Nagumo, Mitio and
Isé, Kusuo
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On the Normal Forms of Differential Equations in the Neighborhood of an Equilibrium Point

By Mitio NAGUMO and Kusuo ISÉ

§ 1. Introduction.

1. In this note we use the notations $\partial_i u$ and $\partial_{ij}^2 u$ for $\dfrac{\partial}{\partial x_i} u$ and $\dfrac{\partial^2}{\partial x_i \partial x_j} u$ respectively. The vectors $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_m)$ in R^m will be denoted briefly by x and y respectively.

Let $A=(a_{ij})$ be a constant real (m, m)-matrix, all of whose characteristic roots λ_i $(i=1, \cdots, m)$ have non-zero real parts, and $f(x)=(f_1(x), \cdots, f_m(x))$ a real vector function of class C^1 on some neighborhood of $x=0$, such that $f(0)=0$ and $|\partial_x f(x)| \leqq K \cdot |x|$ with a constant $K>0$ where

$$|x|=(\sum_i x_i^2)^{\frac{1}{2}}, \quad |\partial_x f(x)|=\{\sum_{i,j} (\partial_i f_j(x))^2\}^{\frac{1}{2}}.$$

We consider the autonomous systems

$$\frac{dx}{dt}=A \cdot x+f(x) \tag{1.1}$$

and

$$\frac{dy}{dt}=A \cdot y, \tag{1.2}$$

regarding x, y and $f(x)$ as the column-vectors. The purpose of this note is to show that, under some conditions on λ_i $(i=1, \cdots, m)$ and $f(x)$, the system (1.1) can be transformed into (1.2) by a change of variables

$$y=x+u(x) \tag{1.3}$$

where $u(x)=(u_1(x), \cdots, u_m(x))$ is a real vector function of class C^1, such that

$$\begin{cases} u(0)=0 \\ |\partial_x u(x)| \leqq L \cdot |x| \end{cases} \tag{1.4}$$

with some constant $L>0$.

When $f(x)$ is analytic regular in x, in order to show the existence of the transformation given by (1.3) with analytic regular $u(x)$, we must necessarily assume that there exist no relations of the form

$$\lambda_i = \sum_{j=1}^{m} n_j \cdot \lambda_j \tag{1.5}$$

where n_j $(j=1, \cdots, m)$ are non-negative integers such that $\sum_{j=1}^{m} n_j > 1$. As to this case, some results were obtained by H. Poincaré, C. L. Siegel, and others, while we obtain the present result for the real systems with a transformation of class C^1 under some weaker conditions.

§ 2. Main Theorem.

2. Theorem. *Assumptions:*

(i) *A is a constant real (m, m)-matrix, all of whose characteristic roots λ_i $(i=1, \cdots, m)$ have non-zero real parts:* $\Re(\lambda_i) \neq 0$ $(i=1, \cdots, m)$.

(ii) *Let*

$$f_i(x) = p_i(x) + q_i(x) \qquad (i = 1, \cdots, m) \tag{2.1}$$

where $p_i(x)$ are polynomials in x with real coefficients such that $p_i(0) = \partial_j p_i(0) = 0$ $(i=1, \cdots, m;\ j=1, \cdots, m)$, *and $q_i(x)$ $(i=1, \cdots, m)$ are real-valued functions of class C^1 satisfying*

$$\begin{cases} q(0) = 0 \\ |\partial_x q(x)| \leqq Q \cdot |x|^h \end{cases} \tag{2.2}$$

with some integer $h>0$ and some constant $Q>0$.

(iii) *There exist no relations of the form*

$$\lambda_i = \sum_{j=1}^{m} n_j \cdot \lambda_j$$

where n_j $(j=1, \cdots, m)$ are non-negative integers such that

$$h > \sum_{j=1}^{m} n_j > 1 .$$

Conclusion: There exists a positive constant h_0, depending only on λ_i $(i=1, \cdots, m)$, with the following property: if $h>h_0$, there exist functions $u_i(x)$ $(i=1, \cdots, m)$ of class C^1 satisfying (1.4), *such that the system* (1.1) *is reduced to the form* (1.2) *by the substitution* (1.3).

3. If (1.1) is transformed into (1.2) by (1.3), $u(x)$ must satisfy the system of partial differential equations

(3.1) $$\sum_{i=1}^{m}\left(\sum_{j=1}^{m} a_{ij}x_j+f_i(x)\right)\cdot\partial_i u_\nu=\sum_{\mu=1}^{m} a_{\nu\mu}u_\mu-f_\nu(x) \qquad (\nu=1,\cdots,m).$$

For we have, by operating $\frac{d}{dt}$ on both sides of (1.3),

$$\frac{dy_i}{dt}=\frac{dx_i}{dt}+\sum_{\nu=1}^{m}\partial_\nu u_i\cdot\frac{dx_\nu}{dt} \qquad (i=1,\cdots,m)$$

from which (3.1) follows immediately by (1.1), (1.2) and (1.3). Conversely, if $u(x)$ is any function satisfying (3.1), then the substitution (1.3) will transform (1.1) into (1.2). Thus we have only to show the existence of $u(x)$ satisfying (1.4) and (3.1), if h is sufficiently large.

