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THE NECESSARY CONDITION ON THE FIBER-SUM DECOMPOSABILITY OF GENUS-2 LEFSCHETZ FIBRATIONS

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Abstract

The fiber-sum construction gives us many interesting examples of Lefschetz fibrations. Which Lefschetz fibrations can be decomposed as fiber-sums? Stipsicz obtained some results on the fiber-sum decomposition, which state about the relationship between the minimality and the fiber-sum decomposability of Lefschetz fibrations. He proved that every Lefschetz fibration with section of self-intersection number -1 cannot be decomposed as any nontrivial fiber-sum. In this paper, we show that the reverse of this theorem does not hold and we characterize genus-2 decomposable Lefschetz fibrations with $b_2^+ = 1$.

1. Introduction

A *Lefschetz fibration* is a smooth map $f: X \rightarrow \Sigma$, where X is a closed connected oriented smooth 4-manifold and Σ is a closed connected oriented surface, such that f has finitely many critical points $C = \{p_1, p_2, \dots, p_m\}$ and around each p_i and $f(p_i)$ there are complex local coordinate neighborhoods compatible with the orientations of X and Σ on which f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$. The *genus* of f is defined to be the genus of a generic fiber of f . The singular fibers of a Lefschetz fibration are obtained from the nearby generic fibers by collapsing a simple closed curve, called the *vanishing cycle*, to a point. A singular fiber is called *reducible* or *irreducible* according to whether the corresponding vanishing cycle separates or does not separate in the generic fiber. A Lefschetz fibration f is *relatively minimal* if there is no fiber containing a smooth sphere of self-intersection number -1 . We will always assume that a Lefschetz fibration f is injective on its critical points set $C = \{p_1, p_2, \dots, p_m\}$ and f is relatively minimal. Moreover, in this paper, we will assume that a base space Σ is a 2-sphere. For the definitions and more details on Lefschetz fibrations, see [4] and [15].

Lefschetz fibrations have been known as important structures on 4-manifolds ever since Donaldson [2] showed that, after blow-ups, every closed symplectic 4-manifold admits Lefschetz fibrations and Gompf [4] showed that closed 4-manifolds with Lefschetz fibrations admit symplectic structures. So, we can study the topology of symplectic 4-manifolds through Lefschetz fibrations.

Many examples of Lefschetz fibrations are given by projective complex surfaces. In particular, they are often constructed from double branched coverings of surface bundles like the Hirzebruch surfaces. On the other hand, since the isomorphism class of a genus- g Lefschetz fibration $X \rightarrow S^2$ is determined by its global monodromy, genus- g Lefschetz fibrations over S^2 can be also constructed from positive relations in the mapping class group Γ_g of genus g corresponding to their global monodromies. Here, a *positive relation* is the relation obtained from a factorization of the identity via positive Dehn twists on a surface Σ_g of genus g . Moreover, we can construct Lefschetz fibrations from given fibrations via the fiber-sum, which is the most topological technique using the “cut-and-paste” method.

It is an important problem whether a Lefschetz fibration is decomposable into fiber-sum. Since the fiber-sum decomposability of a Lefschetz fibration $f: X \rightarrow S^2$ implies that the corresponding positive relation $w = 1$ can be written as the product $w_1 w_2 = 1$ of nontrivial positive relations $w_1 = 1$ and $w_2 = 1$ up to elementary transformations and conjugations, the fiber-sum decomposability problem is important from the viewpoint of the mapping class group as well. For more details on positive relations, see [3] and [4].

QUESTION 1.1. Which Lefschetz fibration over S^2 can be decomposed as the fiber-sum of nontrivial Lefschetz fibrations over S^2 ?

Stipsicz proved the following result on the fiber-sum decomposition, which starts studying the relationship between the minimality and the fiber-sum decomposition of Lefschetz fibrations.

Theorem 1.1 (Stipsicz [19], Smith [17]). *If a Lefschetz fibration $f: X \rightarrow S^2$ admits a section with self-intersection number -1 , then it cannot be decomposed as any nontrivial fiber-sum.*

Under this theorem, Stipsicz proposed the following question in [19].

QUESTION 1.2. For a Lefschetz fibration $f: X \rightarrow S^2$ with nontrivial fiber-sum decomposition, is X minimal?

For this question, Stipsicz conjectured that any nontrivial fiber-sum $X_1 \#_F X_2$ is minimal, and Usher proved this conjecture in [23]. Question 1.2 was solved affirmatively. On the other hand, Theorem 1.1 naturally raises the following question:

QUESTION 1.3 (Conjecture 2.4 [19]). Does the converse of Theorem 1.1 hold? Namely, does every Lefschetz fibration over S^2 without nontrivial fiber-sum decomposition admit a section with self-intersection number -1 ?

In this paper, we deal with Question 1.3 and Question 1.1 for genus-2 Lefschetz fibrations with $b_2^+ = 1$, and we will prove the following:

Theorem 1.2. Question 1.3 has a counterexample. There is a Lefschetz fibration $f: X \rightarrow S^2$ such that

- (1) f cannot be decomposed as any nontrivial fiber-sum;
- (2) f admits no section with self-intersection number -1 .

Theorem 1.3. Let $f: X \rightarrow S^2$ be a genus-2 Lefschetz fibration with $b_2^+(X) = 1$. If $f: X \rightarrow S^2$ is decomposed as the nontrivial fiber-sum $X = X_1 \#_F X_2$, then we have the following:

- (1) X is not a rational surface nor a ruled surface.
- (2) Each X_i ($i = 1, 2$) is diffeomorphic to $S^2 \times T^2 \# \overline{3\mathbb{C}P^2}$ or $S^2 \times T^2 \# \overline{4\mathbb{C}P^2}$.

The organization of this paper is as follows: In §3, we introduce a genus-2 Lefschetz fibration over S^2 constructed by Auroux [1] and we prove that it provides a counterexample to Question 1.3. In §4, we consider decomposable genus-2 Lefschetz fibrations with $b_2^+ = 1$ and prove Theorem 1.3.

