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ON A TRANSITIVE TRANSFORMATION GROUP OF
A COMPACT GROUP MANIFOLD

HIDEKI OZEKI

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1. Introduction. Let $K$ be a connected compact Lie group and $H$ a closed subgroup of $K$. Suppose a connected Lie subgroup $G$ of $K$ acts simply transitively on the coset space $K/H$ by the left translation. Then the composition mapping

$$F: G \times H \rightarrow K$$

defined by $F(g, h) = gh \ (g \in G, \ h \in H)$ gives rise to a diffeomorphism of the product manifold $G \times H$ onto $K$. Consequently, for their Lie algebras, we have

$$\mathfrak{f} = \mathfrak{g} + \mathfrak{h} \quad \text{(direct sum of vector spaces)}.$$

We shall prove in this paper the following:

**Theorem 1.** Let $\mathfrak{f}$ be a compact Lie algebra. Suppose there exist two subalgebras $\mathfrak{g}$ and $\mathfrak{h}$ of $\mathfrak{f}$ such that

$$\mathfrak{f} = \mathfrak{g} + \mathfrak{h} \quad \text{(direct sum of vector spaces)}.$$

Then there exist a direct sum decomposition

$$\mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

of Lie algebras and Lie algebra homomorphisms

$$\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1 \quad \text{and} \quad \psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$$

with the following properties:

(i) $\mathfrak{g} = \{X, \varphi(X)\mid X \in \mathfrak{g}_1\}$.

(ii) $\mathfrak{h} = \{\psi(Y), \ Y \in \mathfrak{h}_1\}$.

(iii) $\psi \circ \varphi$ has no non-zero fixed vector.

As a result we see that the Lie algebra $\mathfrak{f}$ is isomorphic with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lie algebras. This theorem gives us an infinitesimal characterization of a homogeneous space of the type mentioned in the above. Some
Such a homogeneous space is related with a study of isometries of a compact group manifold. Let $G$ be a connected compact Lie group and choose a left invariant Riemannian metric $ds^2$ on $G$. Denote by $K$ the identity component of the isometry group of $(G, ds^2)$. We identify an element $g$ of $G$ with its left translation $L_g$ on $G$. Ochiai-Takahashi [2] proved that if $G$ is simple then $G$ is normal in $K$. Their theorem follows immediately from our Theorem 1. The conclusion of their theorem does not hold in general if $G$ is not simple, as our example shows. However, our Theorem 3 asserts that if $G$ is simply connected then we have a similar conclusion by a suitable change of the action of $G$ on the space.

2. Recall that a Lie algebra $\mathfrak{f}$ is said to be compact if it can be represented as a Lie algebra of a compact Lie group. For a compact Lie algebra $\mathfrak{f}$, we denote by $c(\mathfrak{f})$ its center and by $s(\mathfrak{f})$ its maximal semi-simple ideal, so that we have $s(\mathfrak{f}) = [\mathfrak{f}, \mathfrak{f}]$ and

$$\mathfrak{f} = s(\mathfrak{f}) \oplus c(\mathfrak{f})$$

(direct sum of Lie algebras). The same notation will be used for a connected Lie group $K$ when the Lie algebra $\mathfrak{f}$ of $K$ is compact. $c(K)$ and $s(k)$ are the connected Lie subgroups of $K$ corresponding to Lie subalgebras $c(\mathfrak{f})$ and $s(\mathfrak{f})$ respectively.

Note that a connected Lie group $K$ has a compact Lie algebra if and only if $K$ has a bi-invariant Riemannian metric and also that any subalgebra of a compact Lie algebra is compact. In the sequel, for a Lie group homomorphism, the induced Lie algebra homomorphism is denoted by the same symbol.

**Lemma 1.** Let $K$, $G$ and $H$ be connected Lie groups with Lie algebras $\mathfrak{k}$, $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose $\mathfrak{k}$ is compact. Let $\phi: G \to K$ and $\psi: H \to K$ be Lie group homomorphisms such that the induced homomorphisms $\phi: \mathfrak{g} \to \mathfrak{k}$ and $\psi: \mathfrak{h} \to \mathfrak{k}$ are both injective and

$$\mathfrak{k} = \phi(\mathfrak{g}) + \psi(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Then the composition mapping

$$F: G \times H \to K$$

defined by $F(g, h) = \phi(g) \cdot \psi(h)$ is a covering map.

