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SPECTRAL ASYMPTOTICS FOR DIRICHLET ELLIPTIC OPERATORS WITH NON-SMOOTH COEFFICIENTS

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Abstract

We consider a $2m$-th-order elliptic operator of divergence form in a domain Ω of $\mathbb{R}^n$, assuming that the coefficients are Hölder continuous of exponent $r \in (0, 1]$. For the self-adjoint operator associated with the Dirichlet boundary condition we improve the asymptotic formula of the spectral function $e(\tau^{2m}, x, y)$ for $x = y$ to obtain the remainder estimate $O(\tau^{n-\theta} + \operatorname{dist}(x, \partial\Omega)^{-1}\tau^{n-1})$ with any $\theta \in (0, r)$, using the L^p theory of elliptic operators of divergence form. We also show that the spectral function is in $C^{m-1,1-\varepsilon}$ with respect to (x, y) for any small $\varepsilon > 0$. These results extend those for the whole space $\mathbb{R}^n$ obtained by Miyazaki [19] to the case of a domain.

Introduction

Let us consider a $2m$-th-order elliptic operator of divergence form

$$Au(x) = \sum_{|\alpha|\leq m, |\beta|\leq m} D^\alpha(a_{\alpha\beta}(x)D^\beta u(x)) \tag{0.1}$$

with $L^\infty(\mathbb{R}^n)$ coefficients in $\mathbb{R}^n$ and assume that the leading coefficients are in $C^{0,r}(\mathbb{R}^n)$ for some $r \in (0, 1]$. Here we use the notation

$$D = (D_1, \dots, D_n), \quad D_j = -i\frac{\partial}{\partial x_j} \quad (j = 1, \dots, n), \quad i = \sqrt{-1}.$$

Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $A_{L^2(\Omega)}$ the self-adjoint realization associated with the Dirichlet boundary condition in Ω, and $e_\Omega(\tau, x, y)$ the spectral function of $A_{L^2(\Omega)}$.

We are interested in obtaining a better estimate for the remainder term of the asymtotic formula of $e_\Omega(\tau, x, x)$ when the smoothness index r of the leading coefficients is given. For simplicity of notation we consider $e_\Omega(\tau^{2m}, x, x)$ instead of $e_\Omega(\tau, x, x)$ when

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we give its asymptotic formulas. In [19] we showed that $e_{\mathbb{R}^n}(\tau, x, y)$ is in $C^{m-1,1-\varepsilon}$ with respect to (x, y) for any small $\varepsilon > 0$ and that the asymptotic formula

$$e_{\mathbb{R}^n}(\tau^{2m}, x, x) = c_A(x)\tau^n + O(\tau^{n-\theta}) \quad \text{as} \quad \tau \to \infty \tag{0.2}$$

holds with any $\theta \in (0, r)$ if $\Omega = \mathbb{R}^n$, where

$$c_A(x) = (2\pi)^{-n} \int_{\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}<1} d\xi,$$

and O-estimate is uniform with respect to x. Formula (0.2) is based on the theorem of L^p resolvents of elliptic operators of divergence form in $\mathbb{R}^n$ [18, Main Theorem] and the asymptotic formula for spectral functions of pseudodifferential operators due to Zielinski [30]. Now that we have established the L^p theory of elliptic operators under the Dirichlet boundary condition in [20, 21, 22], it is natural to try to extend the results for $\mathbb{R}^n$ to the case $\Omega \neq \mathbb{R}^n$. Accordingly, the purpose of this paper is to show that $e_\Omega(\tau, x, y)$ is in $C^{m-1,1-\varepsilon}$ with respect to (x, y) for any small $\varepsilon > 0$ and to derive the asymptotic formula

$$e_\Omega(\tau^{2m}, x, x) = c_A(x)\tau^n + O(\tau^{n-\theta} + \operatorname{dist}(x, \partial\Omega)^{-1}\tau^{n-1}) \quad \text{as} \quad \tau \to \infty \tag{0.3}$$

with any $\theta \in (0, r)$.

To contrast with known results we set $\delta(x) = \min\{1, \operatorname{dist}(x, \partial\Omega)\}$ and note that (0.3) remains unchanged if we replace $\operatorname{dist}(x, \partial\Omega)$ by $\delta(x)$. In [10, 11, 17, 26] the asymptotic formula for $e_\Omega(\tau^{2m}, x, x)$ was obtained with the remainder term of the form $O(\delta(x)^{-\theta}\tau^{n-\theta})$, where one can take any $\theta \in (0, r/(r+3))$ in [10], $\theta \in (0, r/(r+2))$ in [11, 26], and $\theta \in (0, r/(r+1))$ in [17]. Our remainder estimate makes the range of θ wider. In addition, $O(\tau^{n-\theta} + \delta(x)^{-1}\tau^{n-1})$ is better than $O(\delta(x)^{-\theta}\tau^{n-\theta})$, since $\delta(x)^{-\theta}\tau^{n-\theta} = \tau^{n-\theta^2}$ and $\delta(x)^{-1}\tau^{n-1} = \tau^{n-\theta}$ if we choose $x \in \Omega$ so that $\delta(x) = \tau^{\theta-1}$. Hence our estimate improves those in [10, 11, 17, 26]. Moreover, it appears that (0.3) splits the remainder term into two parts: one depending on the smoothness of the coefficients and one influenced by the boundary. When the coefficients are in C^∞, it was proved independently by Brüning [4] and Tsujimoto [27] that (0.3) holds with $\theta = 1$ (see also [13]).

In this paper, we derive (0.3) with any $\theta \in (0, r)$ for a given $r \in (0, 1]$ as a corollary of the proposition stating that if $A_{L^2(\mathbb{R}^n)}$ satisfies (0.2) with some $\theta \in (0, 1]$ then $A_{L^2(\Omega)}$ satisfies (0.3) with the same θ. In order to prove this proposition we follow the spirit of Hörmander [5] and Brüning [4]. We first estimate the difference between the resolvent kernel for $A_{L^2(\Omega)}$ and that for $A_{L^2(\mathbb{R}^n)}$, then show that the kernel of $\exp\bigl(-zA_{L^2(\Omega)}^{1/(2m)}\bigr) - \exp\bigl(-zA_{L^2(\mathbb{R}^n)}^{1/(2m)}\bigr)$, which is defined for $\operatorname{Re} z > 0$, is analytically continued to some disk with center 0, and finally apply a Fourier Tauberian theorem.

We would like to emphasize that our results can be obtained without assuming $2m > n$. In most papers the assumption $2m > n$ was essential, since the resolvent kernel has singularities on the diagonal when $2m \le n$. Otherwise, extra assumptions were needed such as $D(A^k_{L^2(\Omega)}) \subset H^{2mk,2}(\Omega)$ for some k with $2mk > n$. Such additional assumptions are, however, not required with the help of the L^p theory for the Dirichlet problem in a domain. Instead of the regularity such as $D(A^k_{L^2(\Omega)}) \subset H^{2mk,2}(\Omega)$, which is impossible in the case of non-smooth coefficients, the L^p theory leads us to $D(A^k_{L^2(\Omega)}) \subset C^{m-1,1-\varepsilon}(\Omega)$ for a small $\varepsilon > 0$ if k is large enough. The idea of using the L^p theory for the case of non-smooth coefficients goes back to Beals [2], who considered elliptic operators of non-divergence form.

When Ω is bounded, the spectrum of $A_{L^2(\Omega)}$ consists only of eigenvalues with finite multiplicities accumulating only at ∞. Let $N_\Omega(\tau)$ denote the number of the eigenvalues of $A_{L^2(\Omega)}$ not exceeding τ. The asymptotic behavior of $N_\Omega(\tau)$ is related to that of the spectral function, for $N_\Omega(\tau)$ is obtained by integrating $e_\Omega(\tau, x, x)$ with respect to x over Ω. Thanks to the min-max principle, the investigation for $N_\Omega(\tau)$ has always been ahead of that for $e_\Omega(\tau, x, x)$. Improving the results in [10, 11, 12, 14, 16, 26], Zielinski [29] obtained the asymptotic formula

$$N_\Omega(\tau^{2m}) = c_{A,\Omega}\tau^n + O(\tau^{n-\theta}) \quad \text{as} \quad \tau \to \infty \tag{0.4}$$

with any $\theta \in (0, r)$ for a general boundary problem when $2m > n$ (see also [28, 30]), where $c_{A,\Omega} = \int_\Omega c_A(x)\,dx$. In some special cases, including the case $n = 1$, Miyazaki [15, 16] showed that (0.4) holds with $\theta = r$. Formula (0.4) can be derived by combining (0.3) with the estimate $|e_\Omega(\tau^{2m}, x, y)| \le C\tau^n$. Accordingly, we could say that the investigation for $e_\Omega(\tau, x, x)$ has caught up with that for $N_\Omega(\tau)$ as long as we treat the Dirichlet boundary condition, a domain with smooth boundary and the remainder term $O(\tau^{n-\theta})$ with $\theta < 1$.

