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COHOMOLOGY OF DISCRETE SUBGROUPS OF $Sp(p, q)$

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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Introduction

Let G be a connected semi-simple Lie group with finite center and no compact factors. Let Γ be a uniform discrete subgroup of G and (ρ, F) be a finite dimensional irreducible representation of G. We are interested in the cohomology space $H^*(\Gamma, F)$. The purpose of this paper is to prove a non-vanishing theorem for $H^*(\Gamma, F)$ in the case of $G=Sp(p, q)$ $(p \geqq q \geqq 1)$.

As it is well-known, we can describe $H^*(\Gamma, F)$ in terms of the relative Lie algebra cohomology. Let $\mathfrak{g}$ be the Lie algebra of G and K be a maximal compact subgroup of G. Denote by $\hat{G}$ the unitary dual of G. For $(U, H_U) \in \hat{G}$, we denote by H_U^0 the space of K-finite vectors in H_U. Then H_U^0 is an irreducible $(\mathfrak{g}, K)$-module. Also $m(U, \Gamma)$ denotes the multiplicity with which U occurs in $L^2(\Gamma \backslash G)$. Define the subset $\hat{G}_\rho$ of $\hat{G}$ as follows;

$$\hat{G}_\rho = \{U \in \hat{G} \mid \chi_U = \chi_{\rho^*}\}$$

where ρ^* is the contragradient representation of ρ and χ_U (resp. χ_{ρ^*}) is the infinitesimal character of U (resp. ρ^*). Then, from the formula of Matsushima-Murakami ([1], VII, Theorem 6.1), we have

$$H^*(\Gamma, F) = \sum_{U \in \hat{G}_\rho} m(U, \Gamma) H^*(\mathfrak{g}, K; H_U^0 \otimes F). \tag{0.1}$$

From now on, we assume that G is simple. Depending on Kumaresan's work, Vogan and Zuckerman obtained the following precise vanishing theorem for the $(\mathfrak{g},K)$-cohomology ([5], Theorem 8.1); if U is non-trivial, we have

$$H^i(\mathfrak{g}, K; H_U^0 \otimes F) = \{0\} \qquad (i<r_G)$$

where r_G is the positive integer determined by G and given by Table 8.2 in [5] for non-complex groups. From this result and (0.1), if F is non-trivial, we have

$$H^i(\Gamma, F) = \{0\} \qquad (i<r_G).$$

Note that r_G depends only on G and, in general, $r_G \geqq \mathrm{rank}_{\boldsymbol{R}} G$. On the other hand, the vanishing of $H^i(\Gamma, F)$ below the $\boldsymbol{R}$-rank has been obtained in some papers ([1], VII, Proposition 6.4). There are some simple groups such that $r_G = \mathrm{rank}_{\boldsymbol{R}} G$. In the case of $G = SU(p, q)$ $(p \geqq q \geqq 1)$, where $r_G = q = \mathrm{rank}_{\boldsymbol{R}} G$, Borel and Wallach showed that this vanishing theorem is best possible ([1], VIII, Corollary 5.9).

We concentrate our attention on the case of $G = Sp(p, q)$. In this case, $r_G = 2q$ and hence $r_G > q = \mathrm{rank}_{\boldsymbol{R}} G$. Therefore it is interesting to ask if the above vanishing theorem is best possible for $G = Sp(p, q)$. In this paper, we show that, in the case of $G = Sp(p, q)$, the first possible non-zero cohomology $H^*(\Gamma, F)$ appears indeed at the degree $2q = r_G$. Main results are Theorem 3.4 and Theorem 4.2. In the case that F is trivial and $q = 1$, Theorem 3.4 is contained in the results of [3], Theorem 3.2 (see Remark 3.5). Also Theorem 4.2 for trivial F improves a part of the results of [4], Theorem 4.1 (see Remark 4.4.).

Our method is similar to that in [1], VIII and depends heavily on the results there.

1. The imbedding of $Sp(p, q)$ into $Sp(2n, \boldsymbol{R})$

1.1. Throughout this paper, G will denote the group $Sp(p, q)$ $(p \geqq q \geqq 1)$. At first we give our realization of G and provide some notations.

We set $n = p + q$. Let $K_{p,q}$ be the $2n \times 2n$ matrix given by

$$K_{p,q} = \left(\begin{array}{cc|cc} I_p & 0 & & \\ 0 & -I_q & \multicolumn{2}{c}{0} \\ \hline & & I_p & 0 \\ \multicolumn{2}{c|}{0} & 0 & -I_q \end{array}\right)$$

where I_m is the $m \times m$ identity matrix. The group G is given by

$$G = \{ g \in Sp(n, \boldsymbol{C}) \mid {}^t g K_{p,q} \bar{g} = K_{p,q} \}.$$

As a maximal compact subgroup of G, we choose $K = G \cap U(2n)$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. For a real Lie algebra $\mathfrak{u}$, denote by $\mathfrak{u}_c$ the complexification of $\mathfrak{u}$.

Let E_{ij} be the square matrix with 1 in the (i, j)-position and 0 elsewhere. For $1 \leqq i \leqq n$, set

$$T_i = \left(\begin{array}{c|c} E_{ii} & 0 \\ \hline 0 & -E_{ii} \end{array}\right)$$

and define

$$\mathfrak{t}=\{\sum_{j=1}^{n}\mu_jT_j\,|\,\mu_j\in\sqrt{-1}\boldsymbol{R}\}.$$

Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t}\subset\mathfrak{k}$. Also define $\lambda_i\in\mathfrak{t}_c^*$ $(1\leqq i\leqq n)$ by

$$\lambda_i(\sum_{j=1}^{n}\mu_jT_j)=\mu_i\,.$$

The root system Δ (resp. $\Delta_{\mathfrak{k}}$) of the pair $(\mathfrak{g}_c, \mathfrak{t}_c)$ (resp. $(\mathfrak{k}_c, \mathfrak{t}_c)$) is given by

$$\Delta=\{\pm\lambda_i\pm\lambda_j\,|\,1\leqq i, j\leqq n\}$$
$$(\text{resp. } \Delta_{\mathfrak{k}}=\{\pm\lambda_i\pm\lambda_j\,|\,1\leqq i, j\leqq p \quad\text{or}\quad p+1\leqq i, j\leqq p+q\})\,.$$

We choose an order of $(\sqrt{-1}\mathfrak{t})^*$ so that the set of simple roots in Δ is $\{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \cdots, \lambda_{n-1}-\lambda_n, 2\lambda_n\}$. Denote by Δ^+ (resp. $\Delta_{\mathfrak{k}}^+$) the set of positive roots in Δ (resp. $\Delta_{\mathfrak{k}}$). Throughout this paper we fix this order.

For later use, we choose root vectors of $\mathfrak{g}_c$ as follows;

$$X_{\lambda_i+\lambda_j}=\left(\begin{array}{c|c}0 & F_{ij}\\ \hline 0 & 0\end{array}\right)\qquad(1\leqq i,\ j\leqq n)$$

$$X_{-\lambda_i-\lambda_j}=\left(\begin{array}{c|c}0 & 0\\ \hline F_{ij} & 0\end{array}\right)\qquad(1\leqq i,\ j\leqq n)$$

$$X_{\lambda_i-\lambda_j}=\left(\begin{array}{c|c}E_{ij} & 0\\ \hline 0 & -E_{ji}\end{array}\right)\qquad(1\leqq i,\ j\leqq n,\ i\neq j)$$

where $F_{ij}=E_{ij}+E_{ji}$ if $i\neq j$ and $F_{ij}=E_{ii}$ if $i=j$. Then $\{T_i\,|\,1\leqq i\leqq n\}\cup\{X_\alpha\,|\,\alpha\in\Delta\}$ is a basis of $\mathfrak{g}_c$.

1.2. Now we construct an imbedding of G into $Sp(2n, \boldsymbol{R})$. Our imbedding is obtained by composing an imbedding of G into $SU(2p, 2q)$ and an imbedding of $SU(2p, 2q)$ into $Sp(2n, \boldsymbol{R})$. From now on, G' denotes the group $SU(2p, 2q)$. As a maximal compact subgroup of G', we choose $K'=G'\cap U(2n)$. Let $\mathfrak{g}'$ be the Lie algebra of G'.

