obtained by Zielinski [31, 32] when the coefficients are in  $C^{1,1}$ , and by Ivrii [8] when the coefficients are in  $C^{1,\varepsilon}$  for any small  $\varepsilon > 0$ . In [3, 9] some elaboration of these results on  $N_\Omega(\tau)$  is given in terms of the modulus of continuity.

## 1. Main results

Let us now state the main results precisely. Throughout this paper we assume the following conditions on the elliptic operator  $A$  defined in (0.1) and a domain  $\Omega \subset \mathbb{R}^n$ :

(H0)  $\Omega$  is a uniform  $C^1$  domain if  $n \geq 2$ , and  $\Omega$  is an interval of  $\mathbb{R}$  if  $n = 1$ ;

(H1) There exists  $\delta_A > 0$  such that the principal symbol  $a(x, \xi)$  satisfies

$$a(x, \xi) := \sum_{|\alpha|+|\beta|=m} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \geq \delta_A |\xi|^{2m} \quad \text{for } x \in \mathbb{R}^n, \xi \in \mathbb{R}^n;$$

(H2)  $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$  and  $a_{\alpha\beta} \in L^\infty(\mathbb{R}^n)$  for  $|\alpha| \leq m, |\beta| \leq m$ . In addition, the leading coefficients  $a_{\alpha\beta}$  with  $|\alpha| = |\beta| = m$  are uniformly continuous in  $\mathbb{R}^n$ .

For the definition of a uniform  $C^1$  domain or a domain having uniform  $C^1$  regularity we refer to [1, 25]. Here are two examples of uniform  $C^1$  domain: a domain with bounded  $C^1$  boundary; the domain defined by the set of points  $x = (x', x_n) \in \mathbb{R}^n$  satisfying  $x_n > \psi(x')$ , where  $\psi \in C^1(\mathbb{R}^{n-1})$  whose first derivatives are bounded and uniformly continuous in  $\mathbb{R}^{n-1}$ .

For  $1 \leq p \leq \infty$  and  $\sigma \in \mathbb{R}$  we denote by  $H^{\sigma,p}(\Omega)$  the  $L^p$  Sobolev space of order  $\sigma$  in  $\Omega$ . In particular, for  $\sigma = -k$  with an integer  $k > 0$ ,  $H^{-k,p}(\Omega)$  is the space of functions  $f$  written as

$$(1.1) \quad f = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad f_\alpha \in L^p(\Omega),$$

and the norm  $\|f\|_{H^{-k,p}(\Omega)}$  is defined by  $\|f\|_{H^{-k,p}(\Omega)} = \inf \sum_{|\alpha| \leq k} \|f_\alpha\|_{L^p(\Omega)}$ , where the infimum is taken over all  $\{f_\alpha\}_{|\alpha| \leq k}$  satisfying (1.1). The space  $H_0^{\sigma,p}(\Omega)$  is defined to be the completion of  $C_0^\infty(\Omega)$  in  $H^{\sigma,p}(\Omega)$ . Then  $A$  defines a bounded linear operator from  $H_0^{m,p}(\Omega)$  to  $H^{-m,p}(\Omega)$ . When we want to stress  $p$  or  $\Omega$ , we write  $A_{p,\Omega}$  or  $A_\Omega$  for  $A$ . The operator  $A_{L^p(\Omega)}$  in  $L^p(\Omega)$  is defined by

$$D(A_{L^p(\Omega)}) = \{u \in H_0^{m,p}(\Omega) : A_\Omega u \in L^p(\Omega)\},$$

$$A_{L^p(\Omega)}u = A_\Omega u \quad \text{for } u \in D(A_{L^p(\Omega)}).$$

As is well known, when  $p = 2$ , the operator  $A_{L^2(\Omega)}$  is a self-adjoint operator, and it is usually defined by a sesquilinear form

$$Q[u, v] = \int_\Omega \sum_{|\alpha| \leq m, |\beta| \leq m} a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha v(x)} dx$$

on  $H_0^{m,2}(\Omega) \times H_0^{m,2}(\Omega)$ .

For an integer  $j \geq 0$  and  $\sigma \in (0, 1]$  we denote by  $C^{j,\sigma}(\Omega)$  the space of  $j$  times continuously differentiable functions  $f$  such that the norm

$$\|f\|_{C^{j,\sigma}(\Omega)} = \sum_{0 \leq |\alpha| \leq j} \|\partial^\alpha f\|_{L^\infty(\Omega)} + \sum_{|\alpha|=j} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\sigma}$$

is finite. For  $h \in \mathbb{R}^n$ , functions  $f(x)$  and  $g(x, y)$  we set

$$\Omega_h = \{x \in \Omega : x + h \in \Omega\}, \quad \Delta_h f(x) = f(x + h) - f(x),$$

$$\Delta_{1,h} g(x, y) = g(x + h, y) - g(x, y), \quad \Delta_{2,h} g(x, y) = g(x, y + h) - g(x, y).$$

We define several constants, constant vectors, functions and a region as follows.

$$M_A = \max_{|\alpha| \leq m, |\beta| \leq m} \|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^n)}, \quad M_{A,r} = \max_{|\alpha|=|\beta|=m} \|a_{\alpha\beta}\|_{C^{0,r}(\mathbb{R}^n)}.$$

$$\zeta_A = (n, m, \delta_A, M_A), \quad \zeta_{A,r} = (n, m, \delta_A, M_A, M_{A,r}),$$

$$c_A(x) = (2\pi)^{-n} \int_{a(x,\xi) < 1} d\xi, \quad c_{A,\Omega} = \int_{\Omega} c_A(x) dx,$$

$$\omega_A(\varepsilon) = \max_{|\alpha|=|\beta|=m} \sup_{|h| \leq \varepsilon} \sup_{x \in \mathbb{R}^n} |a_{\alpha\beta}(x+h) - a_{\alpha\beta}(x)|,$$

$$\Lambda(R, \eta) = \{\lambda \in \mathbb{C} : |\lambda| \geq R, \eta \leq \arg \lambda \leq 2\pi - \eta\} \quad \text{for } R \geq 0, \eta \in \left(0, \frac{\pi}{2}\right).$$

By definition  $\omega_A(\varepsilon) \leq M_{A,r} \varepsilon^r$  holds if the leading coefficients are in  $C^{0,r}(\mathbb{R}^n)$ .

**Theorem 1.1.** *Assume (H0)–(H2). Then for  $|\alpha| < m, |\beta| < m$  the derivatives  $\partial_x^\alpha \partial_y^\beta e_\Omega(\tau, x, y)$  are Hölder continuous of exponent  $\sigma$  with respect to  $(x, y)$  for any  $\sigma \in (0, 1)$ . There exist  $C_1 = C(\zeta_A, \omega_A, \Omega)$  and  $C_2 = C(\sigma, \zeta_A, \omega_A, \Omega)$  such that*

$$(1.2) \quad |\partial_x^\alpha \partial_y^\beta e_\Omega(\tau^{2m}, x, y)| \leq C_1 \tau^{n+|\alpha|+|\beta|}$$

for  $(x, y) \in \Omega \times \Omega, \tau \geq 1,$

$$(1.3) \quad |\Delta_{1,h} \partial_x^\alpha \partial_y^\beta e_\Omega(\tau^{2m}, x, y)| \leq C_2 \tau^{n+|\alpha|+|\beta|+\sigma} |h|^\sigma$$

for  $h \in \mathbb{R}^n, (x, y) \in \Omega_h \times \Omega, \tau \geq 1,$

$$(1.4) \quad |\Delta_{2,h} \partial_x^\alpha \partial_y^\beta e_\Omega(\tau^{2m}, x, y)| \leq C_2 \tau^{n+|\alpha|+|\beta|+\sigma} |h|^\sigma$$

for  $h \in \mathbb{R}^n, (x, y) \in \Omega \times \Omega_h, \tau \geq 1.$

Theorem 1.1 will be proved in Section 2.