§ 3. Auxiliary Theorem.

4. In this section we consider the system of semi-linear partial differential equations

(4.1) $$\sum_{i=1}^{m} P_i(x)\cdot\partial_i u_\nu=Q_\nu(x,u) \qquad (\nu=1,\cdots,l)$$

where $x=(x_1,\cdots,x_m)$ and $u=(u_1,\cdots,u_l)$ denote real vectors in R^m and R^l respectively. Let $P_i(x)$ be real-valued functions of class C^1 in an open domain $D\subset R^m$, such that

(4.2) $$(P_1(x),\cdots,P_m(x))\neq(0,\cdots,0) \qquad (x\in D).$$

And $Q_\nu(x,u)$ be real-valued functions of class C^1 in

$$\Omega=\{(x,u)\in R^{m+l}:\ x\in D,\ |u|\leqq\omega(x)\}$$

where $\omega(x)$ is some positive-valued function of class C^1 in D. A curve $x=x(t)$ in R^m is said to be a *base characteristic* of (4.1) if $x(t)$ satisfies the following system of ordinary differential equations:

(4.3) $$\frac{dx_i}{dt}=P_i(x) \qquad (i=1,\cdots,m).$$

Let an $(m-1)$-dimensional manifold M in R^m be given by

(4.4) $$M:\ x_i=A_i(s_1,\cdots,s_{m-1}) \qquad (i=1,\cdots,m)$$

where $A_i(s)$ are functions of class C^1 in some domain $S\subset R^{m-1}$ such that $A(s)=(A_1(s),\cdots,A_m(s))\in D$ for $s=(s_1,\cdots,s_{m-1})\in S$. We assume that

$$(4.5)\qquad \left| P_i(A(s))\,,\ \partial_j A_i(s) \quad \begin{matrix} i\downarrow 1,\cdots,m \\ j\to 1,\cdots,m-1 \end{matrix} \right|^{1)} \neq 0 \quad \text{for} \quad s\in S$$

and that any base characteristic

$$(4.6)\qquad x=x(t,\ s)\,,$$

issuing from a point of M so that $x(0, s)=A(s)$, exists on the interval: $0\leqq t<\tau(s)$ where $\tau(s)$ is a continuous function on S, and that the set $X=\{x=x(t,\ s): 0\leqq t<\tau(s),\ s\in S\}$ is filled up only onefold with all those curves $x=x(t,\ s)(s\in S)$, i.e. to any point $x\in X$ there corresponds just one $(t,\ s)$ such that $x=x(t,\ s)$, $0\leqq t<\tau(s)$, $s\in S$. Then we have easily

$$\frac{\partial(x_1,\cdots,x_m)}{\partial(t,\ s_1,\cdots,s_{m-1})} = \left| P_i(A(s))\,,\ \partial_j A_i(s) \quad \begin{matrix} i\downarrow 1,\cdots,m \\ j\to 1,\cdots,m-1 \end{matrix} \right| \cdot \exp\left(\int_0^t \sum_{i=1}^m \partial_i P_i(x)_{x=x(t,\ s)}\,dt\right) \neq 0\,,$$

which shows that the $1-1$ mapping (4.6) from $\{(t,\ s): s\in S,\ 0\leqq t<\tau(s)\}$ onto X and its inverse are both of class C^1.

By (4.6) the system (4.1) is reduced to the following system of ordinary differential equations, s being a parameter:

$$(4.7)\qquad \frac{du_\nu}{dt}=Q_\nu(x(t,\ s),\ u) \qquad (\nu=1,\cdots,l)\,.$$

We have then

$$\partial_t\omega(x(t,\ s))=\Big[\sum_{i=1}^m P_i(x)\cdot\partial_i\omega(x)\Big]_{x=x(t,\ s)}$$

and

$$\partial_t|u(t,\ s)|\cdot|u(t,\ s)|=\sum_{\nu=1}^l u_\nu(t,\ s)\cdot Q_\nu(x(t,\ s),\ u(t,\ s))$$

for any solution $u(t,\ s)$ of (4.7). Hence, we obtain easily the following auxiliary theorem which is our principal tool.

Auxiliary theorem. *Under the conditions mentioned above, let the inequality*

$$(4.8)\qquad \sum_{i=1}^m P_i(x)\,\partial_i\omega(x)\geqq\frac{1}{\omega(x)}\sum_{\nu=1}^m Q_\nu(x,\ u)\cdot u_\nu$$

hold for any $x\in X$ *such that* $|u|=\omega(x)$. *Then, for any function* $B(s)=(B_1(s),\cdots,B_l(s))$ *of class* C^1 *on* S *such that*

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1) an (m, m)-determinant.

