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CERTAIN ASPECTS OF TWISTED LINEAR ACTIONS

Dedicated to Professor Hiroshi Toda on his 60th birthday

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0. Introduction

In the previous paper [2], we have introduced the concept of a twisted linear action which is an analytic action of a non-compact Lie group on a sphere, and we have shown as an example that there have been uncountably many topologically distinct analytic actions of $SL(n, \mathbf{R})$ on the $(2n-1)$ -sphere.

In this paper, we shall show another aspect of twisted linear actions. In particular, we shall show that there are uncountably many C^1 -differentiably distinct but topologically equivalent analytic actions of $SL(n, \mathbf{R})$ on a k -sphere for each $k \geq n \geq 2$.

1. Twisted linear actions

Throughout this paper, a matrix means only the one with real coefficients.

1.1. Let $\mathbf{u}=(u_i)$ and $\mathbf{v}=(v_i)$ be column vectors in \mathbf{R}^n . As usual, we define their inner product by $\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i$ and the length of \mathbf{u} by $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$. Let $M=(m_{ij})$ be a square matrix of degree n . We say that M satisfies the condition (T) if the quadratic form

$$\mathbf{x} \cdot M \mathbf{x} = \sum_{i,j} m_{ij} x_i x_j$$

is positive definite. It is easy to see that M satisfies (T) if and only if

$$(T') \quad \frac{d}{dt} \|\exp(tM) \mathbf{x}\| > 0 \quad \text{for each } \mathbf{x} \in \mathbf{R}_0^n = \mathbf{R}^n - \{0\}, t \in \mathbf{R}.$$

If M satisfies (T') , then

$$\lim_{t \rightarrow +\infty} \|\exp(tM) \mathbf{x}\| = +\infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} \|\exp(tM) \mathbf{x}\| = 0$$

for each $\mathbf{x} \in \mathbf{R}_0^n$, and hence there exists a unique real valued analytic function τ

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on \mathbf{R}_0^n such that

$$\|\exp(\tau(\mathbf{x})M)\mathbf{x}\| = 1 \quad \text{for } \mathbf{x} \in \mathbf{R}_0^n.$$

Therefore, we can define an analytic mapping π^M of \mathbf{R}_0^n onto the unit $(n-1)$ -sphere S^{n-1} by

$$\pi^M(\mathbf{x}) = \exp(\tau(\mathbf{x})M)\mathbf{x} \quad \text{for } \mathbf{x} \in \mathbf{R}_0^n,$$

if M satisfies the condition **(T)**.

1.2. Let G be a Lie group, $\rho: G \rightarrow \mathbf{GL}(n, \mathbf{R})$ a matricial representation, and M a square matrix of degree n satisfying **(T)**. We call (ρ, M) a *TC-pair* of degree n , if $\rho(g)M = M\rho(g)$ for each $g \in G$. For a *TC-pair* (ρ, M) of degree n , we can define an analytic mapping

$$\xi: G \times S^{n-1} \rightarrow S^{n-1} \quad \text{by } \xi(g, x) = \pi^M(\rho(g)x),$$

and we see that ξ is an analytic G -action on S^{n-1} . We call $\xi = \xi^{(\rho, M)}$ a twisted linear action of G on S^{n-1} determined by the *TC-pair* (ρ, M) , and we say that ξ is associated to the matricial representation ρ .

1.3. For a given Lie group G , we introduce certain equivalence relations on *TC-pairs*. Let (ρ, M) and (σ, N) be *TC-pairs* of degree n . We say that (ρ, M) is algebraically equivalent to (σ, N) if there exist $A \in \mathbf{GL}(n, \mathbf{R})$ and a positive real number c satisfying

$$(*) \quad cN = AMA^{-1} \quad \text{and} \quad \sigma(g) = A\rho(g)A^{-1} \quad \text{for each } g \in G.$$

We say that (ρ, M) is C^r -equivalent to (σ, N) if there exists a C^r -diffeomorphism f of S^{n-1} onto itself such that the following diagram is commutative:

$$\begin{array}{ccc} G \times S^{n-1} & \xrightarrow{1 \times f} & G \times S^{n-1} \\ \downarrow \xi^{(\rho, M)} & & \downarrow \xi^{(\sigma, N)} \\ S^{n-1} & \xrightarrow{f} & S^{n-1}. \end{array}$$

We call f a G -equivariant C^r -diffeomorphism.

