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<thead>
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<th>Title</th>
<th>On the normal forms of differential equations in the neighborhood of an equilibrium point</th>
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</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nagumo, Mitio; Isé, Kusuo</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Mathematical Journal. 9(2) P.221-P.234</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1957</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/7848">https://doi.org/10.18910/7848</a></td>
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<tr>
<td>DOI</td>
<td>10.18910/7848</td>
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Osaka University
§ 1. Introduction.

1. In this note we use the notations $\partial u$ and $\partial^2 u$ for $\frac{\partial}{\partial x_i} u$ and $\frac{\partial^2}{\partial x_i \partial x_j} u$ respectively. The vectors $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_m)$ in $\mathbb{R}^m$ will be denoted briefly by $x$ and $y$ respectively.

Let $A=(a_{ij})$ be a constant real $(m, m)$-matrix, all of whose characteristic roots $\lambda_i$ ($i=1, \cdots, m$) have non-zero real parts, and $f(x)=(f_1(x), \cdots, f_m(x))$ a real vector function of class $C^1$ on some neighborhood of $x=0$, such that $f(0)=0$ and $|\partial_x f(x)| \leq K \cdot |x|$ with a constant $K>0$ where

$$|x| = \left( \sum_i x_i^2 \right)^{\frac{1}{2}}, \quad |\partial_x f(x)| \leq \left\{ \sum_{i,j} (\partial_{x_i} f_j(x))^2 \right\}^{\frac{1}{2}}.$$

We consider the autonomous systems

\begin{align*}
1.1 & \quad \frac{dx}{dt} = A \cdot x + f(x) \\
1.2 & \quad \frac{dy}{dt} = A \cdot y,
\end{align*}

regarding $x$, $y$ and $f(x)$ as the column-vectors. The purpose of this note is to show that, under some conditions on $\lambda_i$ ($i=1, \cdots, m$) and $f(x)$, the system (1.1) can be transformed into (1.2) by a change of variables

\begin{align*}
1.3 & \quad y = x + u(x)
\end{align*}

where $u(x)=(u_1(x), \cdots, u_m(x))$ is a real vector function of class $C^1$, such that

\begin{align*}
1.4 & \quad \begin{cases} u(0) = 0 \\ |\partial_x u(x)| \leq L \cdot |x| \end{cases}
\end{align*}

with some constant $L>0$. 

\textbf{On the Normal Forms of Differential Equations in the Neighborhood of an Equilibrium Point}

By Mitio Nagumo and Kusuo Ise

Nagumo, Mitio and Isé, Kusuo
Osaka Math. J.
9 (1957), 221–234
When $f(x)$ is analytic regular in $x$, in order to show the existence of the transformation given by (1.3) with analytic regular $u(x)$, we must necessarily assume that there exist no relations of the form

$$\lambda_i = \sum_{j=1}^{m} n_j \lambda_j$$

where $n_j (j=1, \ldots, m)$ are non-negative integers such that $\sum_{j=1}^{m} n_j > 1$. As to this case, some results were obtained by H. Poincaré, C. L. Siegel, and others, while we obtain the present result for the real systems with a transformation of class $C^1$ under some weaker conditions.

§ 2. Main Theorem.

2. Theorem. Assumptions:

(i) $A$ is a constant real $(m,m)$-matrix, all of whose characteristic roots $\lambda_i (i=1, \ldots, m)$ have non-zero real parts: $\Re(\lambda_i) \neq 0$ $(i=1, \ldots, m)$.

(ii) Let

$$f_i(x) = p_i(x) + q_i(x) \quad (i=1, \ldots, m)$$

where $p_i(x)$ are polynomials in $x$ with real coefficients such that $p_i(0) = \partial_j p_i(0) = 0$ $(i=1, \ldots, m; j=1, \ldots, m)$, and $q_i(x) (i=1, \ldots, m)$ are real-valued functions of class $C^1$ satisfying

$$q(0) = 0 \quad |\partial_x q(x)| \leq Q \cdot |x|^h$$

with some integer $h > 0$ and some constant $Q > 0$.

(iii) There exist no relations of the form

$$\lambda_i = \sum_{j=1}^{m} n_j \lambda_j$$

where $n_j (j=1, \ldots, m)$ are non-negative integers such that

$$h > \sum_{j=1}^{m} n_j > 1.$$  

Conclusion: There exists a positive constant $h_0$, depending only on $\lambda_i (i=1, \ldots, m)$, with the following property: if $h > h_0$, there exist functions $u_i(x) (i=1, \ldots, m)$ of class $C^1$ satisfying (1.4), such that the system (1.1) is reduced to the form (1.2) by the substitution (1.3).

