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AUTOMORPHISM INVARIANT INNER PRODUCT IN HILBERT SPACES OF HOLOMORPHIC FUNCTIONS ON THE UNIT BALL OF C^n

Dedicated to Professor Hideki Ozeki on his 60th birthday

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1. Introduction

Let B be the open unit ball in C^n and $\mathrm{Aut}(B)$ the group of holomorphic automorphisms of B. When $n=1$, B is the unit disc in C and the space $\mathscr{H}$ consisting of holomorphic functions f on B such that

$$\|f\| = \left(\iint_B |f'(z)|^2 dx\, dy\right)^{1/2} < \infty$$

is called the Dirichlet space. $\mathscr{H}$ is characterized as the unique Hilbert space of holomorphic functions on the unit disc which is $\mathrm{Aut}(B)$ invariant, i.e.,

$$\|f \circ \varphi\| = \|f\|$$

for all $f \in \mathscr{H}$ and $\varphi \in \mathrm{Aut}(B)$ [1]. The inner product in $\mathscr{H}$ is given by

$$(*) \qquad \langle f_1, f_2 \rangle = \iint_B f_1'(z)\overline{f_2'(z)}\, dx\, dy.$$

Strictly speaking this is a semi-inner product and $\mathscr{H}/C$ is a Hilbert space.

For $n > 1$, Zhu[5] proved that there exists a unique Hilbert space of holomorphic functions on B which is $\mathrm{Aut}(B)$ invariant. His description is in terms of the power series expansions of the holomorphic functions, and although several trials of finding a natural analog of the inner product $(*)$ are made, it is also shown that none of them generalizes to higher dimensions.

In this paper we give two explicit integral formulas for $\mathrm{Aut}(B)$ invariant inner product, both of them are derived from the analytic continuation of unitary representations of $\mathrm{Aut}(B)$ as in Wallach [4].

2. Preliminaries

Let $G = SU(n,1)$, i.e., the Lie group of linear transformations of determinant 1 in C^{n+1} which preserves the hermitian form

$$|z_1|^2+\cdots+|z_n|^2-|z_{n+1}|^2.$$

Hence the group G consists of all $(n+1)\times(n+1)$ complex matrices g of determinant 1 such that

$$g\begin{bmatrix}1_n & 0\\ 0 & -1\end{bmatrix}g^*=\begin{bmatrix}1_n & 0\\ 0 & -1\end{bmatrix}$$

where $*$ denotes the conjugate transpose and 1_n is the $n\times n$ identity matrix. Let us write $g\in G$ in block form as $g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ where a, b, c are $n\times n$, $n\times 1$, $1\times n$ matrices, respectively and $d\in C$. Then G consists of all matrices $g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ of determinant 1 such that

(2.1a) $$a^*a-c^*c=1_n,\quad a^*b=c^*d,\quad |d|^2-b^*b=1,$$

or equivalently

(2.1b) $$aa^*-bb^*=1_n,\quad ac^*=bd^*,\quad |d|^2-cc^*=1,$$

where (2.1b) is obtained by replacing g by g^{-1} in (2.1a). Throughout this paper we regard the points in C^n as column vectors. Then G acts transitively on B by

(2.2) $$z\to g\cdot z=(az+b)(cz+d)^{-1}\quad \text{if } g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in G.$$

Holomorphic automorphism groups of bounded symmetric domains are known (see [2]), and in the case of the unit ball B of C^n we have

$$\mathrm{Aut}(B)=G/(\text{center of } G).$$

Therefore every holomorphic automorphism of B can be represented by $g\in G$. For other description of Aut (B), see [3].

Let ν be Lebesgue measure on C^n, so normalized that $\nu(B)=1$, and let μ be the measure on B defined by

(2.3a) $$d\mu(z)=\frac{1}{(1-|z|^2)^{n+1}}d\nu(z).$$

Then (see [3])

(2.3b) the measure μ is invariant under the action of G.