Lemma. *If (ρ, M) is algebraically equivalent to (σ, N) , then (ρ, M) is C^ω -equivalent to (σ, N) .*

Proof. It has been proved in the previous paper [2], but we give a proof for completeness. Suppose that there exist $A \in \mathbf{GL}(n, \mathbf{R})$ and a positive real number c satisfying (*). Define analytic mappings h_A and k_A of S^{n-1} into itself by

$$h_A(x) = \pi^N(Ax) \quad \text{and} \quad k_A(y) = \pi^M(A^{-1}y).$$

Then the composites $h_A k_A$ and $k_A h_A$ are the identity mapping on S^{n-1} by the condition $cN=AMA^{-1}$, and hence h_A is a C^ω -diffeomorphism. Furthermore, the equality

$$h_A(\xi^{(\rho, M)}(g, x)) = \xi^{(\sigma, N)}(g, h_A(x))$$

holds for each $g \in G$ and $x \in S^{n-1}$, by the condition (*). q.e.d.

Theorem ([2], Theorem 3.3). *Let G be a compact Lie group and $\rho: G \rightarrow GL(n, \mathbf{R})$ a matricial representation. Then any TC-pairs (ρ, M) and (ρ, N) are C^ω -equivalent.*

2. First typical examples

Here we shall study twisted linear actions of $G=SL(n, \mathbf{R})$ on the $(nk-1)$ -sphere associated to a representation $\rho = \rho_n \otimes I_k$, that is, $\rho(A) = A \otimes I_k$.

2.1. Let A and $B = (b_{ij})$ be square matrices of degrees n and k , respectively. Denote by $A \otimes B$ the Kronecker product written in the form

$$A \otimes B = \begin{pmatrix} b_{11}A & \cdots & b_{1k}A \\ \vdots & & \vdots \\ b_{k1}A & \cdots & b_{kk}A \end{pmatrix}.$$

Let u_1, \dots, u_k be column vectors in \mathbf{R}^n . Then the correspondence

$$(u_1, \dots, u_k) \rightarrow \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix}$$

defines a linear isomorphism $\iota: M(n, k; \mathbf{R}) \rightarrow \mathbf{R}^{nk}$. Let X and Y be $n \times k$ matrices. As usual, we define their inner product by

$$\langle X, Y \rangle = \text{trace}({}^tXY),$$

and the length of X by $\|X\| = \sqrt{\langle X, X \rangle}$. Then ι is an isometry. Furthermore, the equality

$$(A \otimes B) \iota(X) = \iota(AX{}^tB)$$

holds, where A and B are square matrices of degrees n and k , respectively, and X is an $n \times k$ matrix. In the following, we shall identify \mathbf{R}^{nk} with $M(n, k; \mathbf{R})$ via the isometry ι .

2.2. We obtain the following lemma directly.

Lemma 2.2. *Let \bar{M} be a square matrix of degree nk . Then*

$$\bar{M}(A \otimes I_k) = (A \otimes I_k) \bar{M}$$

for each $A \in \mathbf{SL}(n, \mathbf{R})$, if and only if $\bar{M} = I_n \otimes M$ for some square matrix M of degree k . Furthermore, $I_n \otimes M$ satisfies the condition **(T)** if and only if M satisfies **(T)**.

Consequently, $(\rho_n \otimes I_k, I_n \otimes M)$ is a *TC*-pair for any square matrix M of degree k satisfying **(T)**, and any *TC*-pair $(\rho_n \otimes I_k, \bar{M})$ is written in such a form. Furthermore, *TC*-pairs $(\rho_n \otimes I_k, I_n \otimes M)$ and $(\rho_n \otimes I_k, I_n \otimes N)$ are algebraically equivalent, if and only if there exist $A \in \mathbf{GL}(k, \mathbf{R})$ and a positive real number c satisfying $cN = AMA^{-1}$.