3. If (1.1) is transformed into (1.2) by (1.3), $u(x)$ must satisfy the system of partial differential equations
On the Normal Forms of Differential Equations in the Neighborhood of an Equilibrium Point

(3.1) \[ \sum_{i=1}^{m} \left( \sum_{j=1}^{m} a_{ij} x_j + f_i(x) \right) \partial_{ij} u = \sum_{\mu=1}^{m} a_{\nu \mu} u_{\mu} - f_\nu(x) \quad (\nu = 1, \ldots, m). \]

For we have, by operating \( \frac{d}{dt} \) on both sides of (1.3),

\[ \frac{dy_i}{dt} = \frac{dx_i}{dt} + \sum_{i=1}^{m} \partial_{ij} u \frac{dx_j}{dt} \quad (i = 1, \ldots, m) \]

from which (3.1) follows immediately by (1.1), (1.2) and (1.3). Conversely, if \( u(x) \) is any function satisfying (3.1), then the substitution (1.3) will transform (1.1) into (1.2). Thus we have only to show the existence of \( u(x) \) satisfying (1.4) and (3.1), if \( h \) is sufficiently large.

\[ \dot{y}_i = \lambda_i y_i + \sum_{j=1}^{m} \partial_{ij} u \frac{dx_j}{dt} \]

§ 3. Auxiliary Theorem.

4. In this section we consider the system of semi-linear partial differential equations

(4.1) \[ \sum_{i=1}^{m} P_i(x) \partial_{ij} u = Q_i(x, u) \quad (\nu = 1, \ldots, l) \]

where \( x = (x_1, \ldots, x_m) \) and \( u = (u_1, \ldots, u_l) \) denote real vectors in \( \mathbb{R}^m \) and \( \mathbb{R}^l \) respectively. Let \( P_i(x) \) be real-valued functions of class \( C^1 \) in an open domain \( D \subset \mathbb{R}^m \), such that

(4.2) \[ \left( P_1(x), \ldots, P_m(x) \right) \neq (0, \ldots, 0) \quad (x \in D). \]

And \( Q_i(x, u) \) be real-valued functions of class \( C^1 \) in

\[ \Omega = \{(x, u) \in \mathbb{R}^{m+l} : x \in D, \ |u| \leq \omega(x)\} \]

where \( \omega(x) \) is some positive-valued function of class \( C^1 \) in \( D \). A curve \( x = x(t) \) in \( \mathbb{R}^m \) is said to be a base characteristic of (4.1) if \( x(t) \) satisfies the following system of ordinary differential equations:

(4.3) \[ \frac{dx_i}{dt} = P_i(x) \quad (i = 1, \ldots, m). \]

Let an \((m-1)\)-dimensional manifold \( M \) in \( \mathbb{R}^m \) be given by

(4.4) \[ M : x_i = A_i(s_1, \ldots, s_{m-1}) \quad (i = 1, \ldots, m) \]

where \( A_i(s) \) are functions of class \( C^1 \) in some domain \( S \subset \mathbb{R}^{m-1} \) such that \( A(s) = (A_1(s), \ldots, A_m(s)) \in D \) for \( s = (s_1, \ldots, s_{m-1}) \in S \). We assume that
and that any base characteristic

\begin{equation}
\tag{4.6}
x = x(t, s),
\end{equation}

issuing from a point of \( M \) so that \( x(0, s) = A(s) \), exists on the interval:
\( 0 \leq t < \tau(s) \) where \( \tau(s) \) is a continuous function on \( S \), and that the set \( X = \{ x = x(t, s) : 0 \leq t < \tau(s), s \in S \} \) is filled up only onefold with all those curves \( x = x(t, s)(s \in S) \), i.e. to any point \( x \in X \) there corresponds just one \( (t, s) \) such that \( x = x(t, s), 0 \leq t < \tau(s), s \in S \). Then we have easily

\[
\frac{\partial(x_1, \cdots, x_m)}{\partial(t, s_1, \cdots, s_{m-1})} \quad i \downarrow 1, \cdots, m \quad j \rightarrow 1, \cdots, m-1 \quad \exp \left( \int_{0}^{\tau(s)} \sum_{i=1}^{m} \partial_{i} P_i(x, x(t, s), s) \, dt \right) = 0,
\]

which shows that the \( 1-1 \) mapping (4.6) from \( \{ (t, s) : s \in S, 0 \leq t < \tau(s) \} \) onto \( X \) and its inverse are both of class \( C^1 \).

By (4.6) the system (4.1) is reduced to the following system of ordinary differential equations, \( s \) being a parameter:

\begin{equation}
\tag{4.7}
d\gamma = Q_\gamma(x(t, s), u) \quad (\nu = 1, \ldots, l).
\end{equation}

We have then

\[
\partial_t \omega(x(t, s)) = \left[ \sum_{i=1}^{m} P_i(x) \cdot \partial_i \omega(x) \right]_{x = x(t, s)}
\]

and

\[
\partial_t |u(t, s)| \cdot |u(t, s)| = \sum_{i=1}^{l} n_i(t, s) \cdot Q_\gamma(x(t, s), u(t, s))
\]

for any solution \( u(t, s) \) of (4.7). Hence, we obtain easily the following auxiliary theorem which is our principal tool.