Proof. In general, we denote the left translation and the right translation of a group induced by an element $x$ in it by $L_x$ and $R_x$ respectively. Then, for the mapping $F$, we have the following commutative diagram:
for \((g, h) \in G \times H\). This gives an identity
\[
F = (L_{\phi(g)} \circ R_{\psi(h)}) \circ F \circ (L_{g^{-1}}, R_{h^{-1}})
\]
Taking the differentials, we have
\[
(dF)_{(e,e)} = (d(L_{\phi(g)} \circ R_{\psi(h)})) \circ (dF)_{(e,e)} \circ (d(L_{g^{-1}}, R_{h^{-1}}))_{(e,e)}
\]
We identify \(T_{(e,e)}(G \times H)\) with \(\Gamma_{\phi}(G) + \Gamma_{\psi}(H)\) (direct sum of vector spaces). Since \((dF)_{(e,e)} \mid T_{e}(G) = \phi \) and \((dF)_{(e,e)} \mid T_{e}(H) = \psi\), our assumption in the lemma implies that \((dF)_{(e,e)}\) gives an isomorphism of \(T_{(e,e)}(G \times H)\) onto \(T_{e}(K) = \mathfrak{f}\). By the above identity, we see that \((dF)_{(e,e)}\) is isomorphic at each point \((g, h)\) of \(G \times H\). Since \(\mathfrak{f}\) is compact, we can choose a bi-invariant Riemannian metric \(d\mathfrak{s}^2\) on \(K\). Then \(d\mathfrak{s}^2 = F^*(d\mathfrak{s}^2)\) gives a Riemannian metric on the manifold \(G \times H\), which is locally isometric with \((K, d\mathfrak{s}^2)\) via \(F\). In virtue of the first commutative diagram, the Riemannian metric \(d\mathfrak{s}^2\) on \(G \times H\) is \(L(G)\) and \(R(H)\)-invariant, and hence it is complete. Thus we see that \(F\) is a locally isometric mapping of a compact Riemannian manifold \((G \times H, d\mathfrak{s}^2)\) into \((K, d\mathfrak{s}^2)\). This proves that \(F\) is a covering map.

**Lemma 2.** Let \(\mathfrak{t}\) be a compact Lie algebra, and let \(\mathfrak{g}\) and \(\mathfrak{h}\) be two subalgebras of \(\mathfrak{t}\) such that

\[
\mathfrak{t} = \mathfrak{g} + \mathfrak{h} \quad \text{(direct sum of vector spaces)}.
\]
Then, \(\mathfrak{t}\) is isomorphic with the direct sum \(\mathfrak{g} \oplus \mathfrak{h}\) of Lie algebras. Consequently, we have

\[
\dim c(\mathfrak{t}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h}).
\]

Proof. For \(\mathfrak{g}, \mathfrak{h}\) and \(\mathfrak{t}\), choose simply connected Lie groups \(G, H\) and \(K\) with the corresponding Lie algebras respectively. Let

\[
\phi : \mathfrak{g} \to \mathfrak{t} \quad \text{and} \quad \psi : \mathfrak{h} \to \mathfrak{t}
\]
be the inclusion mappings. They induce Lie group homomorphisms

\[
\phi : G \to K \quad \text{and} \quad \psi : H \to K.
\]
The composition mapping \(F\) of the product manifold \(G \times H\) into \(K\) defined by

\[
F(g, h) = \phi(g) \psi(h)
\]
is a covering map by Lemma 1. Since \(K\) is assumed to be simply connected, we
have a diffeomorphism of $G \times H$ onto $K$. $\mathfrak{f}$ is compact and hence $\mathfrak{g}$ and $\mathfrak{h}$ are compact. Since $G$, $H$ and $K$ are simply connected and their Lie algebras are compact, we see $G = s(G) \times c(G)$, $H = s(H) \times c(H)$ and $K = s(K) \times c(K)$. Since $F$ is a diffeomorphism of the product manifold $G \times H$ onto $K$ we see

$$\dim c(K) = \dim c(G) + \dim c(H)$$

and hence

$$\dim c(\mathfrak{f}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h}).$$

Note that $s(K)$ is a maximal compact subgroup of $K$. Also we see that $F$ induces a homotopy equivalence between $s(G) \times s(H)$ and $s(K)$.

