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AN INTEGRO-DIFFERENTIAL EQUATION FOR A COMPOUND POISSON PROCESS WITH DRIFT AND THE INTEGRAL EQUATION OF H. CRAMER

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1. Introduction

Let $(Y(t))_{t\geq 0}$ be a compound Poisson process on $\mathbf{R}=(-\infty, \infty)$ with the characteristic function

$$\mathbf{E}(e^{iyY(t)}) = \exp\left\{t\int_{-\infty}^{\infty}(e^{iyu}-1)\nu(du)\right\}, \tag{1.1}$$

where ν is a finite measure. For short we assume that

$$\int_{-\infty}^{\infty}\nu(du) = 1\,. \tag{1.2}$$

Let $(X(t))_{t\geq 0}$ be the compound Poisson process with a drift term at $(a\in\mathbf{R})$:

$$X(t) = at+Y(t)\,. \tag{1.3}$$

It is known that, if f is a bounded function with a second continuous derivative, $t^{-1}[\mathbf{E}\{f(x+X(t))\}-f(x)]$ converges uniformly on every bounded intervals to

$$Af(x) := af'(x)+\int_{-\infty}^{\infty}[f(x+y)-f(x)]\,\nu(dy)\,. \tag{1.4}$$

One now enlarges the domain of A as follows. Let G be an open subset of $\mathbf{R}$. Denote by $\mathcal{B}$ the class of all bounded, measurable and real-valued functions on $\mathbf{R}$. Define

$$\begin{aligned}\mathcal{D}(A;G) :&= \{f\in\mathcal{B};\, f \text{ is absolutely continuous in } G\} \qquad \text{if } a\neq 0,\\ &= \mathcal{B} \qquad \text{if } a=0.\end{aligned} \tag{1.5}$$

For $f\in\mathcal{D}(A;G)$, Af is defined almost everywhere in G by (1.4).

The main result of this note is Theorem 2 in section 2 which describes, for each $\lambda\geq 0$, a natural class of functions in $\mathcal{D}(A;G)$ satisfying the equation

$$(\lambda-A)f = 0 \qquad \text{almost everywhere in } G. \tag{1.6}$$

From the point of view ot potential theory, equation (1.6) may be regarded as the infinitesimal expression of the property that f is " λ-harmonic in G" for the Markov process associated with the compound Poisson process $(X(t))$.

In section 3 we give a proof of the integral equation of H. Cramér [2] as an application of Theorem 2.

REMARK. In a forthcoming paper [6] we will discuss a generalization of Theorem 2 to the most general process with stationary independent increments, using the Schwartz distribution theory.

2. Harmonic functions

Here and after we follow the usual notation and terminology of Markov processes without further reference [1], [5]. There would be no confusion in using the same symbols $Y(t)$ and $X(t)$ as the single compound Poisson processes (section 1) to denote the associated Markov processes.

Let then

$$(\Omega, \mathcal{F}, \mathcal{F}_t, Y(t), \mathbf{P}^x, \theta_t)$$

be a standard realization of the compound Poisson process defined by (1.1). That is, the process $(Y(t))_{t\geq 0}$ with respect to $\mathbf{P}^x$ represents the same compound Poisson process starting at x. $(X(t))_{t\geq 0}$ is defined by (1.3) as before:

$$X(t) = at + Y(t), \qquad a \in \mathbf{R}. \tag{2.1}$$

For a stopping time T and $\lambda \geq 0$, define

$$H_T^\lambda(x, E) := \mathbf{E}^x(e^{-\lambda T}; X(T) \in E), \qquad E \in \mathcal{B}(\mathbf{R}). \tag{2.2}$$

For each $B \in \mathcal{B}(\mathbf{R})$,

$$H_B^\lambda := H_{T_B}^\lambda \quad \text{with} \quad T_B := \inf\{t>0;\, X(t) \in B\}. \tag{2.3}$$

A finite function f which is $\lambda(\geq 0)$-excessive for $(X(t))$ is said to be *λ-harmonic on an open set* G if, for every compact set $K \subset G$,

$$f = H_{\complement K}^\lambda f, \tag{2.4}$$

where $\complement K = \mathbf{R} \setminus K$. Let $x \in \mathbf{R}$ and let T be a stopping time such that $T \leq T_{\complement K}$ $\mathbf{P}^x$-almost surely for some compact $K \subset G$. Since f is supposed to be λ-excessive, one has

$$f(x) = H_T^\lambda f(x). \tag{2.5}$$

We give some examples of λ-harmonic function. Let f be a finite λ-excessive function and let $B \in \mathcal{B}(\mathbf{R})$. Then the function $H_{\complement B}^\lambda f$ is λ-harmonic in $\operatorname{int} B$ (=the interior of B). In particular, the λ-hitting probability