\textbf{Auxiliary theorem.} \textit{Under the conditions mentioned above, let the inequality}

\begin{equation}
\tag{4.8}
\sum_{i=1}^{m} P_i(x) \partial_{i} \omega(x) \geq \frac{1}{\omega(x)} \sum_{i=1}^{m} Q_i(x, u) \cdot u_i
\end{equation}

\textit{hold for any} \( x \in X \) \textit{such that} \( |u| = \omega(x) \). \textit{Then, for any function} \( B(s) = (B_1(s), \cdots, B_l(s)) \) \textit{of class} \( C^1 \) \textit{on} \( S \) \textit{such that}

\begin{itemize}
  \item[1)] an \((m, m)\)-determinant.
\end{itemize}
there exists a unique solution $u(x)$ of (4.1) on $X$, such that

$$u(A(s)) = B(s)$$

and

$$|u(x)| \leq o(x)$$

for $x \in X$.

§ 4. Estimation of $u(x)$.

5. Consider the system of partial differential equations

$$(5.1) \quad \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_{j} + f_{i}(x) \right) \frac{\partial u_{\nu}}{\partial x_{\nu}} = \sum_{\mu=1}^{m} a_{\nu\mu} u_{\mu} + g_{\nu}(x) \quad (\nu = 1, \ldots, m)$$

for which we have the following lemma.

**Lemma.** Let $A = (a_{ij})$ and $f(x)$ satisfy the assumptions (i), (ii) and (iii) in the theorem. Let $g_{\nu}(x) \ (\nu = 1, \ldots, m)$ be real-valued functions of class $C^1$ on some neighborhood of $0$, such that

$$(5.2) \quad \begin{cases} g(0) = 0 \\ |\partial_{x}g(x)| \leq G |x|^p \end{cases}$$

where $G$ and $p$ are positive constants.

Then there exists a constant $h_{0} > 0$, which depends only on $\lambda_{i} \ (i = 1, \ldots, m)$, with the following property: if $p > h_{0}$, the system (5.1) has a unique solution $u(x)$ in a neighborhood of $0$, such that

$$(5.3) \quad u(x) = 0 \quad \text{on the cone} \quad \sum_{i=1}^{k} x_{i}^2 = \sum_{i=k+1}^{m} x_{i}^2 > 0$$

and

$$(5.4) \quad |\partial_{x}u(x)| \leq C \cdot G |x|^p$$

where $C$ is a positive constant depending only on $\lambda_{i} \ (i = 1, \ldots, m)$ and $p$.

6. **Proof.** By setting $P_{i}(x) = \sum_{j=1}^{n} a_{ij} x_{j} + f_{i}(x)$ and $Q_{i}(x, u) = \sum_{\mu=1}^{m} a_{\nu\mu} u_{\mu} + g_{\nu}(x)$, the system (5.1) has the form (4.1). Without loss of generality we assume that $A = (a_{ij})$ has the following form:

$$(i) \quad a_{ii} = \Re(\lambda_{i}) \quad (i = 1, \ldots, m),$$

$$\Re(\lambda_{i}) > 0 \quad \text{for} \ i \leq k,$$

$$\Re(\lambda_{i}) < 0 \quad \text{for} \ i > k.$$
(ii) \[ a_{ij} = 0 \quad \text{for } i \leq k \text{ and } j > k, \]
\[ a_{ij} = 0 \quad \text{for } i > k \text{ and } j \leq k. \]

(6.1) \[ |\sum_{i \neq j} a_{ij} x_i x_j| \leq \delta |x|^2 \]
where \( \delta \) is any prescribed positive number.

In what follows, we write \( \sum = \sum_a \) and \( \sum = \sum_{\beta = k+1}^{\alpha} \). We suppose that \( f \) and \( g \) are functions of class \( C^1 \) on \( U = \{ x : \sum_a x_a^2 \leq r^2, \sum_{\beta} x_{\beta}^2 \leq r^2 \} \) where \( r \) is a positive constant. We consider the case \( 0 < k < m \). Because, if \( k = 0 \) or \( k = m \), the proof of the lemma will be simpler.

With sufficiently small \( \varepsilon > 0 \) we set\(^3\)

(6.2) \[ S_\varepsilon(x) = \begin{cases} \sum_a x_a^2 - \sum_{\beta} x_{\beta}^2 & \text{when } \sum_a x_a^2 \geq \varepsilon^2 \\ \sum_a x_a^2 - \sum_{\beta} x_{\beta}^2 - \frac{1}{2\varepsilon^2} (\varepsilon^2 - \sum_a x_a^2)^2 & \text{when } \sum_a x_a^2 < \varepsilon^2, \end{cases} \]
and define a bounded region \( U_\varepsilon \) by \( U_\varepsilon = \{ x \in U : S_\varepsilon(x) \geq 0 \} \).