For $g \in G$ and $z \in B$, let $Jac(g,z)$ denote the holomorphic Jacobian matrix of the mapping $w \to g \cdot w$ at the point z.

Lemma 2.4. *Let* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ *and* $z \in B$. *Then*

$$Jac(g,z) = (a - (g \cdot z)c)(cz + d)^{-1},$$

where $g \cdot z$ *is as in* (2.2).

Proof. For any column vector $v \in \mathbf{C}^n$, we have

$$\begin{aligned} Jac(g,z)v &= \lim_{h \to 0} \frac{1}{h}(g \cdot (z + hv) - g \cdot z) \\ &= av(cz+d)^{-1} - (az+b)(cz+d)^{-1}cv(cz+d)^{-1} \\ &= (a - (g \cdot z)c)(cz+d)^{-1}v. \end{aligned}$$

This implies the lemma. ■

Define $J_1 : G \times B \to GL(n,\mathbf{C})$ and $K_1 : B \times B \to GL(n,\mathbf{C})$ by

$$J_1(g,z) = a - (g \cdot z)c \quad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G,$$

$$K_1(z,w) = 1_n - zw^*.$$

Similarly define $J_2 : G \times B \to \mathbf{C}^\times (= GL(1,\mathbf{C}))$ and $K_2 : B \times B \to \mathbf{C}^\times$ by

$$J_2(g,z) = cz + d \quad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G,$$

$$K_2(z,w) = (1 - w^*z)^{-1}.$$

Note that

$$J_1(g,z)^{-1} = zb^* + a^*, \quad J_2(g,z)^{-1} = -b^*(g \cdot z) + \bar{d};$$

this follows from (2.1).

Lemma 2.5. *For* $i = 1,2$, *we have*

$$J_i(g_1 g_2, z) = J_i(g_1, g_2 \cdot z) J_i(g_2, z) \quad \textit{for } g_1, g_2 \in G \textit{ and } z \in B,$$

and

$$K_i(g \cdot z, g \cdot w) = J_i(g,z) K_i(z,w) J_i(g,w)^* \quad \textit{for } g \in G \textit{ and } z,w \in B.$$

Proof. It follows from (2.1) that

$$(zb^*+a^*)(g\cdot z)=zd^*+c^* \quad \text{for } g=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in G \text{ and } z\in B.$$

The lemma then follows from direct computations. ■

Lemma 2.6. *For every $z\in B$, $K_1(z,z)$ is a positive definite matrix.*

Proof. For $z\in B$, choose $g\in G$ so that $z=g\cdot 0$. Then Lemma 2.5 implies that

$$K_1(z,z)=K_1(g\cdot 0,g\cdot 0)=J_1(g,0)J_1(g,0)^*.$$

Since $J_1(g,0)$ is nonsingular, $K_1(z,z)$ is positive definite. ■

3. Integral formulas for the invariant inner product

For $\lambda\in C$, put

$$c(\lambda)=\frac{1}{n!}\lambda\prod_{i=2}^{n}(\lambda-i)=\frac{1}{n!}\lambda(\lambda-2)(\lambda-3)\cdots(\lambda-n).$$

Let $H(B,C^n)$ be the space of holomorphic functions on B with values in C^n. If F_1, $F_2\in H(B,C^n)$, regarding $F_1(z)$, $F_2(z)$ as row vectors, let for $\lambda\in C$

$$\langle F_1,F_2\rangle_\lambda=c(\lambda)\int_B F_1(z)(1_n-zz^*)F_2(z)^*(1-|z|^2)^\lambda d\mu(z) \tag{3.1}$$

provided the integral converges absolutely. Since $d\mu(z)=(1-|z|^2)^{-(n+1)}dv(z)$ and 1_n-zz^* is positive definite by Lemma 2.6, it is clear that if $\lambda\geq n+1$ and if F is bounded on B, then $\langle F,F\rangle_\lambda<\infty$; furthermore the function $\lambda\to\langle F,F\rangle_\lambda$ extends to a holomorphic function on the region $\{z\in C;\ \mathrm{Re}(z)>\lambda\}$. Let for $\lambda\in C$

$$H_\lambda(B,C^n)=\{F\in H(B,C^n);\ \langle F,F\rangle_{\mathrm{Re}(\lambda)}<\infty\}.$$