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2. Preliminaries

Let $f_i: X_i \rightarrow S^2$ ($i = 1, 2$) be a genus- g Lefschetz fibration. Removing regular neighborhoods $N(F_1), N(F_2)$ of generic fibers F_1, F_2 in each, we glue these open remainders along their boundaries by using a fiber-preserving diffeomorphism $\varphi: \partial(X_1 - \text{Int}N(F_1)) \rightarrow \partial(X_2 - \text{Int}N(F_2))$ with $f_2 \circ \varphi = f_1$ on $\partial(X_1 - \text{Int}N(F_1))$. We denote the resultant 4-manifold by $X_1 \#_F X_2$, that is, $X_1 \#_F X_2 = (X_1 - \text{Int}N(F_1)) \cup_\varphi (X_2 - \text{Int}N(F_2))$. Then $X_1 \#_F X_2$ admits a genus- g Lefschetz fibration $X_1 \#_F X_2 \rightarrow S^2$ associated to f_1 and f_2 . We call the genus- g Lefschetz fibration $X_1 \#_F X_2 \rightarrow S^2$ the *fiber-sum* of f_1 and f_2 . The diffeomorphism type of $X_1 \#_F X_2$ might depend on the choice of the gluing diffeomorphism φ . Indeed, in [14] Ozbagci and Stipsicz construct infinitely many Lefschetz fibrations as the fiber-sums from the same building blocks by using various gluing diffeomorphisms. For the sake of brevity, we do not record those dependencies.

We begin with the following formulas for classical invariants of the fiber-sums.

Lemma 2.1. Let $f_i: X_i \rightarrow S^2$ ($i = 1, 2$) be a genus- g Lefschetz fibration. Then, for a fiber-sum $X_1 \#_F X_2 \rightarrow S^2$ of f_1 and f_2 , we have the following, where we denote the Euler characteristic of X by $e(X)$:

- (1) $e(X_1 \#_F X_2) = e(X_1) + e(X_2) + 4(g - 1)$,
- (2) $b_2^+(X_1 \#_F X_2) - b_1(X_1 \#_F X_2) = 2g - 1 - (b_1(X_1) + b_1(X_2)) + (b_2^+(X_1) + b_2^+(X_2))$,
- (3) $c_1^2(X_1 \#_F X_2) = c_1^2(X_1) + c_1^2(X_2) + 8(g - 1)$.

Proof. (1) Let $N(F_i)$ be the tubular neighborhood of a generic fiber F_i of f_i ($i = 1, 2$). Then, $X_1 \#_F X_2 = (X_1 - \text{Int}N(F_1)) \cup_\varphi (X_2 - \text{Int}N(F_2))$, where $\varphi: \partial N(F_1) \rightarrow \partial N(F_2)$ is the gluing diffeomorphism. Hence, we can get the formula by calculating the Euler characteristic straightly.

(2) Since $e(X_i) = 2 - 2b_1(X_i) + b_2^+(X_i) + b_2^-(X_i)$ and $\sigma(X_i) = b_2^+(X_i) - b_2^-(X_i)$, we get $e(X_i) + \sigma(X_i) = 2 - 2b_1(X_i) + 2b_2^+(X_i)$. Hence, by the Novikov additivity of signatures, we have that

$$\begin{aligned} & 2 - 2b_1(X_1 \#_F X_2) + 2b_2^+(X_1 \#_F X_2) \\ &= e(X_1 \#_F X_2) + \sigma(X_1 \#_F X_2) \\ &= e(X_1) + e(X_2) + 4(g - 1) + \sigma(X_1) + \sigma(X_2) \\ &= (e(X_1) + \sigma(X_1)) + (e(X_2) + \sigma(X_2)) + 4(g - 1) \\ &= 4g - 2(b_1(X_1) + b_1(X_2)) + 2(b_2^+(X_1) + b_2^+(X_2)). \end{aligned}$$

(3) By the Hirzebruch’s signature theorem, we have that

$$\begin{aligned} c_1^2(X_1 \#_F X_2) &= 3\sigma(X_1 \#_F X_2) + 2e(X_1 \#_F X_2) \\ &= 3\sigma(X_1) + 3\sigma(X_2) + 2e(X_1) + 2e(X_2) + 8(g - 1) \\ &= c_1^2(X_1) + c_1^2(X_2) + 8(g - 1). \end{aligned} \quad \square$$

REMARK 2.1. Let $f: X \rightarrow S^2$ be a genus- g (≥ 2) Lefschetz fibration.

(1) Since X admits a symplectic structure, we have $b_2^+(X) \geq 1$. Every nontrivial genus- g Lefschetz fibration $X \rightarrow S^2$ has irreducible singular fibers and so has nonseparating vanishing cycles [18]. Hence, we have $b_1(X) < 2g$. If we choose the identity map as the gluing map φ for the self fiber-sum $X \#_F X$, then we have $b_1(X \#_F X) = b_1(X)$. Thus, because of $b_2^+(X) \geq 1$ and $b_1(X) < 2g$, it follows from Lemma 2.1 that we have $b_2^+(X \#_F X) > 1$ for the self fiber-sum $X \#_F X$ with the identity map as the gluing map φ .

(2) Since the self fiber-sum $X \#_F X$ is a minimal symplectic 4-manifold with $b_2^+(X \#_F X) > 1$ by Theorem 1.5 in [18] or the Stipsicz conjecture [23], it follows from [22] that we have $c_1^2(X \#_F X) \geq 0$. Hence, by Lemma 2.1 we have $2c_1^2(X) + 8(g - 1) = c_1^2(X \#_F X) \geq 0$, i.e. $c_1^2(X) \geq 4(1 - g)$.