A theorem in homotopy theory ([3], [4]) states that if two simply connected compact Lie groups are homotopically equivalent then they are isomorphic as Lie groups. Thus, we see that the Lie group $s(K)$ is isomorphic with the direct product $s(G) \times s(H)$ of Lie groups. Finally we can conclude that the Lie algebra $\mathfrak{f}$ is isomorphic with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lie algebras. q.e.d.

**Corollary 1.** Under the same assumption as above, we have

$$s(\mathfrak{f}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Proof. Since $\mathfrak{f}$ and $\mathfrak{g} \oplus \mathfrak{h}$ are isomorphic, $s(\mathfrak{f})$ and $s(\mathfrak{g}) \oplus s(\mathfrak{h})$ are isomorphic. Especially, we have

$$\dim s(\mathfrak{f}) = \dim s(\mathfrak{g}) + \dim s(\mathfrak{h}).$$

On the other hand, we know

$$s(\mathfrak{f}) = [\mathfrak{f}, \mathfrak{f}], \quad s(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}], \quad s(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{h}].$$

Thus, we have

$$s(\mathfrak{f}) \supset s(\mathfrak{g}) \quad \text{and} \quad s(\mathfrak{f}) \supset s(\mathfrak{h}).$$

The assumption $\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{h}$ (direct sum of vector spaces) shows that $s(\mathfrak{g}) \oplus s(\mathfrak{h})$ is a direct sum of vector spaces in $s(\mathfrak{f})$. The first equality on dimension proves our corollary. q.e.d.

3. Theorem 1 will follow easily from the following:

**Proposition 1.** Let $\mathfrak{f}$ be a compact Lie algebra and let $\mathfrak{g}$ and $\mathfrak{h}$ be its subalgebras such that

$$\mathfrak{f} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

Then $\mathfrak{f}$ has a direct sum decomposition of Lie algebras
$\mathfrak{t} = \mathfrak{l} \oplus \mathfrak{l}'$

with the following properties:

(i) The projection $\pi$ of $\mathfrak{t}$ onto $\mathfrak{l}$ with respect to the above decomposition induces an isomorphism of $\mathfrak{g}$ onto $\mathfrak{l}$.

(ii) $\mathfrak{t} = \mathfrak{t} + \mathfrak{h}$ (direct sum of vector spaces).

Proof. We prove the proposition by induction on $\dim \mathfrak{t}$. When $\dim \mathfrak{t} = 1$, the proposition holds since $\mathfrak{t} = \mathfrak{g}$ or $\mathfrak{t} = \mathfrak{h}$. Now assume that the proposition holds when $\dim \mathfrak{t} < N$. Let $\dim \mathfrak{t} = N$. To simplify the argument we prepare the following:

**Sublemma.** Suppose $\mathfrak{t}$ has a non-trivial proper ideal $\mathfrak{t}_1$ such that

$$\mathfrak{t}_1 = (\mathfrak{g} \cap \mathfrak{t}_1) + (\mathfrak{h} \cap \mathfrak{t}_1) \quad (\text{direct sum of vector spaces}).$$

Then the assertion of Proposition 1 holds for $\mathfrak{t}$, $\mathfrak{g}$ and $\mathfrak{h}$.

Proof. For $\mathfrak{t}_1$, we choose a complementary ideal $\mathfrak{t}_2$ so that we have a direct sum decomposition

$$\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2.$$  

Let $\pi_2$ be the projection of $\mathfrak{t}$ onto $\mathfrak{t}_2$. We have

$$\dim \mathfrak{g} = \dim \mathfrak{g} \cap \mathfrak{t}_1 + \dim \pi_2(\mathfrak{g}),$$

$$\dim \mathfrak{h} = \dim \mathfrak{h} \cap \mathfrak{t}_1 + \dim \pi_2(\mathfrak{h}).$$