(4.9) $$|B(s)| \leqq \omega(A(s)),$$

there exists a unique solution $u(x)$ *of* (4.1) *on* X, *such that*

$$u(A(s)) = B(s)$$

and

(4.10) $$|u(x)| \leqq \omega(x)$$

for $x \in X$.

§ 4. Estimation of $u(x)$.

5. Consider the system of partial differential equations

(5.1) $$\sum_{i=1}^{m} \left(\sum_{j=1}^{m} a_{ij}x_j + f_i(x)\right) \partial_i u_\nu = \sum_{\mu=1}^{m} a_{\nu\mu} \cdot u_\mu + g_\nu(x) \qquad (\nu = 1, \cdots, m)$$

for which we have the following lemma.

Lemma. *Let* $A = (a_{ij})$ *and* $f(x)$ *satisfy the assumptions* (i), (ii) *and* (iii) *in the theorem. Let* $g_\nu(x)$ $(\nu = 1, \cdots, m)$ *be real-valued functions of class* C^1 *on some neighborhood of* 0, *such that*

(5.2) $$\begin{cases} g(0) = 0 \\ |\partial_x g(x)| \leqq G|x|^p \end{cases}$$

where G *and* p *are positive constants.*

Then there exists a constant $h_0 > 0$, *which depends only on* λ_i $(i = 1, \cdots, m)$, *with the following property*: *if* $p > h_0$, *the system* (5.1) *has a unique solution* $u(x)$ *in a neighborhood of* 0, *such that*

(5.3) $$u(x) = 0 \quad \textit{on the cone} \quad \sum_{i=1}^{k} x_i^2 = \sum_{i=k+1}^{m} x_i^2$$ [2)]

and

(5.4) $$|\partial_x u(x)| \leqq C \cdot G|x|^p$$

where C *is a positive constant depending only on* λ_i $(i = 1, \cdots, m)$ *and* p.

6. *Proof.* By setting $P_i(x) = \sum_{j=1}^{m} a_{ij}x_j + f_i(x)$ and $Q_\nu(x, u) = \sum_{\mu=1}^{m} a_{\nu\mu}u_\mu + g_\nu(x)$, the system (5.1) has the form (4.1). Without loss of generality we assume that $A = (a_{ij})$ has the following form:

(i) $$\begin{array}{ll} a_{ii} = \Re(\lambda_i) & (i = 1, \cdots, m), \\ \Re(\lambda_i) > 0 & \text{for } i \leqq k, \\ \Re(\lambda_i) < 0 & \text{for } i > k. \end{array}$$

2) Cf. (i) $k=0$ means $\Re(\lambda_i) < 0$ for all i, and $k=m$ means $\Re(\lambda_i) > 0$ for all i.

(ii) $a_{ij}=0$ for $i\leqq k$ and $j>k$,

$a_{ij}=0$ for $i>k$ and $j\leqq k$.

(iii)

$$|\sum_{i\neq j} a_{ij}x_i x_j|\leqq\delta|x|^2 \tag{6.1}$$

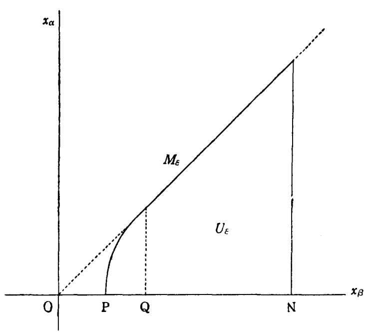
where δ is any prescribed positive number.

In what follows, we write $\sum_\alpha=\sum_{\alpha=1}^k$ and $\sum_\beta=\sum_{\beta=k+1}^m$. We suppose that f and g are functions of class C^1 on $U=\{x:\sum_\alpha x_\alpha^2\leqq r^2, \sum_\beta x_\beta^2\leqq r^2\}$ where r is a positive constant. We consider the case $0<k<m$. Because, if $k=0$ or $k=m$, the proof of the lemma will be simpler.

With sufficiently small $\varepsilon>0$ we set[3)]

$$S_\varepsilon(x)=\begin{cases}\sum_\alpha x_\alpha^2-\sum_\beta x_\beta^2 & \text{when } \sum_\alpha x_\alpha^2\geqq\varepsilon^2\\ \sum_\alpha x_\alpha^2-\sum_\beta x_\beta^2-\frac{1}{2\varepsilon^2}(\varepsilon^2-\sum_\alpha x_\alpha^2)^2 & \text{when } \sum_\alpha x_\alpha^2<\varepsilon^2,\end{cases} \tag{6.2}$$

and define a bounded region U_ε by $U_\varepsilon=\{x\in U: S_\varepsilon(x)\geqq 0\}$.



