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***On the Normal Forms of Differential Equations in
the Neighborhood of an Equilibrium Point***

By Mitio NAGUMO and Kusuo ISÉ

§ 1. Introduction.

1. In this note we use the notations $\partial_i u$ and $\partial_{ij}^2 u$ for $\frac{\partial}{\partial x_i} u$ and $\frac{\partial^2}{\partial x_i \partial x_j} u$ respectively. The vectors (x_1, \dots, x_m) and (y_1, \dots, y_m) in R^m will be denoted briefly by x and y respectively.

Let $A=(a_{ij})$ be a constant real (m, m) -matrix, all of whose characteristic roots λ_i ($i=1, \dots, m$) have non-zero real parts, and $f(x)=(f_1(x), \dots, f_m(x))$ a real vector function of class C^1 on some neighborhood of $x=0$, such that $f(0)=0$ and $|\partial_x f(x)| \leq K \cdot |x|$ with a constant $K > 0$ where

$$|x| = \left(\sum_i x_i^2 \right)^{\frac{1}{2}}, \quad |\partial_x f(x)| = \left\{ \sum_{i,j} (\partial_i f_j(x))^2 \right\}^{\frac{1}{2}}.$$

We consider the autonomous systems

$$(1.1) \quad \frac{dx}{dt} = A \cdot x + f(x)$$

and

$$(1.2) \quad \frac{dy}{dt} = A \cdot y,$$

regarding x, y and $f(x)$ as the column-vectors. The purpose of this note is to show that, under some conditions on λ_i ($i=1, \dots, m$) and $f(x)$, the system (1.1) can be transformed into (1.2) by a change of variables

$$(1.3) \quad y = x + u(x)$$

where $u(x)=(u_1(x), \dots, u_m(x))$ is a real vector function of class C^1 , such that

$$(1.4) \quad \begin{cases} u(0) = 0 \\ |\partial_x u(x)| \leq L \cdot |x| \end{cases}$$

with some constant $L > 0$.

When $f(x)$ is analytic regular in x , in order to show the existence of the transformation given by (1.3) with analytic regular $u(x)$, we must necessarily assume that there exist no relations of the form

$$(1.5) \quad \lambda_i = \sum_{j=1}^m n_j \cdot \lambda_j$$

where n_j ($j=1, \dots, m$) are non-negative integers such that $\sum_{j=1}^m n_j > 1$.

As to this case, some results were obtained by H. Poincaré, C. L. Siegel, and others, while we obtain the present result for the real systems with a transformation of class C^1 under some weaker conditions.

§ 2. Main Theorem.

2. Theorem. Assumptions:

(i) A is a constant real (m, m) -matrix, all of whose characteristic roots λ_i ($i=1, \dots, m$) have non-zero real parts: $\Re(\lambda_i) \neq 0$ ($i=1, \dots, m$).

(ii) Let

$$(2.1) \quad f_i(x) = p_i(x) + q_i(x) \quad (i=1, \dots, m)$$

where $p_i(x)$ are polynomials in x with real coefficients such that $p_i(0) = \partial_j p_i(0) = 0$ ($i=1, \dots, m$; $j=1, \dots, m$), and $q_i(x)$ ($i=1, \dots, m$) are real-valued functions of class C^1 satisfying

$$(2.2) \quad \begin{cases} q(0) = 0 \\ |\partial_x q(x)| \leq Q \cdot |x|^h \end{cases}$$

with some integer $h > 0$ and some constant $Q > 0$.

(iii) There exist no relations of the form

$$\lambda_i = \sum_{j=1}^m n_j \cdot \lambda_j$$

where n_j ($j=1, \dots, m$) are non-negative integers such that

$$h > \sum_{j=1}^m n_j > 1.$$

Conclusion: There exists a positive constant h_0 , depending only on λ_i ($i=1, \dots, m$), with the following property: if $h > h_0$, there exist functions $u_i(x)$ ($i=1, \dots, m$) of class C^1 satisfying (1.4), such that the system (1.1) is reduced to the form (1.2) by the substitution (1.3).

3. If (1.1) is transformed into (1.2) by (1.3), $u(x)$ must satisfy the system of partial differential equations

$$(3.1) \quad \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \cdot \partial_i u_\nu = \sum_{\mu=1}^m a_{\nu\mu} u_\mu - f_\nu(x) \quad (\nu = 1, \dots, m).$$

For we have, by operating $\frac{d}{dt}$ on both sides of (1.3),

$$\frac{dy_i}{dt} = \frac{dx_i}{dt} + \sum_{\nu=1}^m \partial_\nu u_i \cdot \frac{dx_\nu}{dt} \quad (i = 1, \dots, m)$$

from which (3.1) follows immediately by (1.1), (1.2) and (1.3). Conversely, if $u(x)$ is any function satisfying (3.1), then the substitution (1.3) will transform (1.1) into (1.2). Thus we have only to show the existence of $u(x)$ satisfying (1.4) and (3.1), if h is sufficiently large.