**Proposition 1.2.** *Assume (H0)–(H2). Then if there exist  $C_0 > 0$  and  $\theta \in (0, 1]$  such that*

$$(1.5) \quad |e_{\mathbb{R}^n}(\tau^{2m}, x, x) - c_A(x)\tau^n| \leq C_0 \tau^{n-\theta}$$

for  $x \in \Omega, \tau \geq 1,$  then there exists  $C = C(C_0, \theta, \zeta_A, \omega_A, \Omega)$  such that

$$(1.6) \quad |e_\Omega(\tau^{2m}, x, x) - c_A(x)\tau^n| \leq C(\tau^{n-\theta} + \text{dist}(x, \partial\Omega)^{-1} \tau^{n-1})$$

for  $x \in \Omega, \tau \geq 1.$

Proposition 1.2 will be proved in Section 4 after estimating the difference between the resolvent kernels for  $\Omega$  and  $\mathbb{R}^n$  in Section 3.

**Theorem 1.3.** *In addition to (H0)–(H2) we assume that the leading coefficients of  $A$  are in  $C^{0,r}(\mathbb{R}^n)$  for some  $r \in (0, 1]$ . Then for any  $\theta \in (0, r)$  there exists  $C = C(\theta, r, \zeta_{A,r}, \Omega)$  such that*

$$(1.7) \quad |e_{\Omega}(\tau^{2m}, x, x) - c_A(x)\tau^n| \leq C(\tau^{n-\theta} + \text{dist}(x, \partial\Omega)^{-1}\tau^{n-1})$$

for  $x \in \Omega, \tau \geq 1$ .

*Proof.* By [19, Theorem 2] estimate (1.5) holds for a given  $\theta \in (0, r)$ . Then Proposition 1.2 yields Theorem 1.3. □

As mentioned in the Introduction, the asymptotic formula for  $N_{\Omega}(\tau)$ , which Zielinski [29] proved, can be derived again as a corollary of Theorems 1.1 and 1.3.

**Corollary 1.4.** *In addition to (H0)–(H2) we assume that the leading coefficients of  $A$  are in  $C^{0,r}(\mathbb{R}^n)$  for some  $r \in (0, 1]$ , and that  $\Omega$  is bounded. Then for any  $\theta \in (0, r)$  there exists  $C = C(\theta, r, \zeta_{A,r}, \Omega)$  such that*

$$(1.8) \quad |N_{\Omega}(\tau^{2m}) - c_{A,\Omega}\tau^n| \leq C\tau^{n-\theta}$$

for  $\tau \geq 1$ .

*Proof.* Set  $\Omega_{\varepsilon} = \{x \in \Omega: \text{dist}(x, \partial\Omega) < \varepsilon\}$  for  $\varepsilon > 0$ . Since  $\Omega$  is a bounded  $C^1$  domain, it follows that  $|\Omega_{\varepsilon}| \leq C\varepsilon$  with some  $C$ . This implies  $\int_{\Omega \setminus \Omega_{\varepsilon}} \delta(x)^{-1} dx \leq C \log \varepsilon^{-1}$  for  $0 < \varepsilon < 1$  (see [14]). We evaluate

$$N_{\Omega}(\tau^{2m}) - c_{A,\Omega}\tau^n = \int_{\Omega} \{e(\tau^{2m}, x, x) - c_A(x)\tau^n\} dx$$

by using (1.7) on  $\Omega \setminus \Omega_{\varepsilon}$  and (1.2) with  $\alpha = \beta = 0$  on  $\Omega_{\varepsilon}$ , and set  $\varepsilon = \tau^{-1}$ . Since  $\tau^{n-1} \log \tau \leq C\tau^{n-\theta}$  for  $\theta < 1$ , we get (1.8). □

## 2. Rough estimates for spectral functions

By (H1) and Gårding’s inequality  $A_{L^2(\Omega)}$  is bounded from below. The assertions of Theorem 1.1 and Proposition 1.2 remain unchanged if we replace  $A$  by  $A + C$  with constant  $C$ . So in the following we may assume that  $A$  is positive without loss of generality. We start with the theorem on  $L^p$  resolvents.

**Theorem 2.1.** *Let  $p \in (1, \infty)$  and  $\eta \in (0, \pi/2)$ . Then there exist  $R = R(\eta, \zeta_A, \omega_A, \Omega) \geq 1$  and  $C = C(p, \eta, \zeta_A, \Omega)$  such that for  $\lambda \in \Lambda(R, \eta)$  the resolvent  $(A_{p,\Omega} - \lambda)^{-1}$  exists and satisfies*

$$(2.1) \quad \|(A_{p,\Omega} - \lambda)^{-1}\|_{H^{-j,p}(\Omega) \rightarrow H^{k,p}(\Omega)} \leq C|\lambda|^{-1+(j+k)/(2m)}$$

for  $0 \leq j \leq m$ ,  $0 \leq k \leq m$ . In addition, the resolvents are consistent in the sense that

$$(A_{p,\Omega} - \lambda)^{-1} f = (A_{q,\Omega} - \lambda)^{-1} f$$

for  $f \in H^{-m,p}(\Omega) \cap H^{-m,q}(\Omega)$ ,  $p \neq q \in (1, \infty)$ .

Proof. See [20, 21] for a domain with bounded  $C^{m+1}$  boundary and [22] for a uniform  $C^1$  domain.  $\square$

**REMARK 2.1.** By the definition of the Sobolev space of negative order (2.1) is equivalent to

$$\|D^\alpha (A_\Omega - \lambda)^{-1} D^\beta\|_{L^p(\Omega) \rightarrow L^p(\Omega)} \leq C' |\lambda|^{-1+(|\alpha|+|\beta|)/(2m)}$$

for  $|\alpha| \leq m$ ,  $|\beta| \leq m$  with some constant  $C' > 0$ .

Now that we have established Theorem 2.1, which is the theorem for a domain, Theorem 1.1 can be proved in the same way as [19, Theorem 1], which dealt with the case  $\Omega = \mathbb{R}^n$ . So we only give the outline of the proof.

**Lemma 2.2.** *Let  $j \geq 0$  be an integer and  $0 < \sigma < 1$ . Assume that  $S$  and  $T$  are bounded linear operators on  $L^2(\Omega)$  satisfying*

$$R(S) \subset C^{j,\sigma}(\Omega), \quad R(T^*) \subset C^{j,\sigma}(\Omega),$$

where  $R(S)$  is the range of  $S$  and  $T^*$  is the adjoint of  $T$ . Then  $ST$  is an integral operator with bounded continuous kernel  $K(x, y)$ . Furthermore, for  $|\alpha| \leq j$  and  $|\beta| \leq j$  the derivatives  $\partial_x^\alpha \partial_y^\beta K(x, y)$  are Hölder continuous of exponent  $\sigma$  and satisfy

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq \|D^\alpha S\|_{L^2(\Omega) \rightarrow L^\infty(\Omega)} \|D^\beta T^*\|_{L^2(\Omega) \rightarrow L^\infty(\Omega)}$$

for  $(x, y) \in \Omega \times \Omega$ ,

$$|\Delta_{1,h} \partial_x^\alpha \partial_y^\beta K(x, y)| \leq \|D^\alpha S\|_{L^2(\Omega) \rightarrow C^{0,\sigma}(\Omega)} \|D^\beta T^*\|_{L^2(\Omega) \rightarrow L^\infty(\Omega)} |h|^\sigma$$

for  $h \in \mathbb{R}^n$ ,  $(x, y) \in \Omega_h \times \Omega$ ,

$$|\Delta_{2,h} \partial_x^\alpha \partial_y^\beta K(x, y)| \leq \|D^\alpha S\|_{L^2(\Omega) \rightarrow L^\infty(\Omega)} \|D^\beta T^*\|_{L^2(\Omega) \rightarrow C^{0,\sigma}(\Omega)} |h|^\sigma$$

for  $h \in \mathbb{R}^n$ ,  $(x, y) \in \Omega \times \Omega_h$ .