N $=(r, 0)$

P $=\left(\frac{\sqrt{6}-\sqrt{2}}{2}\varepsilon, 0\right)$

Q $=(\varepsilon, 0)$

First we consider the solution of (5.1) in U_ε vanishing on the $(m-1)$-dimensional manifold $M_\varepsilon=\{x\in U: S_\varepsilon(x)=0\}$. For the base characteristic $x=x(t)$ of (5.1), we have

$$\begin{aligned}\frac{1}{2}\cdot\frac{d}{dt}S_\varepsilon(x(t))&=\sum_\alpha x_\alpha\cdot\dot{x}_\alpha-\sum_\beta x_\beta\cdot\dot{x}_\beta\\ &=\sum_\alpha(\sum_{i=1}^m a_{\alpha i}x_i+f_\alpha(x))x_\alpha-\sum_\beta(\sum_{i=1}^m a_{\beta i}x_i+f_\beta(x))x_\beta\\ &=(\sum_\alpha\sum_{i=1}^m a_{\alpha i}x_\alpha x_i-\sum_\beta\sum_{i=1}^m a_{\beta i}x_\beta x_i)+(\sum_\alpha f_\alpha(x)x_\alpha-\sum_\beta f_\beta(x)x_\beta)>0\end{aligned}$$

3) For the case $k=0$ or $k=m$, we have to set $S_\varepsilon(x)=\sum_{i=1}^m x_i^2$.

when $\mathcal{E}^2 < \sum_\alpha x_\alpha^2 \leqq r^2$, by taking r small enough, and also when $\sum_\alpha x_\alpha^2 \leqq \mathcal{E}^2$

$$\frac{1}{2}\cdot\frac{d}{dt}S_\varepsilon(x(t)) = \left\{1+\frac{1}{\mathcal{E}^2}(\mathcal{E}^2-\sum_\alpha x_\alpha^2)\right\}\sum_\alpha x_\alpha\cdot\dot{x}_\alpha-\sum_\beta x_\beta\cdot\dot{x}_\beta > 0\,.$$

From these inequalities we see that, if $r>0$ is taken small enough, every base characteristic of (5.1) meeting M_ε is transverse to M_ε, and that (4.5) will hold for this case with $M=M_\varepsilon$. In addition, since we have

$$\frac{1}{2}\frac{d}{dt}\sum_\alpha (x_\alpha(t))^2 = \sum_\alpha(\sum_i a_{\alpha i}x_i+f_\alpha(x))x_\alpha > 0$$

for any base characteristic $x(t)$, when r is small enough, we see that U_ε is filled up only onefold with the base characteristics issuing from M_ε. Therefore, we apply the auxiliary theorem to this case, setting $U_\varepsilon = X$.

We set

$$\varphi(x) = (1+\gamma)\sum_\alpha x_\alpha^2 - \sum_\beta x_\beta^2 \tag{6.3}$$

and

$$\omega(x) = W\cdot G\cdot\varphi(x)^{\frac{p+1}{2}} \tag{6.4}$$

where $\gamma>0$ and $W>0$ will be determined later. Then

$$\frac{\gamma}{2}|x|^2 \leqq \varphi(x) \leqq (1+\gamma)|x|^2 \tag{6.5}$$

for every $x\in U_\varepsilon$. Thus we obtain

$$\begin{aligned}&\sum_{i=1}^m P_i(x)\partial_i\omega(x)\\ &= (p+1)\cdot W\cdot G\cdot\varphi(x)^{\frac{p-1}{2}}\{(1+\gamma)\sum_\alpha\sum_j a_{\alpha j}x_\alpha x_j-\sum_\beta\sum_j a_{\beta j}x_\beta x_j\\ &\qquad\qquad + (1+\gamma)\sum_\alpha x_\alpha f_\alpha(x)-\sum_\beta x_\beta f_\beta(x)\}\end{aligned}$$

and

$$\sum_{\nu=1}^m Q_\nu(x,u)\cdot u_\nu = \sum_{\nu=1}^m\sum_{\mu=1}^m a_{\nu\mu}u_\nu u_\mu - \sum_{\nu=1}^m g_\nu(x)u_\nu\,.$$

Now we set

$$\Lambda_0 = \min_{1\leqq i\leqq m}|\Re(\lambda_i)|\,,\quad \Lambda_1 = \max_{1\leqq i\leqq m}|\Re(\lambda_i)|\,. \tag{6.6}$$