§ 3. Auxiliary Theorem.

4. In this section we consider the system of semi-linear partial differential equations

$$(4.1) \quad \sum_{i=1}^m P_i(x) \cdot \partial_i u_\nu = Q_\nu(x, u) \quad (\nu = 1, \dots, l)$$

where $x = (x_1, \dots, x_m)$ and $u = (u_1, \dots, u_l)$ denote real vectors in R^m and R^l respectively. Let $P_i(x)$ be real-valued functions of class C^1 in an open domain $D \subset R^m$, such that

$$(4.2) \quad (P_1(x), \dots, P_m(x)) \neq (0, \dots, 0) \quad (x \in D).$$

And $Q_\nu(x, u)$ be real-valued functions of class C^1 in

$$\Omega = \{(x, u) \in R^{m+l} : x \in D, |u| \leq \omega(x)\}$$

where $\omega(x)$ is some positive-valued function of class C^1 in D . A curve $x = x(t)$ in R^m is said to be a *base characteristic* of (4.1) if $x(t)$ satisfies the following system of ordinary differential equations:

$$(4.3) \quad \frac{dx_i}{dt} = P_i(x) \quad (i = 1, \dots, m).$$

Let an $(m-1)$ -dimensional manifold M in R^m be given by

$$(4.4) \quad M: x_i = A_i(s_1, \dots, s_{m-1}) \quad (i = 1, \dots, m)$$

where $A_i(s)$ are functions of class C^1 in some domain $S \subset R^{m-1}$ such that $A(s) = (A_1(s), \dots, A_m(s)) \in D$ for $s = (s_1, \dots, s_{m-1}) \in S$. We assume that

$$(4.5) \quad \left| P_i(A(s)), \partial_j A_i(s) \begin{matrix} i \downarrow 1, \dots, m \\ j \rightarrow 1, \dots, m-1 \end{matrix} \right| \stackrel{1)}{=} 0 \quad \text{for } s \in S$$

and that any base characteristic

$$(4.6) \quad x = x(t, s),$$

issuing from a point of M so that $x(0, s) = A(s)$, exists on the interval: $0 \leq t < \tau(s)$ where $\tau(s)$ is a continuous function on S , and that the set $X = \{x = x(t, s) : 0 \leq t < \tau(s), s \in S\}$ is filled up only onefold with all those curves $x = x(t, s) (s \in S)$, i.e. to any point $x \in X$ there corresponds just one (t, s) such that $x = x(t, s)$, $0 \leq t < \tau(s)$, $s \in S$. Then we have easily

$$\frac{\partial(x_1, \dots, x_m)}{\partial(t, s_1, \dots, s_{m-1})} = \left| P_i(A(s)), \partial_j A_i(s) \begin{matrix} i \downarrow 1, \dots, m \\ j \rightarrow 1, \dots, m-1 \end{matrix} \right| \cdot \exp \left(\int_0^t \sum_{i=1}^m \partial_i P_i(x)_{x=x(t, s)} dt \right) \neq 0,$$

which shows that the 1-1 mapping (4.6) from $\{(t, s) : s \in S, 0 \leq t < \tau(s)\}$ onto X and its inverse are both of class C^1 .

By (4.6) the system (4.1) is reduced to the following system of ordinary differential equations, s being a parameter:

$$(4.7) \quad \frac{du_\nu}{dt} = Q_\nu(x(t, s), u) \quad (\nu = 1, \dots, l).$$

We have then

$$\partial_i \omega(x(t, s)) = \left[\sum_{i=1}^m P_i(x) \cdot \partial_i \omega(x) \right]_{x=x(t, s)}$$

and

$$\partial_i |u(t, s)| \cdot |u(t, s)| = \sum_{\nu=1}^l u_\nu(t, s) \cdot Q_\nu(x(t, s), u(t, s))$$

for any solution $u(t, s)$ of (4.7). Hence, we obtain easily the following auxiliary theorem which is our principal tool.

Auxiliary theorem. *Under the conditions mentioned above, let the inequality*

$$(4.8) \quad \sum_{i=1}^m P_i(x) \partial_i \omega(x) \geq \frac{1}{\omega(x)} \sum_{\nu=1}^m Q_\nu(x, u) \cdot u_\nu$$

hold for any $x \in X$ such that $|u| = \omega(x)$. Then, for any function $B(s) = (B_1(s), \dots, B_l(s))$ of class C^1 on S such that

1) an (m, m) -determinant.

$$(4.9) \quad |B(s)| \leq \omega(A(s)),$$

there exists a unique solution $u(x)$ of (4.1) on X , such that

$$u(A(s)) = B(s)$$

and

$$(4.10) \quad |u(x)| \leq \omega(x)$$

for $x \in X$.