Let $\tilde{G}$ be the universal covering group of G with covering map $p:\tilde{G}\to G$. Since $\tilde{G}\times B$ is simply connected, we can uniquely define, for each $\lambda\in C$ and $\tilde{g}\in\tilde{G}$, the power $J_2(p(\tilde{g}),z)^\lambda$ with $J_2(p(\tilde{e}),z)^\lambda=1$ ($\tilde{e}$=identity element of $\tilde{G}$) for all $z\in B$. Similarly we can define $K_2(z,w)^\lambda$ so that $K_2(0,0)^\lambda=1$. For $\lambda\in C$, define $j_\lambda:\tilde{G}\times B\to C^\times$ by

$$j_\lambda(\tilde{g},z)=J_2(p(\tilde{g}),z)^\lambda.$$

Then in view of Lemma 2.5 we have

$$j_\lambda(\tilde{g}_1\tilde{g}_2,z)=j_\lambda(\tilde{g}_1,p(\tilde{g}_2)\cdot z)j_\lambda(\tilde{g}_2,z), \tag{3.2a}$$

$$K_2(p(\tilde{g})\cdot z,p(\tilde{g})\cdot w)^{\lambda}=j_{\lambda}(\tilde{g},z)K_2(z,w)^{\lambda}\overline{j_{\lambda}(\tilde{g},w)}. \tag{3.2b}$$

For $F\in H(B,C^n)$ and $\tilde{g}\in\tilde{G}$ with $p(\tilde{g})=g$, we set

$$(U_{\lambda}(\tilde{g})F)(z)=F(g^{-1}\cdot z)J_1(g^{-1},z)j_{\lambda}(\tilde{g}^{-1},z)^{-1}. \tag{3.3}$$

Then Lemma 2.5 and (3.2a) imply that U_{λ} is a(n algebraic) representation of $\tilde{G}$ on $H(B,C^n)$.

Lemma 3.4. *If $F_1,F_2\in H_{\lambda}(B,C^n)$, then*

$$\langle U_{\lambda}(\tilde{g})F_1,U_{\lambda}(\tilde{g})F_2\rangle_{\lambda}=\langle F_1,F_2\rangle_{\lambda}$$

for all $\tilde{g}\in\tilde{G}$.

Proof. Letting $p(\tilde{g})=g$ and using Lemma 2.5 and (3.2), we have

$$\begin{aligned}&\langle U_{\lambda}(\tilde{g})F_1,U_{\lambda}(\tilde{g})F_2\rangle_{\lambda}\\ &=c(\lambda)\int_B F_1(g^{-1}\cdot z)K_1(g^{-1}\cdot z,g^{-1}\cdot z)F_2(g^{-1}\cdot z)^*K_2(g^{-1}\cdot z,g^{-1}\cdot z)^{-\lambda}d\mu(z)\\ &=c(\lambda)\int_B F_1(z)K_1(z,z)F_2(z)^*K_2(z,z)^{-\lambda}d\mu(z) \qquad \text{by (2.3)}\\ &=\langle F_1,F_2\rangle_{\lambda}.\end{aligned}$$

■

For a holomorphic function $f\colon B\to C$, let $f'(z)$ denote the holomorphic Jacobian matrix of f at z, i.e., $f'(z)=(D_1f(z),\cdots,D_nf(z))$, where $D_i=\partial/\partial z_i$. Let $\mathscr{P}(B)$ be the space of holomorphic polynomial functions from B to C. Note that if $f\in\mathscr{P}(B)$ and $\lambda\geq n+1$, then $f'\in H_{\lambda}(B,C^n)$.