Thus,

$$\dim \mathfrak{t}_2 = \dim \mathfrak{t} - \dim \mathfrak{t}_1$$

$$= \dim \pi_2(\mathfrak{g}) + \dim \pi_2(\mathfrak{h}).$$

Since $\mathfrak{t} = \mathfrak{g} + \mathfrak{h}$, $\mathfrak{t}_2 = \pi_2(\mathfrak{t})$ is spanned by $\pi_2(\mathfrak{g})$ and $\pi_2(\mathfrak{h})$, and hence we have

$$\mathfrak{t}_2 = \pi_2(\mathfrak{g}) + \pi_2(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Consider $\mathfrak{t}_1$ and its subalgebras $\mathfrak{g} \cap \mathfrak{t}_1$ and $\mathfrak{h} \cap \mathfrak{t}_1$, and also $\mathfrak{t}_2$ and its subalgebras $\pi_2(\mathfrak{g})$ and $\pi_2(\mathfrak{h})$. By the inductive hypothesis, we have direct sum decompositions

$$\mathfrak{t}_1 = \mathfrak{I}_1 \oplus \mathfrak{I}_1' \quad \text{and} \quad \mathfrak{t}_2 = \mathfrak{I}_2 \oplus \mathfrak{I}_2'$$

with the properties:

i. The projections $\mathfrak{g} \cap \mathfrak{t}_1 \rightarrow \mathfrak{I}_1$ and $\pi_2(\mathfrak{g}) \rightarrow \mathfrak{I}_2$ are isomorphisms.

ii. $\mathfrak{t}_1 = \mathfrak{I}_1 + \mathfrak{h} \cap \mathfrak{t}_1$, $\mathfrak{t}_2 = \mathfrak{I}_2 + \pi_2(\mathfrak{h})$ (direct sums of vector spaces).

Let

$$\mathfrak{I} = \mathfrak{I}_1 \oplus \mathfrak{I}_2 \quad \text{and} \quad \mathfrak{I}' = \mathfrak{I}_1' \oplus \mathfrak{I}_2'.$$
We claim that the direct sum decomposition

\[ \mathfrak{g} = I \oplus I' \]

satisfies the required properties.

First suppose \( X \in g \cap I' \). Then \( \pi_2(X) \in \pi_2(g) \cap I'_2 \). However \( \pi_2(g) \cap I'_2 = \{0\} \) from the assumption. Thus \( \pi_2(X) = 0 \), and hence \( X \in I \). Then \( X \in (g \cap I) \cap I' = \{0\} \). Consequently we have \( g \cap I' = \{0\} \). This shows that the projection of \( g \) into \( I \) with respect to \( I \oplus I' \) is injective. Since they have the same dimension, we have the property (i). Next suppose \( I \cap h \in X \). \( \pi_2(X) \in \pi_2(h) \cap I_2 = \{0\} \), and hence \( X \in I \). We see that \( X \in I \cap (h \cap I) = \{0\} \). Thus, we have \( I \cap h = \{0\} \). Since \( \dim I = \dim I + \dim h \), we see \( I = I + h \) (direct sum of vector spaces). Thus we have the property (ii) also. q.e.d.

We continue our proof of Proposition 1. First consider easy cases.

1. **Suppose** \( \mathfrak{g} \) is abelian.
   Then \( I = g \) and \( I' = h \) satisfy the required properties.

2. **Suppose** \( \mathfrak{g} \) is simple.
   Then, by Lemma 2 we see \( I = g \) or \( I = h \). Thus our assertion holds trivially.

3. **Suppose** \( g \) contains a non-trivial proper ideal, say \( I_1 \), of \( I \).
   Then choose a complementary ideal \( I_2 \) of \( I_1 \) in \( I \), so that we have \( I = I_1 \oplus I_2 \).

   Clearly, \( I_1 \cap g = I_1 \), \( I_1 \cap h \subseteq g \cap h = \{0\} \). Applying the above sublemma we see that Proposition 1 holds in this case.

4. **Suppose** \( h \) contains a non-trivial proper ideal, say \( I_1 \), of \( I \).
   Then again we have \( I_1 \cap g = \{0\} \), and \( I_1 \cap h = I_1 \). Thus we can apply the sublemma in this case also.

5. **Suppose** \( \mathfrak{g} \) is not semi-simple.
   We may suppose \( \mathfrak{g} \) is not abelian. Then the semi-simple part \( s(g) \) is a non-trivial proper ideal of \( \mathfrak{g} \). By Corollary 1, we have

   \[ s(\mathfrak{g}) = s(g) + s(h) \quad \text{(direct sum of vector spaces).} \]

   Since \( s(g) \subseteq g \cap s(\mathfrak{g}) \), \( s(h) \subseteq h \cap s(\mathfrak{g}) \) and \( (g \cap s(\mathfrak{g})) \cap (h \cap s(\mathfrak{g})) = \{0\} \), we have \( s(g) = g \cap s(\mathfrak{g}) \) and \( s(h) = h \cap s(\mathfrak{g}) \). Thus

   \[ s(\mathfrak{g}) = g \cap s(\mathfrak{g}) + h \cap s(\mathfrak{g}) \]

   is a direct sum of vector spaces, and hence we can apply our sublemma.
The above argument shows that we may suppose \( f \) is semi-simple and not simple.