The group G is naturally imbedded into the uintary group of the hermitian form on $\boldsymbol{C}^{2n}$ defined by $K_{p,q}$. We put

$$Z = \left(\begin{array}{c|cc|c} I_p & \multicolumn{2}{c|}{0} & 0 \\ \hline 0 & 0 & I_q & 0 \\ \cline{2-3} & I_p & 0 & \\ \hline 0 & \multicolumn{2}{c|}{0} & I_q \end{array}\right)$$

Then ${}^tZK_{p,q}Z$ gives the standard hermitian form with signature $(2p, 2q)$. So, if we define

$$\psi(g) = {}^tZgZ \qquad (g \in G),$$

we obtain an imbedding ψ; $G \to G'$. Clearly we have $\psi(K) \subset K'$.

Moreover we will imbed G' into $Sp(2n, \boldsymbol{R})$. Naturally we consider $GL(2n, \boldsymbol{C})$, and hence G', as to be the subgroups of $GL(4n, \boldsymbol{R})$. Define the orthogonal matrix Z' by

$$Z' = \left(\begin{array}{cc|c} I_{2p} & 0 & \\ \cline{1-2} 0 & -I_{2q} & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{2n} \end{array}\right)$$

Then it is easily checked that, if we define

$$\psi'(g) = {}^tZ'gZ' \qquad (g \in G'),$$

we obtain an imbedding ψ'; $G' \to Sp(2n, \boldsymbol{R})$. This is the same imbedding that is constructed in [1], VIII, § 2.

In this way we obtain the imbedding

$$\iota = \psi' \circ \psi;\ G \to Sp(2n, \boldsymbol{R}).$$

These imbeddings ψ, ψ' and ι induce the imbeddings of Lie algebras and we use the same letters for them;

$$\begin{aligned} \psi\ &;\ \mathfrak{g}_c \to \mathfrak{g}'_c \\ \psi' &;\ \mathfrak{g}'_c \to \mathfrak{sp}(2n, \boldsymbol{C}) \\ \iota\ &;\ \mathfrak{g}_c \to \mathfrak{sp}(2n, \boldsymbol{C}). \end{aligned}$$

1.3. Here we give the explicit form of the image of ι. It will be used in § 2. For this, we choose a basis of $\mathfrak{sp}(2n, \boldsymbol{C})$ as follows;

$$S_i = \sqrt{-1}\left(\begin{array}{c|c} 0 & E_{ii} \\ \hline -E_{ii} & 0 \end{array}\right) \qquad (1 \leqq i \leqq 2n),$$

$$Y^{+}_{ij} = \frac{1}{2}\left(\begin{array}{c|c} F_{ij} & -\sqrt{-1}F_{ij} \\ \hline -\sqrt{-1}F_{ij} & -F_{ij} \end{array}\right) \qquad (1 \leqq i,\ j \leqq 2n)$$

$$Y^{-}_{ij} = \frac{1}{2}\left(\begin{array}{c|c} F_{ij} & \sqrt{-1}F_{ij} \\ \hline \sqrt{-1}F_{ij} & -F_{ij} \end{array}\right) \qquad (1 \leqq i,\ j \leqq 2n)$$

$$Z^{+}_{ij} = \frac{1}{2}\left(\begin{array}{c|c} E_{ij}-E_{ji} & \sqrt{-1}F_{ij} \\ \hline -\sqrt{-1}F_{ij} & E_{ij}-E_{ji} \end{array}\right) \qquad (1 \leqq i < j \leqq 2n)$$

$$Z^{-}_{ij} = \frac{1}{2}\left(\begin{array}{c|c} E_{ji}-E_{ij} & \sqrt{-1}F_{ji} \\ \hline -\sqrt{-1}F_{ji} & E_{ji}-E_{ij} \end{array}\right) \qquad (1 \leqq i < j \leqq 2n)$$

where $F_{ij}=E_{ij}+E_{ji}$ if $i \neq j$ and $F_{ij}=E_{ii}$ if $i=j$. By straightforward computations we obtain the following explicit description for the image of ι; for $1 \leqq i<j \leqq p$ and $p+1 \leqq k<l \leqq p+q$,

$$(1.1)\qquad \begin{cases} \iota(T_i) = S_i - S_{p+i} \\ \iota(T_k) = -S_{p+k} + S_{p+q+k} \\ \iota(X_{\pm(\lambda_i+\lambda_j)}) = Z^{\pm}_{i,p+j} + Z^{\pm}_{j,p+i} \\ \iota(X_{\pm(\lambda_i+\lambda_k)}) = -Y^{\pm}_{i,p+q+k} + Y^{\mp}_{p+i,p+k} \\ \iota(X_{\pm(\lambda_k+\lambda_l)}) = -Z^{\mp}_{p+k,p+q+l} - Z^{\mp}_{p+l,p+q+k} \\ \iota(X_{\pm 2\lambda_i}) = Z^{\pm}_{i,p+i} \\ \iota(X_{\pm 2\lambda_k}) = -Z^{\mp}_{p+k,p+q+k} \\ \iota(X_{\pm(\lambda_i-\lambda_j)}) = Z^{\pm}_{i,j} - Z^{\mp}_{p+i,p+j} \\ \iota(X_{\pm(\lambda_i-\lambda_k)}) = -Y^{\pm}_{i,p+k} + Y^{\mp}_{p+i,p+q+k} \\ \iota(X_{\pm(\lambda_k-\lambda_l)}) = -Z^{\mp}_{p+k,p+l} + Z^{\pm}_{p+q+k,p+q+l}\,. \end{cases}$$

2. The construction of unitary representations

In this section, we construct a certain series of irreducible unitary representations of G. In [1] Borel and Wallach constructed some irreducible representations of G' by using the oscillator representation. Our representations are obtained from these representations through the imbedding ψ; $G \to G'$. We will often use the results and notations in [1], VIII.

2.1. First we sketch briefly the results in [1], VIII, § 2. Let $Mp(2n, \boldsymbol{R})$ be the Metaplectic group and $(W, L^2(\boldsymbol{R}^{2n}))$ be the oscillator representation of

$Mp(2n, \boldsymbol{R})$. The imbedding ψ'; $G' \to Sp(2n, \boldsymbol{R})$ lifts to an injective homomorphism $\tilde{\psi}'$; $G' \to Mp(2n, \boldsymbol{R})$ ([1], VIII, Lemma 2.9). Define the unitary representation $(V, L^2(\boldsymbol{R}^{2n}))$ of G' by

$$V(g) = W(\tilde{\psi}'(g)) \qquad (g \in G').$$

Then $(V, L^2(\boldsymbol{R}^{2n}))$ decomposes into the direct sum of irreducible representations of G'. In fact, for $r \in \boldsymbol{Z}$, define the subspace H_r of $L^2(\boldsymbol{R}^{2n})$ by

$$H_r = \{\phi \in L^2(\boldsymbol{R}^{2n}) \mid W(\operatorname{Exp} tJ_{2p,2q})(\phi) = \exp(-\sqrt{-1}(p-q+r)t)\phi\}$$

where Exp is the exponential mapping of $\mathfrak{sp}(2n, \boldsymbol{R})$ into $Mp(2n, \boldsymbol{R})$ and

$$J_{2p,2q} = \left(\begin{array}{c|c} 0 & \begin{array}{c|c} -I_{2p} & 0 \\ \hline 0 & I_{2q} \end{array} \\ \hline \begin{array}{c|c} I_{2p} & 0 \\ \hline 0 & -I_{2q} \end{array} & 0 \end{array}\right) \in \mathfrak{sp}(2n, \boldsymbol{R}).$$

Then H_r is stable under G' and so we put

$$V_r(g) = V(g)|_{H_r} \qquad (g \in G').$$

From [1], VIII, Lemma 2.8, for each $r \in \boldsymbol{Z}$, (V_r, H_r) is an irreducible unitary representation of G' and we have

$$L^2(\boldsymbol{R}^{2n}) = \bigoplus_{r \in \boldsymbol{Z}} H_r.$$

In the remainder of this section, we fix $r \in \boldsymbol{Z}$. Denote by $\mathcal{S}(\boldsymbol{R}^{2n})$ the Schwartz space on $\boldsymbol{R}^{2n}$ with the Schwartz topology and set $H_r^\infty = H_r \cap \mathcal{S}(\boldsymbol{R}^{2n})$. Then H_r^∞ is the space of C^∞-vectors for V_r in H_r ([1], VIII. Lemma 1.11). Also, we denote by H_r^0 the space of K'-finite vectors for V_r in H_r. The space H_r^0 is an irreducible admissible $(\mathfrak{g}', K')$-module.