$$(2.6)\qquad \mathbf{E}^x(e^{-\lambda T_{\complement B}}) = H^\lambda_{\complement B}1(x)$$

is λ-harmonic on int B. Let $g\in p\mathcal{B}(\mathbf{R})$ (=the class of non-negative Borel measurable functions). Let $(U_\lambda)_{\lambda>0}$ be the resolvent of $(X(t))$ and $U=U_0$, the potential kernel;

$$(2.7)\qquad U_\lambda g(x) := \mathbf{E}^x\left(\int_0^\infty e^{-\lambda t} g\circ X(t)\,dt\right), \qquad \lambda>0,$$

$$(2.8)\qquad Ug(x) = U_0 g(x) := \mathbf{E}^x\left(\int_0^\infty g\circ X(t)dt\right).$$

If g is supported in $\complement B$, $U_\lambda g$ is $\lambda(\geq 0)$-harmonic on int B as far as it is finite.

Suppose that the drift coefficient $a=0$. Let f be a λ-harmonic function on an open set G. Let σ be the time of first jump of the process $(Y(t))$. For each $x\in G$, choose a compact set K such that $x\in K\subset G$. Then, $\sigma\leq T_{\complement K}$ $\mathbf{P}^x$-almost surely by virtue of $X(t)=Y(t)$.

By (2.5),

$$(2.9)\qquad f(x) = H^\lambda_\sigma f(x) = (\lambda+1)^{-1}\int_{-\infty}^\infty f(x+y)\nu(dy),$$

which is easily seen to be equivalent to

$$(2.10)\qquad (\lambda-A)f(x) = 0, \qquad x\in G.$$

Theorem 1. *Suppose that* $a\neq 0$. *Let* f *be a bounded function which is uniformly* $\lambda(\geq 0)$*-excessive* (i.e. $\lim_{t\to 0}\uparrow H^\lambda_t f(x)=f(x)$ *uniformly in* x) *and* λ*-harmonic on an open set* G. *Then,* $f(x)$, $f'(x)$ *and* $\int_{-\infty}^\infty f(x+y)\nu(dy)$ *are continuous for every* $x\in G$ *and*

$$(2.11)\qquad (\lambda-A)f = 0 \qquad on \quad G.$$

In particular, for every interval $I=[x_0, x]\subset G$,

$$(2.21)\qquad \lambda\int_{x_0}^x f(z)dz - a[f(x)-f(x_0)] - \int_{x_0}^x\left\{\int_{-\infty}^\infty (f(z+y)-f(z))\,\nu(dy)\right\}dz = 0.$$

Proof. Let σ be the time of first jump of $(Y(t))$. One has

$$\begin{aligned} H^\lambda_t f(x) &= \mathbf{E}^x(e^{-\lambda t}f\circ X(t);\ t<\sigma)+\mathbf{E}^x(e^{-\lambda t}f\circ X(t);\ t\geq\sigma)\\ &= I_1+I_2,\\ I_1 &= e^{-\lambda t}\mathbf{E}^x(f(at+x);\ t<\sigma) = e^{-\lambda t}e^{-t}f(at+x),\\ |I_2| &\leq \|f\|(1-e^{-t}) \to 0 \qquad \text{as} \quad t\to 0, \end{aligned}$$

where $\|f\|=\sup|f(x)|$. Since $H^\lambda_t f(x)$ is supposed to converge uniformly in x to $f(x)$ as $t\to 0$, it follows that

$$\lim_{t\to 0} f(at+x) = f(x) \qquad \text{uniformly in } x, \tag{2.13}$$

which implies that f is continuous.

Let K, K' be compact sets such that $K\subset \operatorname{int} K'\subset K'\subset G$. If t is small enough, for every $x\in K$

$$\sigma\wedge t \leq T_{\complement K'} \qquad \mathbf{P}^x\text{-almost surely}, \tag{2.14}$$

so that

$$\lim_{t\to 0} \frac{H^\lambda_{\sigma\wedge t} f(x) - f(x)}{t} = 0 \qquad \text{uniformly in } x\in K. \tag{2.15}$$

On the other hand,

$$\begin{aligned} H^\lambda_{\sigma\wedge t} f(x) &= \mathbf{E}^x(e^{-\lambda t} f\circ X(t);\, t<\sigma) + \mathbf{E}^x(e^{-\lambda\sigma} f\circ X(\sigma);\, \sigma\leq t) \\ &= I_3 + I_4\,, \\ I_3 &= I_1 = e^{-\lambda t} e^{-t} f(at+x)\,, \\ I_4 &= \mathbf{E}^x[e^{-\lambda\sigma} f(a\sigma + Y(\sigma)) I_{[0,t]}(\sigma)]\,. \end{aligned}$$

