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AN APPROXIMATION THEOREM FOR MARKOV PROCESSES

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1. Introduction

In [1], Watanabe proved that for every Markov process X, under some conditions, there exists a sequence of regular step processes (R.S.P.) X^n such that the resolvents of X^n converge weakly to the resolvent of X. Under some supplementary conditions we shall prove that the distributions of X^n converge to the distribution of X. An intuitive description of X^n is as follows: X and X^n start frcm the same state x_0 (we mean that X_0 and X_0^n have the same distribution). If X remains closed to x_0 for a time T_n (that is, $d(x_0, X_t)<\frac{1}{n}$ for all $t<T_n$ and $X_{T_n}=x_1$, with $d(x_0, x_1)\geqq\frac{1}{n}$), then $X_t^n=x_0$ for all $t<D_n$, with D_n an exponentially distributed holding time with same mean value as T_n (T_n is generally not exponentially distributed). Then X^n jumps in x_1 (we mean that X_{T_n} and $X_{D_n}^n$ have the same distribution), and so on.

The rigorous construction of X^n and Watanabe's result are presented in the beginning of the paper. The theorem following this construction is the main result of the paper.

2. Main results

Let E be a locally compact with countable base space (L.C.C.B.), $\mathcal{U}$ an open base and d any metric of E. For each n we can choose the system U_i^n, $i\in N$ and V_i^n, $i\in N$ of sets in $\mathcal{U}$ satisfying the following conditions:

(1) Each $\bar{U}_i^n$ is compact and $d(U_i^n)<\frac{1}{n}$ $(d(A)=\sup(d(x,y);\, x, y\in A))$;

(2) $V_i^n\subseteq U_i^n$;

(3) $\bigcup_i V_i^n=E$;

(4) For every compact set K only a finite number of V_i^n intersect with K.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a standard process with state space E and $(U_\alpha)_{\alpha>0}$ be the resolvent of X. We now define σ_k^n by

$$\sigma_1^n(\omega) = \inf(t;\ X_t(\omega) \notin U_i^n) \qquad \text{if} \quad X_0(\omega) \in V_i^n - \bigcup_{j<i} V_j^n$$
$$\sigma_k^n(\omega) = \sigma_{k-1}^n + \sigma_1^n \circ \theta_{\sigma_{k-1}^n} \qquad \text{for} \quad k > 1 .$$

The following result is Lemma 3.3 in Watanabe's work. Let X be a standard process such that $U_\alpha(C_b(E)) \subseteq C_b(E)$ for $\alpha > 0$ and $\sup_{x \in E} U_0(x, E) < \infty$. Then the following assertions hold:

(i) For each n, $q_n(x) = [E^x(\sigma_1^n)]^{-1}$ and $\Pi_n(x, A) = P^x(X_{\sigma_1^n} \in A)$ represents the parameters of a R.S.P.

The corresponding R.S.P. are denoted by X^n and the resolvent of X^n by $U_\alpha^{(n)}$.

(ii) X^n is an approximation of X in the following sense:

$$\lim_n U_\alpha^{(n)} f(x) = U_\alpha f(x) \quad \text{for every} \quad x \in E \quad \text{and} \quad f \in C_b(E) . \tag{5}$$

We shall need instead of (5) a stronger result. For any compact set K, $\alpha > 0$ and $f \in C_b(E)$

$$\lim_n U_\alpha^{(n)} f(x) = U_\alpha f(x) \quad \text{uniformly for} \quad x \in K . \tag{6}$$

Considering the proof of Lemma 3.1 in [1] it is obvious that to obtain (6) it is sufficient to prove the following condition:

(c) For any compact set $L \subseteq E$, $\alpha > 0$ and $\varepsilon > 0$, we may choose a compact set K such that $L \subseteq K$ and

$$\overline{\lim_n} \sup_{x \in L} U_\alpha^{(n)}(x;\ CK) \leqslant \varepsilon . \tag{7}$$

Through this paper we shall consider on E a metric d such that $B_h(x) = (y;\ d(x, y) < h)$ has compact closure for any $h > 0$ and $x \in E$. For $h > 0$ we define T_h^k, $k \in N$ by

$$T_h = \inf(t;\ d(X_0, X_t) > h) ,$$
$$T_h^1 = T_h \quad \text{and} \quad T_h^{k+1} = T_h^k + T_h \circ \theta_{T_h^k} .$$

We shall also consider the function

$$q(h) = \sup_{x \in E} [E^x(T_h)]^{-1} .$$

We note that $q(h)$ and $q_n(x)$ are distinct notations. The function $h \to q(h)$ is monotone, so we may choose, for every n, h_n and d_n such that $\lim_n h_n = 0$, $d_n < h_n$ and

$$\lim_n \frac{q(h_n - d_n)}{q(h_n + d_n)} = 1 . \tag{8}$$

Now we shall choose the above mentioned U_i^n and V_i^n in the following particular form: $U_i^n = B_{h_n}(x_i)$ and $V_i^n = B_{d_n}(x_i)$ with x_i, $i \in N$, chosen such that condition (4) is fulfilled. σ_i^n will be defined like above with respect to this system of sets. The following two inequalities will be useful in what it follows

(9)
$$T_{h_n - d_n} \leqslant \sigma_1^n \leqslant T_{h_n + d_n} .$$

Now we assume that the following condition holds for X. There is some $a > 0$ such that for every $h > 0$

$$\sup_{x \in E} E^x(T_h) \leqslant a \inf_{x \in E} E^x(T_h) .$$

Then, by (9) we may conclude that

(10)
$$aq(h_n - d_n) \geqslant q_n(x) \geqslant q(h_n + d_n) .$$

Now we are able to formulate the result of the paper.

Theorem. *Let X be a standard process with state space E such that*

(i) $U_\alpha(C_b(E)) \subseteq C_b(E)$ *for every* $\alpha > 0$.

(ii) $\lim_{t \to 0} P_t f(x) = f(x)$ *uniformly on E, for every* $f \in C_c(E)$.

(iii) $\sup_{x \in E} U_0(x, E) < \infty$.

(iv) *There is some $a > 0$ such that for every $h > 0$*

$$\sup_{x \in E} E^x(T_h) \leqslant a \inf_{x \in E} E^x(T_h) .$$

(v) *There is some $c > 0$ such that for every $x \in E$ and $h > 0$*

$$E^x(D_h^x) \geqslant c \| E^{(\cdot)}(D_h^x) \|$$

with

$$\| E^{(\cdot)}(\varphi) \| = \sup_{y \in E} E^y(\varphi) \qquad \textit{and}$$

(11)
$$D_h^x(\omega) = \inf(t;\ X_t(\omega) \notin B_h(x)) .$$

Then, $\lim_n P_n^\mu = P$ for every μ (probability measure on E). (We denote by P_n^μ the distribution of the R.S.P. X^n which has initial measure μ).

We note that condition (iv) implies

(vi)
$$\lim_{h \to 0} \sup_{x \in E} E^x(T_h) = 0 .$$

That is because for any $x \in E$, $\lim_{h \to 0} E^x(T_h) = 0$.

The proof will go as follows: In the first part we establish the similarities between X and X^n. We refer to Appendix 1 which presents the law of large numbers in two forms which are appropriate to our deal. The first two Lemmas assure that we may use the results in Appendix 1. We use it in Lemma

3 which is essential for the whole proof. Roughly speaking this lemma establishes the similarity between the "time" of X^n and the time of X. Lemma 4 is a simple remark which assures that the "space" of X^n and the "space" of X coincide. These similarities are used in all the following, in order to evaluate quantities referring to X^n by their analogoues with respect to X.

In the second part of the proof we establish the tightness of the sequence P_n^μ, $n\in N$. The last part deals with the convergence of the marginal distributions. We use here Watanabe's result in his stronger form (6). To do it we prove first (c), and then we refer to Appendix 3 which enables us to check the convergence of the marginal distributions by the convergence of the resolvents.