**Lemma 2.3.** *For an integer  $k > 1 + n/(2m)$ ,  $\sigma \in (0, 1)$  and  $\eta \in (0, \pi/2)$  there exist  $R = R(k, \sigma, \eta, \zeta_A, \omega_A, \Omega) \geq 1$  and  $C = C(k, \sigma, \eta, \zeta_A, \Omega)$  such that*

$$\|D^\alpha (A - \lambda)^{-k}\|_{L^2(\Omega) \rightarrow L^\infty(\Omega)} \leq C |\lambda|^{-k+n/(4m)+|\alpha|/(2m)},$$



$$\|\Delta_h D^\alpha (A - \lambda)^{-k}\|_{L^2(\Omega) \rightarrow L^\infty(\Omega_h)} \leq C |\lambda|^{-k+n/(4m)+(|\alpha|+\sigma)/(2m)} |h|^\sigma$$

for  $h \in \mathbb{R}^n$ ,  $|\alpha| < m$  and  $\lambda \in \Lambda(R, \eta)$ .

Lemmas 2.2 and 2.3 are essentially the same as [19, Lemma 2.3] and [19, Lemma 3.1], respectively, which dealt with the case  $\Omega = \mathbb{R}^n$ . Lemma 2.2 is a slight extension of [25, Lemma 5.10].

Proof of Theorem 1.1. Let  $\{E_\tau\}$  be the spectral resolution of identity for  $A$ :

$$A = \int_0^\infty \tau dE_\tau.$$

Let  $k$  be as in Lemma 2.3. Since  $R(E_\tau) \subset D(A^k)$  and

$$\|(A - \lambda)^k E_\tau\|_{L^2(\Omega) \rightarrow L^2(\Omega)} = \max_{0 \leq s \leq \tau} (s - \lambda)^k \leq (\tau + |\lambda|)^k$$

for  $\tau \geq 0$  and  $\lambda < 0$ , we see from Lemma 2.3 and the equality  $D^\alpha E_\tau = D^\alpha (A - \lambda)^{-k} (A - \lambda)^k E_\tau$  that for any  $\sigma \in (0, 1)$  there is  $R \geq 1$  such that

$$(2.2) \quad \|D^\alpha E_\tau\|_{L^2(\Omega) \rightarrow L^\infty(\Omega)} \leq C |\lambda|^{-k+n/(4m)+|\alpha|/(2m)} (\tau + |\lambda|)^k,$$

$$(2.3) \quad \|\Delta_h D^\alpha E_\tau\|_{L^2(\Omega) \rightarrow L^\infty(\Omega_h)} \leq C |\lambda|^{-k+n/(4m)+(|\alpha|+\sigma)/(2m)} (\tau + |\lambda|)^k |h|^\sigma$$

for  $h \in \mathbb{R}^n$ ,  $|\alpha| < m$ ,  $\tau \geq 0$  and  $\lambda \leq -R$ . Applying Lemma 2.2 to  $E_\tau = E_\tau E_\tau^*$  and using (2.2), (2.3) with  $\lambda = -\max\{\tau, R\}$ , we obtain Theorem 1.1.  $\square$

### 3. Estimates for resolvent kernels

In this section we estimate the difference between the kernels of  $(A_{L^2(\Omega)}^k - \lambda)^{-1}$  and  $(A_{L^2(\mathbb{R}^n)}^k - \lambda)^{-1}$ , assuming that  $k$  is an integer satisfying

$$(3.1) \quad (k + 1)m > n.$$

As stated in the beginning of Section 2, we may assume that  $A$  is positive. So by Theorem 1.1 we have

$$(3.2) \quad |e_\Omega(\tau^{2m}, x, y)| \leq C \tau^n \quad \text{for } \tau \geq 0, \quad e_\Omega(\tau^{2m}, x, y) = 0 \quad \text{for } \tau < 0.$$

**Lemma 3.1.** *Let  $\sigma > n/(2m)$ , and assume that  $f \in C^1[0, \infty)$  satisfies*

$$(3.3) \quad |f(\tau)| \leq C(1 + \tau)^{-\sigma}, \quad |f'(\tau)| \leq C(1 + \tau)^{-\sigma-1}$$

for  $\tau \geq 0$  with some constant  $C$ . Then  $f(A_{L^2(\Omega)})$  is an integral operator with bounded and continuous kernel, which can be written as

$$(3.4) \quad \int_0^\infty f(\tau) d_\tau e_\Omega(\tau, x, y).$$

Proof. See [19, Lemma 3.2]. □

Let  $\lambda \in \mathbb{C} \setminus [0, \infty)$ . We note that  $k > n/(2m)$  if  $k$  satisfies (3.1). So by Lemma 3.1  $(A_{L^2(\Omega)}^k - \lambda)^{-1}$  is an integral operator with bounded and continuous kernel  $G_{\Omega, \lambda}^k(x, y)$ , which can be written as

$$(3.5) \quad G_{\Omega, \lambda}^k(x, y) = \int_0^\infty (\tau^k - \lambda)^{-1} d_\tau e_\Omega(\tau, x, y).$$

Integration by parts and (3.2) give

$$(3.6) \quad |G_{\Omega, \lambda}^k(x, y)| \leq C \int_0^\infty \frac{\tau^{k-1+n/(2m)}}{|\tau^k - \lambda|^2} d\tau = \frac{C}{k} \int_0^\infty \frac{s^{n/(2mk)}}{|s - \lambda|^2} ds \leq C \frac{|\lambda|^{n/(2mk)}}{d(\lambda)},$$

where  $d(\lambda) = \text{dist}(\lambda, [0, \infty))$ . Needless to say, here and in what follows the statements for  $\Omega$  are also valid for  $\mathbb{R}^n$ . For simplicity we write  $G_\lambda^k(x, y)$  for  $G_{\mathbb{R}^n, \lambda}^k(x, y)$ .

In order to evaluate  $G_{\Omega, \lambda}^k(x, y) - G_\lambda^k(x, y)$  we fix  $x_0 \in \Omega$  and  $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$  satisfying  $\text{supp } \varphi_0 \subset \{x \in \mathbb{R}^n : |x| < 1\}$ ,  $\varphi_0(x) = 1$  for  $|x| \leq 2^{-1}$ , and set

$$\varphi(x) = \varphi_0\left(\frac{x - x_0}{\delta(x_0)}\right).$$

Remember  $\delta(x) = \min\{1, \text{dist}(x, \partial\Omega)\}$ . Clearly,  $\text{supp } \varphi \subset \{x \in \mathbb{R}^n : |x - x_0| < \delta(x_0)\} \subset \Omega$ . For  $\lambda \in \mathbb{C} \setminus [0, \infty)$  let  $\mu_1, \dots, \mu_k$  be the distinct roots of the equation  $w^k = \lambda$  for  $w$ . For simplicity we set  $\mu = \mu_1$ . It is clear that  $|\mu_j| = |\mu|$  and  $\mu_j \in \Lambda(R^{1/k}, \eta/k)$  for  $j = 1, \dots, k$  if  $\lambda \in \Lambda(R, \eta)$  with some  $R > 0$  and  $\eta \in (0, \pi/2)$ . For  $1 \leq l \leq k$  we set

$$(3.7) \quad S_l(A_\Omega) = \prod_{j=1}^l (A_\Omega - \mu_j)^{-1}, \quad T_l(A) = \prod_{j=l}^k (A - \mu_j)^{-1}.$$

Remember that we simply write  $A$  for  $A_{\mathbb{R}^n}$ . Let  $R_\Omega: \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\Omega)$  be the restriction.