Then we have

$$\sum_{i=1}^m P_i(x)\cdot\partial_i\omega(x) > (p+1)(\Lambda_0-2\delta)W\cdot G|x|^2\varphi(x)^{\frac{p-1}{2}} \tag{6.7}$$

and

(6. 8) $$\sum_{\nu=1}^{m} Q_\nu(x, u)\cdot u_\nu < (\Lambda_1+\delta)|u|^2+|g(x)|\cdot|u|$$

on U_ε where δ is given by (6. 1), taking r small enough. From (6. 8) it follows that, if $|u|=\omega(x)$ for some $x\in U_\varepsilon$, then

$$\frac{1}{\omega(x)}\sum_{\nu=1}^{m} Q_\nu(x, u)u_\nu < (\Lambda_1+\delta)\omega(x)+G|x|^{p+1}$$

and so

(6. 9) $$\frac{1}{\omega(x)}\sum_{\nu=1}^{m} Q_\nu(x, u)u_\nu < W\cdot G(\Lambda_1+\delta)\left\{1+\gamma+\frac{\left(\frac{2}{\gamma}\right)^{\frac{p-1}{2}}}{\Lambda_1\cdot W}\right\}|x|^2\varphi(x)^{\frac{p-1}{2}}$$

by virtue of (6. 4) and (6. 5). Thus, if we assume

(6. 10) $$(p+1)\Lambda_0>\Lambda_1,$$

we have

(6. 11) $$\sum_{i=1}^{m} P_i(x)\,\partial_i\omega(x)>\frac{1}{\omega(x)}\sum_{\nu=1}^{m} Q_\nu(x, u)\cdot u_\nu$$

for any $x\in U_\varepsilon$ such that $|u|=\omega(x)$, from (6. 7) and (6. 9), by taking $\delta>0$ and $\gamma>0$ small enough and then W large enough.

Let us assume hereafter that (6. 10) holds. Then it follows from the auxiliary theorem that, when r is small enough, there exists a unique solution $u(x;\varepsilon)$ of (5. 1) on U_ε which vanishes on M_ε, and that it satisfies

(6. 12) $$|u(x;\varepsilon)|\leqq\omega(x)=W\cdot G\cdot\varphi(x)^{\frac{p+1}{2}}\leqq G\cdot K\cdot|x|^{p+1}\quad\text{for}\quad x\in U_\varepsilon$$

where $K=W(1+\gamma)^{\frac{p+1}{2}}$. Notice that W and r are taken independent of $\varepsilon>0$ in the above arguement.

§ 5. Continuation of the Proof of the Lemma. Estimation of $|\partial_x u(x)|$.

7. Next, let us prove that the inequality (5. 4) holds for $u=u(x;\varepsilon)$ on U_ε with some constant $C>0$ independent of ε. In this paragraph we fix $\varepsilon>0$ and write $u_\nu(x)$ in place of $u_\nu(x;\varepsilon)$ for simplicity.

In order to estimate $|\partial_x u(x)|$ on the manifold M_ε, we reduce the system (5. 1) into

(7. 1) $$\frac{du_\nu}{dt}=\sum_{\mu=1}^{m} a_{\nu\mu}u_\mu+g_\nu(x(t, s))\qquad(\nu=1,\cdots,m)$$

by the change of variables given by (4.6). Thus, if we set $u_\nu(t, s) = u_\nu(x(t, s))$, $u=u(t, s)$ is the solution of (7.1) with the initial condition

$$u(0, s) \equiv 0 . \tag{7.2}$$

Therefore,

$$\begin{cases} \partial_t u(0, s) \equiv g(x(0, s)) \\ \partial_s u(0, s) \equiv 0 \end{cases} \tag{7.3}$$

on M_ε. Let $\partial_n u(s)$ denote the normal derivative of $u(t, s)$ at $x=x(0, s)$ on M_ε, then we have

$$\left[\frac{\partial_t u(t, s)}{\{\sum_{i=1}^{m} (\partial_t x_i(t, s))^2\}^{\frac{1}{2}}}\right]_{t=0} = \partial_n u(s) \cdot \cos \theta(s) \tag{7.4}$$

where $\theta(s)$ represents the angle between the base characteristic $x=x(t, s)$ and the normal of M_ε at $x(0, s)$, i.e.

$$\cos \theta(s) = \frac{(1+\sigma(x)) \cdot \sum_\alpha x_\alpha \cdot \partial_t x_\alpha - \sum_\beta x_\beta \cdot \partial_t x_\beta}{\{\sum_{i=1}^{m} (\partial_t x_i)^2\}^{\frac{1}{2}} \{(1+\sigma(x))^2 \sum_\alpha x_\alpha^2 + \sum_\beta x_\beta^2\}^{\frac{1}{2}}} \tag{7.5}$$

where

$$\sigma(x) = \begin{cases} 0 & \text{when } \sum_\alpha x_\alpha^2 \geqq \varepsilon^2 \\ \frac{1}{\varepsilon^2} (\varepsilon^2 - \sum_\alpha x_\alpha^2) & \text{when } \sum_\alpha x_\alpha^2 < \varepsilon^2 . \end{cases}$$