2.3. Let M be a square matrix of degree k satisfying **(T)**. Denote by ζ^M the twisted linear $\mathbf{SL}(n, \mathbf{R})$ action on the $(nk-1)$ -sphere determined by the *TC*-pair $(\rho_n \otimes I_k, I_n \otimes M)$. Identifying \mathbf{R}^{nk} with $M(n, k; \mathbf{R})$ via the isometry ι , we can describe

$$\zeta^M : \mathbf{SL}(n, \mathbf{R}) \times S^{nk-1} \rightarrow S^{nk-1}$$

as follows. That is, S^{nk-1} can be viewed as the set of all $n \times k$ matrices X with $\|X\|=1$, and ζ^M is written in the form

$$\zeta^M(A, X) = AX \exp(\theta^t M)$$

for a real number θ which is uniquely determined by the condition

$$\|AX \exp(\theta^t M)\| = 1.$$

Let $I(M)$ and $O(M)$ denote the isotropy group at

$$\frac{1}{\sqrt{k}} \begin{pmatrix} I_k \\ 0 \end{pmatrix}$$

and the orbit through that point, respectively, with respect to the twisted linear action ζ^M . We obtain the following lemma.

Lemma 2.3. *Suppose $n > k \geq 2$. Then the isotropy group $I(M)$ is written in the form*

$$I(M) = \left\{ \left(\begin{array}{c|c} \exp(\theta^t M) & * \\ \hline 0 & * \end{array} \right) : \theta \in \mathbf{R} \right\}$$

and the orbit $O(M)$ is an open dense subset consisting of all $n \times k$ matrices X with $\text{rank } X = k$ and $\|X\|=1$.

2.4. Suppose that $n > k \geq 2$ and there exists an $\mathbf{SL}(n, \mathbf{R})$ -equivariant homeomorphism f of S^{nk-1} with a twisted linear action ζ^M onto S^{nk-1} with a twisted linear action ζ^N . Then we obtain $f(O(M)) = O(N)$, and hence $I(M)$ and $I(N)$

are conjugate in $SL(n, \mathbf{R})$. Finally, we see that there exist $A \in GL(k, \mathbf{R})$ and a positive real number c satisfying $cN = AMA^{-1}$, by making use of the fact that M and N satisfy the condition (T) and the group $I(M)$ contains a subgroup written in the form

$$\left\{ \left(\begin{array}{c|c} I_k & * \\ \hline 0 & I_{n-k} \end{array} \right) \right\}.$$

Summing up the above discussion, we obtain the following result.

Theorem 2.4. *Suppose $n > k \geq 2$. Then any two of TC-pairs in the form $(\rho_n \otimes I_k, \bar{M})$ are algebraically equivalent if and only if they are C^0 -equivalent.*

Consequently, we see that if $n > k \geq 2$ then there are uncountably many topologically distinct twisted linear actions of $SL(n, \mathbf{R})$ on S^{n+k-1} associated to the matricial representation $\rho_n \otimes I_k$. This is a generalization of a result studied in the previous paper [2].

3. Second typical examples

Here we shall study twisted linear actions of $G = SL(n, \mathbf{R})$ on the $(n+k-1)$ -sphere associated to a representation $\rho = \rho_n \oplus I_k$, that is, $\rho(A) = A \oplus I_k$.

3.1. Let A and B be square matrices of degrees n and k , respectively. We denote by $A \oplus B$ the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

of degree $n+k$. We obtain the following lemma.

Lemma 3.1. *Let $n \geq 2$ and $k \geq 1$. Let \bar{M} be a square matrix of degree $n+k$. Then*

$$\bar{M}(A \oplus I_k) = (A \oplus I_k) \bar{M}$$

for each $A \in SL(n, \mathbf{R})$, if and only if $\bar{M} = cI_n \oplus M$ for some square matrix M of degree k and a real number c . Furthermore, $\bar{M} = cI_n \oplus M$ satisfies the condition (T) , if and only if c is positive and M satisfies (T) .