In order to choose an orthogonal basis of H_r^0, we need some notations. Let $(x_1, \cdots, x_{2n})$ be the coordinates of $\boldsymbol{R}^{2n}$. Following [1], VIII, 1.16, for $1 \leqq j \leqq 2n$, define the operator D_j and $A_j^\pm$ by

$$D_j = \frac{1}{2}\left(\frac{\partial^2}{\partial x_j^2} - x_j^2\right), \quad A_j^\pm = \frac{1}{2}\left(\frac{\partial}{\partial x_j} \pm x_j\right).$$

Denote by $\boldsymbol{Z}_+$ the set of non-negative integers. For $m = (m_1, \cdots, m_{2n}) \in (\boldsymbol{Z}_+)^{2n}$, we set

$$\phi_m = (A_1^-)^{m_1}(A_2^-)^{m_2} \cdots (A_{2n}^-)^{m_{2n}}\phi_0$$

where ϕ_0 is the C^∞-function on $\boldsymbol{R}^{2n}$ defined by

$$\phi_0(x) = (2\pi)^{-n} \exp\left(-\frac{1}{2}\sum_{i=1}^{2n} x_i^2\right) \qquad (x \in \boldsymbol{R}^{2n}).$$

(Note that ϕ_m is equal to ψ_m in [1], VIII 1.16, up to the multiplication by a constant.) Then, by [1], VIII, Lemma 1.17, $\{\phi_m \mid m \in (\boldsymbol{Z}_+)^{2n}\}$ are mutually orthogonal in $L^2(\boldsymbol{R}^{2n})$ and we have

$$H_r^0 = \bigoplus_{m \in \Phi_r} \boldsymbol{C}\phi_m \tag{2.1}$$

where $\Phi_r = \{m \in (\boldsymbol{Z}_+)^{2n} \mid \sum_{i=1}^{2p} m_i - \sum_{i=2p+1}^{2n} m_i = r\}$.

2.2. Now we construct unitary representations of G. Using the imbedding ψ; $G \to G'$, we define

$$U_r(g) = V_r(\psi(g)) \qquad (g \in G).$$

Then we obtain the unitary representation (U_r, H_r) of G. Clearly, the subspace H_r^0 of H_r is included in the space of K-finite vectors for U_r in H_r and stable under $\mathfrak{g}$ and K. Thus H_r^0 is a $(\mathfrak{g}, K)$-module. The infinitesimal representation of $\mathfrak{g}_c$ on H_r^0 induced from U_r is denoted by the same letter U_r.

We will examine the $(\mathfrak{g}, K)$-module H_r^0 in detail. First we consider the infinitesimal representation $(W, \mathcal{S}(\boldsymbol{R}^{2n}))$ of $\mathfrak{sp}(2n, \boldsymbol{C})$ induced from $(W, L^2(\boldsymbol{R}^{2n}))$. By [2], p. 232, Theorem 5.4, the action of $\mathfrak{sp}(2n, \boldsymbol{C})$ on $\mathcal{S}(\boldsymbol{R}^{2n})$ is explicitly given as follows;

$$\begin{cases} W(S_i) = D_i & (1 \leqq i \leqq 2n) \\ W(Y_{ij}^{\pm}) = \pm 2A_i^{\pm}A_j^{\pm} & (1 \leqq i,\ j \leqq 2n,\ i \neq j) \\ W(Y_{ii}^{\pm}) = \pm A_i^{\pm}A_i^{\pm} & (1 \leqq i \leqq 2n) \\ W(Z_{ij}^{\pm}) = 2A_i^{\pm}A_j^{\mp} & (1 \leqq i < j \leqq 2n). \end{cases} \tag{2.2}$$

Using the relation formulas among D_j and $A_j^{\pm}$ in [1], VIII, 1.16, we obtain

$$\begin{cases} D_j(\phi_m) = -\frac{1}{2}(2m_j+1)\phi_{m_1,\cdots,m_{2n}} \\ A_i^+A_j^+(\phi_m) = \frac{1}{4}m_im_j\phi_{m_1,\cdots,m_i-1,\cdots,m_j-1,\cdots,m_{2n}} \\ A_i^+A_i^+(\phi_m) = \frac{1}{4}m_i(m_i-1)\phi_{m_1,\cdots,m_i-2,\cdots,m_{2n}} \\ A_i^-A_j^-(\phi_m) = \phi_{m_1,\cdots,m_i+1,\cdots,m_j+1,\cdots,m_{2n}} \\ A_i^-A_i^-(\phi_m) = \phi_{m_1,\cdots,m_i+2,\cdots,m_{2n}} \\ A_i^+A_j^-(\phi_m) = -\frac{1}{2}m_i\phi_{m_1,\cdots,m_i-1,\cdots,m_j+1,\cdots m_{2n}} \end{cases} \tag{2.3}$$

where $m\in(\boldsymbol{Z}_+)^{2n}$, $1\leqq i<j\leqq 2n$ and $\phi_{k_1,\cdots,k_{2n}}$ is considered to be 0 if $k_i<0$ for some i. Therefore, combining (1.1), (2.2) and (2.3), we have the following formulas; for $1\leqq i, j\leqq p$ and $p+1\leqq k, l\leqq p+q$,

$$\tag{2.4} \begin{cases} U_r(T_i)(\phi_m)=(m_{p+i}-m_i)\phi_m \\ U_r(T_k)(\phi_m)=(m_{p+k}-m_{p+q+k})\phi_m \end{cases}$$

$$\tag{2.5} \begin{cases} U_r(X_{\lambda_i+\lambda^j})(\phi_m)=-m_i\phi_{m_1,\cdots,m_i-1,\cdots,m_{p+j}+1,\cdots,m_{2n}} \\ \qquad\qquad -m_j\phi_{m_1,\cdots,m_j-1,\cdots,m_{p+i}+1,\cdots,m_{2n}} \\ U_r(X_{2\lambda_i})(\phi_m)=-m_i\phi_{m_1,\cdots,m_i-1,\cdots,m_{p+i}+1,\cdots,m_{2n}} \\ U_r(X_{\lambda_k+\lambda_l})(\phi_m)=m_{p+q+l}\phi_{m_1,\cdots,m_{p+k}+1,\cdots,m_{p+q+l}-1,\cdots,m_{2n}} \\ \qquad\qquad +m_{p+q+k}\phi_{m_1,\cdots,m_{p+l}+1,\cdots,m_{p+q+k}-1,\cdots,m_{2n}} \\ U_r(X_{2\lambda_k})(\phi_m)=m_{p+q+k}\phi_{m_1,\cdots,m_{p+k}+1,\cdots,m_{p+q+k}-1,\cdots,m_{2n}} \\ U_r(X_{\lambda_i-\lambda_j})(\phi_m)=-m_i\phi_{m_1,\cdots,m_i-1,\cdots,m_j+1,\cdots,m_{2n}} \\ \qquad\qquad +m_{p+j}\phi_{m_1,\cdots,m_{p+i}+1,\cdots,m_{p+j}-1,\cdots,m_{2n}} \\ U_r(X_{\lambda_k-\lambda_l})(\phi_m)=m_{p+l}\phi_{m_1,\cdots,m_{p+k}+1,\cdots,m_{p+l}-1,\cdots,m_{2n}} \\ \qquad\qquad -m_{p+q+k}\phi_{m_1,\cdots,m_{p+q+k}-1,\cdots,m_{p+q+l}+1,\cdots,m_{2n}} \end{cases}$$