§ 4. Estimation of $u(x)$.

5. Consider the system of partial differential equations

$$(5.1) \quad \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_i u_\nu = \sum_{\mu=1}^m a_{\nu\mu} u_\mu + g_\nu(x) \quad (\nu = 1, \dots, m)$$

for which we have the following lemma.

Lemma. Let $A = (a_{ij})$ and $f(x)$ satisfy the assumptions (i), (ii) and (iii) in the theorem. Let $g_\nu(x)$ ($\nu = 1, \dots, m$) be real-valued functions of class C^1 on some neighborhood of 0, such that

$$(5.2) \quad \begin{cases} g(0) = 0 \\ |\partial_x g(x)| \leq G|x|^p \end{cases}$$

where G and p are positive constants.

Then there exists a constant $h_0 > 0$, which depends only on λ_i ($i = 1, \dots, m$), with the following property: if $p > h_0$, the system (5.1) has a unique solution $u(x)$ in a neighborhood of 0, such that

$$(5.3) \quad u(x) = 0 \quad \text{on the cone} \quad \sum_{i=1}^k x_i^2 = \sum_{i=k+1}^m x_i^2$$

and

$$(5.4) \quad |\partial_x u(x)| \leq C \cdot G |x|^p$$

where C is a positive constant depending only on λ_i ($i = 1, \dots, m$) and p .

6. *Proof.* By setting $P_i(x) = \sum_{j=1}^m a_{ij} x_j + f_i(x)$ and $Q_\nu(x, u) = \sum_{\mu=1}^m a_{\nu\mu} u_\mu + g_\nu(x)$, the system (5.1) has the form (4.1). Without loss of generality we assume that $A = (a_{ij})$ has the following form:

$$(i) \quad \begin{aligned} a_{ii} &= \Re(\lambda_i) & (i = 1, \dots, m), \\ \Re(\lambda_i) &> 0 & \text{for } i \leq k, \\ \Re(\lambda_i) &< 0 & \text{for } i > k. \end{aligned}$$

2) Cf. (i) $k=0$ means $\Re(\lambda_i) < 0$ for all i , and $k=m$ means $\Re(\lambda_i) > 0$ for all i .

$$(ii) \quad \begin{aligned} a_{ij} &= 0 & \text{for } i \leq k \text{ and } j > k, \\ a_{ij} &= 0 & \text{for } i > k \text{ and } j \leq k. \end{aligned}$$

(iii)

$$(6.1) \quad \left| \sum_{i \neq j} a_{ij} x_i x_j \right| \leq \delta |x|^2$$

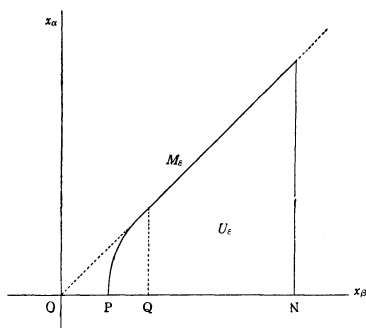
where δ is any prescribed positive number.

In what follows, we write $\sum_{\alpha} = \sum_{\alpha=1}^k$ and $\sum_{\beta} = \sum_{\beta=k+1}^m$. We suppose that f and g are functions of class C^1 on $U = \{x: \sum_{\alpha} x_{\alpha}^2 \leq r^2, \sum_{\beta} x_{\beta}^2 \leq r^2\}$ where r is a positive constant. We consider the case $0 < k < m$. Because, if $k=0$ or $k=m$, the proof of the lemma will be simpler.

With sufficiently small $\varepsilon > 0$ we set³⁾

$$(6.2) \quad S_{\varepsilon}(x) = \begin{cases} \sum_{\alpha} x_{\alpha}^2 - \sum_{\beta} x_{\beta}^2 & \text{when } \sum_{\alpha} x_{\alpha}^2 \geq \varepsilon^2 \\ \sum_{\alpha} x_{\alpha}^2 - \sum_{\beta} x_{\beta}^2 - \frac{1}{2\varepsilon^2} (\varepsilon^2 - \sum_{\alpha} x_{\alpha}^2)^2 & \text{when } \sum_{\alpha} x_{\alpha}^2 < \varepsilon^2, \end{cases}$$

and define a bounded region U_{ε} by $U_{\varepsilon} = \{x \in U: S_{\varepsilon}(x) \geq 0\}$.