Proposition 3.5. *If $f_1,f_2\in\mathscr{P}(B)$, then the function $\lambda\to\langle f_1',f_2'\rangle_{\lambda}$, which is initially defined by a convergent integral for* $\mathrm{Re}(\lambda)>n$, *extends to a meromorphic function on C, which is moreover holomorphic on the region $\{z\in C;\ \mathrm{Re}(z)>-1\}$.*

Proof. For a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $z\in C^n$, define

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\quad \alpha!=\alpha_1!\cdots\alpha_n!,\quad z^{\alpha}=z_1^{\alpha_1}\cdots z_n^{\alpha_n}.$$

Let ε_i be the multi-index that has 1 in the ith place and 0 elsewhere. Then for multi-indices α and β

$$\begin{aligned}(z^{\alpha})'(1_n-zz^*)(z^{\beta})'^*&=\sum_i\alpha_iz^{\alpha-\varepsilon_i}\left(\sum_j(\delta_{ij}-z_i\bar{z}_j)\beta_j\bar{z}^{\beta-\varepsilon_j}\right)\\ &=\sum_{i,j}\alpha_i\beta_j(z^{\alpha-\varepsilon_i}\bar{z}^{\beta-\varepsilon_j}\delta_{ij}-z^{\alpha}\bar{z}^{\beta})\end{aligned} \tag{3.6}$$

$$=\sum_i \alpha_i\beta_i z^{\alpha-\varepsilon_i}\bar{z}^{\beta-\varepsilon_i}-(\sum_{i,j}\alpha_i\beta_j)z^{\alpha}\bar{z}^{\beta}.$$

If $\lambda>n$, we heve (see [5], p.840)

$$\int_B z^{\alpha}\bar{z}^{\beta}(1-|z|^2)^{\lambda}d\mu(z)=\begin{cases}\dfrac{n!\alpha!\Gamma(\lambda-n)}{\Gamma(\lambda+|\alpha|)} & \text{if } \alpha=\beta\\ 0 & \text{if } \alpha\neq\beta,\end{cases} \tag{3.7}$$

where Γ is the classical gamma function. Therefore if $\lambda>n$, (3.1), (3.6) and (3.7) imply that

$$\begin{aligned}\langle(z^{\alpha})',(z^{\alpha})'\rangle_{\lambda}&=c(\lambda)n!\Gamma(\lambda-n)\\ &\quad\times\left(\sum_i\alpha_i^2\frac{(\alpha-\varepsilon_i)!}{\Gamma(\lambda+|\alpha|-1)}-(\sum_{i,j}\alpha_i\alpha_j)\frac{\alpha!}{\Gamma(\lambda+|\alpha|)}\right)\\ &=\frac{c(\lambda)n!\Gamma(\lambda-n)\alpha!}{\Gamma(\lambda+|\alpha|)}(\sum_i\alpha_i(\lambda+|\alpha|-1)-|\alpha|^2)\\ &\qquad\qquad(\text{since } \alpha_i(\alpha-\varepsilon_i)!=\alpha!)\\ &=\frac{c(\lambda)n!\Gamma(\lambda-n)\alpha!|\alpha|(\lambda-1)}{\Gamma(\lambda+|\alpha|)}\\ &=\frac{\alpha!|\alpha|}{(\lambda+1)(\lambda+2)\cdots(\lambda+|\alpha|-1)}\\ &\qquad\qquad(\text{since } \Gamma(\lambda+|\alpha|)=\Gamma(\lambda-n)\Pi_{j=-n}^{|\alpha|-1}(\lambda+j)).\end{aligned} \tag{3.8a}$$