$$\tag{2.6} \begin{cases} U_r(X_{-\lambda_i-\lambda_j})(\phi_m)=-m_{p+i}\phi_{m_1,\cdots,m_j+1,\cdots,m_{p+i}-1,\cdots,m_{2n}} \\ \qquad\qquad -m_{p+j}\phi_{m_1,\cdots,m_i+1,\cdots,m_{p+j}-1,\cdots,m_{2n}} \\ U_r(X_{-2\lambda_i})(\phi_m)=-m_{p+i}\phi_{m_1,\cdots,m_i+1,\cdots,m_{p+i}-1,\cdots,m_{2n}} \\ U_r(X_{-\lambda_k-\lambda_l})(\phi_m)=m_{p+l}\phi_{m_1,\cdots,m_{p+l}-1,\cdots,m_{p+q+k}+1,\cdots,m_{2n}} \\ \qquad\qquad +m_{p+k}\phi_{m_1,\cdots,m_{p+k}-1,\cdots,m_{p+q+l}+1,\cdots,m_{2n}} \\ U_r(X_{-2\lambda_k})(\phi_m)=m_{p+k}\phi_{m_1,\cdots,m_{p+k}-1,\cdots,m_{p+q+k}+1,\cdots,m_{2n}} \\ U_r(X_{-\lambda_i+\lambda_j})(\phi_m)=-m_j\phi_{m_1,\cdots,m_i+1,\cdots,m_j-1,\cdots,m_{2n}} \\ \qquad\qquad +m_{p+i}\phi_{m_1,\cdots,m_{p+i}-1,\cdots,m_{p+j}+1,\cdots,m_{2n}} \\ U_r(X_{-\lambda_k+\lambda_l})(\phi_m)=m_{p+k}\phi_{m_1,\cdots,m_{p+k}-1,\cdots,m_{p+l}+1,\cdots,m_{2n}} \\ \qquad\qquad -m_{p+q+l}\phi_{m_1,\cdots,m_{p+q+k}+1,\cdots,m_{p+q+l}-1,\cdots,m_{2n}} \end{cases}$$

$$\tag{2.7} \begin{cases} U_r(X_{\lambda_i+\lambda_k})(\phi_m)=-\frac{1}{2}m_im_{p+q+k}\phi_{m_1,\cdots,m_i-1,\cdots,m_{p+q+k}-1,\cdots,m_{2n}} \\ \qquad\qquad +2\phi_{m_1,\cdots,m_{p+i}+1,\cdots,m_{p+k}+1,\cdots,m_{2n}} \\ U_r(X_{\lambda_i-\lambda_k})(\phi_m)=-\frac{1}{2}m_im_{p+k}\phi_{m_1,\cdots,m_i-1,\cdots,m_{p+k}-1,\cdots,m_{2n}} \\ \qquad\qquad -2\phi_{m_1,\cdots,m_{p+i}+1,\cdots,m_{p+q+k}+1,\cdots,m_{2n}} \end{cases}$$

$$\tag{2.8} \begin{cases} U_r(X_{-\lambda_i-\lambda_k})(\phi_m)=-\frac{1}{2}m_{p+i}m_{p+k}\phi_{m_1,\cdots,m_{p+i}-1,\cdots,m_{p+k}-1,\cdots,m_{2n}} \\ \qquad\qquad +2\phi_{m_1,\cdots,m_i+1,\cdots,m_{p+q+k}+1,\cdots,m_{2n}} \end{cases}$$

$$\left| \begin{aligned} U_r(X_{-\lambda_i+\lambda_k})(\phi_m) &= \frac{1}{2} m_{p+i} m_{p+q+k} \phi_{m_1,\cdots,m_{p+i}-1,\cdots,m_{p+q+k}-1,\cdots,m_{2n}} \\ &\quad + 2\phi_{m_1,\cdots,m_i+1,\cdots,m_{p+k}+1,\cdots,m_{2n}} \end{aligned} \right.$$

Of course, in these formulas, $\phi_{k_1,\cdots,k_{2n}}$ should be considered as to be 0 if $k_i<0$ for some i.

Now we can determine the set of weights of the $\mathfrak{g}_c$-module H_r^0. Let ϕ_m be in H_r^0. By (2.4) we have

$$U_r(\sum_{i=1}^{n} \mu_i T_i)(\phi_m) = \{\sum_{i=1}^{p} (m_{p+i}-m_i)\mu_i + \sum_{k=p+1}^{p+q} (m_{p+k}-m_{p+q+k})\mu_k\} \phi_m$$

From this, the following lemma immediately follows.

Lemma 2.1. *Let $m=(m_1, \cdots, m_{2n})$ be in Φ_r. In the $\mathfrak{g}_c$-module H_r^0, ϕ_m is a weight vector corresponding to the weight*

$$\Lambda_m = \sum_{i=1}^{p} (m_{p+i}-m_i)\lambda_i + \sum_{k=p+1}^{p+q} (m_{p+k}-m_{p+q+k})\lambda_k \,.$$

We remark that the multiplicity of Λ_m in H_0^r is not finite.

2.3. Here we determine the K-spectrum of H_r^0. Let $\hat{K}$ be the set of all equivalence classes of irreducible representations of K. Define the subset D_K of $\mathfrak{t}_c^*$ by

$$D_K = \left\{ \lambda = \sum_{i=1}^{n} a_i\lambda_i \;\middle|\; \begin{array}{l} a_i \in \boldsymbol{Z} \\ a_1 \geqq a_2 \geqq \cdots \geqq a_p \geqq 0 \\ a_{p+1} \geqq a_{p+2} \geqq \cdots \geqq a_n \geqq 0 \end{array} \right\}.$$

Then there is the bijective correspondence between $\hat{K}$ and D_K. That is, $\lambda \in D_K$ corresponds to the irreducible K-module with highest weight λ. We denote by E_λ this K-module.

Let $s \in \boldsymbol{Z}_+$ and $s \geqq -r$. We define the finite dimensional subspace $H_{r,s}^0$ of H_r^0 by

$$H_{r,s}^0 = \bigoplus_{m \in \Phi_{r,s}} \boldsymbol{C}\phi_m \,,$$

where the subset $\Phi_{r,s}$ of Φ_r is given by

$$\Phi_{r,s} = \{m \in (\boldsymbol{Z}_+)^{2n} \mid \sum_{i=1}^{2p} m_i = r+s, \sum_{i=2p+1}^{2n} m_i = s\} \,.$$

From (2.1), we have

$$H_r^0 = \bigoplus_{s \in \boldsymbol{Z}_+, s \geqq -r} H_{r,s}^0 \,.$$

Proposition 2.2. *Let* $s \in \boldsymbol{Z}_+$ *and* $s \geqq -r$. *Then* $H^0_{r,s}$ *is the irreducible* K-*submodule of* H^0_r *with highest weight* $(r+s)\lambda_1+s\lambda_{p+1} \in D_K$. Hence we have

$$H^0_r = \bigoplus_{s\in \boldsymbol{Z}_+, s\geqq -r} E_{(r+s)\lambda_1+s\lambda_{p+1}}$$

as K-*modules.*

Proof. Put $E_s = E_{(r+s)\lambda_1+s\lambda_{p+1}}$. Let X be in $\mathfrak{k}_c$. By (2.4), (2.5) and (2.6), $U_r(X)(\phi_m)$ is a linear combination of $\phi_{m'} = \phi_{m'_1,\cdots,m'_{2n}}$ such that

$$\sum_{i=1}^{2p} m'_i = \sum_{i=1}^{2p} m_i\,, \quad \sum_{i=2p+1}^{2n} m'_i = \sum_{i=2p+1}^{2n} m_i\,.$$

Therefore $H^0_{r,s}$ is stable under $\mathfrak{k}_c$.

Now we put $\phi = \phi_{0,\cdots,0,r+s,0,\cdots,0,s,0,\cdots,0}$, where $r+s$ (resp. s) appears in the $(p+1)$-th (resp. $(2p+1)$-th) position. Then $\phi \in H^0_{r,s}$ and, by Lemma 2.1, ϕ is a weight vector corresponding to the weight $(r+s)\lambda_1+s\lambda_{p+1}$. It is easy to see that this weight is the highest among all the weights for $H^0_{r,s}$. Hence E_s certainly occurs in $H^0_{r,s}$.