$$N = (r, 0)$$

$$P = \left(\frac{\sqrt{6} - \sqrt{2}}{2} \varepsilon, 0 \right)$$

$$Q = (\varepsilon, 0)$$

First we consider the solution of (5.1) in U_{ε} vanishing on the $(m-1)$ -dimensional manifold $M_{\varepsilon} = \{x \in U: S_{\varepsilon}(x) = 0\}$. For the base characteristic $x = x(t)$ of (5.1), we have

$$\begin{aligned} \frac{1}{2} \cdot \frac{d}{dt} S_{\varepsilon}(x(t)) &= \sum_{\alpha} x_{\alpha} \cdot \dot{x}_{\alpha} - \sum_{\beta} x_{\beta} \cdot \dot{x}_{\beta} \\ &= \sum_{\alpha} \left(\sum_{i=1}^m a_{\alpha i} x_i + f_{\alpha}(x) \right) x_{\alpha} - \sum_{\beta} \left(\sum_{i=1}^m a_{\beta i} x_i + f_{\beta}(x) \right) x_{\beta} \\ &= \left(\sum_{\alpha} \sum_{i=1}^m a_{\alpha i} x_{\alpha} x_i - \sum_{\beta} \sum_{i=1}^m a_{\beta i} x_{\beta} x_i \right) + \left(\sum_{\alpha} f_{\alpha}(x) x_{\alpha} - \sum_{\beta} f_{\beta}(x) x_{\beta} \right) > 0 \end{aligned}$$

3) For the case $k=0$ or $k=m$, we have to set $S_{\varepsilon}(x) = \sum_{i=1}^m x_i^2$.

when $\varepsilon^2 < \sum_{\alpha} x_{\alpha}^2 \leq r^2$, by taking r small enough, and also when

$$\sum_{\alpha} x_{\alpha}^2 \leq \varepsilon^2$$

$$\frac{1}{2} \cdot \frac{d}{dt} S_{\varepsilon}(x(t)) = \left\{ 1 + \frac{1}{\varepsilon^2} (\varepsilon^2 - \sum_{\alpha} x_{\alpha}^2) \right\} \sum_{\alpha} x_{\alpha} \cdot \dot{x}_{\alpha} - \sum_{\beta} x_{\beta} \cdot \dot{x}_{\beta} > 0.$$

From these inequalities we see that, if $r > 0$ is taken small enough, every base characteristic of (5.1) meeting M_{ε} is transverse to M_{ε} , and that (4.5) will hold for this case with $M = M_{\varepsilon}$. In addition, since we have

$$\frac{1}{2} \frac{d}{dt} \sum_{\alpha} (x_{\alpha}(t))^2 = \sum_{\alpha} (\sum_i a_{\alpha i} x_i + f_{\alpha}(x)) x_{\alpha} > 0$$

for any base characteristic $x(t)$, when r is small enough, we see that U_{ε} is filled up only onefold with the base characteristics issuing from M_{ε} . Therefore, we apply the auxiliary theorem to this case, setting $U_{\varepsilon} = X$.

We set

$$(6.3) \quad \varphi(x) = (1 + \gamma) \sum_{\alpha} x_{\alpha}^2 - \sum_{\beta} x_{\beta}^2$$

and

$$(6.4) \quad \omega(x) = W \cdot G \cdot \varphi(x)^{\frac{p+1}{2}}$$

where $\gamma > 0$ and $W > 0$ will be determined later. Then

$$(6.5) \quad \frac{\gamma}{2} |x|^2 \leq \varphi(x) \leq (1 + \gamma) |x|^2$$

for every $x \in U_{\varepsilon}$. Thus we obtain

$$\begin{aligned} & \sum_{i=1}^m P_i(x) \partial_i \omega(x) \\ &= (p+1) \cdot W \cdot G \cdot \varphi(x)^{\frac{p-1}{2}} \left\{ (1 + \gamma) \sum_{\alpha} \sum_j a_{\alpha j} x_{\alpha} x_j - \sum_{\beta} \sum_j a_{\beta j} x_{\beta} x_j \right. \\ & \quad \left. + (1 + \gamma) \sum_{\alpha} x_{\alpha} f_{\alpha}(x) - \sum_{\beta} x_{\beta} f_{\beta}(x) \right\} \end{aligned}$$

and

$$\sum_{\nu=1}^m Q_{\nu}(x, u) \cdot u_{\nu} = \sum_{\nu=1}^m \sum_{\mu=1}^m a_{\nu \mu} u_{\nu} u_{\mu} - \sum_{\nu=1}^m g_{\nu}(x) u_{\nu}.$$

Now we set

$$(6.6) \quad \Lambda_0 = \min_{1 \leq i \leq m} |\Re(\lambda_i)|, \quad \Lambda_1 = \max_{1 \leq i \leq m} |\Re(\lambda_i)|.$$