3. An indecomposable Lefschetz fibration which cannot admit sections with self-intersection number -1

We consider fiber-sums of genus-2 Lefschetz fibrations in this section and §4. Let $f: X \rightarrow S^2$ be a non-minimal genus-2 Lefschetz fibration. The total space X admits a symplectic structure such that fibers are symplectic submanifolds. Then, by the (-1) -curve theorem [22, 9, 15] we may assume that smooth 2-spheres in X with self-intersection number -1 are pseudo-holomorphic (-1) -curves in the symplectic manifold X . By using the (-1) -curve theorem and the theory of pseudo-holomorphic curves effectively, the author proved the following theorem. See also [7] and [8].

Table 1. Possible pairs (n, s) as geography

b_2^+	Possible pairs (n, s)	\mathcal{E}_X
$b_2^+ > 1$	$(16, 2), (30, 0)$	Type (1, 1)
	$(28, 1), (40, 0)$	Type (1)
	$(14, 3), (28, 1)$	Type (2)
$b_2^+ = 1$	$n + 2s = 20, n > 0, s \geq 0$	
	$n + 2s = 10, n > 0, s \geq 0$	

Theorem 3.1 ([15]). *Let $f: X \rightarrow S^2$ be a non-minimal genus-2 Lefschetz fibration. Suppose that X is not rational nor ruled. Then, smooth 2-spheres in X with self-intersection number -1 are \mathbb{Z} -homologous to pseudo-holomorphic (-1) -curves by changing orientations of spheres if necessary. Moreover, let \mathcal{E}_X be the set of all classes represented by pseudo-holomorphic (-1) -curves in X . Then, \mathcal{E}_X consists of at most two classes and \mathcal{E}_X is one of the following three:*

- Type (1, 1): $\mathcal{E}_X = \{E_1, E_2\}, E_1 \cdot F = E_2 \cdot F = 1$.
- Type (1): $\mathcal{E}_X = \{E\}, E \cdot F = 1$.
- Type (2): $\mathcal{E}_X = \{E\}, E \cdot F = 2$.

In the first and the second cases, spheres representing \mathcal{E}_X are sections of $f: X \rightarrow S^2$. Note that $E_1 \cdot E_2 = 0$ for E_1 and E_2 in the case of Type (1, 1), which follows from the proof of Corollary 3 in [7]. If a sphere E with self-intersection number -1 is of Type (2), that is, E intersects any generic fiber in two points, then we call E a *double section*.

The Stipsicz conjecture, which was proved by Usher [23, 16], implies that non-minimal Lefschetz fibrations over S^2 are indecomposable into fiber-sum, namely, irreducible. In [15], the author proved the following theorem on the geography of non-minimal genus-2 Lefschetz fibrations over S^2 :

Theorem 3.2 ([15]). *Only finitely many pairs (n, s) can be realized as the pairs of the numbers of singular fibers in non-minimal genus-2 Lefschetz fibrations over S^2 . Here, n and s are the numbers of irreducible and reducible singular fibers, respectively. Furthermore, we have the table of possible pairs of the numbers of singular fibers in non-minimal genus-2 Lefschetz fibrations over S^2 . See Table 1.*

In fact, there is a non-minimal genus-2 Lefschetz fibration over S^2 of Type (2) realizing $(n, s) = (28, 1)$ in Table 1, which is constructed by Auroux [1]. We introduce this fibration and prove that it is a counterexample to Question 1.3: Consider a curve C of degree 7 in $\mathbb{C}P^2$ with two triple points p_1 and p_2 . Then, we may assume that the three branches of C through p_i intersect each other transversely. Let L_0 be the line through p_1 and p_2 . Since $[C] \cdot [L_0] = 7$, the line L_0 intersects C transversely in another point p . Next blow up $\mathbb{C}P^2$ at p and let B be the proper transform of C in

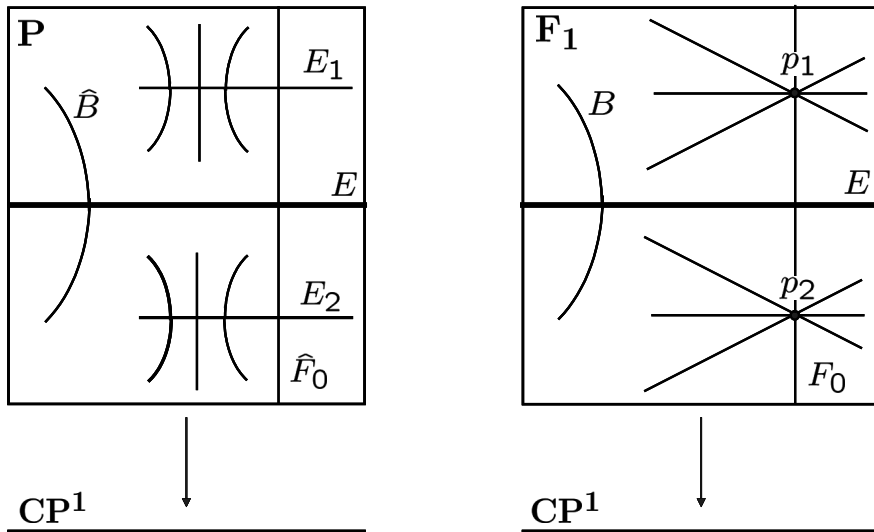


Fig. 1.

the Hirzebruch surface $\mathbb{F}_1 = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. Let L be a line in $\mathbb{C}P^2$, F a fiber of the $\mathbb{C}P^1$ -bundle $\mathbb{F}_1 \rightarrow \mathbb{C}P^1$ and E the exceptional curve of the blow-up. We have that $[E] = [L] - [F]$ and $[B] = [C] - [E] = 6[L] + [F]$. The proper transform F_0 of L_0 is the fiber of \mathbb{F}_1 through two triple points p_1 and p_2 . The exceptional curve E intersects the curves B and F_0 in one point each transversely.

Next blow up \mathbb{F}_1 at p_1 and p_2 and let P be the resultant complex surface. We denote the proper transforms of B and F_0 in P by \hat{B} and \hat{F}_0 , respectively. See Fig. 1.