3. Proofs

We first define, for all $h>0$,

$$\tilde{F}_h(t) = \inf_{x\in E} P^x(T_h<t) = 1-\sup_{x\in E} P^x(T_h>t)\,. \tag{12}$$

$\tilde{F}_h$ is infimum of a family of increasing functions which are right continuous and have left hand limits, then so is $\tilde{F}_h$. Next, it is obvious that $\tilde{F}_h(0)=0$, and so, in order to show that $\tilde{F}_h$ is a distribution function on R_+, it will suffice to see that

$$\lim_{t\to\infty} \tilde{F}_h(t) = 1\,. \tag{13}$$

By Chebyshev's inequality $P^x(T_h>t)\leqslant\frac{1}{t}E^x(T_h)$ and so

$$\tilde{F}_h(t)\geqslant 1-\frac{1}{t}\sup_{x\in E} E^x(T_h)\,.$$

Because $U_0(x, E)=E^x\left(\int_0^\infty \mathrm{I}_E\circ X_t\,dt\right)=E^x(\zeta)$ with $\zeta=\inf(t;\, X_t=\Delta)$, by (ii), $\sup_{x\in E} E^x(T_h)\leqslant \sup_{x\in E} E^x(\zeta)<\infty$ and so (13) is proved.

We denote by F_h the distribution on R_+ corresponding to $\tilde{F}_h$. It is obvious that for any t and x

$$F_h([0, t])\leqslant P^x(T_h\leqslant t) \quad \text{and} \quad F_h((t, \infty)) = \sup_{x\in E} P^x(T_h>t)\,. \tag{14}$$

In order to simplify notations we shall denote

$$Y_k^n = X_{\sigma_k^n} \quad \text{and} \quad Z_k^n = X_{\tau_k}^n\,. \tag{15}$$

with $\tau_1=\inf(t;\, X_t^n\neq X_0^n)$ and $\tau_k=\tau_{k-1}+\tau_1\circ\theta_{\tau_{k-1}}$. Next, let us put

$$t_n = 2 \sup_{x\in E} ||E^{(\cdot)}(D^x_{h_n})|| \qquad (\text{see } (11)) .$$

The following relations will be used to prove Lemma 1

$$(16) \qquad \begin{aligned} &(a) \quad \lim_n t_n = 0 , \\ &(b) \quad F_{h_n}((k\cdot t_n, \infty)) \leqslant \left(\frac{1}{2}\right)^k . \end{aligned}$$

Because $T_h = D^x_h$ P^xa.s., (a) is a consequence of (v) and (vi). To prove (b) we shall use exercise (10.25) in [2]: If $\alpha(t) = \sup_{x\in E} P^x(D_U > t)$, then

$$(17) \qquad P^x(D_U > k\cdot t) \leqslant \alpha(t)^k$$

(U is a measurable set and $D_U = \inf(t; X_t \in U)$). Next we consider $t = t_n$ and $U = CB_{h_n}(x)$ then $D_U = D^x_{h_n}$ and by Chebyshev's inequality we obtain

$$P^y(D^x_{h_n} > t_n) \leqslant \frac{1}{t_n} E(D^x_{h_n}).$$

We take the supremum over all $y \in E$ and considering the definition of t_n and α we conclude that

$$\alpha(t_n) \leqslant \frac{1}{2} .$$

By (14) and (17) we get (b)

$$F_{h_n}((k\cdot t_n, \infty)) = \sup_{x\in E} P^x(D^x_{h_n} > k\cdot t_n) \leqslant \left(\frac{1}{2}\right)^k .$$

Lemma 1. (a) *For every $a > 0$*

$$\lim_n \int_a^\infty zF_{h_n}(dz) \cdot \left[\int_0^\infty zF_{h_n}(dz)\right]^{-1} = 0 .$$

$$(b) \quad \int_0^\infty zF_{h_n}(dz) \leqslant \frac{4}{c} \sup_{x\in E} E^x(T_{h_n}) = \frac{4}{c}[q(h_n)]^{-1}$$

where c is defined in (v).

Proof. Let be $k_n \in N$ such that $k_n \leqslant \frac{a}{t_n} < k_n + 1$; we have

$$\int_a^\infty zF_{h_n}(dz) \leqslant (k_n+1)F_{h_n}([k_n\cdot t_n, \infty)) + t_n \sum_{k>k_n} F_{h_n}([k\cdot t_n, \infty)) .$$

By (16) we get

$$(18) \qquad \int_a^\infty zF_{h_n}(dz) \leqslant t_n(k_n+2)2^{-k_n} .$$

It is obvious that for every $x \in E$

$$\int_0^\infty z F_{h_n}(dz) \geqslant \int T_{h_n} P^x(d\omega)$$

and so, by (v)

$$\int_0^\infty z F_{h_n}(dz) \geqslant \sup_{x \in E} E^x(T_{h_n}) \geqslant c \sup_{x \in E} \|E^{(\cdot)}(D^x_{h_n})\| = \frac{c}{2} t_n .$$

The right continuity of the trajectories assures that $D^x_{h_n} > 0$ P^xa.s. (see (11)) and then $t_n > 0$ and we may write

$$\Big[\int_a^\infty z F_{h_n}(dz)\Big] \cdot \Big[\int_0^\infty z F_{h_n}(dz)\Big]^{-1} \leqslant t_n (k_n+2) 2^{-k_n} \cdot \Big[\frac{c}{2} t_n\Big]^{-1} = \frac{c}{2}(k_n+2) 2^{-k_n} .$$

The last term vanishes when $n \to \infty$ and (a) is proved. Now we have

$$\int_0^\infty z F_{h_n}(dz) \leqslant t_n F_{h_n}((0, \infty)) + t_n \sum_{k=1}^\infty F_{h_n}((k \cdot t_n, \infty)) \leqslant \frac{4}{c} \sup_{x \in E} E^x(T_{h_n})$$

and (b) is also proved.

Lemma 2. *For every* $k \in N$ *and* $l > 0$,

(a) $P^\mu(\sigma^n_k > l) \leqslant F^{*k}_{h_n+d_n}(l, \infty)$,

(b) $P^\mu_n(\tau_k \leqslant l) \leqslant e^{*k}_{aq(h_n-d_n)}(0, l)$,

(*a is defined in* (iv) *and e_α is the exponential distribution with parameter α*).

Proof. For F and G distributions on R_+

$$F * G(l, \infty) = \int_0^\infty F(l-t, \infty)\, G(dt)$$

and so, if F and F' are such that $F(s, \infty) \leqslant F'(s, \infty)$ for every $s \in R_+$, then

$$F * G(l, \infty) \leqslant F' * G(l, \infty) . \tag{19}$$

To prove (a) we proceed by induction on k. For $k=1$, (a) is (12). Using the strong Markov property for two variables functions we get

$$P^\mu(\sigma^n_k > l) = P(\sigma^n_{k-1} + \sigma^n_1 \circ \theta_{\sigma^n_{k-1}} > l) = \int P^{Y^n_k(\omega)}(\sigma^n_1 > l - \sigma^n_{k-1}(\omega)) P(d\omega)$$

(see (15)). By (9) and (12), for every fixed ω

$$P^{Y^n_{k-1}(\omega)}(\sigma^n_1 > l - \sigma^n_{k-1}(\omega)) \leqslant P^{Y^n_{k-1}(\omega)}(T_{h_n+d_n} > l - \sigma^n_{k-1}(\omega)) \leqslant F_{h_n+d_n}(l - \sigma^n_{k-1}(\omega), \infty)$$

and so

$$P(\sigma^n_k > l) \leqslant P^\mu \circ (\sigma^n_{k-1})^{-1} * F_{h_n+d_n}(l, \infty) .$$

Now, using the induction hypothesis and (19) we finish the proof.