3.2. Let M be a square matrix of degree k satisfying (T) . Denote by χ^M the twisted linear $SL(n, \mathbf{R})$ action on the $(n+k-1)$ -sphere determined by the TC-pair $(\rho_n \oplus I_k, I_n \oplus M)$. Then χ^M is written in the form

$$\chi^M(A, \mathbf{u} \oplus \mathbf{v}) = e^\theta A \mathbf{u} \oplus e^{\theta M} \mathbf{v}$$

for a real number θ which is uniquely determined by the condition

$$\|e^\theta A\mathbf{u}\|^2 + \|e^{\theta M}\mathbf{v}\|^2 = 1,$$

where \mathbf{u} is a column vector in \mathbf{R}^n and \mathbf{v} is a column vector in \mathbf{R}^k satisfying $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 1$.

3.3. Let us define closed subgroups $L(n)$ and $N(n)$ of $SL(n, \mathbf{R})$ by the forms

$$L(n) = \left\{ \left(\begin{array}{c|ccc} 1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & * \end{array} \right) \right\}, \quad N(n) = \left\{ \left(\begin{array}{c|ccc} \lambda & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & * \end{array} \right) : \lambda > 0 \right\}.$$

Denote by $F(M)$ the fixed point set of $L(n)$ with respect to the twisted linear action χ^M . Then we obtain the following lemma.

Lemma 3.3. *With respect to the twisted linear action χ^M ,*

$$F(M) = \{a\mathbf{e}_1 \oplus \mathbf{v} : a^2 + \|\mathbf{v}\|^2 = 1\},$$

where $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbf{R}^n$. *The isotropy group at $0 \oplus \mathbf{v}$ coincides with $SL(n, \mathbf{R})$, the one at $\pm \mathbf{e}_1 \oplus 0$ coincides with $N(n)$, and if $a\|\mathbf{v}\| \neq 0$, then the one at $a\mathbf{e}_1 \oplus \mathbf{v}$ coincides with $L(n)$.*

3.4. Notice that the normalizer $N(L(n))$ of $L(n)$ acts on $F(M)$ naturally via χ^M , the identity component of $N(L(n))$ coincides with $N(n)$, and the factor group $N(L(n))/L(n)$ is naturally isomorphic to the multiplicative group \mathbf{R}^\times consisting of non-zero real numbers.

Let us investigate the induced $N(L(n))/L(n)$ action on $F(M)$ via χ^M . Leaving fixed any point $a\mathbf{e}_1 \oplus \mathbf{v}$ of $F(M)$ satisfying $a\|\mathbf{v}\| \neq 0$, we have a real valued analytic function $\theta = \theta(\alpha)$ on \mathbf{R}^\times determined by

$$\chi^M \left(\left(\begin{array}{c|ccc} \alpha & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & * \end{array} \right), a\mathbf{e}_1 \oplus \mathbf{v} \right) = e^\theta \alpha a\mathbf{e}_1 \oplus e^{\theta M}\mathbf{v}$$

and $(e^\theta \alpha)^2 + \|e^{\theta M}\mathbf{v}\|^2 = 1$. Then $\theta(-\alpha) = \theta(\alpha)$ and

$$\frac{d\theta}{d\alpha} < 0 < \frac{d}{d\alpha} (e^\theta \alpha)$$

for $\alpha > 0$. Furthermore, we obtain

$$\lim_{\alpha \rightarrow +\infty} \theta(\alpha) = -\infty, \quad \lim_{\alpha \rightarrow +\infty} e^\theta \alpha = |a|^{-1}, \quad \lim_{\alpha \rightarrow +\infty} \|e^{\theta M}\mathbf{v}\| = 0,$$

and

$$\lim_{\alpha \rightarrow 0^+} e^\theta \alpha = 0, \lim_{\alpha \rightarrow 0^+} e^{\theta M} v = \pi^M(v).$$

3.5. Here we shall show the following result.

Theorem 3.5. *Let M, N be any square matrices of degree k satisfying the condition (T). Then there exists an $SL(n, \mathbf{R})$ -equivariant homeomorphism f of S^{n+k-1} with a twisted linear action χ^M onto S^{n+k-1} with a twisted linear action χ^N .*