If we let E_1 and E_2 be the exceptional curves of the two blow-ups, then we have the relations $[\hat{B}] = [B] - 3[E_1] - 3[E_2] = 6[L] + [F] - 3[E_1] - 3[E_2]$ and $[\hat{F}_0] = [F] - [E_1] - [E_2]$. Since $[\hat{B}] + [\hat{F}_0] = 2(3[L] + [F] - 2[E_1] - 2[E_2])$ is divisible by 2, we can consider the double cover $\pi: \hat{Y} \rightarrow P$ branched along $\hat{B} \cup \hat{F}_0$. Because of $[F] \cdot [E_1] = [F] \cdot [E_2] = 0$, we have $[\hat{F}_0]^2 = -2$ and so $[\pi^{-1}(\hat{F}_0)]^2 = -2/2 = -1$. Hence, $\pi^{-1}(\hat{F}_0)$ is a rational curve with self-intersection number -1 . Let $\hat{f}: \hat{Y} \rightarrow \mathbb{C}P^1$ be the fibration obtained by composing the double cover $\pi: \hat{Y} \rightarrow P$ with the projection $P \rightarrow \mathbb{C}P^1$ induced from the bundle projection $\mathbb{F}_1 \rightarrow \mathbb{C}P^1$. Because of $([\hat{B}] + [\hat{F}_0]) \cdot [F] = 6$, a fiber of \hat{f} is a closed surface of genus 2 obtained as the double cover of $\mathbb{C}P^1$ branched at 6 points. Namely, \hat{f} is a genus-2 fibration. Then, the fiber of \hat{f} corresponding to F_0 is $\pi^{-1}(\hat{F}_0 \cup E_1 \cup E_2) = \pi^{-1}(\hat{F}_0) \cup \pi^{-1}(E_1) \cup \pi^{-1}(E_2)$. Here, the preimages $\pi^{-1}(E_1)$ and $\pi^{-1}(E_2)$ are elliptic curves with self-intersection number -2 , for these are obtained as the double covers of spheres E_1 and E_2 branched at 4 points each.

By blowing down $\pi^{-1}(\hat{F}_0)$ in \hat{Y} , we obtain a holomorphic genus-2 fibration $f: Y \rightarrow \mathbb{C}P^1$ induced from the genus-2 fibration $\hat{f}: \hat{Y} \rightarrow \mathbb{C}P^1$. This fibration f has one re-

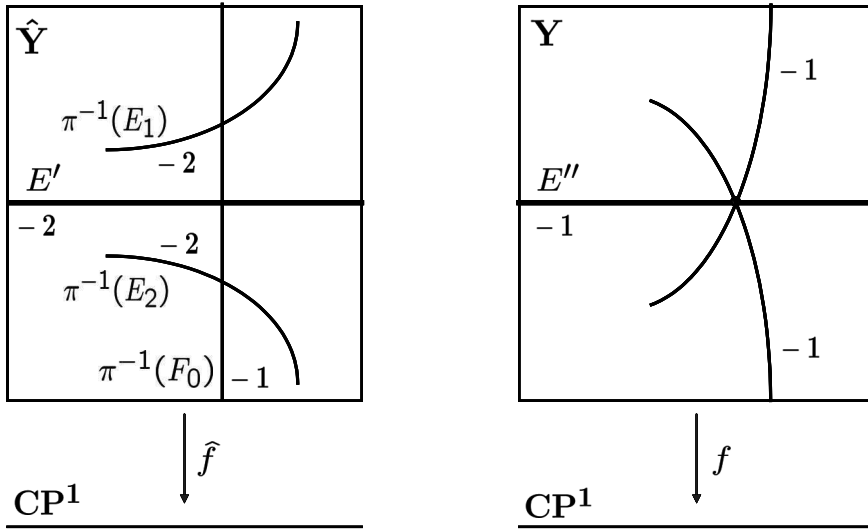


Fig. 2.

ducible singular fiber consisting of two elliptic curves with self-intersection number -1 . See Fig. 2. Then, Auroux proved in [1] that the complex surface Y admits a holomorphic genus-2 Lefschetz fibration $\bar{f}: Y \rightarrow \mathbb{C}P^1$ with $(n, s) = (28, 1)$.

Let E'' be the resultant curve in Y obtained from the exceptional curve E in \mathbb{F}_1 via these blow-ups/down. See Fig. 2. By chasing the exceptional curve E through the blow-ups/down, we can show that E'' is a rational curve in Y with self-intersection number -1 . For more details, see [15].

The exceptional curve E'' passes through the singular point of the reducible singular fiber which is the intersection between two elliptic curves induced from $\pi^{-1}(E_1)$ and $\pi^{-1}(E_2)$, and E'' comes from a section of the Hirzebruch surface \mathbb{F}_1 . Hence, the intersection number $[E''] \cdot [F]$ of E'' with a generic fiber F in Y is 2.

Therefore, the holomorphic genus-2 Lefschetz fibration $\bar{f}: Y \rightarrow \mathbb{C}P^1$ admits a double section E'' with self-intersection number -1 , that is, \bar{f} represents $(n, s) = (28, 1)$ of Type (2) in Table 1.

We need the following proposition in order to prove that the Auroux's genus-2 Lefschetz fibration \bar{f} is indecomposable into fiber-sum.

Proposition 3.1. *If a genus- g (≥ 2) Lefschetz fibration $f: X \rightarrow S^2$ admits a double section with self-intersection number -1 , then it cannot be decomposed as any nontrivial fiber-sum.*

Proof. Let E be a double section of f with self-intersection number -1 . Suppose that $f: X \rightarrow S^2$ is decomposed as a nontrivial fiber-sum $X = X_1 \#_F X_2$. Note that

$X_1 \#_F X_2 = (X_1 - \text{Int } N(F_1)) \cup_\varphi (X_2 - \text{Int } N(F_2))$, where $\varphi: \partial N(F_1) \rightarrow \partial N(F_2)$ is the gluing diffeomorphism. Here F_i is a generic fiber of the Lefschetz fibration $f_i: X_i \rightarrow S^2$ and $N(F_i)$ is the tubular neighborhood of F_i ($i = 1, 2$). We may assume that the intersection $E \cap \partial(X_i - \text{Int } N(F_i))$ consists of two circles by taking a sufficiently small disk $D_i = f_i(N(F_i)) \subset S^2$ if necessary.