In order to prove (b) we obtain in the same way as above

$$P_n^\mu(\tau_k \leqslant l) = \int P_n^{Z_{k-1}^n(\omega)}(\tau_1 \leqslant l - \tau_{k-1}(\omega)) P_n^\mu(d\omega)$$

see ((15)). With respect to $P_n^{Z_{k-1}^n(\omega)}$, τ_1 is exponentially distributed with parameter $q_n(Z_{k-1}^n(\omega)) \leqslant aq(h_n - d_n)$ (see (10)). Then

$$P^{Z_{k-1}^n(\omega)}(\tau_1 \leqslant l - \tau_{k-1}(\omega)) \leqslant e_{dq(h_n-d_n)}(0, l - \tau_{k-1}(\omega)) \qquad \text{for every } \omega .$$

We conclude that

$$P_n^\mu(\tau_k \leqslant l) \leqslant P_n^\mu \circ \tau_k^{-1} * e_{aq(h_n-d_n)}(0, l)$$

and the proof fiinishes like above.

Lemma 3. *For fixed $k>0$ and $\delta>0$ we define*

$k' = 16kac^{-1}$, $\delta' = 32\delta ac^{-1}$ (*c defined in* (v) *and a in* (iv)),

$k_n \in N$ *such that* $k_n \leqslant 2kq(h_n - d_n)a < k_n + 1$,

$l_n \in N$ *such that* $l_n \leqslant 2q(h_n - d_n)a < l_n + 1$,

$A_n = \{\omega;\ \tau_i - \tau_j > \delta$ *for every i, j such that* $l_n \leqslant i - j \leqslant k_n\}$,

$B_n = \{\omega;\ \sigma_i^n - \sigma_j^n < \delta'$ *for every i, j such that* $0 < i - j \leqslant l_n\}$,

(*We shall use these notations throughout all the rest of the paper*). *Then*

(a) $\lim_n P_n^\mu(\tau_{k_n} < k) = 0$,

(b) $\lim_n P^\mu(\sigma_{k_n}^n > k') = 0$,

(c) $\lim_n P_n^\mu(CA_n) = 0$,

(d) $\lim_n P^\mu(CB_n) = 0$.

We note that all these limits are uniform with respect to the family $\{\mu;\ \mu$ probability measure on $E\}$.

(e) $\overline{\lim_n} \sup_{x \in E} E^x(\sigma_{k_n}^n;\ \sigma_{k_n}^n > k') = 0$.

The idea of this lemma is that both $\sigma_{k_n}^n$ and τ_{k_n} are sums of little quantities with the same mean value. If we take k_n (the number of terms in the sum) such that $k_n a_n \sim l$ (a_n is the mean value), then $\sigma_{k_n}^n \sim l$ and $\tau_{k_n} \sim l$. This is the idea of the law of large numbers and to prove the lemma we refer to Appendix 1, which presents appropriate forms of this law.

Proof. (a) By Lemma 2.b

$$P_n^{\mu}(\tau_{k_n}<k)\leqslant e^{*k_n}_{dq(h_n-d_n)}(0, k)\,.$$

$$\lim_n aq(h_n-d_n)^{-1}k_n = 2k$$

and so, by Lemma 3, Appendix 1 we get

$$\lim_n e^{*k_n}_{aq(h_n-d_n)}(0, k)=0 \quad \text{(independent of } \mu)\,.$$

(b) By Lemma 2.a,

$$P_n^{\mu}(\sigma^n_{k_n}>k')\leqslant F^{*k_n}_{h_n+d_n}(k', \infty)\,.$$

By Lemma 1.b,

$$M(F_{h_n+d_n})\leqslant \frac{4}{c}(q(h_n+d_n))^{-1}$$

and so

$$M(F_{h_n+d_n})\leqslant \frac{4}{c}(2kq(h_n-d_n)a)(q(h_n+d_n))^{-1}$$

and by (8) we obtain

$$\overline{\lim_n}\, M(F(h_n+d_n))k_n\leqslant \frac{4}{c}\cdot 2ka = \frac{8ka}{c} = \frac{k'}{2}\,.$$

Lemma 1.a assures that conditions in Lemma 1 and Corollary 2, Appendix 1 are fulfilled and so $\overline{\lim_n}\, F_{h_n+d_n}(k', \infty)=0$ and (b) is proved.

To prove (c) we note that for i, j with $i-j\geqslant l_n$ $\tau_i-\tau_j\geqslant \tau_{j+l_n}-\tau_j$ and so $CA_n\subseteq \bigcup_{j\leqslant k_n}(\tau_{j+l_n}-\tau_j<\delta)$. Then $P_n^{\mu}(CA_n)\leqslant \sum_{j\leqslant k_n} P_n^{\mu}(\tau_{j+l_n}-\tau_j<\delta)$. By the Markov property

$$P_n^{\mu}(\tau_{j+l_n}-\tau_j<\delta) = E_n^{\mu}(P^{Z_j^n}(\tau_{l_n}<\delta))$$

which is dominated by $e^{*l_n}_{dq(h_n-d_n)}(0, \delta)$ (see Lemma 2.b) and so

$$P_n^{\mu}(CA_n)\leqslant k_n e^{*l_n}_{dq(h_n-d_n)}(0, \delta)$$

$$\lim_n \frac{k_n}{l_n} = \frac{k}{\delta}<\infty \quad \text{and} \quad \lim_n \frac{l_n}{aq(h_n-d_n)} = 2\delta\,,$$

and so, by Lemma 3, Appendix 1 the term in the right of the above inequality vanishes when $n\to\infty$.

To prove (d) we note that $\omega\in CB_n$ implies that there is some $i, j\leqslant k_n$ such that $0\leqslant i-j\leqslant l_n+1$ and $\sigma_i^n-\sigma_j^n>\delta'$.

Then, there is some $p\leqslant \dfrac{k_n}{l_n+1}$ such that

$$\sigma^n_{p(l_n+1)}-\sigma^n_{(p-1)(l_n+1)})>\frac{\delta'}{2}\,.$$

We conclude that

$$P^{\mu}(CB_n)\leqslant \sum P^{\mu}\Big((\sigma^n_{p(l_n+1)}-\sigma^n_{(p-1)(l_n+1)})>\frac{\delta'}{2}\Big)$$

with the sum over $p<\frac{k_n}{l_n+1}$. By the strong Markov property and Lemma 2.a we obtain

$$P^{\mu}\Big((\sigma^n_{p(l_n+1)}-\sigma^n_{(p-1)(l_n+1)})>\frac{\delta'}{2}\Big)\leqslant F^{*(l_n+1)}_{h_n+d_n}\Big(\frac{\delta'}{2},\infty\Big)$$

for every p, and so

$$P^{\mu}(CB_n)\leqslant \frac{k_n}{l_n+1}F^{*(l_n+1)}_{h_n+d_n}\Big(\frac{\delta'}{2},\infty\Big).$$

The proof ends like to the points (b) and (c).

To prove (e) we note that Lemma 2.a implies that

$$E^x(\sigma^n_{k_n};\ \sigma^n_{k_n}>k')=\int_{(z>k')} zP^x\circ\sigma^{n-1}_{k_n}(dz)\leqslant\int_{(z>k')} zF^{*k_n}_{h_n+d_n}(dz).$$

Because

$$\lim_n F^{*k_n}_{h_n+d_n}=\varepsilon_k \qquad \text{with} \quad k<k'$$

the term in the right of the above inequality vanishes under $\overline{\lim_n}\ \sup_x$ and (e) is proved.

Lemma 4. $(Y^n_k, k\in N)$ *has, with respect to* P^{μ} *the same distribution as* $(Z^n_k, k\in N)$ *with respect to* P^{μ}_n *(see* (15)).