Proof. By the above investigation, we can construct uniquely an $N(L(n))/L(n)$ -equivariant homeomorphism f_0 of $F(M)$ onto $F(N)$ satisfying the following conditions

$$f_0(ae_1 \oplus v) = ae_1 \oplus v \quad \text{for } |a| = 1 \quad \text{or } 1/\sqrt{2},$$

and

$$f_0(0 \oplus \pi^M(v)) = 0 \oplus \pi^N(v) \quad \text{for } \|v\| = 1/\sqrt{2}.$$

Next we consider the following diagram

$$\begin{array}{ccc} \mathbf{SO}(n) \times F(M) & \xrightarrow{\psi_1} & S^{n+k-1} \\ \downarrow 1 \times f_0 & & \downarrow f \\ \mathbf{SO}(n) \times F(N) & \xrightarrow{\psi_2} & S^{n+k-1}, \end{array}$$

where

$$\begin{aligned} \psi_1(K, x) &= \chi^M(K, x) = (K \oplus I_k) x, \\ \psi_2(K, x) &= \chi^N(K, x) = (K \oplus I_k) x. \end{aligned}$$

By the construction of f_0 , we see that $\psi_1(K, x) = \psi_1(K', x')$ if and only if $\psi_2(K, f_0(x)) = \psi_2(K', f_0(x'))$, and hence we obtain a unique bijection f of S^{n+k-1} onto itself satisfying

$$f \circ \psi_1 = \psi_2 \circ (1 \times f_0).$$

Then f is a homeomorphism, because ψ_1 and ψ_2 are closed continuous mappings. Finally, we show that f is $SL(n, \mathbf{R})$ -equivariant. Let $A \in SL(n, \mathbf{R})$, $K \in \mathbf{SO}(n)$ and $x \in F(M)$. Then, there are $B \in \mathbf{SO}(n)$ and $U \in N(n)$ such that $AK = BU$, and hence

$$\begin{aligned} f(\chi^M(A, \psi_1(K, x))) &= f(\chi^M(AK, x)) = f(\chi^M(BU, x)) \\ &= f(\psi_1(B, \chi^M(U, x))) = \psi_2(B, f_0(\chi^M(U, x))) \\ &= \psi_2(B, \chi^N(U, f_0(x))) = \chi^N(BU, f_0(x)) \\ &= \chi^N(AK, f_0(x)) = \chi^N(A, \psi_2(K, f_0(x))) \\ &= \chi^N(A, f(\psi_1(K, x))). \end{aligned}$$

Consequently, we see that f is an $SL(n, \mathbf{R})$ -equivariant homeomorphism of S^{n+k-1} with the action χ^M onto S^{n+k-1} with the action χ^N . q.e.d.

3.6. Next we shall show the following result.

Theorem 3.6. *Let M, N be square matrices of degree k satisfying the condition (T). If there exists an $SL(n, \mathbf{R})$ -equivariant C^1 -diffeomorphism f of S^{n+k-1} with a twisted linear action χ^M onto S^{n+k-1} with a twisted linear action χ^N , then*

$$N = PMP^{-1}$$

for some $P \in GL(k, \mathbf{R})$.

Proof. By the existence of such an equivariant C^1 -diffeomorphism f , we obtain an $N(L(n))/L(n)$ -equivariant C^1 -diffeomorphism $f_0: F(M) \rightarrow F(N)$. Considering points whose isotropy groups coincide with $N(n)/L(n)$, we can assume

$$f_0(e_1 \oplus 0) = e_1 \oplus 0.$$

Then we obtain an isomorphism

$$df_0: T_{e_1 \oplus 0} F(M) \rightarrow T_{e_1 \oplus 0} F(N)$$

of tangential representation spaces of the isotropy group $N(n)/L(n)$.

Here we consider the representation space $T_{e_1 \oplus 0} F(M)$. Denote by $F(M)_+$ an open subset of $F(M)$ consisting of $ae_1 \oplus v$ with $a > 0$, and define

$$\psi^M: F(M)_+ \rightarrow \mathbf{R}^k \text{ by } \psi^M(ae_1 \oplus v) = \exp(-(\log a) M) v.$$

Then ψ^M is a C^ω -diffeomorphism satisfying $\psi^M(e_1 \oplus 0) = 0$. Considering the C^ω -diffeomorphism ψ^M , we see that the tangential representation of the isotropy group $N(n)/L(n) \cong \mathbf{R}$ on $T_{e_1 \oplus 0} F(M)$ is equivalent to a representation

$$\sigma^M: \mathbf{R} \rightarrow GL(k, \mathbf{R}) \text{ defined by } \sigma^M(\lambda) = \exp(-\lambda M).$$

The existence of the isomorphism df_0 of tangential representation spaces assures that the representations σ^M and σ^N are equivalent, and hence the equality $N = PMP^{-1}$ holds for some $P \in GL(k, \mathbf{R})$. q.e.d.