Since a sphere is separated into an annulus and two disks by two circles, one of $E \cap (X_1 - \text{Int } N(F_1))$ and $E \cap (X_2 - \text{Int } N(F_2))$ is an annulus and the other consists of two disks. We assume that of $E \cap (X_1 - \text{Int } N(F_1))$ consists of two disks. We consider the fiber-sum $X_i \#_F X_i$ of two copies of X_i ($i = 1, 2$). Let α_i be the homology class in $X_i \#_F X_i$ coming from the double section E . Since $E \cap (X_1 - \text{Int } N(F_1))$ consists of two disks, α_1 is represented by two spheres C_1, C_2 in $X_1 \#_F X_1$, which are sections of $X_1 \#_F X_1$. On the other hand, since $E \cap (X_2 - \text{Int } N(F_2))$ is an annulus, α_2 is represented by a torus T in $X_2 \#_F X_2$. Then, we can see that $\alpha_2^2 \leq 0$. Remark 2.1 (1) shows that $b_2^+(X_i \#_F X_i) > 1$ ($i = 1, 2$). By applying the adjunction inequality [5, 12] to $X_2 \#_F X_2$ with $b_2^+(X_2 \#_F X_2) > 1$, we have that $0 = 2g(T) - 2 \geq |(K, \alpha_2)| + \alpha_2^2 \geq \alpha_2^2$. Here, K denotes the canonical class of the symplectic manifold $X_2 \#_F X_2$. Hence, $\alpha_2^2 \leq 0$. Moreover, since $\alpha_1^2 + \alpha_2^2 = 2[E]^2 = -2$ and both α_1^2 and α_2^2 are even, we have that $\alpha_1^2 \geq 0$ or $\alpha_1^2 = -2$.

The case of $\alpha_1^2 \geq 0$: Since $C_1 \cap C_2 = \emptyset$ and $\alpha_1^2 = [C_1]^2 + [C_2]^2$, we have that $[C_1]^2$ or $[C_2]^2$ is non-negative. Hence, it follows from the vanishing theorem of Seiberg–Witten invariants [4] that the Seiberg–Witten invariant $SW_{X_1 \#_F X_1}$ of $X_1 \#_F X_1$ is trivial. However, $X_1 \#_F X_1$ is a symplectic manifold and so it follows from Taubes’ theorem [21] that the Seiberg–Witten invariant $SW_{X_1 \#_F X_1}$ is nontrivial. This is a contradiction.

The case of $\alpha_1^2 = -2$: By the above argument, it does not come about that either of the self-intersection numbers of two spheres C_1 and C_2 in $X_1 \#_F X_1$ is non-negative. Thus, we get that $[C_1]^2 = [C_2]^2 = -1$. Then, the nontrivial fiber-sum $X_1 \#_F X_1 \rightarrow S^2$ has a section with self-intersection number -1 , and so this is in contradiction to Theorem 1.1.

Hence, the fact that a nontrivial fiber-sum $X_1 \#_F X_2$ has a double section with self-intersection number -1 contradicts itself. This completes the proof. □

Therefore, we have the following theorem:

Theorem 3.3. *There exists a holomorphic genus-2 Lefschetz fibration $f: X \rightarrow \mathbb{C}P^1$ such that*

- (1) *f cannot be decomposed as any nontrivial fiber-sum;*
- (2) *f cannot admit any section with self-intersection number -1 ;*
- (3) *X is not minimal and f admits only one double section with self-intersection number -1 , which intersects any generic fiber in two points.*

We can consider the Auroux’s genus-2 Lefschetz fibration as an example of fibrations in Theorem 3.3.

4. fiber-sums with $b_2^+ = 1$

Suppose that a genus-2 Lefschetz fibration $f: X \rightarrow S^2$ has n irreducible singular fibers and s reducible singular fibers. Since the abelianization Γ_2^{ab} of the mapping class group Γ_2 is isomorphic to $\mathbb{Z}/10\mathbb{Z}$, we have $n + 2s \equiv 0 \pmod{10}$ [11]. Since every singular fiber contributes 1 to the Euler characteristic $e(X)$, we have $e(X) = n + s - 4$. Moreover, for the signature $\sigma(X)$, we have $\sigma(X) = -3n/5 - s/5$ by the Matsumoto's local signature formula [11].

Ozbagci proved in [13] that the minimal number of singular fibers in a genus-2 Lefschetz fibration over S^2 is 7 or 8. Then, we can characterize a genus-2 Lefschetz fibration with seven or eight singular fibers.

Proposition 4.1. *Let $f: X \rightarrow S^2$ be a genus-2 Lefschetz fibration.*

- (1) *If $f: X \rightarrow S^2$ has seven singular fibers, then X is diffeomorphic to $S^2 \times T^2 \# \overline{3CP^2}$.*
- (2) *If $f: X \rightarrow S^2$ has eight singular fibers, then X is diffeomorphic to $S^2 \times T^2 \# \overline{4CP^2}$.*

Proof. Let $f: X \rightarrow S^2$ be a genus-2 Lefschetz fibration with n irreducible singular fibers and s reducible singular fibers.