Proof. That is because both of them are Markov chains with initial distribution μ and kernel

$$\Pi_n(x, dy)=P^x\circ X^{-1}_{\sigma^n_1}(dy).$$

Now, in order to prove the relative compactness of the sequence P^{μ}_n, $n\in N$, we shall use Theorem 2, page 429 in [3] which we write down for processes with time $[0,\infty)$. The tightness of P_n, $n\in N$, is equivalent to the following conditions

(1) for every $\varepsilon>0$ and $k>0$ there is some compact set $K\subseteq E$ such that $\lim_n P^{\mu}_n$ (ω; there is some $t<k$ such that $X_t\notin K)\leqslant\varepsilon$ (We shall shorten the above expression by writing "(ω; $(\exists)t\leqslant k$, $X_t\notin K$)".)

(2) $\lim_{\delta\to 0}\overline{\lim_n}\, P^{\mu}_n(\omega;\ W''_{k',\delta}(X^x)>\varepsilon)=0$ for every $k>0$, $\varepsilon>0$,

(3) $\lim_{\delta\to 0}\overline{\lim_n}\, P^{\mu}_n(\omega;\ W_{[0,\delta)}(X^n)>\varepsilon)=0$ for every $\varepsilon>0$,

(4) $\lim_{\delta\to 0}\overline{\lim_n}\, P^{\mu}_n(\omega;\ W_{[k-\delta,k)}(X^n)>\varepsilon)=0$ for every $\varepsilon>0$, $k>0$,

$$W_{[a,b)}(X) = \sup\,(d(X_t, X_s);\ a \leqslant t < s < b)\,,$$
$$W''_{k,\delta}(X) = \sup\,(\min\,(d(X_{t'}, X_t),\ d(X_t, X_{t''})))$$

with the supremum taken over all t', t, t'' such that $0 \vee (t-\delta) \leqslant t' < t < t'' \leqslant k \wedge (t+\delta)$.

To prove (1) it will suffice to show that for any $k>0$, $\varepsilon>0$ and any compact set $K \subseteq E$, there is some $k'>0$ such that:

$$\overline{\lim_n}\, P_n^\mu(\omega;\ (\exists)t<k,\ X_t^n \notin K) \leqslant P^\mu(\omega;\ (\exists)t \leqslant k',\ X_t \notin K)$$

and then we refer to the tightness of P^μ itself. By Lemma 3, (a) we know that

$$\overline{\lim_n}\, P_n^\mu(\omega;\ (\exists)t<k,\ X_t^n \notin K)$$
$$= \overline{\lim_n}\, P_n^\mu(\omega;\ (\exists)t \leqslant k,\ X_t^n \notin K,\ \tau_{k_n} > k) \leqslant \overline{\lim_n}\, P_n^\mu(\omega;\ (\exists)j < k_n,\ Z_j^n \notin K)$$

(k_n is chosen, with respect to k, like in Lemma 3).

By Lemma 4 we know that the last term in the above inequality is equal to $\overline{\lim}_n P^\mu(\omega;\ (\exists)j \leqslant k_n,\ Y_j^n \notin K)$ and by Lemma 3, (b) that is

$$\overline{\lim_n}\, P^\mu(\omega;\ (\exists)j \leqslant k_n,\ Y_j^n \notin K,\ \sigma_j^n < k')\,,$$

with k' chosen in Lemma 3. Because the terms under the limit are dominated by $P^\mu(\omega;\ (\exists)t<k',\ X_t \notin K)$, the proof of (1) is complete. (In what will follow we shall frequently use the same way of passing from X^n to X).

To prove (4) we note first that

$$W_{[k-\delta,k)}(X) \leqslant 2 \sup_{k-\delta \leqslant t < k} d(X_t, X_{k-\delta})\,.$$

We have to show that for every $k>0$ and $\varepsilon>0$

$$\lim_{\delta\to 0} \overline{\lim_n}\, P_n^\mu\big(\sup_{k-\delta \leqslant t < k} d(X_t, X_{k-\delta}) > \varepsilon\big) = 0\,. \tag{20}$$

By the Markov property

$$P_n^\mu\big(\sup_{k-\delta \leqslant t < k} d(X_t^n, X_{k-\delta}^n) \geqslant \varepsilon\big) = \int P_n^x\big(\sup_{0 \leqslant t < \delta} d(X_0^n, X_t^n) > \varepsilon\big) P_n^\mu \circ (X_{k-\delta}^n)^{-1}(dx)\,.$$

Using Lemma 4, we obtain in the same way like above

$$P_n^x\big(\sup_{0 \leqslant t < \delta} d(X_0^n, X_t^n) > \varepsilon\big) \leqslant P^x\big(\sup_{0 \leqslant t \leqslant \delta'} d(X_0, X_t) > \varepsilon\big) + P_n^\mu(\tau_{l_n} > \delta) + P^\mu(\sigma_{l_n}^n < \delta')$$

with δ and l_n like in Lemma 3. Because the convergence in Lemma 3, (a), (b) is uniform with respect to $x \in E$, the two last terms vanish under $\overline{\lim}_n \int$. $\delta \to 0$ implies that $\delta' \to 0$ and so we have to prove that

$$\lim_{\delta'\to 0}\overline{\lim_n}\int P^x(\sup_{0\leq t<\delta'} d(X_t, X_0)>\varepsilon)P_n^\mu\circ(X_{k-\delta}^n)^{-1}(dx)=0\,.$$

For a fixed $\eta>0$, (1) assures that we may choose a compact set K_η such that

$$\overline{\lim_n}\, P_n^\mu(\omega;\,(\exists)t\leq k,\ X_t^n\notin K_\eta)\leq\eta\,. \tag{21}$$

We dominate the above integral by

$$\sup_{x\in K_\eta} E^x(\sup_{0\leq t\leq\delta'} d(X_t, X_0)>\varepsilon)+P_n^\mu\circ(X_{k-\delta}^n)^{-1}(CK_\eta)\,.$$

Proposition 1, Appendix 2 assures that the first term vanishes under $\lim_{\delta'\to 0}$. By (21), for every $\delta>0$

$$\lim_n P_n^\mu\circ(X_{k-\delta}^n)^{-1}(CK_\eta)\leq\eta\,.$$

Because η is arbitrary small, the proof is complete.

An analogous proof goes for (3) and also for

$$\lim_{\delta\to 0}\overline{\lim_n}\, P_n^\mu(\sup_{t\leq s\leq t+\delta} d(X_t, X_s)>\varepsilon)=0 \tag{22}$$

for every $t>0$, $\varepsilon>0$.

The last relation will be used later.

To prove (2) we define for a fixed $k>0$ and $\varepsilon>0$ a discrete correspondent of $W''_{k,\delta}$, that is

$$T_n\colon E^{k_n}\to R$$

$$T_n(x_1,\cdots,x_{k_n})=\sup\min\,(d(x_i, x_j),\ d(x_i, x_p))$$

with $i, j, p\in N$ such that

$$i-(l_n+1)\leq j<i<p\leq k\wedge(i+(l_n+1))$$

(k_n and l_n are chosen with respect to k and δ like in Lemma 3). By Lemma 3, (a), (c)

$$\overline{\lim_n}\, P_n^\mu(W''_{k,\delta}(X^n)>\varepsilon)=\overline{\lim_n}\, P^\mu(W''_{k,\delta}>\varepsilon,\ A_n,\ \tau_{k_n}>k)\,.$$

We note that for $\omega\in A_n\cap(\tau_{k_n}>k)$

$$W''_{k,\delta}(X^n(\omega))>\varepsilon\Rightarrow T_n(Z_1^n(\omega),\cdots,Z_{k_n}^n(\omega))>\varepsilon\,.$$

Let be $0\leq t',\ t,\ t''\leq k$ such that $t-\delta\leq t'\leq t\leq t''\leq t+\delta$ and $d(X_{t'}^n, X_t^n)>\varepsilon$, $d(X_t^n, X_{t''}^n)<\varepsilon$. We define i by $\tau_i\leq t<\tau_{i+1}$ and j and p the corresponding integers with respect to t' and t''

$$t-t'<\delta\Rightarrow\tau_i(\omega)-\tau_{j+1}(\omega)<\delta\,.$$

Because $\omega \in A_n$, it follows that $i-(j+1) \leqslant l_n$, that is $i-j \leqslant l_n+1$. Because $\omega \in (\tau_{k_n} > k)$, $t'' \leqslant k \Rightarrow \tau_p(\omega) \leqslant k \Rightarrow p \leqslant k_n$.