For the case of C^∞ coefficients we refer to [6, 7, 23], where the two-term asymptotic formula for $N_\Omega(\tau)$ is also considered. It is known that $\theta = 1$ is the best possible in (0.4) for the case of C^∞ coefficients. It is remarkable that (0.4) with $\theta = 1$ was obtained by Zielinski [31, 32] when the coefficients are in $C^{1,1}$, and by Ivrii [8] when the coefficients are in $C^{1,\varepsilon}$ for any small $\varepsilon > 0$. In [3, 9] some elaboration of these results on $N_\Omega(\tau)$ is given in terms of the modulus of continuity.

1. Main results

Let us now state the main results precisely. Throughout this paper we assume the following conditions on the elliptic operator A defined in (0.1) and a domain $\Omega \subset \mathbb{R}^n$:
(H0) Ω is a uniform C^1 domain if $n \ge 2$, and Ω is an interval of $\mathbb{R}$ if $n = 1$;
(H1) There exists $\delta_A > 0$ such that the principal symbol $a(x, \xi)$ satisfies

$$a(x,\xi) := \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \ge \delta_A|\xi|^{2m} \quad \text{for} \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n;$$

(H2) $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$ and $a_{\alpha\beta} \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \leq m$, $|\beta| \leq m$. In addition, the leading coefficients $a_{\alpha\beta}$ with $|\alpha| = |\beta| = m$ are uniformly continuous in $\mathbb{R}^n$.

For the definition of a uniform C^1 domain or a domain having uniform C^1 regularity we refer to [1, 25]. Here are two examples of uniform C^1 domain: a domain with bounded C^1 boundary; the domain defined by the set of points $x = (x', x_n) \in \mathbb{R}^n$ satisfying $x_n > \psi(x')$, where $\psi \in C^1(\mathbb{R}^{n-1})$ whose first derivatives are bounded and uniformly continuous in $\mathbb{R}^{n-1}$.

For $1 \leq p \leq \infty$ and $\sigma \in \mathbb{R}$ we denote by $H^{\sigma,p}(\Omega)$ the L^p Sobolev space of order σ in Ω. In particular, for $\sigma = -k$ with an integer $k > 0$, $H^{-k,p}(\Omega)$ is the space of functions f written as

$$f = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad f_\alpha \in L^p(\Omega), \tag{1.1}$$

and the norm $\| f \|_{H^{-k,p}(\Omega)}$ is defined by $\| f \|_{H^{-k,p}(\Omega)} = \inf \sum_{|\alpha| \leq k} \| f_\alpha \|_{L^p(\Omega)}$, where the infimum is taken over all $\{f_\alpha\}_{|\alpha| \leq k}$ satisfying (1.1). The space $H_0^{\sigma,p}(\Omega)$ is defined to be the completion of $C_0^\infty(\Omega)$ in $H^{\sigma,p}(\Omega)$. Then A defines a bounded linear operator from $H_0^{m,p}(\Omega)$ to $H^{-m,p}(\Omega)$. When we want to stress p or Ω, we write $A_{p,\Omega}$ or A_Ω for A. The operator $A_{L^p(\Omega)}$ in $L^p(\Omega)$ is defined by

$$D(A_{L^p(\Omega)}) = \{u \in H_0^{m,p}(\Omega) \colon A_\Omega u \in L^p(\Omega)\},$$
$$A_{L^p(\Omega)} u = A_\Omega u \quad \text{for} \quad u \in D(A_{L^p(\Omega)}).$$

As is well known, when $p = 2$, the operator $A_{L^2(\Omega)}$ is a self-adjoint operator, and it is usually defined by a sesquilinear form

$$Q[u, v] = \int_\Omega \sum_{|\alpha| \leq m,\ |\beta| \leq m} a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha v(x)}\, dx$$

on $H_0^{m,2}(\Omega) \times H_0^{m,2}(\Omega)$.

For an integer $j \geq 0$ and $\sigma \in (0, 1]$ we denote by $C^{j,\sigma}(\Omega)$ the space of j times continuously differentiable functions f such that the norm

$$\| f \|_{C^{j,\sigma}(\Omega)} = \sum_{0 \leq |\alpha| \leq j} \| \partial^\alpha f \|_{L^\infty(\Omega)} + \sum_{|\alpha| = j} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\sigma}$$

is finite. For $h \in \mathbb{R}^n$, functions $f(x)$ and $g(x, y)$ we set

$$\Omega_h = \{x \in \Omega \colon x + h \in \Omega\}, \quad \Delta_h f(x) = f(x + h) - f(x),$$
$$\Delta_{1,h} g(x, y) = g(x + h, y) - g(x, y), \quad \Delta_{2,h} g(x, y) = g(x, y + h) - g(x, y).$$

We define several constants, constant vectors, functions and a region as follows.

$$M_A = \max_{|\alpha|\le m,\ |\beta|\le m} \|a_{\alpha\beta}\|_{L^\infty(\mathbb{R}^n)}, \quad M_{A,r} = \max_{|\alpha|=|\beta|=m} \|a_{\alpha\beta}\|_{C^{0,r}(\mathbb{R}^n)}.$$

$$\zeta_A = (n, m, \delta_A, M_A), \quad \zeta_{A,r} = (n, m, \delta_A, M_A, M_{A,r}),$$

$$c_A(x) = (2\pi)^{-n} \int_{a(x,\xi)<1} d\xi, \quad c_{A,\Omega} = \int_\Omega c_A(x)\,dx,$$

$$\omega_A(\varepsilon) = \max_{|\alpha|=|\beta|=m} \sup_{|h|\le\varepsilon} \sup_{x\in\mathbb{R}^n} |a_{\alpha\beta}(x+h) - a_{\alpha\beta}(x)|,$$

$$\Lambda(R, \eta) = \{\lambda \in \mathbb{C} \colon |\lambda| \ge R,\ \eta \le \arg\lambda \le 2\pi - \eta\} \quad \text{for} \quad R \ge 0,\ \eta \in \left(0, \frac{\pi}{2}\right).$$

By definition $\omega_A(\varepsilon) \le M_{A,r}\varepsilon^r$ holds if the leading coefficients are in $C^{0,r}(\mathbb{R}^n)$.

Theorem 1.1. *Assume* (H0)–(H2). *Then for* $|\alpha| < m$, $|\beta| < m$ *the derivatives* $\partial_x^\alpha \partial_y^\beta e_\Omega(\tau, x, y)$ *are Hölder continuous of exponent* σ *with respect to* (x, y) *for any* $\sigma \in (0, 1)$. *There exist* $C_1 = C(\zeta_A, \omega_A, \Omega)$ *and* $C_2 = C(\sigma, \zeta_A, \omega_A, \Omega)$ *such that*

$$|\partial_x^\alpha \partial_y^\beta e_\Omega(\tau^{2m}, x, y)| \le C_1 \tau^{n+|\alpha|+|\beta|} \tag{1.2}$$

for $(x, y) \in \Omega \times \Omega$, $\tau \ge 1$,

$$|\Delta_{1,h} \partial_x^\alpha \partial_y^\beta e_\Omega(\tau^{2m}, x, y)| \le C_2 \tau^{n+|\alpha|+|\beta|+\sigma} |h|^\sigma \tag{1.3}$$

for $h \in \mathbb{R}^n$, $(x, y) \in \Omega_h \times \Omega$, $\tau \ge 1$,

$$|\Delta_{2,h} \partial_x^\alpha \partial_y^\beta e_\Omega(\tau^{2m}, x, y)| \le C_2 \tau^{n+|\alpha|+|\beta|+\sigma} |h|^\sigma \tag{1.4}$$

for $h \in \mathbb{R}^n$, $(x, y) \in \Omega \times \Omega_h$, $\tau \ge 1$.

Theorem 1.1 will be proved in Section 2.