We compare the dimension of $H^0_{r,s}$ with that of E_s. Since $\{\phi_m \mid m \in \Phi_{r,s}\}$ is a basis of $H^0_{r,s}$, we have

$$\begin{aligned}\dim H^0_{r,s} &= \#\Phi_{r,s} \\ &= \binom{2p+r+s-1}{r+s}\cdot\binom{2q+s-1}{s} \\ &= \frac{(2p+r+s-1)!(2q+s-1)!}{(2p-1)!(r+s)!(2q-1)!s!}\,.\end{aligned}$$

On the other hand, Weyl's dimension formula gives the dimension of E_s. Denote by $(\ ,\ )_{\mathfrak{k}}$ the inner product in $(\sqrt{-1}\mathfrak{t})^*$ induced from the Killing form of $\mathfrak{k}_c$. Recall that

$$(\lambda_i, \lambda_j)_{\mathfrak{k}} = 0 \qquad \text{if } i \neq j\,,$$

$$(\lambda_i, \lambda_i)_{\mathfrak{k}} = \begin{cases} (4p+4)^{-1} & \text{if } 1 \leqq i \leqq p\,, \\ (4q+4)^{-1} & \text{if } p+1 \leqq i \leqq p+q\,. \end{cases}$$

Also put $\delta_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha\in\Delta^+_{\mathfrak{k}}} \alpha$. Then we have

$$\delta_{\mathfrak{k}} = \sum_{i=1}^{p} (p-i+1)\lambda_i + \sum_{k=p+1}^{p+q} (p+q-k+1)\lambda_k\,.$$

From these formulas, easy calculations yield

$$\dim E_s = \frac{\prod_{\alpha\in\Delta_{\mathfrak{k}}^+}((r+s)\lambda_1+s\lambda_{p+1}+\delta_{\mathfrak{k}},\alpha)_{\mathfrak{k}}}{\prod_{\alpha\in\Delta_{\mathfrak{k}}^+}(\delta_{\mathfrak{k}},\alpha)_{\mathfrak{k}}}$$
$$= \frac{(2p+r+s-1)!(2q+s-1)!}{(2p-1)!(r+s)!(2q-1)!s!}$$
$$= \dim H^0_{r,s}\,.$$

Hence $H^0_{r,s}$ is equivalent to E_s.

2.4. In this stage, we must determine the space of K-finite vectors in H_r for U_r.

Lemma 2.3. *The space of K-finite vectors in H_r for U_r coincides with H^0_r.*

Proof. For $\tau\in\hat{K}$, let $H_r(\tau)$ be the isotypic K-submodule of H_r of type τ. Clearly H^0_r is stable under K and $H^0_r\subset \bigoplus_{\tau\in\hat{K}} H_r(\tau)$. Hence we have $H^0_r=\bigoplus_{\tau\in\hat{K}} H^0_r\cap H_r(\tau)$. Since H^0_r is dense in H_r, by [7], Chapter 4, Proposition 4.4.3.4, the closure of $H^0_r\cap H_r(\tau)$ is $H_r(\tau)$. By Proposition 2.2, $H^0_r\cap H_r(\tau)$ is finite dimensional. Therefore we have $H^0_r\cap H_r(\tau)=H_r(\tau)$ and hence $H^0_r=\bigoplus_{\tau\in\hat{K}} H_r(\tau)$. The lemma is proved.

Together with Proposition 2.2, this lemma shows that (U_r, H_r) is admissible. Moreover we have the following proposition.

Proposition 2.4. *For $r\in\boldsymbol{Z}$, the unitary representation (U_r, H_r) of G is irreducible.*

Proof. From [7], Chapter 4, Theorem 4.5.5.4, it is sufficient to prove that the $\mathfrak{g}$-module H^0_r is algebraically irreducible. Let H be a non-zero $\mathfrak{g}$-stable subspace of H^0_r. Since H is stable under $\mathfrak{k}$, by Proposition 2.2, we have

$$H=\bigoplus_{s\in S(H)} H^0_{r,s}\,,$$

where $S(H)$ is a non-empty subset of $\boldsymbol{Z}_+$. Suppose $s_0\in S(H)$, that is, $H^0_{r,s_0}\subset H$. We take a particular element

$$\phi=\phi_{0,\cdots,0,r+s_0,0,\cdots,0,s_0,0,\cdots,0}$$

in H^0_{r,s_0}, where $r+s_0$ (resp. s_0) appears in the $(p+1)$-th (resp. $(2p+1)$-th) position. Then, by (2.7), we have

$$U_r(X_{\lambda_1+\lambda_{p+1}})(\phi)=2\phi_{0,\cdots,0,r+s_0+1,0,\cdots,0,s_0+1,0\cdots,0}\,.$$

Here the left hand side belongs to H and the right hand side belongs to H^0_{r,s_0+1}.

This implies $H \cap H^0_{r,s_0+1} \neq \{0\}$. Therefore we have $H^0_{r,s_0+1} \subset H$, that is, $s_0+1 \in S(H)$.

Similarly, if $s_0 > \max\{0, -r\}$, we have

$$U_r(X_{-\lambda_1-\lambda_{p+1}})(\phi) = -\frac{1}{2}(r+s_0)s_0\,\phi_{0,\cdots,0,r+s_0-1,0,\cdots,0,s_0-1,0,\cdots,0} + 2\phi_{1,0,\cdots,0,r+s_0,0,\cdots,0,s_0,0,\cdots,0,1,0,\cdots,0}\,,$$

where 1 appears in the first and $(2p+q+1)$-th position. In this formula, the first term of the right hand side belongs to H^0_{r,s_0-1} and the second term belongs to H^0_{r,s_0+1}. Since $H^0_{r,s_0+1} \subset H$, we have $H \cap H^0_{r,s_0-1} \neq \{0\}$ and hence $s_0-1 \in S(H)$.

By the induction, we have $S(H) = \{s \in \boldsymbol{Z}_+ \mid s \geqq -r\}$, that is, $H = H^0_r$. This proves the proposition.

After all we obtain a series of irreducible unitary representations of G; $\{(U_r, H_r) \mid r \in \boldsymbol{Z}\}$.

3. The $(\mathfrak{g}, K)$-cohomology

In this section, we study the $(\mathfrak{g}, K)$-cohomology space of the $(\mathfrak{g}, K)$-module H^0_r $(r \in \boldsymbol{Z})$.

3.1. First of all we recall a known result which is our starting point. Let (U, H_U) be in $\hat{G}$ and (ρ, F) be a finite dimensional irreducible representation of G. Denote by $\mathfrak{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_c$. The representation of $\mathfrak{U}(\mathfrak{g})$ induced by U (resp. ρ) is denoted by the same letter U (resp. ρ). Let C be the Casimir element of $\mathfrak{g}_c$. Then both the operators $U(C)$ and $\rho(C)$ are the scalar operators. Put $U(C) = c_U \cdot \mathrm{Id}$ and $\rho(C) = c_\rho \cdot \mathrm{Id}$, where $c_U, c_\rho \in \boldsymbol{C}$ and Id denotes the identity operator. If we note that K is connected, we have the following lemma.

Lemma 3.1. ([1], II, Proposition 3.1)
(1). *If $c_U \neq c_\rho$, then $H^j(\mathfrak{g}, K; H^0_U \otimes F) = \{0\}$ for all $j \in \boldsymbol{Z}_+$.*
(2). *If $c_U = c_\rho$, then $H^j(\mathfrak{g}, K; H^0_U \otimes F) = \mathrm{Hom}_K(\wedge^j \mathfrak{p}, H^0_U \otimes F)$ for all $j \in \boldsymbol{Z}_+$.*

3.2. For $(U_r, H_r) \in \hat{G}$, we will calculate the operator $U_r(C)$.

Proposition 3.2. *For $r \in \boldsymbol{Z}$, we have*

$$U_r(C) = (4n+4)^{-1}(r+2p)(r-2q) \cdot \mathrm{Id}\,.$$

Proof. We use a concrete realization of C and calculate explicitly the action of $U_r(C)$ on a particular element in H^0_r.