Likewise if $\lambda>n$ and $\alpha\neq\beta$, then

$$\langle(z^{\alpha})',(z^{\beta})'\rangle_{\lambda}=0. \tag{3.8b}$$

Now if $f_1(z)=\Sigma_{\alpha}a_{\alpha}z^{\alpha}$, $f_2(z)=\Sigma_{\beta}b_{\beta}z^{\beta}\in\mathscr{P}(B)$, then by (3.8)

$$\langle f_1',f_2'\rangle_{\lambda}=\sum_{|\alpha|>0}a_{\alpha}\bar{b}_{\alpha}\frac{\alpha!|\alpha|}{(\lambda+1)(\lambda+2)\cdots(\lambda+|\alpha|-1)}\quad\text{(finite sum)}, \tag{3.9}$$

and the proposition follows. ■

We define a representation T of G on holomorphic functions on B by

$$(T(g)f)(z)=f(g^{-1}\cdot z).$$

Then the chain rule and Lemma 2.4 imply that

$$(T(g)f)'(z)=f'(g^{-1}\cdot z)Jac(g^{-1},z)$$

$$=f'(g^{-1}\cdot z)J_1(g^{-1},z)J_2(g^{-1},z)^{-1}.$$

Hence if $\tilde{g}\in\tilde{G}$ with $p(\tilde{g})=g$, then by (3.3)

$$(T(g)f)'(z)=(U_1(\tilde{g})f')(z). \tag{3.10}$$

Note that Proposition 3.5 ensures that if $f_1,f_2\in\mathscr{P}(B)$, then $\lim_{\lambda\to1}\langle f_1',f_2'\rangle_\lambda$ exists.

Theorem 3.11. *If $f_1,f_2\in\mathscr{P}(B)$, then*

$$\langle\!\langle f_1,f_2\rangle\!\rangle=\lim_{\lambda\to1}c(\lambda)\int_B f_1'(z)(1_n-zz^*)f_2'(z)^*(1-|z|^2)^\lambda d\mu(z)$$

defines an (a semi-) inner product on $\mathscr{P}(B)$. Let $\mathscr{H}$ be the Hilbert space completion of $\mathscr{P}(B)$. Then $\mathscr{H}$ consists of holomorphic functions on B and $\mathscr{H}$ is a G invariant Hilbert space; that is, $T(g)f\in\mathscr{H}$ for $g\in G$, $f\in\mathscr{H}$, and

$$\langle\!\langle T(g)f_1,T(g)f_2\rangle\!\rangle=\langle\!\langle f_1,f_2\rangle\!\rangle$$

for all $g\in G$, $f_1,f_2\in\mathscr{H}$.

Proof. Suppose $f_1(z)=\Sigma_\alpha a_\alpha z^\alpha$ and $f_2(z)=\Sigma_\beta b_\beta z^\beta$. Then by (3.1) and (3.9)

$$\begin{aligned}\langle\!\langle f_1,f_2\rangle\!\rangle&=\lim_{\lambda\to1}\langle f_1',f_2'\rangle_\lambda\\&=\sum_\alpha a_\alpha\bar{b}_\alpha\frac{\alpha!|\alpha|}{|\alpha|!}\quad\text{(finite sum)},\end{aligned}\tag{3.12}$$

and the first assertion follows.

To show that $\mathscr{H}$ consists of holomorphic functions, we first show that if $f\in\mathscr{P}(B)$ and $f(0)=0$, then

$$|f(z)|\le\left(\log\frac{1}{1-|z|^2}\right)^{1/2}\|f\| \tag{3.13}$$

for all $z\in B$, where $\|f\|=\langle\!\langle f,f\rangle\!\rangle^{1/2}$. Indeed if $f(z)=\Sigma_\alpha a_\alpha z^\alpha\in\mathscr{P}(B)$, then

$$\begin{aligned}|f(z)|&\le\sum_{|\alpha|>0}|a_\alpha||z^\alpha|\\&\le\left(\sum_{|\alpha|>0}\frac{|\alpha|!}{\alpha!|\alpha|}|z^\alpha|^2\right)^{1/2}\left(\sum_{|\alpha|>0}|a_\alpha|^2\frac{\alpha!|\alpha|}{|\alpha|!}\right)^{1/2}.\end{aligned}$$