Proposition 1.2. *Assume* (H0)–(H2). *Then if there exist* $C_0 > 0$ *and* $\theta \in (0, 1]$ *such that*

$$|e_{\mathbb{R}^n}(\tau^{2m}, x, x) - c_A(x)\tau^n| \le C_0 \tau^{n-\theta} \tag{1.5}$$

for $x \in \Omega$, $\tau \ge 1$, *then there exists* $C = C(C_0, \theta, \zeta_A, \omega_A, \Omega)$ *such that*

$$|e_\Omega(\tau^{2m}, x, x) - c_A(x)\tau^n| \le C(\tau^{n-\theta} + \operatorname{dist}(x, \partial\Omega)^{-1}\tau^{n-1}) \tag{1.6}$$

for $x \in \Omega$, $\tau \ge 1$.

Proposition 1.2 will be proved in Section 4 after estimating the difference between the resolvent kernels for Ω and $\mathbb{R}^n$ in Section 3.

Theorem 1.3. *In addition to* (H0)–(H2) *we assume that the leading coefficients of* A *are in* $C^{0,r}(\mathbb{R}^n)$ *for some* $r \in (0, 1]$. *Then for any* $\theta \in (0, r)$ *there exists* $C = C(\theta, r, \zeta_{A,r}, \Omega)$ *such that*

$$|e_\Omega(\tau^{2m}, x, x) - c_A(x)\tau^n| \le C(\tau^{n-\theta} + \mathrm{dist}(x, \partial\Omega)^{-1}\tau^{n-1}) \tag{1.7}$$

for $x \in \Omega, \ \tau \ge 1$.

Proof. By [19, Theorem 2] estimate (1.5) holds for a given $\theta \in (0, r)$. Then Proposition 1.2 yields Theorem 1.3. □

As mentioned in the Introduction, the asymptotic formula for $N_\Omega(\tau)$, which Zielinski [29] proved, can be derived again as a corollary of Theorems 1.1 and 1.3.

Corollary 1.4. *In addition to* (H0)–(H2) *we assume that the leading coefficients of* A *are in* $C^{0,r}(\mathbb{R}^n)$ *for some* $r \in (0,1]$, *and that* Ω *is bounded. Then for any* $\theta \in (0,r)$ *there exists* $C = C(\theta, r, \zeta_{A,r}, \Omega)$ *such that*

$$|N_\Omega(\tau^{2m}) - c_{A,\Omega}\tau^n| \le C\tau^{n-\theta} \tag{1.8}$$

for $\tau \ge 1$.

Proof. Set $\Omega_\varepsilon = \{x \in \Omega\colon \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ for $\varepsilon > 0$. Since Ω is a bounded C^1 domain, it follows that $|\Omega_\varepsilon| \le C\varepsilon$ with some C. This implies $\int_{\Omega\setminus\Omega_\varepsilon} \delta(x)^{-1}\,dx \le C\log\varepsilon^{-1}$ for $0 < \varepsilon < 1$ (see [14]). We evaluate

$$N_\Omega(\tau^{2m}) - c_{A,\Omega}\tau^n = \int_\Omega \{e(\tau^{2m}, x, x) - c_A(x)\tau^n\}\,dx$$

by using (1.7) on $\Omega \setminus \Omega_\varepsilon$ and (1.2) with $\alpha = \beta = 0$ on Ω_ε, and set $\varepsilon = \tau^{-1}$. Since $\tau^{n-1}\log\tau \le C\tau^{n-\theta}$ for $\theta < 1$, we get (1.8). □

2. Rough estimates for spectral functions

By (H1) and Gårding's inequality $A_{L^2(\Omega)}$ is bounded from below. The assertions of Theorem 1.1 and Proposition 1.2 remain unchanged if we replace A by $A + C$ with constant C. So in the following we may assume that A is positive without loss of generality. We start with the theorem on L^p resolvents.

Theorem 2.1. *Let* $p \in (1, \infty)$ *and* $\eta \in (0, \pi/2)$. *Then there exist* $R = R(\eta, \zeta_A, \omega_A, \Omega) \ge 1$ *and* $C = C(p, \eta, \zeta_A, \Omega)$ *such that for* $\lambda \in \Lambda(R, \eta)$ *the resolvent* $(A_{p,\Omega} - \lambda)^{-1}$ *exists and satisfies*

$$\|(A_{p,\Omega} - \lambda)^{-1}\|_{H^{-j,p}(\Omega)\to H^{k,p}(\Omega)} \le C|\lambda|^{-1+(j+k)/(2m)} \tag{2.1}$$

for $0 \le j \le m,\ 0 \le k \le m$. *In addition, the resolvents are consistent in the sense that*

$$(A_{p,\Omega} - \lambda)^{-1} f = (A_{q,\Omega} - \lambda)^{-1} f$$

for $f \in H^{-m,p}(\Omega) \cap H^{-m,q}(\Omega),\ p \neq q \in (1, \infty)$.

Proof. See [20, 21] for a domain with bounded C^{m+1} boundary and [22] for a uniform C^1 domain. □

REMARK 2.1. By the definition of the Sobolev space of negative order (2.1) is equivalent to

$$\|D^\alpha (A_\Omega - \lambda)^{-1} D^\beta\|_{L^p(\Omega) \to L^p(\Omega)} \le C' |\lambda|^{-1+(|\alpha|+|\beta|)/(2m)}$$

for $|\alpha| \le m,\ |\beta| \le m$ with some constant $C' > 0$.

Now that we have established Theorem 2.1, which is the theorem for a domain, Theorem 1.1 can be proved in the same way as [19, Theorem 1], which dealt with the case $\Omega = \mathbb{R}^n$. So we only give the outline of the proof.

Lemma 2.2. *Let* $j \ge 0$ *be an integer and* $0 < \sigma < 1$. *Assume that* S *and* T *are bounded linear operators on* $L^2(\Omega)$ *satisfying*

$$R(S) \subset C^{j,\sigma}(\Omega), \quad R(T^*) \subset C^{j,\sigma}(\Omega),$$

where $R(S)$ *is the range of* S *and* T^* *is the adjoint of* T. *Then* ST *is an integral operator with bounded continuous kernel* $K(x, y)$. *Furthermore, for* $|\alpha| \le j$ *and* $|\beta| \le j$ *the derivatives* $\partial_x^\alpha \partial_y^\beta K(x, y)$ *are Hölder continuous of exponent* σ *and satisfy*

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \le \|D^\alpha S\|_{L^2(\Omega) \to L^\infty(\Omega)} \|D^\beta T^*\|_{L^2(\Omega) \to L^\infty(\Omega)}$$

for $(x, y) \in \Omega \times \Omega$,

$$|\Delta_{1,h} \partial_x^\alpha \partial_y^\beta K(x, y)| \le \|D^\alpha S\|_{L^2(\Omega) \to C^{0,\sigma}(\Omega)} \|D^\beta T^*\|_{L^2(\Omega) \to L^\infty(\Omega)} |h|^\sigma$$

for $h \in \mathbb{R}^n,\ (x, y) \in \Omega_h \times \Omega$,

$$|\Delta_{2,h} \partial_x^\alpha \partial_y^\beta K(x, y)| \le \|D^\alpha S\|_{L^2(\Omega) \to L^\infty(\Omega)} \|D^\beta T^*\|_{L^2(\Omega) \to C^{0,\sigma}(\Omega)} |h|^\sigma$$

for $h \in \mathbb{R}^n,\ (x, y) \in \Omega \times \Omega_h$.

Lemma 2.3. *For an integer* $k > 1 + n/(2m)$, $\sigma \in (0, 1)$ *and* $\eta \in (0, \pi/2)$ *there exist* $R = R(k, \sigma, \eta, \zeta_A, \omega_A, \Omega) \ge 1$ *and* $C = C(k, \sigma, \eta, \zeta_A, \Omega)$ *such that*

$$\|D^\alpha (A - \lambda)^{-k}\|_{L^2(\Omega) \to L^\infty(\Omega)} \le C |\lambda|^{-k+n/(4m)+|\alpha|/(2m)},$$

$$\|\Delta_h D^\alpha (A-\lambda)^{-k}\|_{L^2(\Omega)\to L^\infty(\Omega_h)} \le C|\lambda|^{-k+n/(4m)+(|\alpha|+\sigma)/(2m)}|h|^\sigma$$

for $h \in \mathbb{R}^n$, $|\alpha| < m$ *and* $\lambda \in \Lambda(R, \eta)$.

Lemmas 2.2 and 2.3 are essentially the same as [19, Lemma 2.3] and [19, Lemma 3.1], respectively, which dealt with the case $\Omega = \mathbb{R}^n$. Lemma 2.2 is a slight extension of [25, Lemma 5.10].