**Lemma 3.2.** *Assume that  $(A_\Omega - \mu_j)^{-1}$  exists for  $j = 1, \dots, k$ . Then it follows that*

$$(3.8) \quad (A_\Omega^k - \lambda)^{-1} \varphi R_\Omega - \varphi R_\Omega (A^k - \lambda)^{-1} = - \sum_{l=1}^k S_l(A_\Omega) R_\Omega [A, \varphi] T_l(A),$$

where  $[A, \varphi] = A\varphi - \varphi A$  and  $\varphi$  stands for the multiplication by the function  $\varphi(x)$ . Furthermore,  $R_\Omega[A, \varphi]$  can be written as

$$(3.9) \quad R_\Omega[A, \varphi] = \sum_{\alpha, \beta, \gamma} D^\alpha b_{\alpha\beta\gamma} \varphi^{(\gamma)} R_\Omega D^\beta$$

with some functions  $b_{\alpha\beta\gamma} \in L^\infty(\Omega)$  satisfying  $\|b_{\alpha\beta\gamma}\|_{L^\infty} \leq C(\zeta_A)$ , where the sum is taken over  $\alpha, \beta, \gamma$  satisfying  $|\alpha| \leq m, |\beta| \leq m, 0 < |\gamma| \leq m, |\alpha + \beta + \gamma| \leq 2m$ .

Proof. Since  $\text{supp } \varphi \subset \Omega$ , we have

$$\begin{aligned} (A_\Omega - \lambda)\varphi R_\Omega(A - \lambda)^{-1} &= R_\Omega(A - \lambda)\varphi(A - \lambda)^{-1} \\ &= R_\Omega\varphi + R_\Omega[A, \varphi](A - \lambda)^{-1}, \end{aligned}$$

which gives

$$(3.10) \quad (A_\Omega - \lambda)^{-1}\varphi R_\Omega = \varphi R_\Omega(A - \lambda)^{-1} - (A_\Omega - \lambda)^{-1}R_\Omega[A, \varphi](A - \lambda)^{-1}.$$

Noting  $(A_\Omega^k - \lambda)^{-1} = \prod_{j=1}^k (A_\Omega - \mu_j)^{-1}$  and using (3.10) repeatedly with  $\lambda = \mu_1, \dots, \mu_k$ , we obtain (3.8). By the Leibniz formula and its variant

$$[D^\beta, \varphi] = \sum_{\beta' < \beta} C_{0\beta\beta'} \varphi^{(\beta-\beta')} D^{\beta'}, \quad [D^\alpha, \varphi] = \sum_{\alpha' < \alpha} C_{1\alpha\alpha'} D^{\alpha'} \varphi^{(\alpha-\alpha')}$$

with some constants  $C_{0\beta\beta'}$  and  $C_{1\alpha\alpha'}$  we have

$$[D^\alpha a_{\alpha\beta} D^\beta, \varphi] = \sum_{\alpha' < \alpha} C_{1\alpha\alpha'} D^{\alpha'} \varphi^{(\alpha-\alpha')} a_{\alpha\beta} D^\beta + \sum_{\beta' < \beta} C_{0\beta\beta'} D^\alpha a_{\alpha\beta} \varphi^{(\beta-\beta')} D^{\beta'}.$$

Then we know that  $R_\Omega[A, \varphi]$  is written in the form of (3.9). □

A useful tool to evaluate the kernel of the right-hand side in (3.8) is the fact that if an operator of the form  $ST$  has a continuous and bounded integral kernel  $K(x, y)$  then it follows that

$$|K(x, y)| \leq \|ST\|_{L^1 \rightarrow L^\infty} \leq \|S\|_{L^p \rightarrow L^\infty} \|T\|_{L^1 \rightarrow L^p}$$

with  $1 < p < \infty$ . In order to apply this fact we shall derive exponential decay estimates for the resolvent kernels and their derivatives.

**Theorem 3.3.** *Let  $p \in (1, \infty), \eta \in (0, \pi/2)$ . Then there exists  $R = R(\eta, \zeta_A, \omega_A, \Omega) \geq 1$  such that for  $\lambda \in \Lambda(R, \eta)$  the resolvent  $(A_{L^p(\Omega)} - \lambda)^{-1}$  exists and it has a kernel  $G_{\Omega, \lambda}(x, y)$  which is independent of  $p$  and satisfies the following. There exist*

$C = C(\eta, \zeta_A, \Omega)$  and  $c = c(\eta, \zeta_A, \Omega)$  such that for  $|\alpha| < m, |\beta| < m$  the derivative  $\partial_x^\alpha \partial_y^\beta G_{\Omega,\lambda}(x, y)$  is continuous off the diagonal in  $\Omega \times \Omega$  and satisfies

$$(3.11) \quad |\partial_x^\alpha \partial_y^\beta G_{\Omega,\lambda}(x, y)| \leq C \Psi_{2m-|\alpha|-|\beta|}(x - y, \lambda, c)$$

for  $x, y \in \Omega$ , where the function  $\Psi_\sigma$  with  $\sigma > 0$  is defined by

$$\Psi_\sigma(x, \lambda, c) = \exp(-c|\lambda|^{1/(2m)}|x|) \times \begin{cases} |x|^{\sigma-n} & (0 < \sigma < n), \\ (1 + \log_+(|\lambda|^{1/(2m)}|x|)^{-1}) & (\sigma = n), \\ |\lambda|^{(n-\sigma)/(2m)} & (\sigma > n), \end{cases}$$

and  $\log_+ s = \max\{0, \log s\}$ . Moreover,  $\partial_x^\alpha \partial_y^\beta G_{\Omega,\lambda}(x, y)$  is also continuous on the diagonal if  $2m - |\alpha| - |\beta| > n$ .

Proof. See [21] for a domain with bounded  $C^{m+1}$  boundary and [22] for a uniform  $C^1$  domain. □

**Lemma 3.4.** Let  $p \in (1, \infty), \eta \in (0, \pi/2), |\alpha| < m, |\beta| < m$ , and set

$$G_{\Omega,\lambda}^{\alpha,\beta}(x, y) = D_x^\alpha (-D_y)^\beta G_{\Omega,\lambda}(x, y).$$

Then there exist  $R = R(\eta, \zeta_A, \omega_A, \Omega) \geq 1, C = C(\eta, \zeta_A, \Omega), c = c(\eta, \zeta_A, \Omega)$  such that for  $\lambda \in \Lambda(R, \eta)$  we have

$$(3.12) \quad D^\alpha (A_\Omega - \lambda)^{-1} D^\beta f(x) = \int_\Omega G_{\Omega,\lambda}^{\alpha,\beta}(x, y) f(y) dy$$

for  $f \in L^p(\Omega)$  and

$$(3.13) \quad |G_{\Omega,\lambda}^{\alpha,\beta}(x, y)| \leq C \Psi_{2m-|\alpha|-|\beta|}(x - y, \lambda, c).$$

Proof. Let  $R$  be the maximum of the  $R$ 's in Theorems 2.1 and 3.3. Then  $(A_\Omega - \lambda)^{-1}$  and  $G_{\Omega,\lambda}^{\alpha,\beta}(x, y)$  exist for  $\lambda \in \Lambda(R, \eta)$ . Estimate (3.13) follows immediately from (3.11).