$$X_t^n = Z_i^n\,, \quad X_{t'}^n = Z_j^n \qquad \text{and} \quad X_{t''}^n = Z_p^n$$

(see (11)) implies that

$$\min\,(d(Z_i^n, Z_j^n),\ d(Z_i^n, Z_p^n)) < \varepsilon$$

and the above implication is proved. We may now conclude that

$$\overline{\lim_n}\, P_n^\mu(W''_{k,\delta}(X^n) > \varepsilon) \leqslant \overline{\lim_n}\, P_n^\mu(T_n(Z_1^n, \cdots, Z_{k_n}^n) > \varepsilon)\,.$$

By Lemma 4 first and then by Lemma 3, (b), (d) the last term is equal to

$$\overline{\lim_n}\, P^\mu(T_n(Y_1^n, \cdots, Y_{k_n}^n) > \varepsilon\,,\ B_n,\ \sigma_{k_n}^n < k')\,.$$

In the same way like above we may dominate this term by

$$P(W''_{k',\delta'}(X) > \varepsilon)\,.$$

So we have proved that

$$\overline{\lim_n}\, P_n^\mu(W''_{k,\delta}(X) > \varepsilon) \leqslant P^\mu(W''_{k',\delta'}(X) > \varepsilon)\,.$$

$$k < \infty \Rightarrow k' < \infty\,, \quad \delta \to 0 \Rightarrow \delta' \to 0\,,$$

and so we may refer to the tightness of X, and the proof of (2) is complete.

To prove the convergence of the marginal distributions we have to verify the hypotheses of Lemma 3, Appendix 3.

The first one is an immediate consequence of (i) in our Theorem. For (ii) we have to verify condition (c) enunciated in the beginning of the paper, i.e.

(c) for any compact set $L \subseteq E$, $\alpha > 0$ and $\varepsilon > 0$, there is some compact set $K \subseteq E$ such that

$$\overline{\lim_n}\, \sup_{x \in L} U_\alpha^{(n)}(x, CK) \leqslant \varepsilon\,.$$

To do it we shall prove that for every compact set $\tilde{K} \subseteq E$ and $k > 0$, we may choose another compact set $K \subseteq E$ such that $\tilde{K} \subseteq K$ and

$$\overline{\lim_n}\, \sup_{x \in L} U_\alpha^{(n)}(x, CK) \leqslant e^{\alpha k'} \sup_{x \in L} U_\alpha(x, C\tilde{K}) + e^{-\alpha k} \tag{23}$$

(k' is defined with respect to k in Lemma 3). If (23) is true, we choose k such that

$$e^{-\alpha k} < \frac{\varepsilon}{2}$$

and ε' such that $e^{\alpha k'}\varepsilon'<\frac{\varepsilon}{2}$. Then, Lemma 2, Appendix 2 assures that there is some compact set $K\subseteq E$ such that

$$\sup_{x\in L} U_\alpha(x, C\tilde{K})<\varepsilon' .$$

So, the compact set K mentioned in (23) is that needed in (c).

If $A^n=\cup U_i^n$ over all i such that $U_i^n\cap\tilde{K}\neq\phi$, then by the definition of U_i^n we may choose a compact set K such that $A^n\subset K$ for every $n\in N$.

Next we prove the following inequality:

$$\text{(24)}\qquad E_n^x\int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t}\mathrm{I}_{CK}\circ X_t^n dt\leqslant E^x\int_{\sigma_i^n}^{\sigma_{i+1}^n}\mathrm{I}_{C\tilde{K}}\circ X_t dt .$$

Because $X_t^n=Z_i^n$ for $\tau_i\leqslant t<\tau_{i+1}$ and $\tau_{i+1}-\tau_i=\tau_1\circ\theta_{\tau_i}$, we have

$$\int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t}\mathrm{I}_{CK}\circ X_t^n dt = \mathrm{I}_{CK}Z_i^n\int_{\tau_i}^{\tau_{i+1}} e^{-dt}dt\leqslant \mathrm{I}_{CK}\circ Z_i^n\tau_1\circ\theta_{\tau_i} .$$

We dominate the term in the left of (24) by $E_n^x(\mathrm{I}_{CK}\circ Z_i^n\tau^1\circ\theta_{\tau_i})$ which by the strong Markov property is

$$E_n^x(\mathrm{I}_{CK}\circ Z_i^n E^{Z_i^n}(\tau_1)) = \Pi_n^i(\mathrm{I}_{CK}q_n^{-1})(x) .$$

By the definition of σ_i^n and K we have

$$E^x\int_{\sigma_i^n}^{\sigma_{i+1}^n}\mathrm{I}_{C\tilde{K}}\circ X_t dt_t\geqslant E^x(\mathrm{I}_{CK}\circ Y_i^n\int_{\sigma_i^n}^{\sigma_{i+1}^n} 1dt)$$

In the same way like above, the last term is equal to

$$\Pi_n^i(\mathrm{I}_{CK}q_n^{-1})(x)$$

and (24) is proved.

Next, to prove (23), we shall change X^n by X in the same way as above:

$$\text{(25)}\qquad U_\alpha^{(n)}(x, CK) = E_n^x\int_0^k e^{-\alpha t}\mathrm{I}_{CK}\circ X_t^n dt+E_n^x\int_k^\infty e^{-\alpha t}\mathrm{I}_{CK}\circ X_t dt .$$

The second term in the sum is dominated by $e^{-\alpha k}$. The first one is dominated by

$$E_n^x\Big(\int_0^k e^{-\alpha t}\mathrm{I}_{CK}\circ X_t^n dt;\ \tau_{k_n}>k\Big)+\frac{1}{\alpha}P_n^x(\tau_{k_n}\leqslant k) .$$

By Lemma 3.a we may ignore the second term in the above sum. The first one is dominated by

$$E_n^x\left(\int_0^{\tau_{k_n}} e^{-\alpha t} \mathrm{I}_{CK} \circ X_t^n \, dt\right)$$

which by (22) is dominated by

$$E^x\left(\int_0^{\sigma_{k_n}^n} \mathrm{I}_{C\tilde{K}} \circ X_t dt\right) \leqslant E^x\left(\int_0^{\sigma_{k_n}^n} \mathrm{I}_{C\tilde{K}} \circ X_t dt;\ \sigma_{k_n}^n < k'\right)$$
$$+ E^x\left(\int_0^{\sigma_{k_n}^n} \mathrm{I}_{C\tilde{K}} \circ X_t dt;\ \sigma_{k_n}^n \geqslant k'\right).$$
$$\int_0^{\sigma_{k_n}^n} \mathrm{I}_{C\tilde{K}} \circ X_t dt \leqslant \sigma_{k_n}^n$$

and therefore we may dominate the second term in the sum by $E^x(\sigma_{k_n}^n;\ \sigma_{k_n}^n \geqslant k')$ which we may ignore (see Lemma 3.e). To dominate the first term we note that on $\sigma_{k_n}^n < k'$

$$\int_0^{\sigma_{k_n}^n} \mathrm{I}_{C\tilde{K}} \circ X_t dt \leqslant \int_0^{k'} \mathrm{I}_{C\tilde{K}} \circ X_t dt \leqslant e^{\alpha k'} \int_0^{k'} e^{-\alpha t} \mathrm{I}_{C\tilde{K}} \circ X_t dt$$

and therefore

$$E^x\left(\int_0^{\sigma_{k_n}^n} \mathrm{I}_{C\tilde{K}} \circ X_t \, dt;\ \sigma_{k_n}^n < k'\right) \leqslant e^{\alpha k'} U_\alpha(x, C\tilde{K}).$$

The proof of (23) is complete and also that of (c).