Proof of Theorem 1.1. Let $\{E_\tau\}$ be the spectral resolution of identity for A:

$$A = \int_0^\infty \tau \, dE_\tau.$$

Let k be as in Lemma 2.3. Since $R(E_\tau) \subset D(A^k)$ and

$$\|(A-\lambda)^k E_\tau\|_{L^2(\Omega)\to L^2(\Omega)} = \max_{0\le s\le\tau}(s-\lambda)^k \le (\tau+|\lambda|)^k$$

for $\tau \ge 0$ and $\lambda < 0$, we see from Lemma 2.3 and the equality $D^\alpha E_\tau = D^\alpha(A-\lambda)^{-k}(A-\lambda)^k E_\tau$ that for any $\sigma \in (0,1)$ there is $R \ge 1$ such that

$$\|D^\alpha E_\tau\|_{L^2(\Omega)\to L^\infty(\Omega)} \le C|\lambda|^{-k+n/(4m)+|\alpha|/(2m)}(\tau+|\lambda|)^k, \tag{2.2}$$

$$\|\Delta_h D^\alpha E_\tau\|_{L^2(\Omega)\to L^\infty(\Omega_h)} \le C|\lambda|^{-k+n/(4m)+(|\alpha|+\sigma)/(2m)}(\tau+|\lambda|)^k|h|^\sigma \tag{2.3}$$

for $h \in \mathbb{R}^n$, $|\alpha| < m$, $\tau \ge 0$ and $\lambda \le -R$. Applying Lemma 2.2 to $E_\tau = E_\tau E_\tau^*$ and using (2.2), (2.3) with $\lambda = -\max\{\tau, R\}$, we obtain Theorem 1.1. □

3. Estimates for resolvent kernels

In this section we estimate the difference between the kernels of $(A^k_{L^2(\Omega)} - \lambda)^{-1}$ and $(A^k_{L^2(\mathbb{R}^n)} - \lambda)^{-1}$, assuming that k is an integer satisfying

$$(k+1)m > n. \tag{3.1}$$

As stated in the beginning of Section 2, we may assume that A is positive. So by Theorem 1.1 we have

$$|e_\Omega(\tau^{2m}, x, y)| \le C\tau^n \quad \text{for} \quad \tau \ge 0, \quad e_\Omega(\tau^{2m}, x, y) = 0 \quad \text{for} \quad \tau < 0. \tag{3.2}$$

Lemma 3.1. *Let* $\sigma > n/(2m)$, *and assume that* $f \in C^1[0,\infty)$ *satisfies*

$$|f(\tau)| \le C(1+\tau)^{-\sigma}, \quad |f'(\tau)| \le C(1+\tau)^{-\sigma-1} \tag{3.3}$$

for $\tau \geq 0$ with some constant C. Then $f(A_{L^2(\Omega)})$ is an integral operator with bounded and continuous kernel, which can be written as

$$\int_0^\infty f(\tau)\, d_\tau e_\Omega(\tau, x, y). \tag{3.4}$$

Proof. See [19, Lemma 3.2]. □

Let $\lambda \in \mathbb{C} \setminus [0, \infty)$. We note that $k > n/(2m)$ if k satisfies (3.1). So by Lemma 3.1 $(A^k_{L^2(\Omega)} - \lambda)^{-1}$ is an integral operator with bounded and continuous kernel $G^k_{\Omega,\lambda}(x, y)$, which can be written as

$$G^k_{\Omega,\lambda}(x, y) = \int_0^\infty (\tau^k - \lambda)^{-1}\, d_\tau e_\Omega(\tau, x, y). \tag{3.5}$$

Integration by parts and (3.2) give

$$|G^k_{\Omega,\lambda}(x, y)| \leq C \int_0^\infty \frac{\tau^{k-1+n/(2m)}}{|\tau^k - \lambda|^2}\, d\tau = \frac{C}{k} \int_0^\infty \frac{s^{n/(2mk)}}{|s - \lambda|^2}\, ds \leq C \frac{|\lambda|^{n/(2mk)}}{d(\lambda)}, \tag{3.6}$$

where $d(\lambda) = \mathrm{dist}(\lambda, [0, \infty))$. Needless to say, here and in what follows the statements for Ω are also valid for $\mathbb{R}^n$. For simplicity we write $G^k_\lambda(x, y)$ for $G^k_{\mathbb{R}^n,\lambda}(x, y)$.

In order to evaluate $G^k_{\Omega,\lambda}(x, y) - G^k_\lambda(x, y)$ we fix $x_0 \in \Omega$ and $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ satisfying $\mathrm{supp}\, \varphi_0 \subset \{x \in \mathbb{R}^n : |x| < 1\}$, $\varphi_0(x) = 1$ for $|x| \leq 2^{-1}$, and set

$$\varphi(x) = \varphi_0\left(\frac{x - x_0}{\delta(x_0)}\right).$$

Remember $\delta(x) = \min\{1, \mathrm{dist}(x, \partial\Omega)\}$. Clearly, $\mathrm{supp}\, \varphi \subset \{x \in \mathbb{R}^n : |x - x_0| < \delta(x_0)\} \subset \Omega$. For $\lambda \in \mathbb{C} \setminus [0, \infty)$ let $\mu_1, \dots, \mu_k$ be the distinct roots of the equation $w^k = \lambda$ for w. For simplicity we set $\mu = \mu_1$. It is clear that $|\mu_j| = |\mu|$ and $\mu_j \in \Lambda(R^{1/k}, \eta/k)$ for $j = 1, \dots, k$ if $\lambda \in \Lambda(R, \eta)$ with some $R > 0$ and $\eta \in (0, \pi/2)$. For $1 \leq l \leq k$ we set

$$S_l(A_\Omega) = \prod_{j=1}^{l} (A_\Omega - \mu_j)^{-1}, \quad T_l(A) = \prod_{j=l}^{k} (A - \mu_j)^{-1}. \tag{3.7}$$

Remember that we simply write A for $A_{\mathbb{R}^n}$. Let $R_\Omega : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\Omega)$ be the restriction.

Lemma 3.2. *Assume that $(A_\Omega - \mu_j)^{-1}$ exists for $j = 1, \dots, k$. Then it follows that*

$$(A^k_\Omega - \lambda)^{-1} \varphi R_\Omega - \varphi R_\Omega (A^k - \lambda)^{-1} = -\sum_{l=1}^{k} S_l(A_\Omega) R_\Omega [A, \varphi] T_l(A), \tag{3.8}$$

where $[A, \varphi] = A\varphi - \varphi A$ *and* φ *stands for the multiplication by the function* $\varphi(x)$*. Furthermore,* $R_\Omega[A, \varphi]$ *can be written as*

$$R_\Omega[A, \varphi] = \sum_{\alpha,\beta,\gamma} D^\alpha b_{\alpha\beta\gamma} \varphi^{(\gamma)} R_\Omega D^\beta \tag{3.9}$$

with some functions $b_{\alpha\beta\gamma} \in L^\infty(\Omega)$ *satistying* $\|b_{\alpha\beta\gamma}\|_{L^\infty} \leq C(\zeta_A)$*, where the sum is taken over* α, β, γ *satisfying* $|\alpha| \leq m$, $|\beta| \leq m$, $0 < |\gamma| \leq m$, $|\alpha + \beta + \gamma| \leq 2m$.

Proof. Since $\operatorname{supp} \varphi \subset \Omega$, we have

$$\begin{aligned}(A_\Omega - \lambda)\varphi R_\Omega (A - \lambda)^{-1} &= R_\Omega (A - \lambda)\varphi (A - \lambda)^{-1} \\ &= R_\Omega \varphi + R_\Omega[A, \varphi](A - \lambda)^{-1},\end{aligned}$$

which gives

$$(A_\Omega - \lambda)^{-1} \varphi R_\Omega = \varphi R_\Omega (A - \lambda)^{-1} - (A_\Omega - \lambda)^{-1} R_\Omega [A, \varphi](A - \lambda)^{-1}. \tag{3.10}$$

Noting $(A_\Omega^k - \lambda)^{-1} = \prod_{j=1}^k (A_\Omega - \mu_j)^{-1}$ and using (3.10) repeatedly with $\lambda = \mu_1, \dots, \mu_k$, we obtain (3.8). By the Leibniz formula and its variant

$$[D^\beta, \varphi] = \sum_{\beta' < \beta} C_{0\beta\beta'} \varphi^{(\beta - \beta')} D^{\beta'}, \quad [D^\alpha, \varphi] = \sum_{\alpha' < \alpha} C_{1\alpha\alpha'} D^{\alpha'} \varphi^{(\alpha - \alpha')}$$

with some constants $C_{0\beta\beta'}$ and $C_{1\alpha\alpha'}$ we have

$$[D^\alpha a_{\alpha\beta} D^\beta, \varphi] = \sum_{\alpha' < \alpha} C_{1\alpha\alpha'} D^{\alpha'} \varphi^{(\alpha - \alpha')} a_{\alpha\beta} D^\beta + \sum_{\beta' < \beta} C_{0\beta\beta'} D^\alpha a_{\alpha\beta} \varphi^{(\beta - \beta')} D^{\beta'}.$$

Then we know that $R_\Omega[A, \varphi]$ is written in the form of (3.9). □

A useful tool to evaluate the kernel of the right-hand side in (3.8) is the fact that if an operator of the form ST has a continuous and bounded integral kernel $K(x, y)$ then it follows that

$$|K(x, y)| \leq \|ST\|_{L^1 \to L^\infty} \leq \|S\|_{L^p \to L^\infty} \|T\|_{L^1 \to L^p}$$

with $1 < p < \infty$. In order to apply this fact we shall derive exponential decay estimates for the resolvent kernels and their derivatives.