Let  $f, g \in C_0^\infty(\Omega)$ . Noting  $(A_\Omega - \lambda)^{-1}|_{L^p(\Omega)} = (A_{L^p(\Omega)} - \lambda)^{-1}$  and using Theorem 3.3, we have

$$(D^\alpha (A_\Omega - \lambda)^{-1} D^\beta f, g)_\Omega = \iint_{\Omega \times \Omega} G_{\Omega,\lambda}(x, y) D_y^\beta f(y) \overline{D_x^\alpha g(x)} dx dy,$$

where we set  $(u, v)_\Omega = \int_\Omega u(x) \overline{v(x)} dx$ . Set  $B_\varepsilon(x) = \{y \in \mathbb{R}^n : |x - y| < \varepsilon\}$  for  $x \in \Omega$

and sufficiently small  $\varepsilon > 0$ . Integrating by parts, we have

$$\begin{aligned} \int_{\Omega} G_{\Omega,\lambda}(x, y) D_{y_j} f(y) dy &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B_{\varepsilon}(x)} G_{\Omega,\lambda}(x, y) D_{y_j} f(y) dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B_{\varepsilon}(x)} (-1) D_{y_j} G_{\Omega,\lambda}(x, y) f(y) dy \\ &\quad + i^{-1} \lim_{\varepsilon \rightarrow 0} \int_{\partial B_{\varepsilon}(x)} G_{\Omega,\lambda}(x, y) f(y) \frac{x_j - y_j}{|x - y|} dS_y \\ &= \int_{\Omega} (-1) D_{y_j} G_{\Omega,\lambda}(x, y) f(y) dy. \end{aligned}$$

Here we used  $G_{\Omega,\lambda}(x, \cdot) \in L^1(\Omega)$ ,  $D_{y_j} G_{\Omega,\lambda}(x, \cdot) \in L^1(\Omega)$  and  $\int_{\partial B_{\varepsilon}(x)} |G_{\Omega,\lambda}(x, y)| dS_y = o(1)$  as  $\varepsilon \rightarrow 0$ , which follow from (3.11).

Repeating this procedure, we get

$$(D^{\alpha}(A_{\Omega} - \lambda)^{-1} D^{\beta} f, g)_{\Omega} = \iint_{\Omega \times \Omega} G_{\Omega,\lambda}^{\alpha,\beta}(x, y) f(y) \overline{g(x)} dx dy.$$

Hence (3.12) holds for  $f \in C_0^{\infty}(\Omega)$ . By Theorem 2.1 and (3.13) we see that the both sides of (3.12) define bounded operators in  $L^p(\Omega)$ . Since  $C_0^{\infty}(\Omega)$  is dense in  $L^p(\Omega)$ , (3.12) also holds for  $f \in L^p(\Omega)$ .  $\square$

For a fixed  $x_0 \in \Omega$  we set

$$B_{x_0} = \left\{ x \in \mathbb{R}^n : |x - x_0| < \frac{\delta(x_0)}{4} \right\}.$$

Let  $R_{x_0}: L^{\infty}(\Omega) \rightarrow L^{\infty}(B_{x_0})$  be the restriction and  $E_{x_0}: L^1(B_{x_0}) \rightarrow L^1(\mathbb{R}^n)$  the extension defined by  $E_{x_0}u(x) = u(x)$  for  $x \in B_{x_0}$  and  $E_{x_0}u(x) = 0$  for  $x \in \mathbb{R}^n \setminus B_{x_0}$ . Obviously we have

$$\|R_{x_0}\|_{L^{\infty}(\Omega) \rightarrow L^{\infty}(B_{x_0})} = 1, \quad \|E_{x_0}\|_{L^1(B_{x_0}) \rightarrow L^1(\mathbb{R}^n)} = 1.$$

**Lemma 3.5.** *Let  $p \in (1, \infty)$ ,  $\eta \in (0, \pi/2)$ ,  $(k + 1)m > n$ ,  $1 \leq l \leq k$ . Then there exist  $R = R(\eta, \zeta_A, \omega_A, \Omega) \geq 1$ ,  $C = C(p, k, \eta, \zeta_A, \Omega)$  and  $c = c(k, \eta, \zeta_A, \Omega)$  such that the following estimates hold for  $\lambda \in \Lambda(R, \eta)$ .*

(i) *If  $|\alpha| \leq m$  and  $p^{-1} < lm/n$ , then*

$$\|R_{x_0} S_l(A_{\Omega}) D^{\alpha}\|_{L^p(\Omega) \rightarrow L^{\infty}(B_{x_0})} \leq C |\mu|^{-l+|\alpha|/(2m)+n/(2mp)}.$$

(ii) *If  $|\alpha| < m$ ,  $0 < |\gamma| \leq m$  and  $p^{-1} < (2ml - |\alpha|)/n$ , then*

$$\begin{aligned} &\|R_{x_0} S_l(A_{\Omega}) D^{\alpha} \varphi^{(\gamma)}\|_{L^p(\Omega) \rightarrow L^{\infty}(B_{x_0})} \\ &\leq C \delta(x_0)^{-|\gamma|} |\mu|^{-l+|\alpha|/(2m)+n/(2mp)} \exp(-c\delta(x_0)|\mu|^{1/(2m)}). \end{aligned}$$

(iii) If  $|\beta| \leq m$  and  $p^{-1} > 1 - (k - l + 1)m/n$ , then

$$\|D^\beta T_l(A)E_{x_0}\|_{L^1(B_{x_0}) \rightarrow L^p(\mathbb{R}^n)} \leq C|\mu|^{-k+l-1+|\beta|/(2m)+(1-1/p)n/(2m)}.$$

(iv) If  $|\beta| < m$ ,  $0 < |\gamma| \leq m$  and  $p^{-1} > 1 - \{2m(k - l + 1) - |\beta|\}/n$ , then

$$\begin{aligned} &\|\varphi^{(\gamma)} D^\beta T_l(A)E_{x_0}\|_{L^1(B_{x_0}) \rightarrow L^p(\mathbb{R}^n)} \\ &\leq C\delta(x_0)^{-|\gamma|} |\mu|^{-k+l-1+|\beta|/(2m)+(1-1/p)n/(2m)} \exp(-c\delta(x_0)|\mu|^{1/(2m)}). \end{aligned}$$

Proof. Let  $R_0$  be the maximum of the  $R$ 's in Theorem 2.1 and Lemma 3.4 for the angle  $\eta/k$ . As will be seen below, Lemma 3.5 holds with  $R = R_0^k$ .

(i) Let  $1 < q < r \leq \infty$  and  $q^{-1} - r^{-1} < m/n$ . Then by Theorem 2.1 and the Sobolev embedding theorem we have

$$\begin{aligned} &\|(A_\Omega - \lambda)^{-1} D^\alpha\|_{L^q(\Omega) \rightarrow L^r(\Omega)} \\ (3.14) \quad &\leq \|(A_\Omega - \lambda)^{-1}\|_{H^{-|\alpha|,q}(\Omega) \rightarrow L^q(\Omega)}^{1-(n/m)(1/q-1/r)} \|(A_\Omega - \lambda)^{-1}\|_{H^{-|\alpha|,q}(\Omega) \rightarrow H^{m,q}(\Omega)}^{(n/m)(1/q-1/r)} \\ &\leq C|\lambda|^{-1+|\alpha|/(2m)+(n/(2m))(1/q-1/r)} \end{aligned}$$

for  $\lambda \in \Lambda(R_0, \eta/k)$ ,  $|\alpha| \leq m$ . In view of  $p^{-1} < lm/n$  we can choose a decreasing sequence  $\{p_j\}_{j=0}^l$  satisfying

$$\infty = p_0 > p_1 > \dots > p_l = p, \quad p_j^{-1} - p_{j-1}^{-1} < \frac{m}{n} \quad (j = 1, \dots, l).$$

Using (3.7), (3.14) and  $|\mu| = |\mu_j|$  for  $j = 1, \dots, k$ , we have

$$\begin{aligned} &\|S_l(A_\Omega)D^\alpha\|_{L^p(\Omega) \rightarrow L^\infty(\Omega)} \\ (3.15) \quad &\leq \prod_{j=1}^{l-1} \|(A_\Omega - \mu_j)^{-1}\|_{L^{p_j}(\Omega) \rightarrow L^{p_{j-1}}(\Omega)} \times \|(A_\Omega - \mu_l)^{-1} D^\alpha\|_{L^{p_l}(\Omega) \rightarrow L^{p_{l-1}}(\Omega)} \\ &\leq C|\mu|^{-l+|\alpha|/(2m)+n/(2mp)} \end{aligned}$$

for  $\lambda \in \Lambda(R_0^k, \eta)$ , which gives (i).