Recall that the Killing form of $\mathfrak{g}_c$ is given by

$$(X, Y) = 2(n+1)\,\mathrm{Tr}\,XY \qquad (X, Y \in \mathfrak{g}_c)\,.$$

Using the basis of $\mathfrak{g}_c$ in 1.1., we have

$$4(n+1)C = \sum_{i=1}^{n} T_i T_i + \sum_{1\leqq i<j\leqq n} (X_{\lambda_i+\lambda_j} X_{-\lambda_i-\lambda_j} + X_{-\lambda_i-\lambda_j} X_{\lambda_i+\lambda_j})$$
$$+2\sum_{i=1}^{n} (X_{2\lambda_i} X_{-2\lambda_i} + X_{-2\lambda_i} X_{2\lambda_i})$$
$$+ \sum_{1\leqq i<j\leqq n} (X_{\lambda_i-\lambda_j} X_{\lambda_j-\lambda_i} + X_{\lambda_j-\lambda_i} X_{\lambda_i-\lambda_j}) .$$

First we consider the case that $r\geqq 0$. Take a particular element $\phi=\phi_{r,0,\cdots,0}\in H_r^0$. Using (2.4), $\cdots$, (2.8), we calculate straightforwardly $4(n+1)\,U_r(C)(\phi)$. Some terms turn out to vanish and the other terms are given as follows;

$$\sum_{i=1}^{n} U_r(T_i T_i)(\phi) = r^2\phi$$

$$U_r(X_{\lambda_i\pm\lambda_k} X_{-\lambda_i\mp\lambda_k})(\phi) = \begin{cases} -(r+1)\phi\pm 4\phi' & \text{if } i=1 \\ -\phi\pm 4\phi'' & \text{if } i\neq 1 \end{cases}$$

$$U_r(X_{-\lambda_i\mp\lambda_j} X_{\lambda_i\pm\lambda_j})(\phi) = \begin{cases} r\phi & \text{if } i=1 \\ 0 & \text{if } i\neq 1 \end{cases}$$

$$U_r(X_{-\lambda_i\mp\lambda_k} X_{\lambda_i\pm\lambda_k})(\phi) = \begin{cases} -\phi\pm 4\phi' & \text{if } i=1 \\ -\phi\pm 4\phi'' & \text{if } i\neq 1 \end{cases}$$

$$2U_r(X_{-2\lambda_i} X_{2\lambda_i})(\phi) = \begin{cases} 2r\phi & \text{if } i=1 \\ 0 & \text{if } i\neq 1 \end{cases}$$

where $1\leqq i<j\leqq p$, $p+1\leqq k\leqq p+q$ and ϕ', ϕ'' are certain elements in H_r^0 determined by ϕ. From these formulas, we can easily show that

$$4(n+1)U_r(C)(\phi) = (r+2p)(r-2q)\phi .$$

In the case that $r<0$, if we take $\phi=\phi_{0,\cdots,0,-r}\in H_r^0$, similar calculations yield the above formula. Thus the proposition is proved.

3.3. Now we will show the non-vanishing of the $(\mathfrak{g}, K)$-cohomology of H_r^0. For this, we need the following lemma.

Lemma 3.3. *For $2q\lambda_1\in D_K$, we have*

$$\dim \mathrm{Hom}_K(\wedge^{2q}\mathfrak{p}, E_{2q\lambda_1}) = 1 .$$

Proof. Any weight of $\wedge^{2q}\mathfrak{p}_c$ is the sum of $2q$ distinct non-compact roots of $\mathfrak{g}_c$. Since we have

$$2q\lambda_1 = \sum_{k=p+1}^{p+q} \{(\lambda_1+\lambda_k)+(\lambda_1-\lambda_k)\} ,$$

$2q\lambda_1$ is a weight of $\wedge^{2q}\mathfrak{p}_c$ with multiplicity 1. It is easy to see that $2q\lambda_1$ is the highest among all the weights of $\wedge^{2q}\mathfrak{p}_c$. The lemma is proved.

For $l \in \boldsymbol{Z}_+$, $l\lambda_1$ is a dominant integral form for $(\mathfrak{g}_c, \mathfrak{t}_c)$. Denote by (ρ_l, F_l) the irreducible finite dimensional representation of G with highest weight $l\lambda_1$; that is, (ρ_l, F_l) is the l-th symmetric tensor product of the standard representation of G on $\boldsymbol{C}^{2n}$. Let (ρ_l^*, F_l^*) be the contragradient representation of (ρ_l, F_l).

Theorem 3.4. *If $r \geqq 2q$, then we have*

$$H^{2q}(\mathfrak{g}, K; H_r^0 \otimes F_{r-2q}^*) \neq \{0\}.$$

Proof. As it is well-known, the operator $\rho_{r-2q}^*(C)$ is given by

$$\rho_{r-2q}^*(C) = \{((r-2q)\lambda_1+\delta, (r-2q)\lambda_1+\delta)-(\delta, \delta)\} \cdot \mathrm{Id},$$

where $(\ , \ )$ is the inner product in $(\sqrt{-1}\mathfrak{t})^*$ induced from the Killing form of $\mathfrak{g}_c$ and $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Note that

$$\delta = \sum_{i=1}^{n} (n-i+1)\lambda_i,$$

$$(\lambda_i, \lambda_j) = (4n+4)^{-1}\delta_{ij} \qquad (1 \leqq i, j \leqq n).$$

By easy computations, we have

$$\rho_{r-2q}^*(C) = (4n+4)^{-1}(r+2p)(r-2q) \cdot \mathrm{Id}.$$

From this and Proposition 3.2, $U_r(C)$ and $\rho_{r-2q}^*(C)$ act as the multiplication by the same scalar. Hence Lemma 3.1 implies that

$$\begin{aligned} \dim H^{2q}(\mathfrak{g}, K; H_r^0 \otimes F_{r-2q}^*) &= \dim \mathrm{Hom}_K (\wedge^{2q}\mathfrak{p}, H_r^0 \otimes F_{r-2q}^*) \\ &= \dim \mathrm{Hom}_K (\wedge^{2q}\mathfrak{p} \otimes F_{r-2q}, H_r^0). \end{aligned}$$

On the other hand, by Proposition 2.2, we have

$$\dim \mathrm{Hom}_K (E_{r\lambda_1}, H_r^0) = 1. \tag{3.1}$$

Also, since $r\lambda_1 = 2q\lambda_1 + (r-2q)\lambda_1$, Lemma 3.3 implies that

$$\dim \mathrm{Hom}_K (E_{r\lambda_1}, \wedge^{2q}\mathfrak{p} \otimes F_{r-2q}) \neq 0. \tag{3.2}$$

Therefore, combining (3.1) and (3.2), we have

$$\dim \mathrm{Hom}_K (\wedge^{2q}\mathfrak{p} \otimes F_{r-2q}, H_r^0) \neq 0.$$

This proves the theorem.

REMARK 3.5. By Theorem 1.4 in [5], there is at most one irreducible unitary representation (U, H_U) such that $U(C)$ acts by the same scalar as

$\rho^*_{r-2q}(C)$ and $E_{r\lambda_1}$ occurs in H^0_U. Our representation (U_r, H_r) is this very representation. Therefore we can determine the position of U_r in the Langlands' classification. In the case of $q=1$, (U_2, H_2) is equivalent to the Langlands' representation $J_{1,2}$ in [3], Theorem 3.2.

4. The imbedding of U_r into $L^2(\Gamma\backslash G)$

In this section, we fix $r\in \boldsymbol{Z}$. We will construct a certain uniform discrete subgroup Γ of G such that $m(U_r, \Gamma)\neq 0$. Together with Theorem 3.4 and (0.1), this will prove the non-vanishing of the cohomology of Γ. The results in this section depend heavily on the results in [1], VIII, § 5.

4.1. Our discrete subgroup will be constructed arithmetically. First we realize G and G' as subgroups of linear algebraic groups.