Theorem 3.3. *Let* $p \in (1, \infty)$, $\eta \in (0, \pi/2)$*. Then there exists* $R = R(\eta, \zeta_A, \omega_A, \Omega) \geq 1$ *such that for* $\lambda \in \Lambda(R, \eta)$ *the resolvent* $(A_{L^p(\Omega)} - \lambda)^{-1}$ *exists and it has a kernel* $G_{\Omega,\lambda}(x, y)$ *which is independent of* p *and satisfies the following. There exist*

$C = C(\eta, \zeta_A, \Omega)$ and $c = c(\eta, \zeta_A, \Omega)$ such that for $|\alpha| < m$, $|\beta| < m$ the derivative $\partial_x^\alpha \partial_y^\beta G_{\Omega,\lambda}(x, y)$ is continuous off the diagonal in $\Omega \times \Omega$ and satisfies

$$|\partial_x^\alpha \partial_y^\beta G_{\Omega,\lambda}(x, y)| \leq C\Psi_{2m-|\alpha|-|\beta|}(x - y, \lambda, c) \tag{3.11}$$

for $x, y \in \Omega$, where the function Ψ_σ with $\sigma > 0$ is defined by

$$\Psi_\sigma(x, \lambda, c) = \exp(-c|\lambda|^{1/(2m)}|x|) \times \begin{cases} |x|^{\sigma-n} & (0 < \sigma < n), \\ (1 + \log_+(|\lambda|^{1/(2m)}|x|)^{-1}) & (\sigma = n), \\ |\lambda|^{(n-\sigma)/(2m)} & (\sigma > n), \end{cases}$$

and $\log_+ s = \max\{0, \log s\}$. Moreover, $\partial_x^\alpha \partial_y^\beta G_{\Omega,\lambda}(x, y)$ is also continuous on the diagonal if $2m - |\alpha| - |\beta| > n$.

Proof. See [21] for a domain with bounded C^{m+1} boundary and [22] for a uniform C^1 domain. □

Lemma 3.4. *Let $p \in (1, \infty)$, $\eta \in (0, \pi/2)$, $|\alpha| < m$, $|\beta| < m$, and set*

$$G_{\Omega,\lambda}^{\alpha,\beta}(x, y) = D_x^\alpha(-D_y)^\beta G_{\Omega,\lambda}(x, y).$$

Then there exist $R = R(\eta, \zeta_A, \omega_A, \Omega) \geq 1$, $C = C(\eta, \zeta_A, \Omega)$, $c = c(\eta, \zeta_A, \Omega)$ such that for $\lambda \in \Lambda(R, \eta)$ we have

$$D^\alpha(A_\Omega - \lambda)^{-1}D^\beta f(x) = \int_\Omega G_{\Omega,\lambda}^{\alpha,\beta}(x, y) f(y)\, dy \tag{3.12}$$

for $f \in L^p(\Omega)$ and

$$|G_{\Omega,\lambda}^{\alpha,\beta}(x, y)| \leq C\Psi_{2m-|\alpha|-|\beta|}(x - y, \lambda, c). \tag{3.13}$$

Proof. Let R be the maximum of the R's in Theorems 2.1 and 3.3. Then $(A_\Omega - \lambda)^{-1}$ and $G_{\Omega,\lambda}^{\alpha,\beta}(x, y)$ exist for $\lambda \in \Lambda(R, \eta)$. Estimate (3.13) follows immediately from (3.11).

Let $f, g \in C_0^\infty(\Omega)$. Noting $(A_\Omega - \lambda)^{-1}|_{L^p(\Omega)} = (A_{L^p(\Omega)} - \lambda)^{-1}$ and using Theorem 3.3, we have

$$(D^\alpha(A_\Omega - \lambda)^{-1}D^\beta f, g)_\Omega = \iint_{\Omega\times\Omega} G_{\Omega,\lambda}(x, y) D_y^\beta f(y)\overline{D_x^\alpha g(x)}\, dx\, dy,$$

where we set $(u, v)_\Omega = \int_\Omega u(x)\overline{v(x)}\, dx$. Set $B_\varepsilon(x) = \{y \in \mathbb{R}^n : |x - y| < \varepsilon\}$ for $x \in \Omega$

and sufficiently small $\varepsilon > 0$. Integrating by parts, we have

$$\begin{aligned}\int_\Omega G_{\Omega,\lambda}(x,y)D_{y_j}f(y)\,dy &= \lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon(x)} G_{\Omega,\lambda}(x,y)D_{y_j}f(y)\,dy\\ &= \lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon(x)}(-1)D_{y_j}G_{\Omega,\lambda}(x,y)f(y)\,dy\\ &\quad + i^{-1}\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} G_{\Omega,\lambda}(x,y)f(y)\frac{x_j-y_j}{|x-y|}\,dS_y\\ &= \int_\Omega (-1)D_{y_j}G_{\Omega,\lambda}(x,y)f(y)\,dy.\end{aligned}$$

Here we used $G_{\Omega,\lambda}(x,\cdot)\in L^1(\Omega)$, $D_{y_j}G_{\Omega,\lambda}(x,\cdot)\in L^1(\Omega)$ and $\int_{\partial B_\varepsilon(x)}|G_{\Omega,\lambda}(x,y)|\,dS_y = o(1)$ as $\varepsilon\to 0$, which follow from (3.11).

Repeating this procedure, we get

$$(D^\alpha(A_\Omega-\lambda)^{-1}D^\beta f, g)_\Omega = \iint_{\Omega\times\Omega} G^{\alpha,\beta}_{\Omega,\lambda}(x,y)f(y)\overline{g(x)}\,dx\,dy.$$

Hence (3.12) holds for $f\in C_0^\infty(\Omega)$. By Theorem 2.1 and (3.13) we see that the both sides of (3.12) define bounded operators in $L^p(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, (3.12) also holds for $f\in L^p(\Omega)$. $\square$

For a fixed $x_0\in\Omega$ we set

$$B_{x_0} = \left\{x\in\mathbb{R}^n : |x-x_0| < \frac{\delta(x_0)}{4}\right\}.$$

Let $R_{x_0}\colon L^\infty(\Omega)\to L^\infty(B_{x_0})$ be the restriction and $E_{x_0}\colon L^1(B_{x_0})\to L^1(\mathbb{R}^n)$ the extension defined by $E_{x_0}u(x)=u(x)$ for $x\in B_{x_0}$ and $E_{x_0}u(x)=0$ for $x\in\mathbb{R}^n\setminus B_{x_0}$. Obviously we have

$$\|R_{x_0}\|_{L^\infty(\Omega)\to L^\infty(B_{x_0})} = 1,\quad \|E_{x_0}\|_{L^1(B_{x_0})\to L^1(\mathbb{R}^n)} = 1.$$

Lemma 3.5. *Let $p\in(1,\infty)$, $\eta\in(0,\pi/2)$, $(k+1)m>n$, $1\le l\le k$. Then there exist $R=R(\eta,\zeta_A,\omega_A,\Omega)\ge 1$, $C=C(p,k,\eta,\zeta_A,\Omega)$ and $c=c(k,\eta,\zeta_A,\Omega)$ such that the following estimates hold for $\lambda\in\Lambda(R,\eta)$.*
(i) *If $|\alpha|\le m$ and $p^{-1}<lm/n$, then*

$$\|R_{x_0}S_l(A_\Omega)D^\alpha\|_{L^p(\Omega)\to L^\infty(B_{x_0})} \le C|\mu|^{-l+|\alpha|/(2m)+n/(2mp)}.$$