We verify now the last condition in Lemma 3, Appendix 3. For every $t \geqslant 0$, $f \in U_b(E)$ (uniformly continuous and bounded) and $\varepsilon > 0$, there is some $\delta_\varepsilon > 0$ such that

$$\overline{\lim_n} \sup E_n^\mu(|f(X_t^n) - f(X_s^n)|) \leqslant \varepsilon \tag{26}$$

where the supremum is over all s such that $t \leqslant s \leqslant t + \delta_\varepsilon$. We choose $\eta_\varepsilon > 0$ such that

$$d(x., y) \leqslant \eta_\varepsilon \Rightarrow |f(x) - f(y)| \leqslant \frac{\varepsilon}{2}$$

$$E_n^\mu(|f(X_t^n) - f(X_s^n)|) \leqslant E_n^\mu(|f(X_t^n) - f(X_s^n)|;\ d(X_t^n, X_s^n) < \eta_\varepsilon)$$
$$+ 2\|f\| E_n^\mu(d(X_s^n, X_t^n) > \eta_\varepsilon).$$

The first term is less as $\frac{\varepsilon}{2}$ and therefore

$$\overline{\lim_n} \sup E_n^\mu(|f(X_t^n) - f(X_s^n)|) \leqslant \frac{\varepsilon}{2} + 2\|f\| \overline{\lim_n} E_n^\mu(\sup d(X_t^n, X_s^n) > \eta_\varepsilon)$$

(the supremum like above). (22) assures that we may choose δ_ε needed in (26) and the proof is complete.

4. Appendices

Appendix 1.

Lemma 1. *Let* $(F_n)_{n \in N}$ *be a sequence of distributions on* R_+, $(k_n)_{n \in N}$ *a* *sequence of positive integers such that* $\lim_n k_n a_n = l$ *with* $a_n = \int_0^\infty z F_n(dz)$. *If*

(a) $\lim_n a_n = 0$,

(b) $\lim_n \frac{1}{a_n} \int_a^\infty z F_n(dz) = 0$ *for every* $a > 0$,

then

$$\lim_n F_n^{*k_n} = \varepsilon_l .$$

Proof. It will suffice to show that $\lim_n \varphi_n(t)^{k_n} = e^{-ilt}$ with

$$\varphi_n(t) = \int_0^\infty e^{-itz} F_n(dz) .$$

By (a), $\lim_n F_n = \varepsilon_0$ and so $\lim_n \varphi_n(t) = 1$. We may organize the above limit in an exponential form and it remains to show that $\lim_n k_n(\varphi_n(t) - 1) = -ilt$. By the choose of k_n, that is $\lim_n \frac{1}{a_n}(1 - \varphi_n(t)) = it$. We write $1 - \varphi_n(t)$ in the following form:

$$it \int_0^\infty z F_n(dz) + \int_0^\infty z\Big(\frac{1 - \cos tz}{z}\Big) F_n(dz) + i \int_0^\infty z\Big(\frac{\sin z}{z} - t\Big) F_n(dz) .$$

Both $z \to \frac{1 - \cos tz}{z}$ and $z \to \frac{\sin z}{z} - t$ are bounded continuous functions which vanishes when $z \to 0$, therefore it will suffice to show that for such a function α, $\lim_n \int_0^\infty z\alpha(z) F_n(dz) = 0$.

Let M be such that $|\alpha(z)| < M$ for $z > 0$, and for a fixed $\varepsilon > 0$, $a_\varepsilon > 0$ such that $z < a_\varepsilon \Rightarrow |\alpha(z)| < \varepsilon$.

$$\frac{1}{a_n} \Big| \int_0^\infty z\alpha(z) F_n(dz) \Big| \leqslant \frac{1}{a_n} \int_0^{a_\varepsilon} z |\alpha(z)| F_n(dz) + \frac{M}{a_n} \int_{a_\varepsilon}^\infty z F_n(dz) .$$

By (b) the second term of the sum vanishes when $n \to \infty$. The first one is dominated by ε, which is arbitrary small, and so the proof is complete.

Corollary 2. $\overline{\lim_n}\, k_n a_n < l \Rightarrow \lim_n F_n^{*k_n}(l + \varepsilon, \infty) = 0$.

Lemma 3. *If* $a_n \uparrow \infty$ *and* $\lim_n \frac{k_n}{a_n} = l + \varepsilon$ *with* $k_n \in N$, $\varepsilon > 0$, *then* $\lim_n k_n e_{a_n}^{*k_n}(0, l) = 0$.

Proof. Let $(E, \mathcal{K}, P)$ be a probability space and f_n: $E \to R$, $n \in N$, a sequence of independent variables, e_1 distributed (e_a is the exponential distribution with parameter a). $\frac{1}{a_n} f_i$, $i \in N$ are independent and e_{a_n} distributed and so

$$e_{a_n}^{*k_n}(0, l) = P\left(\frac{1}{a_n} f_1 + \cdots + \frac{1}{a_n} f_{k_n} < l\right) = P\left(\frac{f_1 + \cdots + f_{k_n}}{k_n} \frac{k_n}{a_n} < l\right).$$

By the choice of k_n, for a sufficiently large n, we may dominate this term by

$$P\left(\frac{f_1 + \cdots + f_{k_n}}{k_n} < \left(1 + \frac{\varepsilon}{l}\right)^{-1}\right) \leqslant P\left(\left|\frac{f_1 + \cdots + f_{k_n}}{k_n} - 1\right| > c\right) = P\left(\left(\sum_{i=}^{k} (f_i - 1)\right)^4 > k_n^4 c^4\right)$$

with

$$c = \frac{\varepsilon}{l}\left(1 + \frac{\varepsilon}{l}\right)^{-1}.$$

By Chebyshev's inequality we may dominate it by $\frac{1}{k_n^4 c^4} \int \left(\sum_{i=1}^{k_n} (f_i - 1)\right)^4 dP$. Because f_i are independent with mean value 0, this is

$$\frac{1}{k_n^4 c^4}\left(\sum_{i=1}^{n} \int (f_i - 1)^4 dP + \sum_{i \neq j} \int (f_i - 1)^2 dP \int (f_j - 1)^2 dP\right).$$

This sum is dominated by $(M k_n^2)(k_n^4 c)^{-1}$ for a sufficiently large M. So,

$$\lim_n k_n e_{a_n}^{*k_n}(0, l) \leqslant \lim_n k_n M (k_n^2 c)^{-1} = 0 .$$

Appendix 2. The first proposition follows from an idea exposed in [4].

Proposition 1. *Let E be a L.C.C.B. space with a metric d such that $B_h(x)$ is relatively compact, and X a standard process with semigroup $(P_t)_{t>0}$. If for every $f \in C_c(E)$, $\lim_{t \to 0} P_t f = f$ uniformly on E, then for every compact set $K \subseteq E$ and $\varepsilon > 0$*

(1) $$\lim_{h \to 0} \sup_{x \in K} P^x\left(\sup_{0 \leqslant t \leqslant h} d(X_0, X_t) > \varepsilon\right) = 0 .$$

Proof. We note first that to prove (1) will suffice to show that for every $L \subseteq G \subseteq E$, L compact set and G relatively compact open set,

(2) $$\lim_{h \to 0} \sup_{x \in L} P^x(D_{CG} \leqslant h) = 0$$

($D_A(\omega) = \inf(t; X_t(\omega) \in A)$ for any measurable set A).