(ii) Using Lemma 3.4 and the inequality

$$\int_{\mathbb{R}^n} \Psi_\sigma(x - z, \lambda, c) \Psi_\rho(z - y, \lambda, c) dz \leq C(\sigma, \rho, n, c) \Psi_{\sigma+\rho}\left(x - y, \lambda, \frac{c}{2}\right)$$

for  $\sigma >, \rho > 0$  (see [14, Lemma 3.2]) repeatedly, we see that  $S_l(A_\Omega)D^\alpha$  is an integral operator with kernel  $S_{l,\alpha}(x, y)$  satisfying

$$|S_{l,\alpha}(x, y)| \leq C\Psi_{2ml-|\alpha|}(x - y, \mu, c)$$

if we replace constants  $C, c$  with other ones.

Let  $p^{-1} + q^{-1} = 1$ ,  $x \in B_{x_0}$  and  $y \in \text{supp } \varphi^{(\gamma)}$ . Then  $|x - x_0| < \delta(x_0)/4$  and  $\delta(x_0)/2 \leq |y - x_0| \leq \delta(x_0)$ . Therefore  $|x - y| \geq \delta(x_0)/4$ . We note that  $\|\varphi^{(\gamma)}\|_{L^\infty(\mathbb{R}^n)} \leq C\delta(x_0)^{-|\gamma|}$ ,  $\Psi_\sigma(x, \mu, c) = \Psi_\sigma(x, \mu, c/2) \exp(-c|\mu|^{1/(2m)}|x|/2)$  and  $\|\Psi_\sigma(\cdot, \mu, c)\|_{L^q(\mathbb{R}^n)} = C|\mu|^{(n-\sigma)/(2m)-n/(2mq)}$  if  $\sigma > 0$  and  $(\sigma - n)q > -n$ . Then we have

$$\begin{aligned} & \|R_{x_0} S_l(A_\Omega) D^\alpha \varphi^{(\gamma)}\|_{L^p(\Omega) \rightarrow L^\infty(B_{x_0})}^q \\ & \leq C \sup_{x \in B_{x_0}} \|\Psi_{2ml-|\alpha|}(x - \cdot, \mu, c) \varphi^{(\gamma)}\|_{L^q(\Omega)}^q \\ & \leq C \delta(x_0)^{-q|\gamma|} \sup_{x \in B_{x_0}} \int_{|x-y| \geq \delta(x_0)/4} \Psi_{2ml-|\alpha|}\left(x - y, \mu, \frac{c}{2}\right)^q \\ & \quad \times \exp\left(\frac{-qc|\mu|^{1/(2m)}\delta(x_0)}{8}\right) dy \\ & \leq C \delta(x_0)^{-q|\gamma|} \exp\left(\frac{-qc|\mu|^{1/(2m)}\delta(x_0)}{8}\right) |\mu|^{(n-2ml+|\alpha|)q/(2m)-n/(2m)} \end{aligned}$$

if  $(2ml - |\alpha| - n)q > -n$ . This yields (ii).

(iii) Let  $p^{-1} + q^{-1} = 1$  and set  $(u, v)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} u(x) \overline{v(x)} dx$  and  $T_l(A)^* = \prod_{j=l}^k (A - \overline{\mu_j})^{-1}$ . Then we have

$$(D^\beta T_l(A)u, v)_{\mathbb{R}^n} = (u, T_l(A)^* D^\beta v)_{\mathbb{R}^n}$$

for  $u, v \in C_0^\infty(\mathbb{R}^n)$  because of the self-adjointness of  $A_{L^2(\mathbb{R}^n)}$  and the relation  $(A - \mu_j)^{-1}|_{L^2(\mathbb{R}^n)} = (A_{L^2(\mathbb{R}^n)} - \mu_j)^{-1}$ . Hence

$$\|D^\beta T_l(A)\|_{L^1(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} = \|T_l(A)^* D^\beta\|_{L^q(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)}.$$

We can evaluate the right-hand side in the same way as in (3.15) to obtain (iii).

(iv) can be treated in the same way as (ii), if we note that

$$\|\varphi^{(\gamma)} D^\beta T_l(A) E_{x_0}\|_{L^1(B_{x_0}) \rightarrow L^p(\mathbb{R}^n)} = \|R_{x_0} T_l(A)^* D^\beta \varphi^{(\gamma)}\|_{L^q(\mathbb{R}^n) \rightarrow L^\infty(B_{x_0})}$$

with  $p^{-1} + q^{-1} = 1$  and that  $T_l(A)^* D^\beta$  is an integral operator with kernel  $T_{l,\beta}(x, y)$  satisfying

$$|T_{l,\beta}(x, y)| \leq C \Psi_{2m(k-l+1)-|\beta|}(x - y, \mu, c). \quad \square$$

**Lemma 3.6.** *Let  $\eta \in (0, \pi/2)$ ,  $(k+1)m > n$ . Then there exists  $C = C(k, \eta, \zeta_A, \omega_A, \Omega)$ ,  $c = c(k, \eta, \zeta_A, \Omega)$  such that*

$$(3.16) \quad |G_{\Omega,\lambda}^k(x, x) - G_\lambda^k(x, x)| \leq C|\lambda|^{-1+n/(2mk)} \exp(-c\delta(x)|\lambda|^{1/(2mk)})$$

for  $x \in \Omega$ ,  $\lambda \in \Lambda(0, \eta)$ .

Proof. Let  $R$  be the  $R$  in Lemma 3.5. First we consider the case  $|\lambda| \leq R\delta(x_0)^{-2mk}$ . Then by (3.6) we have  $|G_{\Omega,\lambda}^k(x_0, x_0)| + |G_\lambda^k(x_0, x_0)| \leq C|\lambda|^{-1+n/(2mk)}$  for  $\lambda \in \Lambda(0, \eta)$ , which implies (3.16).