Since σ and $Y(\sigma)$ are independent,

$$\begin{aligned} &\mathbf{E}^x[e^{-\lambda\sigma} f(a\sigma+Y(\sigma)) I_{[0,t]}(\sigma) \mid Y(\sigma) = b] \\ &= \int_0^t e^{-\lambda s} f(as+b) e^{-s}\, ds\,, \end{aligned}$$

so that, by virtue of (2.13),

$$\lim_{t\to 0} \frac{I_4}{t} = \mathbf{E}^x(f\circ Y(\sigma)) = \int_{-\infty}^{\infty} f(x+y)\,\nu(dy) \qquad \text{uniformly in } x. \tag{2.16}$$

By (2.15), (2.16) it follows that

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{H^\lambda_{\sigma\wedge t} f(x) - f(x)}{t} \\ &= \int_{-\infty}^{\infty} f(x+y)\,\nu(dy) + \lim_{t\to 0} \frac{e^{-\lambda t} e^{-t} f(at+x) - f(x)}{t} \\ &= \int_{-\infty}^{\infty} f(x+y)\,\nu(dy) + \lim_{t\to 0} \frac{f(at+x)-f(x)}{t} - (\lambda+1) f(x)\,. \end{aligned} \tag{2.17}$$

All the limits in the above display are uniform for $x\in K$.

Suppose now that $a>0$. It then follows from (2.17) that the right derivative

$$D^+f(x) := \lim_{\Delta\downarrow 0} \frac{f(x+\Delta)-f(x)}{\Delta} \tag{2.18}$$

exists uniformly for $x\in K$. Therefore, $D^+f(x)$ is continuous in K, so that f' exists and equals D^+f in K. Again, by (2.17),

$$0 = \int_{-\infty}^{\infty} f(x+y)\nu(dy)+af'(x)-(\lambda+1)f(x)$$
$$= (A-\lambda)f(x)\,, \qquad x\in K.$$

The same argument is valid for $a<0$.

Theorem 2. *Let f be a bounded, $\lambda(\geq 0)$-harmonic function on an open set G. Then, $f\in\mathcal{D}(A;G)$ and f satisfies*

$$(2.19) \qquad (\lambda-A)f=0 \qquad \textit{almost everywhere on } G,$$

or equivalently, for every interval $[x_0, x]\subset G$,

$$(2.20) \quad \lambda\int_{x_0}^{x} f(z)dz-a[f(x)-f(x_0)]-\int_{x_0}^{x}\left\{\int_{-\infty}^{\infty}(f(z+y)-f(z))\nu(dy)\right\}dz=0\,.$$

Proof. It is enough to consider the case $a\neq 0$.

Let $\lambda>0$. Let K be a compact set $\subset G$ and $I=[x_0, x]\subset \operatorname{int} K$. Since $f=H^{\lambda}_{\complement K}f$, it follows from a theorem of Hunt [4; p. 75] that $f=\lim_{n}\uparrow U_\lambda g_n$ with $g_n\geq 0$ being bounded and supported in $\complement K$. Since each $f_n=U_\lambda g_n$ satisfies those conditions in Theorem 1 for $G=\operatorname{int} K$, f_n satisfies (2.20). Letting $n\to\infty$, one sees that f satisfies (2.20).

Next let $\lambda=0$. Take K as before and define $f_\lambda:=H^{\lambda}_{\complement K}f$ for $\lambda>0$. By the above, f_λ satisfies (2.20). Therefore, $\lim_{\lambda\to 0} f_\lambda=H^{0}_{\complement K}f=f$ satisfies (2.20) for $\lambda=0$.

3. A proof of the integral equation of H. Cramér

Let us now introduce the following objects;

$$(3.1) \qquad S(u):=\nu((-\infty, u]) \qquad \text{for } u<0$$
$$\qquad\qquad =\nu((-\infty, u])-1 \qquad \text{for } u\geq 0,$$

$$(3.2) \qquad T_u:=\inf\{t>0;\, X(t)>u\}\,,$$

$$(3.3) \qquad \psi(u, \xi):=\mathbf{E}^0(e^{-\xi T_u})\,, \qquad u\in\mathbf{R}\,,$$

where ξ is a complex number with $Re\,\xi\geq 0$.

Theorem 3 (H. Cramér [2; p. 61]). *Suppose that*

$$(3.4) \qquad \int_{-\infty}^{0}|y|\nu(dy)+\int_{0}^{\infty}e^{sy}\nu(dy)<\infty \qquad \textit{for some } s>0,$$

$$(3.5) \qquad a+\int_{-\infty}^{\infty}y\nu(dy)<0\,.$$

Then the function $\psi(u, \xi)$ satisfies the integral equation

$$(3.6) \quad a\psi(u, \xi)=\int_{u}^{\infty}S(v)dv+\xi\int_{u}^{\infty}\psi(v, \xi)dv+\int_{0}^{\infty}\psi(v, \xi)S(u-v)dv \qquad \textit{for } u\geq 0.$$

Proof. We will show that (3.6) is a variant of the harmonic equation (2.19) applied to a specific function. This gives an improvement of Cramér's original proof.