Since

$$(1+\sigma(x)) \cdot \sum_\alpha x_\alpha \cdot \partial_t x_\alpha - \sum_\beta x_\beta \cdot \partial_t x_\beta > \frac{1}{2} \Lambda_0 |x|^2 \qquad (x \in U_\varepsilon) \tag{7.6}$$

and

$$(1+\sigma(x))^2 \cdot \sum_\alpha x_\alpha^2 + \sum_\beta x_\beta^2 < 2|x|^2 , \tag{7.7}$$

when r is sufficiently small, we obtain, from (7.3), (7.4), (7.5), (7.6) and (7.7),

$$|\partial_n u| < \frac{2\sqrt{2}}{\Lambda_0} \cdot \frac{|g(x)|}{|x|} .$$

Hence

$$|\partial_n u| < K' \cdot G |x|^p \tag{7.8}$$

on M_ε with some constant $K' > 0$ independent of ε. Thus, from (7.3) and (7.8), we see

$$|\partial_x u(x ; \varepsilon)| \leqq K' \cdot G |x|^p \qquad (x \in M_\varepsilon) . \tag{7.9}$$

Now, operating ∂_μ on both sides of (5.1) and setting $\partial_\mu u_\nu = u^{\nu\mu}$, we have

$$\sum_{i=1}^{m}\Big(\sum_{j=1}^{m} a_{ij}x_j + f_i(x)\Big)\partial_i u^{\nu\mu} = \sum_{j=1}^{m} a_{\nu j}u^{j\mu} - \sum_{i=1}^{m}(a_{i\mu} + \partial_\mu f_i(x))\,u^{\nu i} + \partial_\mu g_\nu(x) \qquad (\nu, \mu = 1, \cdots, m) \tag{7.10}$$

which has also the form (4.1) with unknown functions $u^{\nu\mu}$. Let us assume first that f and g are functions of class C^2 in U and apply the auxiliary theorem to (7.10). We set

$$u' = (u^{1,1}, u^{1,2}, \cdots, u^{m,m}) \qquad (\in R^{m^2})$$

$$P_i(x) = \sum_{j=1}^{m} a_{ij}x_j + f_i(x)$$

$$Q^{\nu\mu}(x, u') = \sum_{j=1}^{m} a_{\nu j}u^{j\mu} - \sum_{i=1}^{m}(a_{i\mu} + \partial_\mu f_i(x))\cdot u^{\nu i} + \partial_\mu g_\nu(x) \qquad (\nu, \mu = 1, \cdots, m)$$

and $\omega'(x) = W'\cdot G\cdot \varphi'(x)^{\frac{p}{2}}$ where $\varphi'(x) = (1+\gamma')\sum_\alpha x_\alpha^2 - \sum_\beta x_\beta^2$.

Then, if we assume

$$p\Lambda_0 > \tilde{\Lambda} \equiv \max_{i,j} |\Re(\lambda_i) - \Re(\lambda_j)|, \tag{7.11}$$

and if we take r small enough, we have

$$\sum_{i=1}^{m} P_i(x)\,\partial_i \omega'(x) > \frac{1}{\omega'(x)} \sum_{\nu=1}^{m}\sum_{\mu=1}^{m} Q^{\nu\mu}(x, u')\cdot u^{\nu\mu}$$

for x such that $\omega'(x) = |u'|$ in U_ε, taking W' and $\gamma' > 0$ appropriately. By the auxiliary theorem and (7.9) we thus get

$$|\partial_x u(x\,;\,\varepsilon)| \leq C\cdot G|x|^p \tag{7.12}$$

in U_ε where $C > 0$ is a constant independent of ε. In the above consideration we can also choose r independent of ε. Let us write

$$h_0 = \max\left(\frac{\Lambda_1}{\Lambda_0} - 1,\ \tilde{\Lambda}\right)$$

and assume $p > h_0$ hereafter, from which (6.10) and (7.11) follow.

We will study in 9. as to the case that f and g are functions of class C^1.

8. Notice that C of (7.12) and r can be chosen independent of $\varepsilon > 0$ which is sufficiently small. Now we consider ε as a variable tending to zero. We see easily $U_\varepsilon \subset U_\varepsilon'$ as $\varepsilon > \varepsilon' > 0$, and $v = u(x\,;\,\varepsilon') - u(x\,;\,\varepsilon)$ must satisfy on U_ε

$$\sum_{i=1}^{m}\Big(\sum_{j=1}^{m} a_{ij}x_j + f_i(x)\Big)\partial_i v_\nu = \sum_{\mu=1}^{m} a_{\nu\mu}v_\mu \qquad (\nu = 1, \cdots, m). \tag{8.1}$$

From (6.12) and (7.12) we see easily that

$$|v| \leqq 2^{\frac{p+3}{2}} K \cdot G \cdot \varepsilon^{p+1}$$

and

$$|\partial_x v| \leqq 2^{\frac{p+2}{2}} C \cdot G \cdot \varepsilon^{p}$$

hold for $v = u(x;\ \varepsilon') - u(x;\ \varepsilon)$ on M_ε. Notice that $\min_{x \in U_\varepsilon} |x| = \frac{\sqrt{6} - \sqrt{2}}{2} \varepsilon$, and we have for any $q > 0$

(8.2)
$$\begin{cases} |v| \leqq K_0 \cdot \varepsilon^{p-q} |x|^{q+1} \\ |\partial_x v| \leqq K_1 \cdot \varepsilon^{p-q} |x|^{q} \end{cases}$$

on M_ε where K_0 and K_1 are constants not depending on ε and ε'.