(ii) *If $|\alpha|<m$, $0<|\gamma|\le m$ and $p^{-1}<(2ml-|\alpha|)/n$, then*

$$\begin{aligned}&\|R_{x_0}S_l(A_\Omega)D^\alpha\varphi^{(\gamma)}\|_{L^p(\Omega)\to L^\infty(B_{x_0})}\\ &\le C\delta(x_0)^{-|\gamma|}|\mu|^{-l+|\alpha|/(2m)+n/(2mp)}\exp(-c\delta(x_0)|\mu|^{1/(2m)}).\end{aligned}$$

(iii) *If* $|\beta| \leq m$ *and* $p^{-1} > 1 - (k - l + 1)m/n$, *then*

$$\|D^\beta T_l(A)E_{x_0}\|_{L^1(B_{x_0}) \to L^p(\mathbb{R}^n)} \leq C|\mu|^{-k+l-1+|\beta|/(2m)+(1-1/p)n/(2m)}.$$

(iv) *If* $|\beta| < m$, $0 < |\gamma| \leq m$ *and* $p^{-1} > 1 - \{2m(k - l + 1) - |\beta|\}/n$, *then*

$$\begin{aligned}&\|\varphi^{(\gamma)} D^\beta T_l(A)E_{x_0}\|_{L^1(B_{x_0}) \to L^p(\mathbb{R}^n)} \\ &\leq C\delta(x_0)^{-|\gamma|}|\mu|^{-k+l-1+|\beta|/(2m)+(1-1/p)n/(2m)} \exp(-c\delta(x_0)|\mu|^{1/(2m)}).\end{aligned}$$

Proof. Let R_0 be the maximum of the R's in Theorem 2.1 and Lemma 3.4 for the angle η/k. As will be seen below, Lemma 3.5 holds with $R = R_0^k$.

(i) Let $1 < q < r \leq \infty$ and $q^{-1} - r^{-1} < m/n$. Then by Theorem 2.1 and the Sobolev embedding theorem we have

$$\begin{aligned}&\|(A_\Omega - \lambda)^{-1}D^\alpha\|_{L^q(\Omega) \to L^r(\Omega)} \\ &\leq \|(A_\Omega - \lambda)^{-1}\|^{1-(n/m)(1/q-1/r)}_{H^{-|\alpha|,q}(\Omega) \to L^q(\Omega)} \|(A_\Omega - \lambda)^{-1}\|^{(n/m)(1/q-1/r)}_{H^{-|\alpha|,q}(\Omega) \to H^{m,q}(\Omega)} \\ &\leq C|\lambda|^{-1+|\alpha|/(2m)+(n/(2m))(1/q-1/r)}\end{aligned} \tag{3.14}$$

for $\lambda \in \Lambda(R_0, \eta/k)$, $|\alpha| \leq m$. In view of $p^{-1} < lm/n$ we can choose a decreasing sequence $\{p_j\}_{j=0}^l$ satisfying

$$\infty = p_0 > p_1 > \cdots > p_l = p, \quad p_j^{-1} - p_{j-1}^{-1} < \frac{m}{n} \quad (j = 1, \ldots, l).$$

Using (3.7), (3.14) and $|\mu| = |\mu_j|$ for $j = 1, \ldots, k$, we have

$$\begin{aligned}&\|S_l(A_\Omega)D^\alpha\|_{L^p(\Omega) \to L^\infty(\Omega)} \\ &\leq \prod_{j=1}^{l-1} \|(A_\Omega - \mu_j)^{-1}\|_{L^{p_j}(\Omega) \to L^{p_{j-1}}(\Omega)} \times \|(A_\Omega - \mu_l)^{-1}D^\alpha\|_{L^{p_l}(\Omega) \to L^{p_{l-1}}(\Omega)} \\ &\leq C|\mu|^{-l+|\alpha|/(2m)+n/(2mp)}\end{aligned} \tag{3.15}$$

for $\lambda \in \Lambda(R_0^k, \eta)$, which gives (i).

(ii) Using Lemma 3.4 and the inequality

$$\int_{\mathbb{R}^n} \Psi_\sigma(x - z, \lambda, c)\Psi_\rho(z - y, \lambda, c)\, dz \leq C(\sigma, \rho, n, c)\Psi_{\sigma+\rho}\Big(x - y, \lambda, \frac{c}{2}\Big)$$

for $\sigma >$, $\rho > 0$ (see [14, Lemma 3.2]) repeatedly, we see that $S_l(A_\Omega)D^\alpha$ is an integral operator with kernel $S_{l,\alpha}(x, y)$ satisfying

$$|S_{l,\alpha}(x, y)| \leq C\Psi_{2ml-|\alpha|}(x - y, \mu, c)$$

if we replace constants C, c with other ones.

Let $p^{-1}+q^{-1}=1$, $x\in B_{x_0}$ and $y\in \operatorname{supp}\varphi^{(\gamma)}$. Then $|x-x_0|<\delta(x_0)/4$ and $\delta(x_0)/2\leq |y-x_0|\leq\delta(x_0)$. Therefore $|x-y|\geq\delta(x_0)/4$. We note that $\|\varphi^{(\gamma)}\|_{L^\infty(\mathbb{R}^n)}\leq C\delta(x_0)^{-|\gamma|}$, $\Psi_\sigma(x,\mu,c)=\Psi_\sigma(x,\mu,c/2)\exp(-c|\mu|^{1/(2m)}|x|/2)$ and $\|\Psi_\sigma(\,\cdot\,,\mu,c)\|_{L^q(\mathbb{R}^n)}=C|\mu|^{(n-\sigma)/(2m)-n/(2mq)}$ if $\sigma>0$ and $(\sigma-n)q>-n$. Then we have

$$\begin{aligned}
&\|R_{x_0}S_l(A_\Omega)D^\alpha\varphi^{(\gamma)}\|^q_{L^p(\Omega)\to L^\infty(B_{x_0})}\\
&\leq C\sup_{x\in B_{x_0}}\|\Psi_{2ml-|\alpha|}(x-\,\cdot\,,\mu,c)\varphi^{(\gamma)}\|^q_{L^q(\Omega)}\\
&\leq C\delta(x_0)^{-q|\gamma|}\sup_{x\in B_{x_0}}\int_{|x-y|\geq\delta(x_0)/4}\Psi_{2ml-|\alpha|}\left(x-y,\mu,\frac{c}{2}\right)^q\\
&\quad\times\exp\left(\frac{-qc|\mu|^{1/(2m)}\delta(x_0)}{8}\right)dy\\
&\leq C\delta(x_0)^{-q|\gamma|}\exp\left(\frac{-qc|\mu|^{1/(2m)}\delta(x_0)}{8}\right)|\mu|^{(n-2ml+|\alpha|)q/(2m)-n/(2m)}
\end{aligned}$$

if $(2ml-|\alpha|-n)q>-n$. This yields (ii).

(iii) Let $p^{-1}+q^{-1}=1$ and set $(u,v)_{\mathbb{R}^n}=\int_{\mathbb{R}^n}u(x)\overline{v(x)}\,dx$ and $T_l(A)^*=\prod_{j=l}^k(A-\overline{\mu_j})^{-1}$. Then we have

$$(D^\beta T_l(A)u,v)_{\mathbb{R}^n}=(u,T_l(A)^*D^\beta v)_{\mathbb{R}^n}$$

for $u,v\in C_0^\infty(\mathbb{R}^n)$ because of the self-adjointness of $A_{L^2(\mathbb{R}^n)}$ and the relation $(A-\mu_j)^{-1}|_{L^2(\mathbb{R}^n)}=(A_{L^2(\mathbb{R}^n)}-\mu_j)^{-1}$. Hence

$$\|D^\beta T_l(A)\|_{L^1(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}=\|T_l(A)^*D^\beta\|_{L^q(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)}.$$

We can evaluate the right-hand side in the same way as in (3.15) to obtain (iii).