Next we consider the case  $|\lambda| \geq R\delta(x_0)^{-2mk}$  ( $\geq R$ ). Since  $G_{\Omega,\lambda}^k(x, y)$  and  $G_\lambda^k(x, y)$  are bounded and continuous, (3.8) gives

$$|G_{\Omega,\lambda}^k(x_0, x_0) - G_\lambda^k(x_0, x_0)| \leq \sum_{l=1}^k \|R_{x_0} S_l(A_\Omega) R_\Omega[A, \varphi] T_l(A) E_{x_0}\|_{L^1(B_{x_0}) \rightarrow L^\infty(B_{x_0})}.$$

The right-hand side can be estimated by using (3.9) and Lemma 3.5. It is important that we always have  $|\alpha| < m$  or  $|\beta| < m$  in the sum in (3.9). Suppose that for each  $l$  with  $1 \leq l \leq k$  we can take  $p \in (1, \infty)$  satisfying the inequalities in (i), (iv) of Lemma 3.5 if  $|\alpha| \leq m$  and  $|\beta| < m$ , and those in (ii), (iii) of Lemma 3.5 if  $|\alpha| < m$  and  $|\beta| \leq m$ . Then we get

$$\begin{aligned} & |G_{\Omega,\lambda}^k(x_0, x_0) - G_\lambda^k(x_0, x_0)| \\ & \leq C \sum_{\alpha, \beta, \gamma} \delta(x_0)^{-|\gamma|} |\mu|^{-k-1+(|\alpha|+|\beta|)/(2m)+n/(2m)} \exp(-c\delta(x_0)|\mu|^{1/(2m)}) \\ & \leq C|\mu|^{-k+n/(2m)} \exp(-c\delta(x_0)|\mu|^{1/(2m)}), \end{aligned}$$

where we have used  $|\alpha + \beta + \gamma| \leq 2m$ ,  $\delta(x_0)^{-1} \leq R^{-1/(2mk)}|\mu|^{1/(2m)}$  and  $|\mu|^{-1} \leq R^{-1/k}$ . This implies (3.16).

So it remains to check that there exists  $p \in (1, \infty)$  satisfying the above-mentioned conditions. In other words, we have only to show that for each integer  $l \in [1, k]$  there exists  $p \in (1, \infty)$  satisfying either of

$$\begin{aligned} I_1(l) & := 1 - \frac{2m(k-l+1) - m}{n} < \frac{1}{p} < \frac{lm}{n} =: I_2(l), \\ I_3(l) & := 1 - \frac{(k-l+1)m}{n} < \frac{1}{p} < \frac{2ml - m}{n} =: I_4(l). \end{aligned}$$

Since  $I_1(l) < 1$ ,  $I_2(l) > 0$ ,  $I_3(l) < 1$  and  $I_4(l) > 0$  always hold, such a  $p$  exists if  $I_1(l) < I_2(l)$  and  $I_3(l) < I_4(l)$ , i.e.,

$$(2k - l + 1)m > n, \quad (k + l)m > n$$

for any  $l \in [1, k]$ . These inequalities hold if  $(k+1)m > n$ . Thus we have shown the existence of  $p$  which has the desired properties.  $\square$

#### 4. Tauberian argument

In order to derive the asymptotic formula for  $e_\Omega(\tau, x, x)$  from that of  $e_{\mathbb{R}^n}(\tau, x, x)$  by using the estimate of  $G_{\Omega,\lambda}^k(x, x) - G_\lambda^k(x, x)$  we prepare the following Tauberian



theorem, which is a modification of Avakumovič's Tauberian theorem [4, Lemma 4]. In the remainder term  $O(\tau^{n-\theta})$  in Lemma 4.1 below we allow the value of  $\theta$  to be not only 1 but also a number in  $(0, 1]$ .

**Lemma 4.1.** *Let  $N(\tau)$  and  $\Lambda(\tau)$  be functions  $\mathbb{R} \rightarrow \mathbb{R}$  satisfying the following conditions:*

- (i)  $N(\tau)$  is non-decreasing;
- (ii) There exist constants  $c_0 > 0$ ,  $\theta \in (0, 1]$  and  $C_1 > 0$  such that

$$|\Lambda(\tau) - c_0\tau^n| \leq C_1\tau^{n-\theta} \quad \text{for } \tau \geq 0, \quad \Lambda(\tau) = 0 \quad \text{for } \tau < 0;$$

- (iii) There exists a constant  $C_2 > 0$  such that

$$|N(\tau)| \leq C_2\tau^n \quad \text{for } \tau \geq 0, \quad N(\tau) = 0 \quad \text{for } \tau < 0;$$

- (iv) If we set

$$F(z) = \int_0^\infty e^{-\tau z} d_\tau(N(\tau) - \Lambda(\tau)),$$

which is analytic for  $\operatorname{Re} z > 0$  by conditions (ii)–(iii), then there exist  $T > 0$  and  $B > 0$  such that  $F(z)$  is analytically continued to the disk  $\{z \in \mathbb{C} : |z| < T\}$  and satisfies

$$(4.1) \quad |F(z)| \leq B \quad \text{for } |z| < T,$$

$$(4.2) \quad F(0) = 0.$$

Then there exists  $C = C(c_0, n, \theta, C_1, C_2)$  such that

$$(4.3) \quad |N(\tau) - c_0\tau^n| \leq C(\tau^{n-\theta} + T^{-1}\tau^{n-1} + B) \quad \text{for } \tau \geq T^{-1}.$$

*Proof.* As in [24], we choose a non-negative-valued function  $\rho \in \mathcal{S}(\mathbb{R})$  such that

$$\rho(\tau) > 0 \quad \text{for } |\tau| \leq 1, \quad \operatorname{supp} \hat{\rho} \subset (-1, 1), \quad \hat{\rho}(0) = \int_{-\infty}^\infty \rho(\tau) d\tau = 1,$$

and set  $\rho_T(\tau) = T\rho(T\tau)$ , where  $\hat{\rho}(t) = \int_{-\infty}^\infty e^{-it\tau} \rho(\tau) d\tau$ . Obviously  $|\hat{\rho}(t)| \leq 1$  and  $\hat{\rho}_T(t) = \hat{\rho}(t/T)$ .

First we shall evaluate  $\rho_T * d\Lambda(\tau)$  and  $\rho_T * \Lambda(\tau)$ . To do so we set

$$h(\tau, T) = \tau^{n-\theta} + T^{-1}\tau^{n-1} + T^{\theta-n} + T^{-n},$$

and  $r(\tau) = \Lambda(\tau) - c\tau^n$  for  $\tau \geq 0$ ,  $r(\tau) = 0$  for  $\tau < 0$ . Then  $|r(\tau)| \leq C_1\tau^{n-\theta}$  for  $\tau \geq 0$ . We often use the inequalities

$$(4.4) \quad \left| \tau - \frac{s}{T} \right|^\kappa \leq C_\kappa \left( \tau^\kappa + \frac{|s|^\kappa}{T^\kappa} \right), \quad \int_{-\infty}^\infty \{\rho(s) + |\rho'(s)|\} |s|^\kappa ds \leq C_\kappa$$

for  $\tau \geq 0$ ,  $\kappa \geq 0$ . Combining

$$\rho_T * d\Lambda(\tau) = nc_0 \int_{-\infty}^{T\tau} \rho(s) \left(\tau - \frac{s}{T}\right)^{n-1} ds + T \int_{-\infty}^{T\tau} \rho'(s) r \left(\tau - \frac{s}{T}\right) ds$$

with (4.4), we have

$$(4.5) \quad |\rho_T * d\Lambda(\tau)| \leq CT h(\tau, T) \quad \text{for } \tau \geq 0.$$

Using

$$\begin{aligned} \rho_T * \Lambda(\tau) &= \Lambda(\tau) - \Lambda(\tau) \int_{T\tau}^{\infty} \rho(s) ds + \int_{-\infty}^{T\tau} \rho(s) \left\{ \Lambda\left(\tau - \frac{s}{T}\right) - \Lambda(\tau) \right\} ds, \\ \left| \int_{T\tau}^{\infty} \rho(s) ds \right| &\leq C(T\tau)^{-1}, \quad \left| \left(\tau - \frac{s}{T}\right)^n - \tau^n \right| = C(|s|\tau^{n-1}T^{-1} + |s|^n T^{-n}) \end{aligned}$$

for  $\tau \geq 0$ , we have

$$(4.6) \quad |\rho_T * \Lambda(\tau) - c_0 \tau^n| \leq C(\tau^{n-\theta} + T^{-1} \tau^{n-1}) \quad \text{for } \tau \geq T^{-1}.$$