One first notes that it is enough to show (3.6) when ξ is a real number $\lambda \geq 0$, for $\psi(u, \xi)$ is analytic on $\{\xi; Re\,\xi > 0\}$ and continuous on $\{\xi; Re\,\xi \geq 0\}$. Henceforth we will write $\lambda(\geq 0)$ for ξ.

For a fixed $\lambda \geq 0$ define

$$(3.7) \qquad f(x) := \mathbf{E}^x(e^{-\lambda T_0}) = \mathbf{E}^0(e^{-\lambda T_{-x}}) = \psi(-x, \lambda), \qquad x \in \mathbf{R}.$$

Equation (3.6) is then transformed into

$$(3.8) \quad af(x) = \int_{-\infty}^{x} S(-z)dz + \lambda \int_{-\infty}^{x} f(z)dz + \int_{-\infty}^{0} f(z)S(z-x)dz, \qquad x \leq 0.$$

Applying Theorem 2 to $f(x) = \mathbf{E}^x(e^{-\lambda T_0})$ (cf. (2.6)), one has

$$(3.9) \qquad (\lambda - A)f(x) = 0 \qquad \text{almost all } x < 0.^{1)}$$

On the other hand, due to Cramér [2; p. 57], condition (3.4) and (3.5) imply that

$$(3.10) \qquad |f(x)| = |\psi(-x, \lambda)| \leq e^{Rx},$$

where R is the supremum of $s > 0$ such that $\int s^{-1}(e^{sy}-1)\nu(dy) + a < 0$ and $\int e^{sy}\nu(dy)$ is analytic in s. One claims that (3.9) and (3.10) imply (3.8). The proof of this part is similar to the original proof of Cramér; he used an approximate equation [2; p. 62, eq. (89)] for the exact equation (3.9). We repeat his argument for the convenience of the reader.

Let $x_0 < x < 0$. Then,

$$(3.11) \quad \lambda \int_{x_0}^{x} f(z)dz - a[f(x) - f(x_0)] - \int_{x_0}^{x} \left\{ \int_{-\infty}^{\infty} (f(z+y) - f(z))\nu(dy) \right\} dz = 0.$$

Let us introduce the following notation;

$$f_1(z) \begin{cases} = f(z) & \text{for } z \leq 0 \\ = 0 & \text{for } z > 0, \end{cases}$$

$$f_2(z) \begin{cases} = 0 & \text{for } z \leq 0 \\ = f(z) = 1 & \text{for } z > 0, \end{cases}$$

$$\varphi(z) = \text{the indicator of the interval } [x_0, x),$$

$$\tilde{\nu}(dy) = \nu(-dy), \quad (f, g) = \int f(z)g(z)dz.$$

It follows that

1) When $a < 0$, an equation similar to (3.9) was obtained by Feller [3; p. 181].

$$\int_{-\infty}^{\infty} f(z+y)\nu(dy) = \tilde{\nu}*f_1(z)+\tilde{\nu}*f_2(z)\,,$$

$$\tilde{\nu}*f_2(z) = \int_{-z}^{\infty} \nu(dy) = -S(-z) \qquad \text{for } z<0,$$

$$\begin{aligned}\int_{x_0}^{x} \nu*f_1(z)dz &= (\tilde{\nu}*f_1,\ \varphi) = (f_1,\ \nu*\varphi) \\ &= \int_{-\infty}^{0} f(z)\cdot\nu*\varphi(z)dz \\ &= \int_{\infty-}^{0} f(z)\,\nu((z-x,\ z-x_0])dz\,.\end{aligned}$$

Therefore,

$$\lambda\int_{x_0}^{x} f(z)dz - a[f(x)-f(x_0)] - \int_{-\infty}^{0} f(z)\nu((z-x,\ z-x_0])dz + \int_{x_0}^{x} f(z)dz + \int_{x_0}^{x} S(-z)dz = 0\,.$$

Letting $x_0 \to -\infty$ and taking account of (3.4) and (3.10),

$$\lambda\int_{x_0}^{x} f(z)dz - af(x) - \int_{-\infty}^{0} f(z)\nu((z-x,\ \infty))dz + \int_{-\infty}^{x} f(z)dz + \int_{-\infty}^{x} S(-z)dz = 0\,,$$

which proves (3.8) by virtue of

$$-\int_{-\infty}^{0} f(z)\,\nu((z-x,\ \infty))\,dz + \int_{-\infty}^{x} f(z)dz = \int_{-\infty}^{0} f(z)S(z-x)dz\,.$$

But $f(x)$ and $S(z-x)$ are left-continuous at $x=0$. Hence, (3.8) is also valid for $x=0$.

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