(iv) can be treated in the same way as (ii), if we note that

$$\|\varphi^{(\gamma)}D^\beta T_l(A)E_{x_0}\|_{L^1(B_{x_0})\to L^p(\mathbb{R}^n)}=\|R_{x_0}T_l(A)^*D^\beta\varphi^{(\gamma)}\|_{L^q(\mathbb{R}^n)\to L^\infty(B_{x_0})}$$

with $p^{-1}+q^{-1}=1$ and that $T_l(A)^*D^\beta$ is an integral operator with kernel $T_{l,\beta}(x,y)$ satisfying

$$|T_{l,\beta}(x,y)|\leq C\Psi_{2m(k-l+1)-|\beta|}(x-y,\mu,c). \qquad \square$$

Lemma 3.6. *Let $\eta\in(0,\pi/2)$, $(k+1)m>n$. Then there exists $C=C(k,\eta,\zeta_A,\omega_A,\Omega)$, $c=c(k,\eta,\zeta_A,\Omega)$ such that*

$$|G^k_{\Omega,\lambda}(x,x)-G^k_\lambda(x,x)|\leq C|\lambda|^{-1+n/(2mk)}\exp(-c\delta(x)|\lambda|^{1/(2mk)}) \tag{3.16}$$

for $x\in\Omega$, $\lambda\in\Lambda(0,\eta)$.

Proof. Let R be the R in Lemma 3.5. First we consider the case $|\lambda| \le R\delta(x_0)^{-2mk}$. Then by (3.6) we have $|G^k_{\Omega,\lambda}(x_0, x_0)| + |G^k_\lambda(x_0, x_0)| \le C|\lambda|^{-1+n/(2mk)}$ for $\lambda \in \Lambda(0, \eta)$, which implies (3.16).

Next we consider the case $|\lambda| \ge R\delta(x_0)^{-2mk}$ ($\ge R$). Since $G^k_{\Omega,\lambda}(x, y)$ and $G^k_\lambda(x, y)$ are bounded and continuous, (3.8) gives

$$|G^k_{\Omega,\lambda}(x_0, x_0) - G^k_\lambda(x_0, x_0)| \le \sum_{l=1}^{k} \| R_{x_0} S_l(A_\Omega) R_\Omega[A, \varphi] T_l(A) E_{x_0} \|_{L^1(B_{x_0}) \to L^\infty(B_{x_0})}.$$

The right-hand side can be estimated by using (3.9) and Lemma 3.5. It is important that we always have $|\alpha| < m$ or $|\beta| < m$ in the sum in (3.9). Suppose that for each l with $1 \le l \le k$ we can take $p \in (1, \infty)$ satisfying the inequalities in (i), (iv) of Lemma 3.5 if $|\alpha| \le m$ and $|\beta| < m$, and those in (ii), (iii) of Lemma 3.5 if $|\alpha| < m$ and $|\beta| \le m$. Then we get

$$\begin{aligned} &|G^k_{\Omega,\lambda}(x_0, x_0) - G^k_\lambda(x_0, x_0)| \\ &\le C \sum_{\alpha,\beta,\gamma} \delta(x_0)^{-|\gamma|} |\mu|^{-k-1+(|\alpha|+|\beta|)/(2m)+n/(2m)} \exp(-c\delta(x_0)|\mu|^{1/(2m)}) \\ &\le C|\mu|^{-k+n/(2m)} \exp(-c\delta(x_0)|\mu|^{1/(2m)}), \end{aligned}$$

where we have used $|\alpha + \beta + \gamma| \le 2m$, $\delta(x_0)^{-1} \le R^{-1/(2mk)}|\mu|^{1/(2m)}$ and $|\mu|^{-1} \le R^{-1/k}$. This implies (3.16).

So it remains to check that there exists $p \in (1, \infty)$ satisfying the above-mentioned conditions. In other words, we have only to show that for each integer $l \in [1, k]$ there exists $p \in (1, \infty)$ satisfying either of

$$I_1(l) := 1 - \frac{2m(k - l + 1) - m}{n} < \frac{1}{p} < \frac{lm}{n} =: I_2(l),$$
$$I_3(l) := 1 - \frac{(k - l + 1)m}{n} < \frac{1}{p} < \frac{2ml - m}{n} =: I_4(l).$$

Since $I_1(l) < 1$, $I_2(l) > 0$, $I_3(l) < 1$ and $I_4(l) > 0$ always hold, such a p exists if $I_1(l) < I_2(l)$ and $I_3(l) < I_4(l)$, i.e.,

$$(2k - l + 1)m > n, \quad (k + l)m > n$$

for any $l \in [1, k]$. These inequalities hold if $(k + 1)m > n$. Thus we have shown the existence of p which has the desired properties. □

4. Tauberian argument

In order to derive the asymptotic formula for $e_\Omega(\tau, x, x)$ from that of $e_{\mathbb{R}^n}(\tau, x, x)$ by using the estimate of $G^k_{\Omega,\lambda}(x, x) - G^k_\lambda(x, x)$ we prepare the following Tauberian theorem, which is a modification of Avakumovič's Tauberian theorem [4, Lemma 4]. In the remainder term $O(\tau^{n-\theta})$ in Lemma 4.1 below we allow the value of θ to be not only 1 but also a number in (0, 1].

Lemma 4.1. *Let $N(\tau)$ and $\Lambda(\tau)$ be functions $\mathbb{R} \to \mathbb{R}$ satisfying the following conditions*:

(i) *$N(\tau)$ is non-decreasing*;

(ii) *There exist constants $c_0 > 0$, $\theta \in (0, 1]$ and $C_1 > 0$ such that*

$$|\Lambda(\tau) - c_0\tau^n| \le C_1\tau^{n-\theta} \quad \textit{for} \quad \tau \ge 0, \quad \Lambda(\tau) = 0 \quad \textit{for} \quad \tau < 0;$$

(iii) *There exists a constant $C_2 > 0$ such that*

$$|N(\tau)| \le C_2\tau^n \quad \textit{for} \quad \tau \ge 0, \quad N(\tau) = 0 \quad \textit{for} \quad \tau < 0;$$

(iv) *If we set*

$$F(z) = \int_0^\infty e^{-\tau z}\, d_\tau(N(\tau) - \Lambda(\tau)),$$

which is analytic for $\operatorname{Re} z > 0$ *by conditions* (ii)–(iii), *then there exist $T > 0$ and $B > 0$ such that $F(z)$ is analytically continued to the disk $\{z \in \mathbb{C} \colon |z| < T\}$ and satisfies*

$$|F(z)| \le B \quad \textit{for} \quad |z| < T, \tag{4.1}$$

$$F(0) = 0. \tag{4.2}$$

Then there exists $C = C(c_0, n, \theta, C_1, C_2)$ such that

$$|N(\tau) - c_0\tau^n| \le C(\tau^{n-\theta} + T^{-1}\tau^{n-1} + B) \quad \textit{for} \quad \tau \ge T^{-1}. \tag{4.3}$$

Proof. As in [24], we choose a non-negative-valued function $\rho \in \mathcal{S}(\mathbb{R})$ such that

$$\rho(\tau) > 0 \quad \text{for} \quad |\tau| \le 1, \quad \operatorname{supp}\hat{\rho} \subset (-1, 1), \quad \hat{\rho}(0) = \int_{-\infty}^\infty \rho(\tau)\, d\tau = 1,$$

and set $\rho_T(\tau) = T\rho(T\tau)$, where $\hat{\rho}(t) = \int_{-\infty}^\infty e^{-i\tau t}\rho(\tau)\, d\tau$. Obviously $|\hat{\rho}(t)| \le 1$ and $\hat{\rho}_T(t) = \hat{\rho}(t/T)$.

First we shall evaluate $\rho_T * d\Lambda(\tau)$ and $\rho_T * \Lambda(\tau)$. To do so we set

$$h(\tau, T) = \tau^{n-\theta} + T^{-1}\tau^{n-1} + T^{\theta-n} + T^{-n},$$

and $r(\tau) = \Lambda(\tau) - c\tau^n$ for $\tau \ge 0$, $r(\tau) = 0$ for $\tau < 0$. Then $|r(\tau)| \le C_1\tau^{n-\theta}$ for $\tau \ge 0$. We often use the inequalities

$$\left|\tau - \frac{s}{T}\right|^\kappa \le C_\kappa\left(\tau^\kappa + \frac{|s|^\kappa}{T^\kappa}\right), \quad \int_{-\infty}^\infty \{\rho(s) + |\rho'(s)|\}|s|^\kappa\, ds \le C_\kappa \tag{4.4}$$

for $\tau \geq 0$, $\kappa \geq 0$. Combining

$$\rho_T * d\Lambda(\tau) = nc_0 \int_{-\infty}^{T\tau} \rho(s)\Big(\tau - \frac{s}{T}\Big)^{n-1} ds + T \int_{-\infty}^{T\tau} \rho'(s) r\Big(\tau - \frac{s}{T}\Big)\, ds$$

with (4.4), we have

$$|\rho_T * d\Lambda(\tau)| \leq CTh(\tau, T) \quad \text{for} \quad \tau \geq 0. \tag{4.5}$$