![Diagram of a bounded region](image)

For the case \( k = 0 \) or \( k = m \), we have to set \( S_\varepsilon(x) = \sum x_i^2 \).

First we consider the solution of (5.1) in \( U_\varepsilon \) vanishing on the \( (m-1) \)-dimensional manifold \( M_\varepsilon = \{ x \in U : S_\varepsilon(x) = 0 \} \). For the base characteristic \( x = x(t) \) of (5.1), we have

\[
\frac{1}{2} \frac{d}{dt} S_\varepsilon(x(t)) = \sum_a x_a \dot{x}_a - \sum_{\beta} x_{\beta} \dot{x}_{\beta} = \sum_a \left( \sum_{i=1}^m a_{ai} x_i + f_a(x) \right) x_a - \sum_{\beta} \left( \sum_{i=1}^m a_{\beta i} x_i + f_{\beta}(x) \right) x_{\beta} = \left( \sum_a \sum_{i=1}^m a_{ai} x_a x_i - \sum_{\beta} \sum_{i=1}^m a_{\beta i} x_i x_{\beta} \right) + \left( \sum_a f_a(x) x_a - \sum_{\beta} f_{\beta}(x) x_{\beta} \right) > 0
\]
when $E^2 < \sum_{a} x_a^2 \leq r^2$, by taking $r$ small enough, and also when
\[ \sum_{a} x_a^2 \leq E^2 \]
\[ \frac{1}{2} \cdot \frac{d}{dt} S_t(x(t)) = \left\{ 1 + \frac{1}{E^2} (E^2 - \sum_{a} x_a^2) \right\} \sum_{a} x_a \cdot \dot{x}_a - \sum_{\beta} x_{\beta} \cdot \dot{x}_{\beta} > 0. \]

From these inequalities we see that, if $r > 0$ is taken small enough, every base characteristic of (5.1) meeting $M_{\varepsilon}$ is transverse to $M_{\varepsilon}$, and that (4.5) will hold for this case with $M = M_{\varepsilon}$. In addition, since we have
\[ \frac{1}{2} \frac{d}{dt} \sum_{a} (x_a(t))^2 = \sum_{a} \left( \sum_{i} a_{ai} x_i + f_a(x) \right) x_a > 0 \]
for any base characteristic $x(t)$, when $r$ is small enough, we see that $U_{\varepsilon}$ is filled up only onefold with the base characteristics issuing from $M_{\varepsilon}$. Therefore, we apply the auxiliary theorem to this case, setting $U_{\varepsilon} = X$

We set
\[ \varphi(x) = (1 + \gamma) \sum_{a} x_a^2 - \sum_{\beta} x_{\beta}^2 \]
and
\[ \omega(x) = W \cdot G \cdot \varphi(x)^{\frac{k+1}{2}} \]
where $\gamma > 0$ and $W > 0$ will be determined later. Then
\[ \frac{\gamma}{2} \left| x \right|^2 \leq \varphi(x) \leq (1 + \gamma) \left| x \right|^2 \]
for every $x \in U_{\varepsilon}$. Thus we obtain
\[ \sum_{t=1}^{m} P_t(x) \partial_{t, \omega}(x) = (p+1) \cdot W \cdot G \cdot \varphi(x)^{\frac{k-1}{2}} \left\{ (1 + \gamma) \sum_{a} \sum_{j} a_{aj} x_a x_j - \sum_{\beta} \sum_{j} a_{\beta j} x_{\beta} x_j + (1 + \gamma) \sum_{a} x_a f_a(x) - \sum_{\beta} x_{\beta} f_{\beta}(x) \right\} \]
and
\[ \sum_{t=1}^{m} Q_t(x, u) \cdot u_{\gamma} = \sum_{t=1}^{m} \sum_{\mu=1}^{m} a_{\mu t} u_{\mu} u_{\gamma} \sum_{t=1}^{m} g_{\gamma}(x) u_{\gamma}. \]

Now we set
\[ \Lambda_0 = \min_{1 \leq i \leq m} |\Re(\lambda_i)|, \quad \Lambda_1 = \max_{1 \leq i \leq m} |\Re(\lambda_i)|. \]