We now choose q so that $p > q > h_0$. The system (8.1) is a special case of (5.1) with $g(x) \equiv 0$, and we get similarly as (6.12) and (7.12)

$$\begin{cases} |u(x;\ \varepsilon') - u(x;\ \varepsilon)| \leqq K' \cdot \varepsilon^{p-q} |x|^{q+1} \\ |\partial_x u(x;\ \varepsilon') - \partial_x u(x;\ \varepsilon)| \leqq K' \cdot \varepsilon^{p-q} |x|^{q} \end{cases}$$

in U_ε where K' is some positive constant independent of ε and ε'. Thus we see that, as $\varepsilon \to 0$, $u_\nu(x;\ \varepsilon)$ and $\partial_\mu u_\nu(x;\ \varepsilon)$ tend to certain functions $u_\nu(x)$ and their derivatives $\partial_\mu u_\nu(x)$ respectively. Clearly this $u(x)$ is a solution of (5.1), vanishing on the manifold: $\sum_\alpha x_\alpha^2 = \sum_\beta x_\beta^2$ and satisfying (5.4) in $U_1 = \{x : \sum_\alpha x_\alpha^2 \leqq r^2,\ \sum_\beta x_\beta^2 \leqq \sum_\alpha x_\alpha^2\}$.

Quite similarly as above, we can prove the existence of a solution $u(x)$ of (5.1) vanishing on $\sum_\alpha x_\alpha^2 = \sum_\beta x_\beta^2$ and satisfying (5.4) in $U_2 = \{x : \sum_\beta x_\beta^2 \leqq r^2,\ \sum_\alpha x_\alpha^2 \leqq \sum_\beta x_\beta^2\}$.

9. Now it remains to prove our lemma when f and g are functions of class C^1. We construct approximation sequences $\{f^n(x)\}_{n=1}^\infty$ and $\{g^n(x)\}_{n=1}^\infty$ of vector functions of class C^2 such that[4)]

(9.1)
$$\begin{cases} f^n(0) = g^n(0) = 0 \\ f^n(x) = p(x) + q^n(x)^{5)} \\ |\partial_x f^n(x) - \partial_x f(x)| \leqq \frac{1}{n} |x|^h \\ |\partial_x g^n(x) - \partial_x g(x)| \leqq \frac{1}{n} |x|^h. \end{cases}$$

4) $f^n(x)$ and $g^n(x)$ have only to be of class C^2 in U excepting $x=0$.

5) c.f. (2.1).

Then there exists a system $u^n(x)=(u_1^n(x), \cdots, u_m^n(x))$ of functions of class C^2 satisfying

$$(9.2) \qquad \sum_{i=1}^{m}(\sum_{j=1}^{m} a_{ij}x_j+f_i^n(x))\,\partial_i u_\nu=\sum_{\mu=1}^{m} a_{\nu\mu}u_\mu+g_\nu^n(x) \qquad (\nu=1,\cdots,m),$$

such that

$$\begin{cases} |u^n(x)| \leqq C\cdot K_2|x|^{p+1} \\ |\partial_x u^n(x)| \leqq C\cdot K_3|x|^p \end{cases}$$

in U where K_2 and K_3 are constants not depending on n. For $u^n(x)-u^{n'}(x)$ $(n\leqq n')$ we have

$$\sum_{i=1}^{m}(\sum_{j=1}^{m} a_{ij}x_j+f_i^n(x))\,\partial_i(u_\nu^n-u_\nu^{n'})=\sum_{\mu=1}^{m} a_{\nu\mu}(u_\mu^n-u_\mu^{n'})+h_\nu^{n,n'}(x),$$

where $\qquad h_\nu^{n,n'}(x)=\{g_\nu^n(x)-g_\nu^{n'}(x)\}+\partial_i u_\nu^{n'}\{f_\nu^{n'}(x)-f_\nu^n(x)\}$

and so $\qquad h^{n,n'}(0)=0 \quad$ and $\quad |\partial_x h^{n,n'}(x)| \leqq \dfrac{H}{n}|x|^p$

with a constant $H>0$ not depending on n and n'. Therefore we see that, as $n\to\infty$, $u^n(x)$ tend to the desired solution of (5.1). The proof of the lemma is thus completed.

§ 6. Proof of the Main Theorem.