If (2) is true, the proof of (1) goes like this: for every $x \in K$ we choose an open set V_x and a compact one K_x such that

$$x \in V_x \subseteq K_x \subseteq B_{\varepsilon/2}(x) .$$

We consider V_{x_i}, $i\in N$, a finite covering for K. Then K_{x_i}, $i\in N$, will also be a covering of K. For any $x\in K$ there is some $i\leqslant n$ such that $x\in K_{x_i}\subseteq B_{\varepsilon/2}(x_i)$, therefore $B_{\varepsilon/2}(x_i)\subseteq B_\varepsilon(x)$ and so $D_{\mathcal{C}B_\varepsilon(x)}\geqslant D_{\mathcal{C}B_{\varepsilon/2}}(x_i)$. Because $(\sup_{t\leqslant h} d(X_0, X_t)>\varepsilon)=(D_{\mathcal{C}B_\varepsilon(x)}\leqslant h)$ P^x a.s., we may conclude that

$$\sup_{x\in K} P^x(\sup_{t\leqslant h} d(X_0, X_t)>\varepsilon)\leqslant \max_{j\leqslant n}\sup_{x\in K_i} P^x(D_{\mathcal{C}B_{\varepsilon/2}(x_i)}\leqslant h)\,.$$

Now we take $L_i=K_i$ and $G_i=B_{\varepsilon/2}(x_i)$ (which is relatively compact), and (2)⇒(1) is proved.

To prove (2) we choose a relatively compact open set U such that $L\subseteq U\subseteq \bar{U}\subset G$ and note that

$$(3)\quad P^x(D_{\mathcal{C}G}\leqslant h)\leqslant P^x(D_{\mathcal{C}G}\leqslant h,\ X_h\in U)+P^x(X_h\in \mathcal{C}U)\,.$$

Let be $f\in C(E)$ such that $I_{\mathcal{C}U}\leqslant f\leqslant I_{\mathcal{C}L}$. Then $P^x(X_h\in\mathcal{C}U)=(P_hI_{\mathcal{C}U})(x)\leqslant P_hf(x)$ and for $x\in L, f(x)=0$. So,

$$\begin{aligned}\sup_{x\in L} P^x(X_h\in\mathcal{C}U)&\leqslant \sup_{x\in L} P_hf(x)\\ &=\sup_{x\in L}|P_hf(x)-f(x)|=\sup_{x\in L}|P_hg(x)-g(x)|\end{aligned}$$

with $g=1-f$. Because $\bar{U}$ is compact, $g\in C_c(E)$ and by our hypothesis this term vanishes when $h\to 0$.

To dominate the first term in (3) we note that $X_{D_{\mathcal{C}G}}\in\mathcal{C}G$ and so if we denote $X_{D_{\mathcal{C}G}}=Y$ and $D_{\mathcal{C}G}=T$ by applying the strong Markov property for two variables functions, we get

$$\begin{aligned}P^x(T\leqslant h,\ X_h\in U)&=\int_{(T\leqslant h,\ Y\in\mathcal{C}G)} P^Y(X_{h-T(\omega)}\in U)P^x(d\omega)\\ &\leqslant\sup_{t\leqslant h}\sup_{y\in\mathcal{C}G} P^y(X_t\in U)\,.\end{aligned}$$

For $f\in C(E)$ such that $I_{\bar{U}}\leqslant f\leqslant I_G$,

$$P^y(X_t\in U)=(P_tI_U)(y)\leqslant P_tf(y)=|P_tf(y)-f(y)|$$

for every $y\in\mathcal{C}G$. supp. $f\subseteq\bar{G}$ which is compact and the proof ends.

Lemma 2. *Let Q be a kernel on E, L.C.C.B. space. If $Q(C_b(E))\subseteq C_b(E)$, then, for every compact $L\subseteq E$ and $\varepsilon>0$, there is some compact set $K_\varepsilon\subseteq E$ such that*

$$\sup_{x\in L} Q(x, \mathcal{C}K_\varepsilon)\leqslant\varepsilon\,.$$

Proof. For every $x\in L$ we choose K_x, K_x' compact sets and $f_x\in C_b(E)$

such that $Q(x, CK_x)<\varepsilon$, $K_x\subseteq \operatorname{Int} K'_x$ and $I_{K_x}\leqslant f_x\leqslant I_{K'_x}$. It is obvious that $Q(I-f_x)(x)<\varepsilon$. $y\to Q(I-f_x)(y)$ is a continuous function, so we may choose V_x such that $Q(I-f_x)(y)<\varepsilon$ for every $y\in V_x$. Let V_{x_i}, $i\in N$ be a finite covering of L. Then, the compact K_ε will be $\bigcup_{i\leqslant n_i} V_{x_i}$. Indeed, for an $x\in L$ there exists i such that $x\in V_{x_i}$

$$Q(x, CK_\varepsilon)\leqslant Q(x, CK'_{x_i})\leqslant Q(x, 1-f_{x_i})<\varepsilon\,.$$

Appendix 3. We introduce first some notations:

$$R^k_+ = ((t_1, \cdots, t_k);\ t_i\geqslant 0)\,,\quad s^k = (s_1, \cdots, s_k)\,,$$
$$ds^k = ds_1 \cdots ds_k$$
$$A(\alpha^k, s^k) = \exp\Big(-\sum_{i=1}^k \alpha_i s_i\Big)\,.$$

For a permutation σ on $(1, 2, \cdots, k)$ we denote

$$\Lambda^k_\sigma = \{(s_1, \cdots, s_k);\ 0\leqslant s_{\sigma(1)}\leqslant\cdots\leqslant s_{\sigma(k)}\}\,.$$

If σ is the identic permutation we ignore it and write Λ^k. We consider a standard process and for $0\leqslant t_1<\cdots\leqslant t_k$ and $f_i\in C_b(E)$ $i\leqslant n$ we define

$$U_{\alpha^k}f_1\cdots f_k = E^\mu\int_{R^k_+} A(\alpha^k, s^k)\prod_{i\leqslant k} f_i(X_{s_i})ds^k\,.$$

We note that $U_{\alpha^k}f_1\cdots f_k$ is the Laplace transform for the distribution

$$F(dt^k) = h(t^k)dt^k \quad\text{with}\quad h(t^k) = E^\mu\Big(\prod_{i\leqslant k} f_i(X_{t_i})\Big)\,.$$

We define also

$$H_{\sigma,\alpha^k}f_1\cdots f_k = E^\mu\int_{\Lambda^k_\sigma} A(\alpha^k, s^k)\prod_{i\leqslant k} f_i(X_{s_i})ds^k\,.$$

If σ is the identic permutation, we ignore it in our notation. Because $m(\partial\Lambda^k_\sigma)=0$ (m is the Lebesgue measure),

$$U_{\alpha^k}f_1\cdots f_k = \sum_\sigma H_{\sigma,\alpha^k}f_1\cdots f_k\,. \tag{1}$$

Lemma 1. *Let X^n $n\in N$ and X be standard processes with state space E, L.C.C.B., such that:*

(i) $U_\alpha(C_b(E))\subseteq C_b(E)$,
(ii) *for every* $f\in C_b(E)$, $\lim_n U^{(n)}_\alpha f=U_\alpha f$ *uniformly on compacts.*

Then, for every $\alpha^k\in R^k_+$ and $f_i\in C_b(E)$ $i\leqslant k$

$$\lim_n U^{(n)}_{\alpha^k}f_1\cdots f_k = U_{\alpha^k}f_1\cdots f_k$$

($U^{(n)}_{\alpha^k}$ *is defined with respect to* X^n *in the same way as* U_{α^k} *with respect to* X. *For* $k=1$ *we shall write* α *instead of* α^1, *so* $U_\alpha(U^{(n)}_\alpha)$ *is the resolvent of* $X(X^n)$).