Then we have

$$(6.7) \quad \sum_{i=1}^m P_i(x) \cdot \partial_i \omega(x) > (p+1)(\Lambda_0 - 2\delta) W \cdot G |x|^2 \varphi(x)^{\frac{p-1}{2}}$$

and

$$(6.8) \quad \sum_{\nu=1}^m Q_{\nu}(x, u) \cdot u_{\nu} < (\Lambda_1 + \delta) |u|^2 + |g(x)| \cdot |u|$$

on U_{ε} where δ is given by (6.1), taking r small enough. From (6.8) it follows that, if $|u| = \omega(x)$ for some $x \in U_{\varepsilon}$, then

$$\frac{1}{\omega(x)} \sum_{\nu=1}^m Q_{\nu}(x, u) u_{\nu} < (\Lambda_1 + \delta) \omega(x) + G |x|^{p+1}$$

and so

$$(6.9) \quad \frac{1}{\omega(x)} \sum_{\nu=1}^m Q_{\nu}(x, u) u_{\nu} < W \cdot G (\Lambda_1 + \delta) \left\{ 1 + \gamma + \left(\frac{2}{\gamma} \right)^{\frac{p-1}{2}} \right\} |x|^2 \varphi(x)^{\frac{p-1}{2}}$$

by virtue of (6.4) and (6.5). Thus, if we assume

$$(6.10) \quad (p+1)\Lambda_0 > \Lambda_1,$$

we have

$$(6.11) \quad \sum_{i=1}^m P_i(x) \partial_i \omega(x) > \frac{1}{\omega(x)} \sum_{\nu=1}^m Q_{\nu}(x, u) \cdot u_{\nu}$$

for any $x \in U_{\varepsilon}$ such that $|u| = \omega(x)$, from (6.7) and (6.9), by taking $\delta > 0$ and $\gamma > 0$ small enough and then W large enough.

Let us assume hereafter that (6.10) holds. Then it follows from the auxiliary theorem that, when r is small enough, there exists a unique solution $u(x; \varepsilon)$ of (5.1) on U_{ε} which vanishes on M_{ε} , and that it satisfies

$$(6.12) \quad |u(x; \varepsilon)| \leq \omega(x) = W \cdot G \cdot \varphi(x)^{\frac{p+1}{2}} \leq G \cdot K \cdot |x|^{p+1} \quad \text{for } x \in U_{\varepsilon}$$

where $K = W(1 + \gamma)^{\frac{p+1}{2}}$. Notice that W and r are taken independent of $\varepsilon > 0$ in the above argument.

§ 5. Continuation of the Proof of the Lemma.

Estimation of $|\partial_x u(x)|$.

7. Next, let us prove that the inequality (5.4) holds for $u = u(x; \varepsilon)$ on U_{ε} with some constant $C > 0$ independent of ε . In this paragraph we fix $\varepsilon > 0$ and write $u_{\nu}(x)$ in place of $u_{\nu}(x; \varepsilon)$ for simplicity.

In order to estimate $|\partial_x u(x)|$ on the manifold M_{ε} , we reduce the system (5.1) into

$$(7.1) \quad \frac{du_{\nu}}{dt} = \sum_{\mu=1}^m a_{\nu\mu} u_{\mu} + g_{\nu}(x(t, s)) \quad (\nu = 1, \dots, m)$$

by the change of variables given by (4.6). Thus, if we set $u_v(t, s) = u_v(x(t, s))$, $u = u(t, s)$ is the solution of (7.1) with the initial condition

$$(7.2) \quad u(0, s) \equiv 0.$$

Therefore,

$$(7.3) \quad \begin{cases} \partial_t u(0, s) \equiv g(x(0, s)) \\ \partial_s u(0, s) \equiv 0 \end{cases}$$

on M_ε . Let $\partial_n u(s)$ denote the normal derivative of $u(t, s)$ at $x = x(0, s)$ on M_ε , then we have

$$(7.4) \quad \left[\frac{\partial_t u(t, s)}{\left\{ \sum_{i=1}^m (\partial_t x_i(t, s))^2 \right\}^{\frac{1}{2}}} \right]_{t=0} = \partial_n u(s) \cdot \cos \theta(s)$$

where $\theta(s)$ represents the angle between the base characteristic $x = x(t, s)$ and the normal of M_ε at $x(0, s)$, i.e.