Notice that the twisted linear actions χ^M are new concrete examples for analytic $SL(n, \mathbf{R})$ -actions on a sphere investigated in [1].

4. Concluding remark

With respect to the first typical examples, we obtain a classification theorem only for the case $n > k \geq 2$ in §2. It seems to be difficult to obtain a similar result for the remaining case $k \geq n \geq 2$ in general. Here we consider the case $n = k = 3$.

The following matrices satisfy the condition (T) .

$$\text{(Type 1) } M_1(a, b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}; 1 \leq a \leq b$$

$$\text{(Type 2) } M_2(a, b) = \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & b \end{pmatrix}; a > 0, b > 0$$

$$\text{(Type 3) } M(a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}; a > 0$$

$$\text{(Type 4) } M_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, if a matrix M of degree 3 satisfy the condition (T) , then M is similar to only one of the above matrices up to positive scalar multiplication. Here we say that M is similar to N up to positive scalar multiplication if there exist a non-singular matrix A and a positive real number c such that $AMA^{-1} = cN$.

Denote by $S(M)$ the 8-sphere with the twisted linear $SL(3, \mathbf{R})$ action ζ^M (see §2.3), where M is a square matrix of degree 3 satisfying the condition (T) . We obtain the following result.

Theorem. (0) *If $S(M)$ and $S(M')$ are equivariantly C^1 -diffeomorphic, then M is similar to M' up to positive scalar multiplication.*

(1) *If $S(M)$ and $S(M')$ are equivariantly homeomorphic, then M and M' have the same type in the above sense.*

(2) *If $S(M_1(a, b))$ and $S(M_1(a', b'))$ are equivariantly homeomorphic, then $(a', b') = (a, b)$ or $(a', b') = (a^{-1}b, b)$.*

(3) *If $S(M_2(a, b))$ and $S(M_2(a', b'))$ are equivariantly homeomorphic, then $a = a'$.*

(4) *If $S(M(a))$ and $S(M(a'))$ are equivariantly homeomorphic, then $a = a'$ or $aa' = 1$.*

Proof. We give only an outline of the proof. The fixed point set $S(M)^{L(3)}$ of the restricted $L(3)$ -action is a 2-sphere and the fixed point set $S(M)^{N(3)}$ of the restricted $N(3)$ -action is a disjoint union of low dimensional spheres, where $L(3)$ and $N(3)$ are closed subgroups of $SL(3, \mathbf{R})$ defined in §3.3.

If we consider homeomorphism classes of $S(M)^{N(3)}$, we can distinguish a matrix of (Type i) from that of (Type j) except for the case $(i, j) = (2, 4)$. Fur-

thermore, we can prove (0) by considering a tangential representation of $N(3)/L(3)$ on the tangent space of the 2-sphere $S(M)^{L(3)}$ at isolated fixed points of the restricted $N(3)$ -action.

Denote by $H(P)$ a closed subgroup of $SL(3, \mathbf{R})$ consisting of all matrices in the form

$$\left(\begin{array}{c|c} e^{\theta P} & * \\ \hline 0 & * \end{array} \right), \theta \in \mathbf{R}$$

where P is a square matrix of degree 2. We can prove the remaining part of the theorem by considering homeomorphism classes of the fixed point sets $S(M)^{H(P)}$ of the restricted $H(P)$ -action. For $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that $S(M_0)^{H(P)}$ is a 1-sphere but $S(M_2(a, b))^{H(P)}$ is a 0-sphere, and hence we can distinguish M_0 from any matrix of (Type 2). By $P = \begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix}$ for $c > 0$, we can prove (3). By $P = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$ for $c > 0$, we can prove (2) and (4). q.e.d.

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