Next we shall evaluate  $\rho_T * dN(\tau)$  and  $\rho_T * N(\tau)$ . Inequality (4.1) implies  $|\widehat{dN}(t) - \widehat{d\Lambda}(t)| \leq B$  for  $|t| < T$ . Hence by (4.5) and

$$\rho_T * dN(\tau) = (2\pi)^{-1} \int_{-T}^T e^{i\tau t} \hat{\rho}_T(t) \{ \widehat{dN}(t) - \widehat{d\Lambda}(t) \} dt + \rho_T * d\Lambda(\tau)$$

we have

$$(4.7) \quad 0 \leq \rho_T * dN(\tau) \leq CT(B + h(\tau, T)) \quad \text{for } \tau \geq 0.$$

Choose  $c_1 > 0$  so that  $\rho(\tau) \geq c_1$  for  $|\tau| \leq 1$ . Since  $N(\tau)$  is non-decreasing, we have

$$(4.8) \quad 0 \leq N(\tau) - N(\tau - T^{-1}) \leq c_1^{-1} T^{-1} \rho_T * dN(\tau) \quad \text{for } \tau \in \mathbb{R}.$$

Dividing the interval  $[0, |s|]$  into at most  $|s|+1$  intervals of length  $\leq 1$ , and using (4.7) and (4.8), we have

$$0 \leq N\left(\tau - \frac{s}{T}\right) - N(\tau) \leq C(1 + |s|) \left( B + h\left(\tau + \frac{|s|}{T}, T\right) \right)$$

when  $s \leq 0$ . Similarly we have

$$0 \leq N(\tau) - N\left(\tau - \frac{s}{T}\right) \leq C(1 + |s|)(B + h(\tau, T))$$

when  $0 \leq s \leq T\tau$ . Then from (iii), the inequality  $|\int_{T\tau}^\infty \rho(s) ds| \leq C(T\tau)^{-1}$  and

$$\rho_T * N(\tau) = N(\tau) - N(\tau) \int_{T\tau}^\infty \rho(s) ds + \int_{-\infty}^{T\tau} \rho(s) \left\{ N\left(\tau - \frac{s}{T}\right) - N(\tau) \right\} ds$$

it follows that

$$(4.9) \quad |\rho_T * N(\tau) - N(\tau)| \leq C(B + h(\tau, T)) \quad \text{for } \tau \geq T^{-1}.$$

Finally we shall evaluate  $\rho_T * N(\tau) - \rho_T * \Lambda(\tau)$ . Since  $F(0) = 0$ , the function  $F(z)/z$  is also analytic in  $|z| < T$ . So (4.1) and the maximum principle give  $|F(z)/z| \leq B/T$  for  $|z| < T$ . On the other hand, integration by parts gives

$$F(z) = z \int_0^\infty e^{-\tau z} (N(\tau) - \Lambda(\tau)) d\tau$$

for  $\text{Re } z > 0$ . Then we have

$$|\hat{N}(t) - \hat{\Lambda}(t)| = \left| \frac{F(it)}{it} \right| \leq \frac{B}{T} \quad \text{for } -T < t < T.$$

Hence

$$(4.10) \quad |\rho_T * (N(\tau) - \Lambda(\tau))| = \left| (2\pi)^{-1} \int_{-T}^T e^{it\tau} \hat{\rho}_T(t) (\hat{N}(t) - \hat{\Lambda}(t)) dt \right| \leq \frac{B}{\pi}$$

for  $\tau \geq 0$ . Combining (4.6), (4.9) and (4.10), we obtain (4.3). □

Proof of Proposition 1.2. For simplicity we write  $e(\tau, x, x)$  for  $e_{\mathbb{R}^n}(\tau, x, x)$ . Let us apply Lemma 4.1 with  $N(\tau) = e_\Omega(\tau^{2m}, x, x)$  and  $\Lambda(\tau) = e(\tau^{2m}, x, x)$ . To do so we shall see that conditions (i)–(iv) in Lemma 4.1 hold. Condition (i) follows from the property of the spectral function. Condition (ii) holds with  $c_0 = c_A(x)$  by assumption (1.5). Condition (iii) follows from (3.2). To check (iv) we set

$$F(z) = \int_0^\infty e^{-\tau z} d_\tau \{e_\Omega(\tau^{2m}, x, x) - e(\tau^{2m}, x, x)\}$$

for  $\text{Re } z > 0$ . By Cauchy's integral theorem and (3.5) we have

$$\begin{aligned} F(z) &= \int_0^\infty e^{-z\tau^{1/(2mk)}} d_\tau \{e_\Omega(\tau^{1/k}, x, x) - e(\tau^{1/k}, x, x)\} \\ &= \frac{-1}{2\pi i} \int_\Gamma e^{-z\lambda^{1/(2mk)}} d\lambda \int_0^\infty (\tau - \lambda)^{-1} d_\tau \{e_\Omega(\tau^{1/k}, x, x) - e(\tau^{1/k}, x, x)\} \\ &= \frac{-1}{2\pi i} \int_\Gamma e^{-z\lambda^{1/(2mk)}} \{G_{\Omega, \lambda}^k(x, x) - G_\lambda^k(x, x)\} d\lambda, \end{aligned}$$

where  $\Gamma$  is the boundary of  $\Lambda(0, \pi/4)$ . Using estimate (3.16) for  $\lambda \in \Lambda(0, \pi/4)$ , we have, for  $|z| < 2^{-1}c\delta(x)$ ,

$$\begin{aligned} & \int_{\Gamma} |e^{-z\lambda^{1/(2mk)}} \{G_{\Omega,\lambda}^k(x, x) - G_{\lambda}^k(x, x)\}| |d\lambda| \\ & \leq C \int_{\Gamma} |\lambda|^{-1+n/(2mk)} \exp\{(|z| - c\delta(x))|\lambda|^{1/(2mk)}\} |d\lambda| \\ & \leq C \int_0^{\infty} r^{-1+n/(2mk)} \exp(-2^{-1}c\delta(x)r^{1/(2mk)}) dr \leq C\delta(x)^{-n}. \end{aligned}$$

Hence  $F(z)$  is analytic in  $\{z \in \mathbb{C}: |z| < 2^{-1}c\delta(x)\}$ , and  $|F(z)| \leq C\delta(x)^{-n}$ . That is, (4.1) is valid with  $T = 2^{-1}c\delta(x)$  and  $B = C\delta(x)^{-n}$ . Equality (4.2) follows from Cauchy's integral theorem and the fact that  $G_{\Omega,\lambda}^k(x, x) - G_{\lambda}^k(x, x)$  is rapidly decreasing as  $|\lambda| \rightarrow \infty$  in  $\Lambda(0, \pi/4)$ . Thus we have checked condition (iv). So we can apply Lemma 4.1 to get

$$\begin{aligned} |e_{\Omega}(\tau^{2m}, x, x) - c_A(x)\tau^n| & \leq C(\tau^{n-\theta} + \delta(x)^{-1}\tau^{n-1} + \delta(x)^{-n}) \\ & \leq C(\tau^{n-\theta} + \delta(x)^{-1}\tau^{n-1}) \end{aligned}$$

for  $\tau \geq 2c^{-1}\delta(x)^{-1}$ . Since  $\delta(x)^{-1} \leq \text{dist}(x, \partial\Omega)^{-1} + 1$ , (1.6) holds for  $\tau \geq 2c^{-1}\delta(x)^{-1}$ . When  $1 \leq \tau \leq 2c^{-1}\delta(x)^{-1}$ , (1.6) follows from (3.2). This completes the proof of Proposition 1.2.  $\square$

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