Then we have
\[ \sum_{t=1}^{m} P_t(x) \cdot \partial_{t, \omega}(x) > (p+1)(\Lambda_0 - 2\delta) W \cdot G \left| x \right|^2 \varphi(x)^{\frac{k-1}{2}} \]
and

\[(6.8) \quad \sum_{k=1}^m Q_k(x, u) \cdot u_k \leq (\Lambda_1 + \delta) |u|^2 + |g(x)| \cdot |u| \]

on \(U_\varepsilon\) where \(\delta\) is given by (6.1), taking \(r\) small enough. From (6.8) it follows that, if \(|u| = \omega(x)\) for some \(x \in U_\varepsilon\), then

\[\frac{1}{\omega(x)} \sum_{k=1}^m Q_k(x, u) u_k \leq (\Lambda_1 + \delta) \omega(x) + G |x|^{p+1} \]

and so

\[(6.9) \quad \frac{1}{\omega(x)} \sum_{k=1}^m Q_k(x, u) \leq W \cdot G (\Lambda_1 + \delta) \left[1 + \gamma + \left(\frac{2}{\Lambda_1 \cdot W}\right)^\frac{p+1}{2}\right] |x|^p \varphi(x)^\frac{p+1}{2} \]

by virtue of (6.4) and (6.5). Thus, if we assume

\[(6.10) \quad (p+1)\Lambda_1 > \Lambda_1, \]

we have

\[(6.11) \quad \sum_{k=1}^m P_k(x) \partial_t \omega(x) \geq \frac{1}{\omega(x)} \sum_{k=1}^m Q_k(x, u) \cdot u_k \]

for any \(x \in U_\varepsilon\) such that \(|u| = \omega(x)\), from (6.7) and (6.9), by taking \(\delta > 0\) and \(\gamma > 0\) small enough and then \(W\) large enough.

Let us assume hereafter that (6.10) holds. Then it follows from the auxiliary theorem that, when \(r\) is small enough, there exists a unique solution \(u(x; \varepsilon)\) of (5.1) on \(U_\varepsilon\) which vanishes on \(M_\varepsilon\), and that it satisfies

\[(6.12) \quad |u(x; \varepsilon)| \leq \omega(x) = W \cdot G \cdot \varphi(x)^\frac{p+1}{2} \leq G \cdot K \cdot |x|^{p+1} \quad \text{for} \quad x \in U_\varepsilon \]

where \(K = W(1+\gamma)^\frac{p+1}{2}\). Notice that \(W\) and \(r\) are taken independent of \(\varepsilon > 0\) in the above argument.

§ 5. Continuation of the Proof of the Lemma.

**Estimation of \(|\partial_x u(x)|\).**

7. Next, let us prove that the inequality (5.4) holds for \(u = u(x; \varepsilon)\) on \(U_\varepsilon\) with some constant \(C > 0\) independent of \(\varepsilon\). In this paragraph we fix \(\varepsilon > 0\) and write \(u_\varepsilon(x)\) in place of \(u(x; \varepsilon)\) for simplicity.

In order to estimate \(|\partial_x u(x)|\) on the manifold \(M_\varepsilon\), we reduce the system (5.1) into

\[(7.1) \quad \frac{du_\varepsilon}{dt} = \sum_{\nu=1}^m a_\nu u_\mu + g_\nu(x(t, s)) \quad (\nu = 1, \ldots, m) \]
by the change of variables given by (4.6). Thus, if we set \( u(t, s) = u(x(t, s)) \), \( u = u(t, s) \) is the solution of (7.1) with the initial condition
\[
(7.2) \quad u(0, s) = 0.
\]
Therefore,
\[
(7.3) \quad \begin{cases} 
\partial_t u(0, s) = g(x(0, s)) \\
\partial_s u(0, s) = 0
\end{cases}
\]
on \( M_\varepsilon \). Let \( \partial_s u(s) \) denote the normal derivative of \( u(t, s) \) at \( x = x(0, s) \) on \( M_\varepsilon \), then we have
\[
(7.4) \quad \frac{\partial_t u(t, s)}{\left\{ \sum_{i=1}^{m} (\partial_i x_i(t, s))^2 \right\}^{\frac{1}{2}}} = \partial_s u(s) \cdot \cos \theta(s)
\]
where \( \theta(s) \) represents the angle between the base characteristic \( x = x(t, s) \) and the normal of \( M_\varepsilon \) at \( x = x(0, s) \), i.e.
\[
\cos \theta(s) = \frac{(1 + \sigma(x)) \cdot \sum_{a} x_a \cdot \partial_a x_a - \sum_{\beta} x_{\beta} \cdot \partial_\beta x_{\beta}}{\left\{ \sum_{i=1}^{m} (\partial_i x_i(t, s))^2 \right\}^{\frac{1}{2}} (1 + \sigma(x))^2 \sum_{a} x_a^2 + \sum_{\beta} x_{\beta}^2}^{\frac{1}{2}}
\]
where
\[
\sigma(x) = \begin{cases} 
0 & \text{when } \sum_{a} x_a^2 \geq \varepsilon^2 \\
\frac{1}{\varepsilon^2} (\varepsilon^2 - \sum_{a} x_a^2) & \text{when } \sum_{a} x_a^2 < \varepsilon^2.
\end{cases}
\]
Since
\[
(7.5) \quad (1 + \sigma(x)) \cdot \sum_{a} x_a \cdot \partial_a x_a - \sum_{\beta} x_{\beta} \cdot \partial_\beta x_{\beta} \geq \frac{1}{2} \Lambda_0 |x|^2 \quad (x \in U_\varepsilon)
\]
and
\[
(7.6) \quad (1 + \sigma(x))^2 \cdot \sum_{a} x_a^2 + \sum_{\beta} x_{\beta}^2 < 2 |x|^2,
\]
when \( r \) is sufficiently small, we obtain, from (7.3), (7.4), (7.5), (7.6) and (7.7),
\[
|\partial_s u| \leq \frac{\sqrt{2}}{\Lambda_0} \cdot \left\{ \frac{g(x)}{|x|} \right\}.
\]
Hence
\[
(7.8) \quad |\partial_s u| \leq K' \cdot G \cdot |x|^{\beta}
\]
on \( M_\varepsilon \) with some constant \( K' > 0 \) independent of \( \varepsilon \). Thus, from (7.3) and (7.8), we see
\[
(7.9) \quad |\partial_s u(x; \varepsilon)| \leq K' \cdot G \cdot |x|^{\beta} \quad (x \in M_\varepsilon).
\]
Now, operating $\partial_\mu$ on both sides of (5.1) and setting $\partial_\mu u_\nu = u^{\mu \nu}$, we have