- (1) Suppose that $n + s = 7$. Because of $n + 2s \equiv 0 \pmod{10}$, we have $(n, s) = (4, 3)$. Since each X satisfies that $n + 2s = 10$, we obtain that

$$\begin{aligned} 2 - 2b_1(X) + 2b_2^+(X) &= e(X) + \sigma(X) = (n + s - 4) + \left(-\frac{3}{5}n - \frac{1}{5}s\right) \\ &= \frac{2(n + 2s)}{5} - 4 = 0 \end{aligned}$$

and so

$$b_1(X) = b_2^+(X) + 1.$$

Let H be the subspace of $H_1(\Sigma_2; \mathbb{R})$ generated by vanishing cycles of X . Here, Σ_2 denotes the reference fiber of genus 2. Since a Lefschetz fibration over S^2 must have a nonseparating vanishing cycle [18], we have $\dim H \geq 1$. Since $H_1(X; \mathbb{R}) = H_1(\Sigma_2; \mathbb{R})/H$, we obtain that $b_1(X) = 4 - \dim H \leq 3$. Thus, we have that $1 \leq b_2^+(X) = b_1(X) - 1 \leq 2$, therefore, $(b_2^+, b_2^-, b_1) = (1, 4, 2)$ or $(2, 5, 3)$.

Suppose that $(b_2^+, b_2^-, b_1) = (2, 5, 3)$. Since $\sigma(X) = -3$ and $e(X) = 3$, we have $K_X^2 = 3\sigma(X) + 2e(X) = -3 < 0$. Hence, it follows from Theorem 0.2 in [22] that X is not minimal, that is, $f: X \rightarrow S^2$ is a non-minimal genus-2 Lefschetz fibration with $(n, s) = (4, 3)$. However, by the table, Table 1, of the geography of non-minimal genus-2 Lefschetz fibrations over S^2 , there is not any non-minimal genus-2 Lefschetz fibration over S^2 with $(n, s) = (4, 3)$. Therefore, a genus-2 Lefschetz fibration $f: X \rightarrow S^2$ with $n + s = 7$ satisfies that $(b_2^+, b_2^-, b_1) = (1, 4, 2)$.

Next we shall prove that X is a ruled surface. Suppose that X is not a ruled surface. Let \tilde{X} be the minimal model of X . Since $b_2^+(\tilde{X}) = 1$ and $b_1(\tilde{X}) = 2$, we have

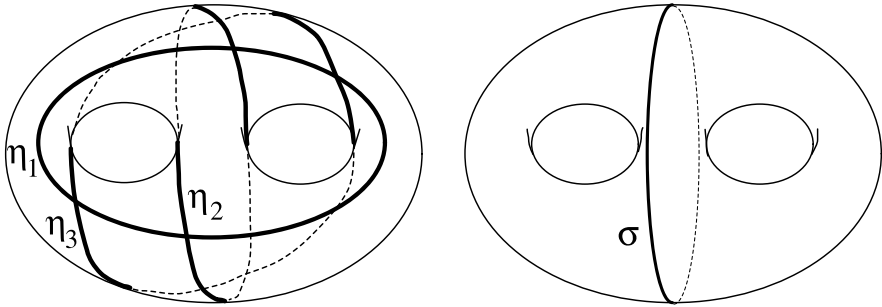


Fig. 3.

that $c_1^2(\tilde{X}) = 3\sigma(\tilde{X}) + 2e(\tilde{X}) = 5b_2^+(\tilde{X}) - b_2^-(\tilde{X}) - 4b_1(\tilde{X}) + 4 = 1 - b_2^-(\tilde{X})$. Moreover, since \tilde{X} is a minimal symplectic 4-manifold with $b_2^+(\tilde{X}) = 1$ and \tilde{X} is not rational nor ruled, it follows from [10] that \tilde{X} satisfies $c_1^2(\tilde{X}) \geq 0$. Hence, we have $b_2^-(\tilde{X}) \leq 1$. Since X is not rational nor ruled and X admits a genus-2 Lefschetz fibration over S^2 , it follows from Theorem 3.1 that X contains at most two 2-spheres with self-intersection number -1 essentially. Therefore, we have that $b_2^-(X) \leq 3$. This is in contradiction with $b_2^-(X) = 4$. Thus, X is a ruled surface. It follows from Corollary 4.1 in [20] that X is the blow-up of a ruled surface over the torus T^2 , and so X is diffeomorphic to $S^2 \times T^2 \# \overline{3\mathbb{C}P^2}$.

(2) Suppose that $n + s = 8$. Because of $n + 2s \equiv 0 \pmod{10}$, we have $(n, s) = (6, 2)$. In the same manner as above, we have $(b_2^+, b_2^-, b_1) = (1, 5, 2)$ or $(2, 6, 3)$. The case of $(b_2^+, b_2^-, b_1) = (2, 6, 3)$ is a contradiction by a Taubes' theorem [22] and the geography of non-minimal genus-2 Lefschetz fibrations over S^2 [15]. Thus, a genus-2 Lefschetz fibration $f: X \rightarrow S^2$ with $n + s = 8$ satisfies that $(b_2^+, b_2^-, b_1) = (1, 5, 2)$. In the same manner as above, it follows that X is the blow-up of a ruled surface over the torus T^2 , and so X is diffeomorphic to $S^2 \times T^2 \# \overline{4\mathbb{C}P^2}$. \square

REMARK 4.1. Matsumoto showed in [11] that $S^2 \times T^2 \# \overline{4\mathbb{C}P^2}$ admits a genus-2 Lefschetz fibration over S^2 with six irreducible singular fibers and two reducible singular fibers and its global monodromy is $(\eta_1 \cdot \sigma \cdot \eta_2 \cdot \eta_3)^2$, where η_1, η_2, η_3 and σ are positive Dehn twists along curves indicated on Fig. 3.

However, the author does not know whether $S^2 \times T^2 \# \overline{3\mathbb{C}P^2}$ admits a Lefschetz fibration over S^2 with seven singular fibers or not.