Since

$$\sum_{|\alpha|>0}\frac{|\alpha|!}{\alpha!|\alpha|}|z^\alpha|^2=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}|z^\alpha|^2$$
$$=\sum_{k=1}^{\infty}\frac{1}{k}|z|^{2k}$$
$$=\log\frac{1}{1-|z|^2},$$

(3.13) follows. Let $\{f_k\}$ be a Cauchy sequence in $\mathscr{P}(B)$. Define $\tilde{f}_k\in\mathscr{P}(B)$ by $\tilde{f}_k(z)=f_k(z)-f_k(0)$. Then $\tilde{f}_k(0)=0$ and, since $\|f_k\|=\|\tilde{f}_k\|$ for all k, $\{\tilde{f}_k\}$ is also a Cauchy sequence. Since (3.13) shows that the norm convergence implies uniform convergence on every compact subset of B, there exists a holomorphic function f on B such that $\tilde{f}_k$ converges uniformly to f on every compact subset of B. Let $f(z)=\Sigma_\alpha a_\alpha z^\alpha$ be the power series expansion and let $\|f\|^2=\sum_\alpha|a_\alpha|^2\frac{\alpha!|\alpha|}{|\alpha|!}$, then $\|f\|<\infty$ and $\lim_{k\to\infty}\|f-f_k\|=0$. Hence we conclude that

$$\mathscr{H}=\left\{f(z)=\sum_\alpha a_\alpha z^\alpha;\sum_\alpha|a_\alpha|^2\frac{\alpha!|\alpha|}{|\alpha|!}<\infty\right\}$$

with inner product

$$\ll f_1,f_2\gg=\sum_\alpha a_\alpha\bar{b}_\alpha\frac{\alpha!|\alpha|}{|\alpha|!}$$

for all $f_1(z)=\Sigma_\alpha a_\alpha z^\alpha$, $f_2(z)=\Sigma_\beta b_\beta z^\beta\in\mathscr{H}$.

Now suppose $f_1,f_2\in\mathscr{P}(B)$, $g\in G$, and take $\tilde{g}\in\tilde{G}$ so that $p(\tilde{g})=g$. Then in view of Lemma 3.4 and Proposition 3.5, it follows by analytic continuation that

$$\begin{aligned}\ll f_1,f_2\gg&=\langle f_1',f_2'\rangle_1(=\langle f_1',f_2'\rangle_\lambda|_{\lambda=1})\\&=\langle U_1(\tilde{g})f_1',U_1(\tilde{g})f_2'\rangle_1\\&=\langle(T(g)f_1)',(T(g)f_2)'\rangle_1\quad\text{by (3.10)}.\end{aligned}$$

This shows that $T(g)f\in\mathscr{H}$ for $g\in G$, $f\in\mathscr{P}(B)$, and $\ll T(g)f_1,T(g)f_2\gg=\ll f_1,f_2\gg$ for $g\in G$, $f_1,f_2\in\mathscr{P}(B)$. Since $\mathscr{P}(B)$ is dense in $\mathscr{H}$, the theorem follows. ∎

We now turn to another description of the G invariant inner product. For $\lambda\in\boldsymbol{C}$, put

$$d(\lambda)=\frac{1}{n!}\prod_{i=0}^{n}(\lambda-i)=\frac{1}{n!}\lambda(\lambda-1)\cdots(\lambda-n).$$

If f_1 and f_2 are holomorphic functions on B, let for $\lambda \in \boldsymbol{C}$,

$$\langle f_1, f_2 \rangle_\lambda = d(\lambda) \int_B f_1(z) \overline{f_2(z)} (1-|z|^2)^\lambda d\mu(z)$$

provided the integral converges absolutely. It is clear that if $\lambda \geq n+1$ and if $f \in \mathscr{P}(B)$, then $\langle f, f \rangle_\lambda < \infty$; furthermore the function $\lambda \to \langle f, f \rangle_\lambda$ extends to a holomorphic function on the region $\{z \in \boldsymbol{C};\ \mathrm{Re}(z) > \lambda\}$.