\begin{equation}
\sum_{i=1}^{m} \left( \sum_{j=1}^{m} a_{ij} x_j + f_i(x) \right) \partial_\mu u^{\mu \nu} = \sum_{i=1}^{m} a_{ij} u^{\mu \nu} - \sum_{i=1}^{m} \left( a_{i\mu} + \partial_\mu f_i(x) \right) u^{\nu i} + \partial_\mu g_\nu(x) \quad (\nu, \mu = 1, \ldots, m)
\end{equation}

which has also the form (4.1) with unknown functions $u^{\mu \nu}$. Let us assume first that $f$ and $g$ are functions of class $C^2$ in $U$ and apply the auxiliary theorem to (7.10). We set

$$P_i(x) = \sum_{j=1}^{m} a_{ij} x_j + f_i(x)$$

$$Q^{\mu \nu}(x, u^\prime) = \sum_{i=1}^{m} a_{ij} u^{\mu \nu} - \sum_{i=1}^{m} \left( a_{i\mu} + \partial_\mu f_i(x) \right) u^{\nu i} + \partial_\mu g_\nu(x) \quad (\nu, \mu = 1, \ldots, m)$$

and $\omega'(x) = W' \cdot G \cdot \varphi'(x)^{\frac{r}{2}}$ where $\varphi'(x) = (1 + \gamma') \sum x_\nu^2 - \sum x_\mu^2$.

Then, if we assume

\begin{equation}
p_A > A \equiv \max \left| \Re(\lambda_\nu) - \Re(\lambda_j) \right|
\end{equation}

and if we take $r$ small enough, we have

$$\sum_{i=1}^{m} P_i(x) \partial_\mu \omega'(x) > \frac{1}{\omega'(x)} \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} Q^{\mu \nu}(x, u^\prime) \cdot u^{\mu \nu}$$

for $x$ such that $\omega'(x) = |u'|$ in $U_*$, taking $W'$ and $\gamma' > 0$ appropriately. By the auxiliary theorem and (7.9) we thus get

\begin{equation}
|\partial_\mu u(x; \varepsilon)| \leq C \cdot G \cdot |x|^p
\end{equation}

in $U_\varepsilon$ where $C > 0$ is a constant independent of $\varepsilon$. In the above consideration we can also choose $r$ independent of $\varepsilon$. Let us write

$$h_0 = \max \left( \frac{A_1}{A_0} - 1, \Lambda \right)$$

and assume $p > h_0$ hereafter, from which (6.10) and (7.11) follow.

We will study in 9. as to the case that $f$ and $g$ are functions of class $C'$.

8. Notice that $C$ of (7.12) and $r$ can be chosen independent of $\varepsilon > 0$ which is sufficiently small. Now we consider $\varepsilon$ as a variable tending to zero. We see easily $U_\varepsilon \subset U_*$ as $\varepsilon > \varepsilon' > 0$, and $v = u(x; \varepsilon') - u(x; \varepsilon)$ must satisfy on $U_\varepsilon$

\begin{equation}
\sum_{i=1}^{m} \left( \sum_{j=1}^{m} a_{ij} x_j + f_i(x) \right) \partial_\mu v_\nu = \sum_{\mu=1}^{m} a_{\mu \nu} v_\mu \quad (\nu = 1, \ldots, m)
\end{equation}
From (6.12) and (7.12) we see easily that

\[ |v| \leq 2^{\frac{b+3}{2}} K \cdot G \cdot \varepsilon^{b+1} \]

and

\[ |\partial_x v| \leq 2^{\frac{b+2}{2}} C \cdot G \cdot \varepsilon^b \]