Let k be a totally real finite extension of $\boldsymbol{Q}$ and d be the degree of k over $\boldsymbol{Q}$. Assume that $d\geqq 2$. Let $\Sigma=\{\sigma_1, \cdots, \sigma_d\}$ be the set of isomorphisms of k into $\boldsymbol{R}$. We regard k as a subfield of $\boldsymbol{R}$ so that σ_1 is the identity mapping. Put $k'=k(\sqrt{-1})$. We extend $\sigma\in\Sigma$ to the imbedding of k' into $\boldsymbol{C}$ which leaves $\sqrt{-1}$ fixed. If $\boldsymbol{H}$ is a linear algebraic group in $GL(l, \boldsymbol{C})$ defined over k or $\boldsymbol{Q}$ and $\boldsymbol{B}$ is a subfield of $\boldsymbol{C}$, we put $\boldsymbol{H}(\boldsymbol{B})=\boldsymbol{H}\cap GL(l, \boldsymbol{B})$.

Denote by $E_{k'}$ the vector space $(k')^{2n}$. We can choose $a\in k$ so that a is positive and the conjugates ${}^\sigma a$ by $\sigma\in\Sigma$ $(\sigma\neq\sigma_1)$ are all negative. Fix such a. Let h (resp. b) be a non-degenerate hermitian form (resp. a non-degenerate skew-symmetric bilinear form) on $E_{k'}$ defined by the matrix

$$\left(\begin{array}{cc|cc} I_p & 0 & & \\ 0 & -aI_q & \multicolumn{2}{c}{0} \\ \hline & & I_p & 0 \\ \multicolumn{2}{c|}{0} & 0 & -aI_q \end{array}\right) \quad \left(\text{resp.}\ \left(\begin{array}{cc|cc} & & I_p & 0 \\ \multicolumn{2}{c|}{0} & 0 & aI_q \\ \hline -I_p & 0 & & \\ 0 & -aI_q & \multicolumn{2}{c}{0} \end{array}\right)\right).$$

Then h is an indefinite hermitian form with signature $(2p, 2q)$ but the conjugates ${}^\sigma h$ by σ $(\sigma\neq\sigma_1)$ are positive definite.

Using h and b, we can construct the linear algebraic group $\boldsymbol{G}$ defined over k such that

$$\boldsymbol{G}(k)=\left\{g\in SL(2n, k') \;\middle|\; \begin{array}{l} h(gz, gw)=h(z, w) \\ b(gz, gw)=b(z, w) \end{array} \quad (z, w\in E_{k'})\right\}.$$

Then $\boldsymbol{G}(\boldsymbol{R})$ is isomorphic to G over $\boldsymbol{R}$. Similarly, using only h, we obtain the linear algebraic group $\boldsymbol{G}'$ defined over k such that

$$G'(k) = \{g \in SL(2n, k') \mid h(gz, gw) = h(z, w) \qquad (z, w \in E_{k'})\} .$$

Also, $\boldsymbol{G}'(\boldsymbol{R})$ is isomorphic to G' over $\boldsymbol{R}$.

Naturally, we have the rational imbedding of $\boldsymbol{G}$ into $\boldsymbol{G}'$ defined over k. We denote by ψ; $\boldsymbol{G} \to \boldsymbol{G}'$ this imbedding. It should be noted that, up to conjugation over $\boldsymbol{R}$, $\psi|_{G(\boldsymbol{R})}$; $\boldsymbol{G}(\boldsymbol{R}) \to \boldsymbol{G}'(\boldsymbol{R})$ coincides with the imbedding ψ; $G \to G'$ in 1.2.

4.2. Now we denote by $\mathrm{Res}_{k/\boldsymbol{Q}}$ the functor of the restriction of scalars from k to $\boldsymbol{Q}$. Let $\mathcal{G} = \mathrm{Res}_{k/\boldsymbol{Q}}\, \boldsymbol{G}$ and $\mathcal{G}' = \mathrm{Res}_{k/\boldsymbol{Q}}\, \boldsymbol{G}'$. Then we have the canonical imbedding $\mathrm{Res}_{k/\boldsymbol{Q}}\, \psi$; $\mathcal{G} \to \mathcal{G}'$ defined over $\boldsymbol{Q}$. Put $\Psi = \mathrm{Res}_{k/\boldsymbol{Q}}\, \psi$.

Over $\boldsymbol{R}$, we have the following isomorphisms ([2], 7.16);

$$\mathcal{G} \cong {}^{\sigma_1}\boldsymbol{G} \times {}^{\sigma_2}\boldsymbol{G} \times \cdots \times {}^{\sigma_d}\boldsymbol{G}$$
$$\mathcal{G}' \cong {}^{\sigma_1}\boldsymbol{G}' \times {}^{\sigma_2}\boldsymbol{G}' \times \cdots \times {}^{\sigma_d}\boldsymbol{G}' ,$$

where, for $\sigma \in \Sigma$, ${}^{\sigma}\boldsymbol{G}$ (resp. ${}^{\sigma}\boldsymbol{G}'$) denotes the conjugate of $\boldsymbol{G}$ (resp. $\boldsymbol{G}'$) by σ. So we have

$$\mathcal{G}(\boldsymbol{R}) \cong G \times Sp(n) \times \cdots \times Sp(n) \tag{4.1}$$

$$\mathcal{G}'(\boldsymbol{R}) \cong G' \times SU(2n) \times \cdots \times SU(2n) . \tag{4.2}$$

Under these isomorphisms, the imbedding Ψ is the product of the conjugations ${}^{\sigma_i}\psi$; ${}^{\sigma_i}\boldsymbol{G} \to {}^{\sigma_i}\boldsymbol{G}'$ $(1 \leq i \leq d)$ of ψ.

As in [1], VIII, 5.3, $\mathcal{G}'$ is naturally imbedded into Sp_N over $\boldsymbol{Q}$ where $N = 2nd$. In fact, consider $E_{k'}$ as to be a $4n$-dimensional vector space over k and write E_k instead of $E_{k'}$. We define the skew-symmetric k-bilinear form β on E_k by

$$h(z, w) = \mu(z, w) + \sqrt{-1}\beta(z, w) \qquad (z, w \in E_k) .$$

Then $\boldsymbol{G}'$ is imbedded into the symplectic group Sp_{2n} defined by β over k. Further, if we consider $E_{\boldsymbol{Q}} = \mathrm{Res}_{k/\boldsymbol{Q}}\, E_k$ and $\beta_{\boldsymbol{Q}} = \mathrm{Res}_{k/\boldsymbol{Q}}\, \beta$, $\mathcal{G}'$ is naturally imbedded into the group Sp_N defined by $\beta_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$. Denote by Ψ'; $\mathcal{G}' \to Sp_N$ this imbedding.

Thus we obtain the imbedding $\Psi' \circ \Psi$; $\mathcal{G} \to Sp_N$ defined over $\boldsymbol{Q}$. We choose a basis of $E_{\boldsymbol{Q}}$ so that $\beta_{\boldsymbol{Q}}$ is of standard form. With respect to this basis, we consider Sp_N as to be the subgroup of $GL(2N, \boldsymbol{C})$. Define

$$\mathcal{G}(\boldsymbol{Z}) = \{g \in \mathcal{G}(\boldsymbol{Q}) \mid (\Psi' \circ \Psi)(g) \in Sp(N, \boldsymbol{Z})\}$$
$$\mathcal{G}'(\boldsymbol{Z}) = \{g \in \mathcal{G}'(\boldsymbol{Q}) \mid \Psi'(g) \in Sp(N, \boldsymbol{Z})\} .$$

Then $\mathcal{G}(\boldsymbol{Z})$ (resp. $\mathcal{G}'(\boldsymbol{Z})$) is an arithmetic subgroup of $\mathcal{G}(\boldsymbol{R})$ (resp. $\mathcal{G}'(\boldsymbol{R})$) ([2], 7.11, 7.12). By a standard argument about arithmetic subgroups, $\mathcal{G}(\boldsymbol{Z})$ (resp. $\mathcal{G}'(\boldsymbol{Z})$) turns out to be a uniform discrete subgroup of $\mathcal{G}(\boldsymbol{R})$ (resp. $\mathcal{G}'(\boldsymbol{R})$)

([1], VIII, 5.4). In the direct product (4.1) (resp. (4.2)), denote by p_1; $\mathcal{G}(\boldsymbol{R})\to G$ (resp. p_1'; $\mathcal{G}'(\boldsymbol{R})\to G'$) the projection to the first component. Define

$$\Gamma_0=p_1(\mathcal{G}(\boldsymbol{Z}))\,,\quad \Gamma_0'=p_1'(\mathcal{G}'(\boldsymbol{Z}))\,.$$

Then Γ_0 (resp. Γ_0') is a uniform discrete subgroup of G (resp. G') ([1], VIII, 5.5). Clearly we have

$$\psi(\Gamma_0)\subset\Gamma_0'\,.$$

As for the group G' and its representation (V_r, H_r), Borel and Wallach obtained the following theorem.