Theorem 3.14. *If $f_1, f_2 \in \mathscr{P}(B)$, then the function $\lambda \to \langle f_1, f_2 \rangle_\lambda$, which is initially defined by a convergent integral for* $\mathrm{Re}(\lambda) > n$, *extends to a meromorphic function on* $\boldsymbol{C}$, *which is moreover holomorphic on the region* $\{z \in \boldsymbol{C};\ \mathrm{Re}(z) > -1\}$.
The pairing

$$\langle\!\langle f_1, f_2 \rangle\!\rangle = \lim_{\lambda \to 0} d(\lambda) \int_B f_1(z) \overline{f_2(z)} (1-|z|^2)^\lambda d\mu(z)$$

defines an (a semi-) inner product on $\mathscr{P}(B)$. This inner product coincides with that in Theorem 3.11. *Therefore the Hilbert space completion is a G invariant Hilbert space that consists of holomorphic functions on B.*

Proof. If $\lambda > n$, then for multi-indices α and β it follows from (3.7) that

$$\begin{aligned}\langle z^\alpha, z^\beta \rangle_\lambda &= \frac{d(\lambda) n! \alpha! \Gamma(\lambda - n)}{\Gamma(\lambda + |\alpha|)} \delta_{\alpha\beta} \\ &= \frac{\alpha!}{(\lambda+1)(\lambda+2)\cdots(\lambda+|\alpha|-1)} \delta_{\alpha\beta}.\end{aligned}$$

Hence if $f_1(z) = \Sigma_\alpha a_\alpha z^\alpha$, $f_2(z) = \Sigma_\beta b_\beta z^\beta \in \mathscr{P}(B)$, then

$$\langle f_1, f_2 \rangle_\lambda = \sum_{|\alpha| > 0} a_\alpha \bar{b}_\alpha \frac{\alpha!}{(\lambda+1)(\lambda+2)\cdots(\lambda+|\alpha|-1)} \quad \text{(finite sum)},$$

and the function $\lambda \to \langle f_1, f_2 \rangle_\lambda$ extends to a meromorphic function on $\boldsymbol{C}$. Moreover

$$\begin{aligned}\langle\!\langle f_1, f_2 \rangle\!\rangle &= \lim_{\lambda \to 0} \langle f_1, f_2 \rangle_\lambda \\ &= \sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha! |\alpha|}{|\alpha|!} \quad \text{(finite sum)},\end{aligned}$$

which coincides with (3.12). ■

REMARK. If f is a holomorphic function on B and $\tilde{g} \in \tilde{G}$ with $p(\tilde{g})=g$, set for $\lambda \in \boldsymbol{C}$

$$(T_\lambda(\tilde{g})f)(z)=j_\lambda(\tilde{g}^{-1},z)^{-1}f(g^{-1}\cdot z).$$

Then by (3.2a) T_λ defines a representation of $\tilde{G}$ and

$$(T_0(\tilde{g})f)(z)=f(g^{-1}\cdot z)=(T(g)f)(z).$$

If f_1 and f_2 are holomorphic functions on B, then it follows from (2.3) and (3.2b)

$$\langle T_\lambda(\tilde{g})f_1, T_\lambda(\tilde{g})f_2\rangle_\lambda=\langle f_1, f_2\rangle_\lambda$$

for all $\tilde{g} \in \tilde{G}$. Consequently, as in the proof of Theorem 3.11, the G invariance of the Hilbert space in Theorem 3.14 may also be proved directly by analytic continuation.

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