hold for \( v = u(x; \varepsilon') - u(x; \varepsilon) \) on \( M_\varepsilon \). Notice that \( \min_{x \in U_\varepsilon} |x| = \frac{\sqrt{6} - \sqrt{2}}{2} \varepsilon \), and we have for any \( q > 0 \)

\begin{align*}
\{ & |v| \leq K_0 \cdot \varepsilon^{b-q} |x|^{q+1} \\
& |\partial_x v| \leq K_1 \cdot \varepsilon^{b-q} |x|^q \}
\end{align*}

on \( M_\varepsilon \) where \( K_0 \) and \( K_1 \) are constants not depending on \( \varepsilon \) and \( \varepsilon' \).

We now choose \( q \) so that \( p > q > h_0 \). The system (8.1) is a special case of (5.1) with \( g(x) = 0 \), and we get similarly as (6.12) and (7.12)

\[ |u(x; \varepsilon') - u(x; \varepsilon)| \leq K' \cdot \varepsilon^{b-q} |x|^{q+1} \]

\[ |\partial_x u(x; \varepsilon') - \partial_x u(x; \varepsilon)| \leq K' \cdot \varepsilon^{b-q} |x|^q \]

in \( U_\varepsilon \) where \( K' \) is some positive constant independent of \( \varepsilon \) and \( \varepsilon' \). Thus we see that, as \( \varepsilon \to 0 \), \( u_\varepsilon(x; \varepsilon) \) and \( \partial_x u_\varepsilon(x; \varepsilon) \) tend to certain functions \( u_\varepsilon(x) \) and their derivatives \( \partial_x u_\varepsilon(x) \) respectively. Clearly this \( u(x) \) is a solution of (5.1), vanishing on the manifold: \( \sum_{a} x_{a}^2 = \sum_{b} x_{b}^2 \) and satisfying

(5.4) in \( U_1 = \{ x : \sum_{a} x_{a}^2 \leq r^2, \sum_{b} x_{b}^2 \leq \sum_{a} x_{a}^2 \} \).

Quite similarly as above, we can prove the existence of a solution \( u(x) \) of (5.1) vanishing on \( \sum_{a} x_{a}^2 = \sum_{b} x_{b}^2 \) and satisfying (5.4) in \( U_2 = \{ x : \sum_{b} x_{b}^2 \leq r^2, \sum_{a} x_{a}^2 \leq \sum_{b} x_{b}^2 \} \).

9. Now it remains to prove our lemma when \( f \) and \( g \) are functions of class \( C^1 \). We construct approximation sequences \( \{ f_n(x) \}_{n=1}^{\infty} \) and \( \{ g_n(x) \}_{n=1}^{\infty} \) of vector functions of class \( C^2 \) such that

\begin{align*}
\{ & f^n(0) = g^n(0) = 0 \\
& f^n(x) = \tilde{f}(x) + q^n(x) \}
\end{align*}

(9.1)

\[ |\partial_x f^n(x) - \partial_x f(x)| \leq \frac{1}{n} |x|^h \]

\[ |\partial_x g^n(x) - \partial_x g(x)| \leq \frac{1}{n} |x|^h \].

4) \( f^n(x) \) and \( g^n(x) \) have only to be of class \( C^2 \) in \( U \) excepting \( x = 0 \).
5) c.f. (2.1).
Then there exists a system \( u^n(x) = (u^n_1(x), \ldots, u^n_m(x)) \) of functions of class \( C^2 \) satisfying

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_j + f^n_i(x) \nabla_i u^n = \sum_{\mu=1}^{n} a_{\mu n} u^n_\mu + g^n_i(x) \quad (\nu = 1, \ldots, m),
\]

such that

\[
\begin{align*}
|u^n(x)| & \leq C \cdot K_3 |x|^{p-1} \\
|\nabla u^n(x)| & \leq C \cdot K_3 |x|^p
\end{align*}
\]
in \( U \) where \( K_2 \) and \( K_3 \) are constants not depending on \( n \). For \( u^n(x) - u''^n(x) \) \((n \leq n')\) we have

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_j + f^n_i(x) \nabla_i (u^n_\nu - u''^n_\nu) = \sum_{\mu=1}^{n} a_{\mu n} (u^n_\mu - u''^n_\mu) + h^n_{\nu,\nu'}(x),
\]

where

\[
h^n_{\nu,\nu'}(x) = \{g^n_\nu(x) - g''^n_\nu(x)\} + \nabla u^n_\nu \{f^n_\nu(x) - f''^n_\nu(x)\}
\]

and so

\[
h^n_{\nu,\nu'}(0) = 0 \quad \text{and} \quad |\nabla h^n_{\nu,\nu'}(x)| \leq \frac{H}{n} |x|^p
\]

with a constant \( H > 0 \) not depending on \( n \) and \( n' \). Therefore we see that, as \( n \to \infty \), \( u^n(x) \) tend to the desired solution of (5.1). The proof of the lemma is thus completed.