10. Let us now turn to the system (3.1),

$$\sum_{i=1}^{m}(\sum_{j=1}^{m} a_{ij}x_j+f_i(x))\,\partial_i u_\nu=\sum_{\mu=1}^{m} a_{\nu\mu}u_\mu-f_\nu(x) \qquad (\nu=1,\cdots,m)$$

where $f_\nu(x)=p_\nu(x)+q_\nu(x)$. First, let us consider

$$(10.1) \qquad \sum_{i=1}^{m}(\sum_{j=1}^{m} a_{ij}x_j+p_i(x))\,\partial_i u_\nu=\sum_{\mu=1}^{m} a_{\nu\mu}u_\mu-p_\nu(x) \qquad (\nu=1,\cdots,m).$$

If there exist no relations of the form $\lambda_i=\sum n_j\lambda_j$ where n_j are non-negative integers such that $\sum_j n_j>1$, we can construct infinite series of the form

$$\sum_{\substack{p_i\geqq 0\\ p_1+\cdots+p_m\geqq 2}} c^\nu_{p_1,\cdots,p_m}\cdot x_1^{p_1}\cdot\cdots\cdot x_m^{p_m}$$

with real coefficients $c^\nu_{p_1,\cdots,p_m}$, such that $u_\nu=\sum c^\nu_{p_1,\cdots,p_m}\cdot x_1^{p_1}\cdots x_m^{p_m}$ $(\nu=1,\cdots,m)$ satisfy (10.1) formally. To see this, make a change of variables, $x=T\cdot y$ and $u=T\cdot w$, by a (complex) matrix T transforming $A=(a_{ij})$ into Jordan's canonical form $T^{-1}\cdot A\cdot T$, and consider about the (complex) system thus obtained.

Setting

$$(10.2)\qquad \acute{u}_\nu(x) = \sum_{2\leq p_1+\cdots+p_m\leq h} c^\nu_{p_1\cdots p_m}\cdot x_1^{p_1}\cdots x_m^{p_m} \qquad (\nu=1,\cdots,m)$$

and

$$(10.3)\qquad \grave{u}_\nu = u_\nu - \acute{u}_\nu(x) \qquad (\nu=1,\cdots,m),$$

we reduce (3. 1) to the system

$$(10.4)\qquad \sum_{i=1}^m \Big(\sum_{j=1}^m a_{ij}x_j + f_i(x)\Big)\partial_i \grave{u}_\nu = \sum_{\mu=1}^m a_{\nu\mu}\grave{u}_\mu + \tilde{f}_\nu(x) \qquad (\nu=1,\cdots,m)$$

where

$$\tilde{f}_\nu(x) = \sum_{\mu=1}^m a_{\nu\mu}\acute{u}_\mu(x) - \sum_{i=1}^m \Big(\sum_{j=1}^m a_{ij}x_j + f_i(x)\Big)\partial_i\acute{u}_\nu(x) - f_\nu(x).$$

In order to define $\acute{u}(x)$ as above, we have only to assume that condition (iii) in the theorem (§ 2) is satisfied.

By (2. 1) we have

$$(10.5)\qquad \tilde{f}_\nu(x) = \tilde{p}_\nu(x) - \tilde{q}_\nu(x)$$

where $\quad \tilde{p}_\nu(x) = \sum_{\mu=1}^m a_{\nu\mu}\acute{u}_\mu(x) - p_\nu(x) - \sum_{i=1}^m \Big(\sum_{j=1}^m a_{ij}x_j + p_i(x)\Big)\partial_i\acute{u}_\nu(x)$

and $\quad \tilde{q}_\nu(x) = q_\nu(x) + \sum_{i=1}^m q_i(x)\,\partial_i\acute{u}_\nu(x),$

and it follows from the definition of $\acute{u}(x)$ that $\tilde{p}_\nu(x)$, polynomials in x_i $(i=1,\cdots,m)$, contain no terms of degree $\leq h$. Therefore, by (2. 2)

$$\begin{cases} \tilde{f}_\nu(0) = 0 \qquad (\nu=1,\cdots,m), \\ |\partial_x \tilde{f}(x)| \leq \tilde{K}\cdot|x|^h \end{cases}$$

where $\tilde{K}$ is a positive constant. Using the lemma, we thus see that there exists a solution $\grave{u}(x) = (\grave{u}_1(x),\cdots,\grave{u}_m(x))$ of (10. 4) in some neighborhood of $x=0$, such that

$$\grave{u}_i(0) = 0 \qquad (i=1,\cdots,m)$$

and

$$|\partial_x\grave{u}(x)| \leq C\cdot\tilde{K}|x|^h$$

with some constant $C>0$. We set

$$u_\nu(x) = \acute{u}_\nu(x) + \grave{u}_\nu(x) \qquad (\nu=1,\cdots,m)$$

and obtain those functions $u_\nu(x)$ $(\nu=1,\cdots,m)$ whose existence was claimed in the theorem. The proof of the theorem is thus completed.

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