Proof. (1) assures that it will suffice to prove

$$\lim_n H^{(n)}_{\sigma,\alpha^k} f_1 \cdots f_k = H_{\sigma,\alpha^k} f_1 \cdots f_k .$$

It is no loss of generality to do it only when σ is the identic permutation. We shall proceed by induction on k. For $k=1$, that is (ii). By the Markov property we obtain

$$E^\mu(\prod_{i\leqslant k} f_i(X_{s_i})) = E^\mu(\prod_{i\leqslant k-1} f_i(X_{s_i}) E^{X_{s_{k-1}}}(f_k(X_{s_k - s_{k-1}}))) .$$

Then, applying twice Fubini's theorem we get

$$H_{\alpha^k} f_1 \cdots f_k = \int_{\Lambda^{k-1}} ds^{k-1} A(\alpha^{k-1}, s^{k-1}) E^\mu \Big\{ \prod_{i\leqslant k-1} f_i(X_{s_i}) E^{X_{k_{s-1}}} \Big(\int_{s_{k-1}}^\infty e^{-\alpha_k s_k} f_k(X_{s_k s_{p-1}}) ds_k \Big) \Big\}.$$

By the changement $s=s_k-s_{k-1}$ we get

$$\int_{s_{k-1}}^\infty e^{\alpha-_k s_k} f_k(X_{s_k - s_{k-1}}) ds = e^{-\alpha_k s_{k-1}} \int_0^\infty e^{-\alpha_k s_k} f_k(X_s) ds$$

and therefore

$$E^{X_{s_{k-1}}} \Big(\int_{s_{k-1}}^\infty e^{-\alpha_k s_k} f(X_{s_k - s_{k-1}}) ds_k \Big) = e^{-\alpha_k s_{k-1}} U_{\alpha_k} f_k(X_{s_{k-1}}) .$$

Condition (i) assures that $U_{\alpha_k} f_k \in C_b(E)$ and we may write

$$H_{\alpha^k} f_1 \cdots f_k = H_{\beta^{k-1}} f_1 \cdots f_{k-2}\, g_{k-1}$$

with

$$\beta^{k-1} = (\beta_1, \cdots, \beta_{k-1}) , \quad \beta_i = \alpha_i \text{ for } i \leqslant k-2 ,$$
$$\beta_{k-1} = \alpha_{k-1} + \alpha_k \quad \text{and} \quad g_{k-1} = f_{k-1} U_{\alpha_k} f_k .$$

We may establish an analogous relation for every $n \in N$. In this case β^{k-1} will be the same, but

$$g^n_{k-1} = f_{k-1} U^{(n)}_{\alpha_k} f_k$$

which is no more continuous. Nevertheless the definition of $H^{(n)}_{\beta^{k-1}}$ makes sense, and we write

$$H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2} g^n_{k-1} - H_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{k-1} = d^n_1 + d^n_2$$

with

$$d^n_1 = H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2} g^n_{k-1} - H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{k-1}$$

$$d_2^n = H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{\beta-1} - H_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{k-1}$$

$g_{k-1} \in C_b(E)$ and by the induction hypothesis we get $\lim_n d_2^n = 0$. For a fixed $\varepsilon > 0$, let $K_\varepsilon \subseteq E$ be a compact such that

$$H_{\beta^{k-1}} f_1 \cdots f_{k-2} I_{CK_\varepsilon} \leqslant \varepsilon .$$

We choose another compact set K' such that $K \subseteq \operatorname{Int} K'$ and a function $\varphi \in C_b(E)$ such that

$$I_{K_\varepsilon} \leqslant \varphi \leqslant I_{K'}$$

$$d_1^n = H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2}((g_{k-1}^n - g_{k-1})\varphi) + H^n_{\beta^{k-1}} f_1 \cdots f_{k-2}((g_{k-1}^n - g_{k-1})(I - \varphi)) .$$

We dominate the first term of the sum by

$$\prod_{i \leqslant k-2} \|f_i\| \sup_{x \in K'} |g_{k-1}^n - g_{k-1}| \leqslant \prod_{i \leqslant k-1} \|f_i\| \sup_{x \in K'} |U^{(n)}_{\alpha_k} f_k - U_{\alpha_k} f_k|$$

which by (ii) vanishes when $n \to \infty$. The second term is dominated by

$$2\|f_k\| \cdot \|f_{k-1}\| H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2}(1 - \varphi) .$$

which by the induction hypothesis converges to

$$2\|f_k\| \, \|f_{k-1}\| H_{\beta^{k-1}} f_1 \cdots f_{k-2}(1 - \varphi) .$$

By the choice of K_ε and φ this term is dominated by $2\|f_k\| f_{k-1}\|\varepsilon$. ε is arbitrary small and so the proof is complete.

Lemma 2. *Let F_n, $n \in N$, and F be distributions on R_+^k of the form*

$$F_n(ds^k) = h_n(s^k) ds^k \quad \textit{and} \quad F(ds^k) = h(s^k) ds^k .$$

If

(i) $\lim F_n = F$,

(ii) h_n, $n \in N$, *are equal right continuous,*

then

$$\lim_n h_n(t^k) = h(t^k) \quad \textit{for every} \quad t^k \in R_+^k .$$

Proof. Let us suppose that there is some $t^k \in R_+^k$ such that $\lim_n h_n(t^k) \neq h(t^k)$. By passing to a subsequence we may assume that for some $\varepsilon > 0$, $h_n(t^k) > h(t^k) + \varepsilon$ for every $n \in N$. Let δ_ε be a constant such that for every $s^k = (s_1, \cdots, s_k)$ with $t_i \leqslant s_i \leqslant t_i + \delta_\varepsilon$ for every $i \leqslant k$, holds that $|h(t^k) - h(s^k)| \leqslant \frac{\varepsilon}{3}$ and $|h_n(t^k) - h_n(s^k)| \leqslant \frac{\varepsilon}{3}$ for every $n \in N$.

We define $A = \prod_{i \leqslant k} [t_i, t_i + \delta_\varepsilon]$. Then $F(\partial A) = 0$ and therefore $\lim_n F_n(A) = F(A)$, that is

$$\lim_n \int_A (h_n(s^k)-h(s^k))ds^k = 0\,, \tag{2}$$

$$h_n(s^k)-h(s^k) = (h_n(s^k)-h_n(t^k))+(h_n(t^k)-h(t^k))+(h(t^k)-h(s^k))\,.$$

For $s^k \in A$ the first and the last term of the above sum are dominated by $\frac{\varepsilon}{3}$, and the middle term is greater than ε. So $h_n(s^k)-h(s^k) \geqslant \frac{\varepsilon}{3}$ for $s^k \in A$, which is in contradictory with (2).

Lemma 3. *Let X_n $n \in N$ and X be standard processes. If*

(i) $U_\alpha(C_b(E)) \subseteq C_b(E)$,

(ii) $\lim_n U_\alpha^{(n)} f = U_\alpha f$ *uniformly on compacts for every* $\alpha > 0$ *and* $f \in C_b(E)$,

(iii) $\lim_{\delta\to 0} \lim_n \sup_{t\leqslant s\leqslant t+\delta} E_n(|f(X_t^n)-f(X_s^n)|)=0$ *for every* $f \in C_b(E)$ *and* $t \geqslant 0$,

then, for every $t_1 < t_2 \cdots < t_k$ *and* $f_i \in U_b(E)$ $i \leqslant k$,

$$\lim_n E_n^\mu \Big(\prod_{i\leqslant k} f_i(X_{t_i}^n)\Big) = E^\mu\Big(\prod_{i\leqslant k} f_i(X_{t_i})\Big)\,. \tag{3}$$

Proof. Lemma 3 is a simple consequence of Lemma 1 and Lemma 2.

REMARK. (3) is sufficient to assure the convergence of the marginal distributions.

References

[1] T. Watanabe: *On the equivalence of excessive functions and superharmonic functions in the theory of Markov processes* I and II, Proc. Japan Acad. **38** (1962), 397–401; 402–407.

[2] R.M. Blumenthal and R.K. Getoor: Markov processes and potential theory, Academic Press, New York, 1958.

[3] I.I. Gihman and A.V. Skorohod: The theory of stochastic processes I, Springer-Verlag, 1974.

[4] P. Courrage and P. Priouret: *Axiomatique du problème de Dirichlet et processus de Markov,* Séminaire de théorie du potentiel, dirigé par M. Brelot et G. Choquet et J. Deny, 8e année 1963/64.

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