$$(7.5) \quad \cos \theta(s) = \frac{(1 + \sigma(x)) \cdot \sum_{\alpha} x_{\alpha} \cdot \partial_t x_{\alpha} - \sum_{\beta} x_{\beta} \cdot \partial_t x_{\beta}}{\left\{ \sum_{i=1}^m (\partial_t x_i)^2 \right\}^{\frac{1}{2}} \left\{ (1 + \sigma(x))^2 \sum_{\alpha} x_{\alpha}^2 + \sum_{\beta} x_{\beta}^2 \right\}^{\frac{1}{2}}}$$

where

$$\sigma(x) = \begin{cases} 0 & \text{when } \sum_{\alpha} x_{\alpha}^2 \geq \varepsilon^2 \\ \frac{1}{\varepsilon^2} (\varepsilon^2 - \sum_{\alpha} x_{\alpha}^2) & \text{when } \sum_{\alpha} x_{\alpha}^2 < \varepsilon^2. \end{cases}$$

Since

$$(7.6) \quad (1 + \sigma(x)) \cdot \sum_{\alpha} x_{\alpha} \cdot \partial_t x_{\alpha} - \sum_{\beta} x_{\beta} \cdot \partial_t x_{\beta} > \frac{1}{2} \Lambda_0 |x|^2 \quad (x \in U_\varepsilon)$$

and

$$(7.7) \quad (1 + \sigma(x))^2 \cdot \sum_{\alpha} x_{\alpha}^2 + \sum_{\beta} x_{\beta}^2 < 2|x|^2,$$

when r is sufficiently small, we obtain, from (7.3), (7.4), (7.5), (7.6) and (7.7),

$$|\partial_n u| < \frac{2\sqrt{2}}{\Lambda_0} \cdot \frac{|g(x)|}{|x|}.$$

Hence

$$(7.8) \quad |\partial_n u| < K' \cdot G |x|^p$$

on M_ε with some constant $K' > 0$ independent of ε . Thus, from (7.3) and (7.8), we see

$$(7.9) \quad |\partial_x u(x; \varepsilon)| \leq K' \cdot G |x|^p \quad (x \in M_\varepsilon).$$

Now, operating ∂_μ on both sides of (5.1) and setting $\partial_\mu u_\nu = u^{\nu\mu}$, we have

$$(7.10) \quad \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_i u^{\nu\mu} \\ = \sum_{j=1}^m a_{\nu j} u^{j\mu} - \sum_{i=1}^m (a_{i\mu} + \partial_\mu f_i(x)) u^{\nu i} + \partial_\mu g_\nu(x) \quad (\nu, \mu = 1, \dots, m)$$

which has also the form (4.1) with unknown functions $u^{\nu\mu}$. Let us assume first that f and g are functions of class C^2 in U and apply the auxiliary theorem to (7.10). We set

$$u' = (u^{1,1}, u^{1,2}, \dots, u^{m,m}) \quad (\in R^{m^2})$$

$$P_i(x) = \sum_{j=1}^m a_{ij} x_j + f_i(x)$$

$$Q^{\nu\mu}(x, u') = \sum_{j=1}^m a_{\nu j} u^{j\mu} - \sum_{i=1}^m (a_{i\mu} + \partial_\mu f_i(x)) \cdot u^{\nu i} + \partial_\mu g_\nu(x) \quad (\nu, \mu = 1, \dots, m)$$

$$\text{and } \omega'(x) = W' \cdot G \cdot \varphi'(x)^{\frac{p}{2}} \quad \text{where } \varphi'(x) = (1 + \gamma') \sum_\alpha x_\alpha^2 - \sum_\beta x_\beta^2.$$

Then, if we assume

$$(7.11) \quad p\Lambda_0 > \tilde{\Lambda} \equiv \max_{i,j} |\Re(\lambda_i) - \Re(\lambda_j)|,$$

and if we take r small enough, we have

$$\sum_{i=1}^m P_i(x) \partial_i \omega'(x) > \frac{1}{\omega'(x)} \sum_{i=1}^m \sum_{\mu=1}^m Q^{\nu\mu}(x, u') \cdot u^{\nu\mu}$$

for x such that $\omega'(x) = |u'|$ in U_ε , taking W' and $\gamma' > 0$ appropriately. By the auxiliary theorem and (7.9) we thus get

$$(7.12) \quad |\partial_x u(x; \varepsilon)| \leq C \cdot G |x|^p$$

in U_ε where $C > 0$ is a constant independent of ε . In the above consideration we can also choose r independent of ε . Let us write

$$h_0 = \max \left(\frac{\Lambda_1}{\Lambda_0} - 1, \tilde{\Lambda} \right)$$

and assume $p > h_0$ hereafter, from which (6.10) and (7.11) follow.

We will study in 9. as to the case that f and g are functions of class C^1 .