§ 6. Proof of the Main Theorem.

10. Let us now turn to the system (3.1),

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_j + f_i(x) \nabla_i u^n = \sum_{\mu=1}^{n} a_{\mu n} u^n_\mu - f_\nu(x) \quad (\nu = 1, \ldots, m)
\]

where \( f_i(x) = p_i(x) + q_i(x) \). First, let us consider

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_j + p_i(x) \nabla_i u^n = \sum_{\mu=1}^{n} a_{\mu n} u^n_\mu - p_\nu(x) \quad (\nu = 1, \ldots, m).
\]

If there exist no relations of the form \( \lambda_i = \sum n_j \lambda_j \) where \( n_j \) are non-negative integers such that \( \sum n_j > 1 \), we can construct infinite series of the form

\[
\sum_{p_1 + \cdots + p_m \geq 2} c^\nu_{p_1, \ldots, p_m} \cdot x_{p_1}^{p_1} \cdots x_{p_m}^{p_m}
\]

with real coefficients \( c^\nu_{p_1, \ldots, p_m} \), such that \( u_\nu = \sum c^\nu_{p_1, \ldots, p_m} \cdot x_{p_1}^{p_1} \cdots x_{p_m}^{p_m} \) \((\nu = 1, \ldots, m)\) satisfy (10.1) formally. To see this, make a change of variables, \( x = T \cdot y \) and \( u = T \cdot w \), by a (complex) matrix \( T \) transforming \( A = (a_{ij}) \) into Jordan's canonical form \( T^{-1} \cdot A \cdot T \), and consider about the (complex) system thus obtained.
Setting

\begin{equation}
\dot{u}_\nu(x) = \sum_{2 \leq p_1 + \cdots + p_m \leq h} c_{p_1}^{\nu} \cdots c_{p_m}^{\nu} x_1^{p_1} \cdots x_m^{p_m} \quad (\nu = 1, \ldots, m)
\end{equation}

and

\begin{equation}
\ddot{u}_\nu = u_\nu - \dot{u}_\nu(x) \quad (\nu = 1, \ldots, m),
\end{equation}

we reduce (3.1) to the system

\begin{equation}
\sum_{i=1}^m \left( \sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_{\nu} \dot{u}_\nu = \sum_{\mu=1}^n \alpha_{\nu\mu} u_\mu + f_{\nu}(x) \quad (\nu = 1, \ldots, m)
\end{equation}

where

\[ f_{\nu}(x) = \sum_{\mu=1}^n a_{\nu\mu} \dot{u}_\mu(x) - \sum_{i=1}^m \left( \sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_{\nu} \dot{u}_\nu(x) - f_\nu(x). \]

In order to define \( \dot{u}(x) \) as above, we have only to assume that condition (iii) in the theorem (§ 2) is satisfied.

By (2.1) we have

\begin{equation}
\tilde{f}_\nu(x) = \tilde{p}_\nu(x) - q_\nu(x)
\end{equation}

where

\[ \tilde{p}_\nu(x) = \sum_{\mu=1}^n a_{\nu\mu} \dot{u}_\mu(x) - p_\nu(x) - \sum_{i=1}^m \left( \sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_{\nu} \dot{u}_\nu(x) \]

and

\[ q_\nu(x) = q_\nu(x) + \sum_{i=1}^m q_i(x) \partial_{\nu} \dot{u}_\nu(x). \]

and it follows from the definition of \( \dot{u}(x) \) that \( \tilde{p}_\nu(x) \), polynomials in \( x_i \) \((i = 1, \ldots, m)\), contain no terms of degree \( \leq h \). Therefore, by (2.2)

\[ \begin{cases} 
\tilde{f}_\nu(0) = 0 & (\nu = 1, \ldots, m), \\
|\partial_x \tilde{f}(x)| \leq \tilde{K} |x|^h
\end{cases} \]

where \( \tilde{K} \) is a positive constant. Using the lemma, we thus see that there exists a solution \( \check{u}(x) = (\check{u}_1(x), \ldots, \check{u}_m(x)) \) of (10.4) in some neighborhood of \( x = 0 \), such that

\[ \check{u}_i(0) = 0 \quad (i = 1, \ldots, m) \]

and

\[ |\partial_x \check{u}(x)| \leq C \cdot \tilde{K} |x|^h \]

with some constant \( C > 0 \). We set

\[ u_\nu(x) = \check{u}_\nu(x) + \dot{u}_\nu(x) \quad (\nu = 1, \ldots, m) \]

and obtain those functions \( u_\nu(x) \) \((\nu = 1, \ldots, m)\) whose existence was claimed.
in the theorem. The proof of the theorem is thus completed.

(Received November 12, 1957)

Literature