Lemma 4.1. *Let $f: X \rightarrow S^2$ be a genus-2 Lefschetz fibration with n irreducible singular fibers and s reducible singular fibers. If $b_2^+(X) = 1$, then we have either (i) $n + 2s = 10, b_1(X) = 2$ or (ii) $n + 2s = 20, b_1(X) = 0$.*

Proof. Since X is a symplectic 4-manifold with $b_2^+(X) = 1$, X is either the blow-up of a ruled surface or $b_1(X) \in \{0, 2\}$ [20]. Moreover, we have that

$$1 - b_2^-(X) = -\frac{3}{5}n - \frac{1}{5}s$$

and

$$3 - 2b_1(X) + b_2^-(X) = n + s - 4,$$

and so we obtain that $n + 2s = 20 - 5b_1(X)$. If X is the blow-up of a ruled surface over the surface Σ_h of genus h , then the genus-2 Lefschetz fibration f must satisfy that $0 \leq 2h \leq 2$ [20]. Moreover, because of $h = 0, 1$, we obtain that $b_1(X) = 0, 2$. Thus, we see that $b_1(X) \in \{0, 2\}$ anyway. If $b_1(X) = 0$, then the above relations imply that $n + 2s = 20$. When $b_1(X) = 2$, $n + 2s = 10$. \square

Since the mapping class group Γ_g is a infinite group, we might construct infinitely many distinct Lefschetz fibrations from given two Lefschetz fibrations via the fiber-sum operation. Hence, it is difficult to decide which Lefschetz fibrations are decomposable into fiber-sum. Thus, we restrict the problem to the case of $b_2^+ = 1$. Then, we have the following theorem:

Theorem 4.1. *Let $f: X \rightarrow S^2$ be a genus-2 Lefschetz fibration with $b_2^+(X) = 1$. If $f: X \rightarrow S^2$ is decomposed as the nontrivial fiber-sum $X = X_1 \#_F X_2$, then we have the following:*

- (1) X is not a rational surface nor a ruled surface.
- (2) Each X_i ($i = 1, 2$) is diffeomorphic to $S^2 \times T^2 \# \overline{3\mathbb{C}P^2}$ or $S^2 \times T^2 \# \overline{4\mathbb{C}P^2}$.

Before giving the proof of Theorem 4.1, we note that one can obtain a lower bound for the number s of reducible singular fibers in genus-2 Lefschetz fibrations. For example, we have that $s \geq 2$ for genus-2 Lefschetz fibrations over S^2 with $n + 2s = 10$, because $c_1^2(X) \geq -4$ by Remark 2.1 (2) and $c_1^2(X) = 3\sigma(X) + 2e(X) = s - 6$.

Proof of Theorem 4.1. Let n and s be the numbers of irreducible and reducible singular fibers of $f: X \rightarrow S^2$, respectively. By Lemma 4.1, we have $n + 2s = 10$ or $n + 2s = 20$. Suppose that each factor $X_i \rightarrow S^2$ has n_i irreducible and s_i reducible singular fibers ($i = 1, 2$). Since $n = n_1 + n_2$, $s = s_1 + s_2$, $n_i + 2s_i \equiv 0 \pmod{10}$ and $n_i > 0$ ($i = 1, 2$), the case of $n + 2s = 10$ does not occur. Hence, we obtain that $n + 2s = 20$, $n_i + 2s_i = 10$ ($i = 1, 2$) and so $b_1(X) = 0$. Since a genus-2 Lefschetz fibration has at least 7 singular fibers [13, 6], we have that $n + s \geq 7$ and $n_i + s_i \geq 7$ ($i = 1, 2$).

Since each X_i satisfies that $n_i + 2s_i = 10$, we have that $b_1(X_i) = b_2^+(X_i) + 1$ ($i = 1, 2$). On the other hand, since $b_2^+(X_i) \geq 1$ and $b_1(X_i) \leq 3$, we get that $2 \leq$

$1 + b_2^+(X_i) = b_1(X_i) \leq 3$. Hence, we obtain that for each i ,

$$(4.1) \quad \begin{cases} b_1(X_i) = 2, \\ b_2^+(X_i) = 1 \end{cases} \quad \text{or} \quad \begin{cases} b_1(X_i) = 3, \\ b_2^+(X_i) = 2. \end{cases}$$

Let H and H_i be subspaces of $H_1(\Sigma_2; \mathbb{R})$ generated by vanishing cycles of X and X_i , respectively. Then, we have that $H = \varphi_*H_1 + H_2$, where φ is the gluing map of the fiber-sum. Since $H_1(X; \mathbb{R}) = H_1(\Sigma_2; \mathbb{R})/H$ and $H_1(X_i; \mathbb{R}) = H_1(\Sigma_2; \mathbb{R})/H_i$ ($i = 1, 2$), we obtain that $b_1(X) = 4 - \dim H$ and $b_1(X_i) = 4 - \dim H_i$ ($i = 1, 2$). Thus, we have that $4 = \dim H = \dim(\varphi_*H_1 + H_2) = \dim H_1 + \dim H_2 - \dim(\varphi_*H_1 \cap H_2)$ and so $\dim H_1 + \dim H_2 = 4 + \dim(\varphi_*H_1 \cap H_2) \geq 4$. Hence, we have that

$$(4.2) \quad b_1(X_1) + b_1(X_2) = 8 - (\dim H_1 + \dim H_2) \leq 4.$$

Therefore, it follows from (4.1) and (4.2) that $b_1(X_1) = b_1(X_2) = 2$ and $b_2^+(X_1) = b_2^+(X_2) = 1$. Because of $n_i + 2s_i = 10$, we have that $s_i \geq 2$. Hence, the pair (n_i, s_i) satisfying that $n_i + 2s_i = 10$, $n_i + s_i \geq 7$ and $s_i \geq 2$ is $(n_i, s_i) = (6, 2)$ or $(n_i, s_i) = (4, 3)$. Therefore, it follows from Proposition 4.1 that each X_i is diffeomorphic to $S^2 \times T^2 \sharp 3\overline{\mathbb{C}P^2}$ or $S^2 \times T^2 \sharp 4\overline{\mathbb{C}P^2}$.