Theorem 4.1 ([1], VIII, Corollary 5.8). *There is a subgroup Γ' of finite index in Γ_0' such that $m(V_r, \Gamma')\neq 0$, where $m(V_r, \Gamma')$ is the multiplicity of V_r in $L^2(\Gamma'\backslash G')$.*

As the proof of this theorem in [1] shows, Γ' is indeed a congruence subgroup of Γ_0'; that is, Γ' is given by

$$\Gamma'=p_1'(\Omega')$$

where Ω' is a congruence subgroup of $\mathcal{G}'(\boldsymbol{Z})$. Using this subgroup Γ', we can construct our desired subgroup of G.

Theorem 4.2. *There is a subgroup Γ of finite index in Γ_0 such that $m(U_r, \Gamma)\neq 0$.*

Proof. Let Γ' and Ω' be as above. There is a congruence subgroup Ω of $\mathcal{G}(\boldsymbol{Z})$ such that $\Psi(\Omega)\subset\Omega'$([2], 7.12). Put $\Gamma=p_1(\Omega)$. Then Γ is a subgroup of finite index in Γ_0 and we have

$$\psi(\Gamma)\subset\Gamma'\,. \tag{4.3}$$

In the following, we will prove that $m(U_r, \Gamma)\neq 0$. As in 2.1, let H_r^∞ be the space of C^∞-vectors in H_r for the representation (V_r, H_r) of G'. Since $m(V_r, \Gamma')\neq 0$, by [1], VIII, Theorem 4.3, there is a non-trivial continuous linear functional λ of H_r^∞ such that

$$\lambda\circ V_r(\gamma)=\lambda$$

for all $\gamma\in\Gamma'$. Using λ, we want to construct a non-trivial intertwining operator of H_r into $L^2(\Gamma\backslash G)$. For $\phi\in H_r^\infty$, define a function $A'(\phi)$; $G'\to\boldsymbol{C}$ by

$$A'(\phi)(g)=\lambda(V_r(g)\phi)\qquad (g\in G')\,.$$

Then $A'(\phi)$ is a C^∞-function on G' and left Γ'-invariant. Since G is imbedded into G' by ψ as a Lie subgroup, $A'(\phi)\circ\psi$; $G\to\boldsymbol{C}$ is a C^∞-function on G. Also, by (4.3), $A'(\phi)\circ\psi_r$ is left Γ-invariant. So we can define a linear mapping A; $H_r^\infty\to C^\infty(\Gamma\backslash G)$ by

$$\begin{aligned} A(\phi)(\Gamma g) &= A'(\phi)(\psi_r(g)) \\ &= \lambda(U_r(g)\phi) \qquad (\phi\in H_r^\infty, g\in G). \end{aligned}$$

Clearly we have

$$A(U_r(g)\phi) = U_\Gamma(g)A(\phi) \qquad (\phi\in H_r^\infty, g\in G)$$

where U_Γ is the right regular representation of G on $L^2(\Gamma\backslash G)$. Moreover, from the continuity of λ, we have

(4.4) $$A(U_r(X)\phi) = U_\Gamma(X)A(\phi) \qquad (X\in\mathfrak{g},\ \phi\in H_r^\infty).$$

Let $\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle_\Gamma$) be the inner product on H_r (resp. $L^2(\Gamma\backslash G)$). For K-finite vectors $\phi_1, \phi_2\in H_r^0$, set

(4.5) $$(\phi_1, \phi_2) = \langle A(\phi_1), A(\phi_2)\rangle_\Gamma .$$

Then $(\ ,\ )$ defines a $\mathfrak{g}$-invariant hermitian form on the $(\mathfrak{g}, K)$-module H_r^0. Here, by Proposition 2.2, H_r^0 decomposes into the sum of the isotypic K-submodules;

$$H_r^0 = \bigoplus_{s\in \boldsymbol{Z}_+, s\geqq -r} H_{r,s}^0 .$$

Since $A|_{H_r^0}$; $H_r^0\to L^2(\Gamma\backslash G)$ is K-equivariant, this decomposition is the orthogonal direct sum with respect to $(\ ,\ )$, too. Also, each $H_{r,s}^0$ is finite dimensional. From these facts, it is easy to see that there is a linear mapping B; $H_r^0\to H_r^0$ such that

(4.6) $$(\phi_1, \phi_2) = \langle B\phi_1, \phi_2\rangle \qquad (\phi_1, \phi_2\in H_r^0).$$

Then, by (4.4), we have

$$B(U_r(X)\phi) = U_r(X)(B(\phi)) \qquad (X\in\mathfrak{g},\ \phi\in H_r^0).$$

Since H_r^0 is an irreducible $(\mathfrak{g}, K)$-module, B is a scalar operator $\nu\cdot\mathrm{Id}$ where $\nu\in\boldsymbol{R}$ and $\nu\geqq 0$. Combining (4.5) and (4.6), we have

$$\langle A(\phi_1), A(\phi_2)\rangle_\Gamma = \nu\langle\phi_1, \phi_2\rangle \qquad (\phi_1, \phi_2\in H_r^0).$$

This implies that $A|_{H_r^0}$ is continuous with respect to the topology of H_r^0 in H_r. Hence the operator $A|_{H_r^0}$ extends to a bounded operator

$$\bar{A};\ H_r\to L^2(\Gamma\backslash G).$$

Note that H_r^0 consists of analytic vectors for U_r and G is connected. Then (4.4) implies that

$$\bar{A}(U_r(g)\phi) = U_\Gamma(g)\bar{A}(\phi) \qquad (\phi \in H_r,\, g \in G)$$

and hence $\bar{A}$ is an intertwining operator of (U_r, H_r) into $(U_\Gamma, L^2(\Gamma\backslash G))$.

On the other hand, λ is non-trivial. From the density of H_r^0 in H_r^∞, $\lambda|_{H_r^0}$ is non-trivial. Hence $\bar{A}$ is non-trivial. The theorem is proved.

Corollary 4.3. *For $l \in \boldsymbol{Z}_+$, there is a uniform discrete subgroup Γ of G such that*

$$H^{2q}(\Gamma, F_l^*) \neq \{0\}.$$

Proof. Theorem 3.4 implies that

$$H^{2q}(\mathfrak{g}, K; H_{l+2q}^0 \otimes F_l^*) \neq \{0\}.$$

Then, by [1], I, Theorem 5.3, the infinitesimal character of H_{l+2q} is equal to that of F_l. Applying Theorem 4.2 to U_{l+2q}, we obtain a uniform discrete subgroup Γ such that $m(U_{l+2q}, \Gamma) \neq 0$. Then, by (0.1), we have

$$H^{2q}(\Gamma, F_l^*) \neq \{0\}.$$

REMARK 4.4. In the above corollary, we consider the case $l=0$. Then we have

$$H^{2q}(\Gamma, \boldsymbol{C}) \neq \{0\}.$$

More precisely, $H^{2q}(\Gamma, \boldsymbol{C})$ contains a cohomology class which corresponds to a non-trivial automorphic representation. This improves the result in [4]. In [4], Millson and Raghunathan showed that, for some Γ, $H^i(\Gamma, \boldsymbol{C})$ contains such a class for any i strictly between 0 and $4pq$ and divisible by either $4p$ or $4q$ ([4], Theorem 4.1).

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