Using

$$\rho_T * \Lambda(\tau) = \Lambda(\tau) - \Lambda(\tau) \int_{T\tau}^{\infty} \rho(s)\, ds + \int_{-\infty}^{T\tau} \rho(s)\Big\{\Lambda\Big(\tau - \frac{s}{T}\Big) - \Lambda(\tau)\Big\}\, ds,$$

$$\left|\int_{T\tau}^{\infty} \rho(s)\, ds\right| \leq C(T\tau)^{-1}, \quad \left|\Big(\tau - \frac{s}{T}\Big)^n - \tau^n\right| = C(|s|\tau^{n-1}T^{-1} + |s|^n T^{-n})$$

for $\tau \geq 0$, we have

$$|\rho_T * \Lambda(\tau) - c_0\tau^n| \leq C(\tau^{n-\theta} + T^{-1}\tau^{n-1}) \quad \text{for} \quad \tau \geq T^{-1}. \tag{4.6}$$

Next we shall evaluate $\rho_T * dN(\tau)$ and $\rho_T * N(\tau)$. Inequality (4.1) implies $|\widehat{dN}(t) - \widehat{d\Lambda}(t)| \leq B$ for $|t| < T$. Hence by (4.5) and

$$\rho_T * dN(\tau) = (2\pi)^{-1} \int_{-T}^{T} e^{i\tau t}\hat{\rho}_T(t)\{\widehat{dN}(t) - \widehat{d\Lambda}(t)\}\, dt + \rho_T * d\Lambda(\tau)$$

we have

$$0 \leq \rho_T * dN(\tau) \leq CT(B + h(\tau, T)) \quad \text{for} \quad \tau \geq 0. \tag{4.7}$$

Choose $c_1 > 0$ so that $\rho(\tau) \geq c_1$ for $|\tau| \leq 1$. Since $N(\tau)$ is non-decreasing, we have

$$0 \leq N(\tau) - N(\tau - T^{-1}) \leq c_1^{-1}T^{-1}\rho_T * dN(\tau) \quad \text{for} \quad \tau \in \mathbb{R}. \tag{4.8}$$

Dividing the interval $[0, |s|]$ into at most $|s|+1$ intervals of length ≤ 1, and using (4.7) and (4.8), we have

$$0 \leq N\Big(\tau - \frac{s}{T}\Big) - N(\tau) \leq C(1 + |s|)\Big(B + h\Big(\tau + \frac{|s|}{T}, T\Big)\Big)$$

when $s \leq 0$. Similarly we have

$$0 \leq N(\tau) - N\Big(\tau - \frac{s}{T}\Big) \leq C(1 + |s|)(B + h(\tau, T))$$

when $0 \le s \le T\tau$. Then from (iii), the inequality $\left|\int_{T\tau}^{\infty} \rho(s)\, ds\right| \le C(T\tau)^{-1}$ and

$$\rho_T * N(\tau) = N(\tau) - N(\tau) \int_{T\tau}^{\infty} \rho(s)\, ds + \int_{-\infty}^{T\tau} \rho(s)\Big\{N\Big(\tau - \frac{s}{T}\Big) - N(\tau)\Big\}\, ds$$

it follows that

$$|\rho_T * N(\tau) - N(\tau)| \le C(B + h(\tau, T)) \quad \text{for} \quad \tau \ge T^{-1}. \tag{4.9}$$

Finally we shall evaluate $\rho_T * N(\tau) - \rho_T * \Lambda(\tau)$. Since $F(0) = 0$, the function $F(z)/z$ is also analytic in $|z| < T$. So (4.1) and the maximum principle give $|F(z)/z| \le B/T$ for $|z| < T$. On the other hand, integration by parts gives

$$F(z) = z \int_0^{\infty} e^{-\tau z}(N(\tau) - \Lambda(\tau))\, d\tau$$

for $\operatorname{Re} z > 0$. Then we have

$$|\hat{N}(t) - \hat{\Lambda}(t)| = \left|\frac{F(it)}{it}\right| \le \frac{B}{T} \quad \text{for} \quad -T < t < T.$$

Hence

$$|\rho_T * (N(\tau) - \Lambda(\tau))| = \left|(2\pi)^{-1} \int_{-T}^{T} e^{i\tau t} \hat{\rho}_T(t)(\hat{N}(t) - \hat{\Lambda}(t))\, dt\right| \le \frac{B}{\pi} \tag{4.10}$$

for $\tau \ge 0$. Combining (4.6), (4.9) and (4.10), we obtain (4.3). □

Proof of Proposition 1.2. For simplicity we write $e(\tau, x, x)$ for $e_{\mathbb{R}^n}(\tau, x, x)$. Let us apply Lemma 4.1 with $N(\tau) = e_\Omega(\tau^{2m}, x, x)$ and $\Lambda(\tau) = e(\tau^{2m}, x, x)$. To do so we shall see that conditions (i)–(iv) in Lemma 4.1 hold. Condition (i) follows from the property of the spectral function. Condition (ii) holds with $c_0 = c_A(x)$ by assumption (1.5). Condition (iii) follows from (3.2). To check (iv) we set

$$F(z) = \int_0^{\infty} e^{-\tau z}\, d_\tau\{e_\Omega(\tau^{2m}, x, x) - e(\tau^{2m}, x, x)\}$$

for $\operatorname{Re} z > 0$. By Cauchy's integral theorem and (3.5) we have

$$\begin{aligned} F(z) &= \int_0^{\infty} e^{-z\tau^{1/(2mk)}}\, d_\tau\{e_\Omega(\tau^{1/k}, x, x) - e(\tau^{1/k}, x, x)\} \\ &= \frac{-1}{2\pi i} \int_\Gamma e^{-z\lambda^{1/(2mk)}}\, d\lambda \int_0^{\infty} (\tau - \lambda)^{-1} d_\tau\{e_\Omega(\tau^{1/k}, x, x) - e(\tau^{1/k}, x, x)\} \\ &= \frac{-1}{2\pi i} \int_\Gamma e^{-z\lambda^{1/(2mk)}}\{G^k_{\Omega,\lambda}(x, x) - G^k_\lambda(x, x)\}\, d\lambda, \end{aligned}$$

where Γ is the boundary of $\Lambda(0, \pi/4)$. Using estimate (3.16) for $\lambda \in \Lambda(0, \pi/4)$, we have, for $|z| < 2^{-1}c\delta(x)$,

$$\begin{aligned}
&\int_\Gamma \left|e^{-z\lambda^{1/(2mk)}}\{G^k_{\Omega,\lambda}(x, x) - G^k_\lambda(x, x)\}\right||d\lambda| \\
&\le C \int_\Gamma |\lambda|^{-1+n/(2mk)} \exp\{(|z| - c\delta(x))|\lambda|^{1/(2mk)}\}|d\lambda| \\
&\le C \int_0^\infty r^{-1+n/(2mk)} \exp(-2^{-1}c\delta(x)r^{1/(2mk)})\, dr \le C\delta(x)^{-n}.
\end{aligned}$$

Hence $F(z)$ is analytic in $\{z \in \mathbb{C}\colon |z| < 2^{-1}c\delta(x)\}$, and $|F(z)| \le C\delta(x)^{-n}$. That is, (4.1) is valid with $T = 2^{-1}c\delta(x)$ and $B = C\delta(x)^{-n}$. Equality (4.2) follows from Cauchy's integral theorem and the fact that $G^k_{\Omega,\lambda}(x, x) - G^k_\lambda(x, x)$ is rapidly decreasing as $|\lambda| \to \infty$ in $\Lambda(0, \pi/4)$. Thus we have checked condition (iv). So we can apply Lemma 4.1 to get

$$\begin{aligned}
|e_\Omega(\tau^{2m}, x, x) - c_A(x)\tau^n| &\le C(\tau^{n-\theta} + \delta(x)^{-1}\tau^{n-1} + \delta(x)^{-n}) \\
&\le C(\tau^{n-\theta} + \delta(x)^{-1}\tau^{n-1})
\end{aligned}$$

for $\tau \ge 2c^{-1}\delta(x)^{-1}$. Since $\delta(x)^{-1} \le \operatorname{dist}(x, \partial\Omega)^{-1} + 1$, (1.6) holds for $\tau \ge 2c^{-1}\delta(x)^{-1}$. When $1 \le \tau \le 2c^{-1}\delta(x)^{-1}$, (1.6) follows from (3.2). This completes the proof of Proposition 1.2. □

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