8. Notice that C of (7.12) and r can be chosen independent of $\varepsilon > 0$ which is sufficiently small. Now we consider ε as a variable tending to zero. We see easily $U_\varepsilon \subset U_{\varepsilon'}$ as $\varepsilon > \varepsilon' > 0$, and $v = u(x; \varepsilon') - u(x; \varepsilon)$ must satisfy on U_ε

$$(8.1) \quad \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_i v_\nu = \sum_{\mu=1}^m a_{\nu\mu} v_\mu \quad (\nu = 1, \dots, m).$$

From (6.12) and (7.12) we see easily that

$$|v| \leq 2^{\frac{p+3}{2}} K \cdot G \cdot \varepsilon^{p+1}$$

and

$$|\partial_x v| \leq 2^{\frac{p+2}{2}} C \cdot G \cdot \varepsilon^p$$

hold for $v = u(x; \varepsilon') - u(x; \varepsilon)$ on M_ε . Notice that $\min_{x \in U_\varepsilon} |x| = \frac{\sqrt{6} - \sqrt{2}}{2} \varepsilon$, and we have for any $q > 0$

$$(8.2) \quad \begin{cases} |v| \leq K_0 \cdot \varepsilon^{p-q} |x|^{q+1} \\ |\partial_x v| \leq K_1 \cdot \varepsilon^{p-q} |x|^q \end{cases}$$

on M_ε where K_0 and K_1 are constants not depending on ε and ε' .

We now choose q so that $p > q > h_0$. The system (8.1) is a special case of (5.1) with $g(x) \equiv 0$, and we get similarly as (6.12) and (7.12)

$$\begin{cases} |u(x; \varepsilon') - u(x; \varepsilon)| \leq K' \cdot \varepsilon^{p-q} |x|^{q+1} \\ |\partial_x u(x; \varepsilon') - \partial_x u(x; \varepsilon)| \leq K' \cdot \varepsilon^{p-q} |x|^q \end{cases}$$

in U_ε where K' is some positive constant independent of ε and ε' . Thus we see that, as $\varepsilon \rightarrow 0$, $u_\nu(x; \varepsilon)$ and $\partial_\mu u_\nu(x; \varepsilon)$ tend to certain functions $u_\nu(x)$ and their derivatives $\partial_\mu u_\nu(x)$ respectively. Clearly this $u(x)$ is a solution of (5.1), vanishing on the manifold: $\sum_\alpha x_\alpha^2 = \sum_\beta x_\beta^2$ and satisfying (5.4) in $U_1 = \{x: \sum_\alpha x_\alpha^2 \leq r^2, \sum_\beta x_\beta^2 \leq \sum_\alpha x_\alpha^2\}$.

Quite similarly as above, we can prove the existence of a solution $u(x)$ of (5.1) vanishing on $\sum_\alpha x_\alpha^2 = \sum_\beta x_\beta^2$ and satisfying (5.4) in $U_2 = \{x: \sum_\beta x_\beta^2 \leq r^2, \sum_\alpha x_\alpha^2 \leq \sum_\beta x_\beta^2\}$.

9. Now it remains to prove our lemma when f and g are functions of class C^1 . We construct approximation sequences $\{f^n(x)\}_{n=1}^\infty$ and $\{g^n(x)\}_{n=1}^\infty$ of vector functions of class C^2 such that⁴⁾

$$(9.1) \quad \begin{cases} f^n(0) = g^n(0) = 0 \\ f^n(x) = p(x) + q^n(x)^{5)} \\ |\partial_x f^n(x) - \partial_x f(x)| \leq \frac{1}{n} |x|^h \\ |\partial_x g^n(x) - \partial_x g(x)| \leq \frac{1}{n} |x|^h. \end{cases}$$

4) $f^n(x)$ and $g^n(x)$ have only to be of class C^2 in U excepting $x=0$.

5) c. f. (2.1).

Then there exists a system $u^n(x) = (u_1^n(x), \dots, u_m^n(x))$ of functions of class C^2 satisfying

$$(9.2) \quad \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i^n(x) \right) \partial_i u_\nu = \sum_{\mu=1}^m a_{\nu\mu} u_\mu + g_\nu^n(x) \quad (\nu = 1, \dots, m),$$

such that

$$\begin{cases} |u^n(x)| \leq C \cdot K_2 |x|^{p+1} \\ |\partial_x u^n(x)| \leq C \cdot K_3 |x|^p \end{cases}$$

in U where K_2 and K_3 are constants not depending on n . For $u^n(x) - u^{n'}(x)$ ($n \leq n'$) we have

$$\sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i^n(x) \right) \partial_i (u_\nu^n - u_\nu^{n'}) = \sum_{\mu=1}^m a_{\nu\mu} (u_\mu^n - u_\mu^{n'}) + h_\nu^{n,n'}(x),$$

where $h_\nu^{n,n'}(x) = \{g_\nu^n(x) - g_\nu^{n'}(x)\} + \partial_i u_\nu^{n'} \{f_i^n(x) - f_i^{n'}(x)\}$

and so $h^{n,n'}(0) = 0$ and $|\partial_x h^{n,n'}(x)| \leq \frac{H}{n} |x|^p$

with a constant $H > 0$ not depending on n and n' . Therefore we see that, as $n \rightarrow \infty$, $u^n(x)$ tend to the desired solution of (5.1). The proof of the lemma is thus completed.