Next we shall prove that X is not rational nor ruled. By Remark 2.1 (2), we have $c_1^2(X_i) \geq -4$. Hence, it follows from Lemma 2.1 that we have $c_1^2(X) = c_1^2(X_1) + c_1^2(X_2) + 8 \geq 0$. Suppose that X is a rational surface. Because of $(n_i, s_i) = (6, 2)$ or $(n_i, s_i) = (4, 3)$, we have $s = s_1 + s_2 \in \{4, 5, 6\}$ and so $c_1^2(X) = s - 4 \in \{0, 1, 2\}$. Hence, X is not diffeomorphic to $S^2 \times S^2$. Thus we set $X = \mathbb{C}P^2 \sharp k\overline{\mathbb{C}P^2}$. Since $c_1^2(X) = 9 - k$, we get that $k \leq 9$. By [9], note that X admits a unique symplectic structure ω essentially. Let K_X be the canonical class of X and let F be the class represented by a generic fiber. Since X is rational and $K_X^2 = c_1^2(X) \geq 0$, we have that $K_X \cdot \omega < 0$. On the other hand, a generic fiber is a ω -symplectic submanifold and so $F \cdot \omega > 0$. Since $(-K_X) \cdot \omega > 0$, $K_X^2 \geq 0$, $F \cdot \omega > 0$ and $F^2 = 0$, the classes $-K_X$ and F belong to the closure of the forward cone $\overline{C}^+ = \{\beta \in H^2(X; \mathbb{R}) \mid \beta^2 \geq 0, \beta \neq 0, \beta \cdot \omega \geq 0\}$. Hence, by the light cone lemma [9], we obtain that $(-K_X) \cdot F \geq 0$, that is, $K_X \cdot F \leq 0$. However, by applying a generic fiber to the adjunction formula, we obtain that

$$2 = 2\text{genus}(F) - 2 = K_X \cdot F + F^2 = K_X \cdot F$$

and this is a contradiction. Therefore, X is not a rational surface. Suppose that X is a ruled surface. Because of $b_1(X) = 0$, X is also a rational surface. Hence, X is not ruled neither. □

REMARK 4.2. When we construct a Lefschetz fibration over S^2 with $b_2^+ = 1$ by using fiber-sum construction, we can not choose any self fiber-sums $X \sharp_F X \rightarrow S^2$ with the identity map as the gluing map because of Remark 2.1 (1). On the other hand, we can choose the gluing map φ of a self fiber-sum $X \sharp_F X \rightarrow S^2$ such that $b_2^+(X \sharp_F X)$

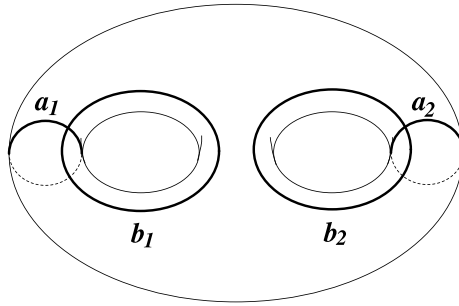


Fig. 4.

$X) = 1$. We will construct such an example from the Matsumoto’s genus-2 Lefschetz fibration $S^2 \times T^2 \natural 4\overline{CP^2} \rightarrow S^2$.

Let a_i and b_i ($i = 1, 2$) be the curves indicated on Fig. 4. Then, we can write $[\eta_1] = [b_1] + [b_2]$, $[\eta_2] = [a_1] + [a_2]$, $[\eta_3] = [a_1] + [a_2] + [b_1] + [b_2]$ and $[\sigma] = 0$ in homology, where each η_i denotes the curve indicated on Fig. 3.

For the positive Dehn twists τ_{a_2} and τ_{b_1} along the curves a_2 and b_1 , we take the diffeomorphism $h = \tau_{b_1} \circ \tau_{a_2}: \Sigma_2 \rightarrow \Sigma_2$. The effect of a positive Dehn twist on $H_1(\Sigma_2; \mathbb{Z})$ is known as the *Picard–Lefschetz formula* and we have that

$$\tau_{C*}(\alpha) = \alpha - (\alpha \cdot C)[C]$$

for the positive Dehn twist τ_C along C . In the case of τ_{a_2} and τ_{b_1} , we have

$$\begin{aligned} \tau_{a_2*}([a_1]) &= [a_1], & \tau_{a_2*}([a_2]) &= [a_2], & \tau_{a_2*}([b_1]) &= [b_1], & \tau_{a_2*}([b_2]) &= [a_2] + [b_2], \\ \tau_{b_1*}([a_1]) &= [a_1] - [b_1], & \tau_{b_1*}([a_2]) &= [a_2], & \tau_{b_1*}([b_1]) &= [b_1], & \tau_{b_1*}([b_2]) &= [b_2]. \end{aligned}$$

Hence, it follows that $h_*([\eta_1]) = [a_2] + [b_1] + [b_2]$, $h_*([\eta_2]) = [a_1] + [a_2] - [b_1]$, $h_*([\eta_3]) = [a_1] + 2[a_2] + [b_2]$ and $h_*([\sigma]) = 0$. Let $X = (S^2 \times T^2 \natural 4\overline{CP^2}) \natural_F (S^2 \times T^2 \natural 4\overline{CP^2})$ be the fiber-sum of two copies of the Matsumoto’s genus-2 Lefschetz fibration $S^2 \times T^2 \natural 4\overline{CP^2} \rightarrow S^2$ with the gluing map $\varphi: \Sigma_2 \times S^1 \rightarrow \Sigma_2 \times S^1$ associated to h . If we let H be the subgroup of $H_1(\Sigma_2; \mathbb{Z})$ generated by $[\eta_1]$, $[\eta_2]$ and $[\eta_3]$, then the first homology group $H_1(X; \mathbb{Z})$ is given by $H_1(X; \mathbb{Z}) \cong H_1(\Sigma_2; \mathbb{Z}) / (H + h_*H)$, and so we have

$$\begin{aligned} H_1(X; \mathbb{Z}) &= [[a_1], [a_2], [b_1], [b_2] \mid [\eta_1] = [\eta_2] = [\eta_3] = 0 \text{ and} \\ & \quad h_*([\eta_1]) = h_*([\eta_2]) = h_*([\eta_3]) = 0] \\ &= 0. \end{aligned}$$

Therefore, we have $b_2^+(X) = 1$ by Lemma 2.1 (2).

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