§ 6. Proof of the Main Theorem.

10. Let us now turn to the system (3.1),

$$\sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_i u_\nu = \sum_{\mu=1}^m a_{\nu\mu} u_\mu - f_\nu(x) \quad (\nu = 1, \dots, m)$$

where $f_\nu(x) = p_\nu(x) + q_\nu(x)$. First, let us consider

$$(10.1) \quad \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + p_i(x) \right) \partial_i u_\nu = \sum_{\mu=1}^m a_{\nu\mu} u_\mu - p_\nu(x) \quad (\nu = 1, \dots, m).$$

If there exist no relations of the form $\lambda_i = \sum_j n_j \lambda_j$ where n_j are non-negative integers such that $\sum_j n_j > 1$, we can construct infinite series of the form

$$\sum_{\substack{p_i \geq 0 \\ p_1 + \dots + p_m \geq 2}} c_{p_1, \dots, p_m}^\nu \cdot x_1^{p_1} \cdot \dots \cdot x_m^{p_m}$$

with real coefficients c_{p_1, \dots, p_m}^ν , such that $u_\nu = \sum c_{p_1, \dots, p_m}^\nu \cdot x_1^{p_1} \cdot \dots \cdot x_m^{p_m}$ ($\nu = 1, \dots, m$) satisfy (10.1) formally. To see this, make a change of variables, $x = T \cdot y$ and $u = T \cdot w$, by a (complex) matrix T transforming $A = (a_{ij})$ into Jordan's canonical form $T^{-1} \cdot A \cdot T$, and consider about the (complex) system thus obtained.

Setting

$$(10.2) \quad \dot{u}_\nu(x) = \sum_{2 \leq p_1 + \dots + p_m \leq h} c_{p_1 \dots p_m}^\nu \cdot x_1^{p_1} \dots x_m^{p_m} \quad (\nu = 1, \dots, m)$$

and

$$(10.3) \quad \dot{u}_\nu = u_\nu - \dot{u}_\nu(x) \quad (\nu = 1, \dots, m),$$

we reduce (3.1) to the system

$$(10.4) \quad \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_i \dot{u}_\nu = \sum_{\mu=1}^m a_{\nu\mu} \dot{u}_\mu + \tilde{f}_\nu(x) \quad (\nu = 1, \dots, m)$$

where

$$\tilde{f}_\nu(x) = \sum_{\mu=1}^m a_{\nu\mu} \dot{u}_\mu(x) - \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + f_i(x) \right) \partial_i \dot{u}_\nu(x) - f_\nu(x).$$

In order to define $\dot{u}(x)$ as above, we have only to assume that condition (iii) in the theorem (§ 2) is satisfied.

By (2.1) we have

$$(10.5) \quad \tilde{f}_\nu(x) = \tilde{p}_\nu(x) - \tilde{q}_\nu(x)$$

$$\text{where} \quad \tilde{p}_\nu(x) = \sum_{\mu=1}^m a_{\nu\mu} \dot{u}_\mu(x) - p_\nu(x) - \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij} x_j + p_i(x) \right) \partial_i \dot{u}_\nu(x)$$

and

$$\tilde{q}_\nu(x) = q_\nu(x) + \sum_{i=1}^m q_i(x) \partial_i \dot{u}_\nu(x),$$

and it follows from the definition of $\dot{u}(x)$ that $\tilde{p}_\nu(x)$, polynomials in x_i ($i=1, \dots, m$), contain no terms of degree $\leq h$. Therefore, by (2.2)

$$\begin{cases} \tilde{f}_\nu(0) = 0 & (\nu = 1, \dots, m), \\ |\partial_x \tilde{f}(x)| \leq \tilde{K} \cdot |x|^h \end{cases}$$

where \tilde{K} is a positive constant. Using the lemma, we thus see that there exists a solution $\dot{u}(x) = (\dot{u}_1(x), \dots, \dot{u}_m(x))$ of (10.4) in some neighborhood of $x=0$, such that

$$\dot{u}_i(0) = 0 \quad (i = 1, \dots, m)$$

and

$$|\partial_x \dot{u}(x)| \leq C \cdot \tilde{K} |x|^h$$

with some constant $C > 0$. We set

$$u_\nu(x) = \dot{u}_\nu(x) + \dot{u}_\nu(x) \quad (\nu = 1, \dots, m)$$

and obtain those functions $u_\nu(x)$ ($\nu=1, \dots, m$) whose existence was claimed

in the theorem. The proof of the theorem is thus completed.

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