(6) Suppose \( f \) is semi-simple and all simple factors of \( f \) are mutually isomorphic with each other.

In this case we shall show that either \( g \) or \( h \) contains a proper ideal of \( f \), so that the proposition holds by (3) or (4). Suppose neither \( g \) nor \( h \) contains a non trivial proper ideal of \( f \). Let

\[
\begin{align*}
  f &= \sum_{i \in I} t_i, \\
  g &= \sum_{j \in J} g_j, \\
  h &= \sum_{k \in K} h_k
\end{align*}
\]

be the decompositions of \( f, g \) and \( h \) into simple factors. By the present assumption, all \( t_i \)'s are mutually isomorphic. By Lemma 2, we see also that all \( g_j \)'s, \( h_k \)'s and \( t_i \)'s are mutually isomorphic, and that

\[
|I| = |J| + |K|
\]

where \( | \cdot | \) indicates the number of elements.

Denote by \( \pi_i \) the projection of \( f \) onto \( t_i \). One sees that

\[
\begin{align*}
  \pi_i(g_j) &= t_i \text{ or } \{0\}, \\
  \pi_i(h_k) &= t_i \text{ or } \{0\}
\end{align*}
\]

for all \( i, j, k \). Put

\[
\begin{align*}
  A_j &= \{i \in I \mid \pi_i(g_j) \neq \{0\}\}, \\
  B_k &= \{i \in I \mid \pi_i(h_k) \neq \{0\}\}
\end{align*}
\]

for each \( j \in J \) and \( k \in K \). Let \( j_1, j_2 \in J \), and \( j_1 \neq j_2 \). Then \( [\pi_i(g_{j_1}), \pi_i(g_{j_2})]=0 \). Thus \( A_{j_1} \cap A_{j_2} = \emptyset \). Hence \( A_j \)'s are mutually disjoint and so are \( B_k \)'s.

Suppose \( A_j \) consists of exactly one element, say \( i \). Then we see \( g_i = t_i \) and hence \( g \) contains a non trivial proper ideal. This is a contradiction. Thus each \( A_j \) contains at least two elements. Similarly we have \( |B_k| \geq 2 \). Thus we have

\[
\sum |A_j| + \sum |B_k| \geq 2(|J| + |K|) = 2|I|
\]

On the other hand,

\[
\sum |A_j| \leq |I| \quad \text{and} \quad \sum |B_k| \leq |I|.
\]

Combining together, we see

\[
\begin{align*}
  |A_j| &= |B_k| = 2
\end{align*}
\]

for every \( j \in J \) and \( k \in K \).

By an elementary combinatorial argument one can decompose the index set \( I \) into two disjoint subsets \( I_1 \) and \( I_2 \) such that, for every \( j, k \), the sets \( A_j \cap I_i \),
A_i \cap I_2, B_k \cap I_1 \text{ and } B_k \cap I_2 \text{ are all non empty. Let}
\alpha_i = \sum \iota_i \text{ and } \alpha_2 = \sum \iota_2,

so that we have \mathfrak{f} = \alpha_1 \oplus \alpha_2. \text{ Denote by } p_i \text{ the projection of } \mathfrak{f} \text{ onto } \alpha_i \text{ (for } i=1, 2). \text{ It follows from our construction that the homomorphisms } p_1 | \mathfrak{g}, p_2 | \mathfrak{g}, p_1 | \mathfrak{h} \text{ and } p_2 | \mathfrak{h} \text{ are all onto isomorphisms. Using the decomposition } \mathfrak{f} = \alpha_1 \oplus \alpha_2, \text{ we can write}
\mathfrak{g} = \{(X, \phi(X)) | X \in \alpha_1\}

and
\mathfrak{h} = \{(\psi(Y), Y) | Y \in \alpha_2\}

by suitable onto isomorphisms \phi: \alpha_1 \rightarrow \alpha_2 \text{ and } \psi: \alpha_2 \rightarrow \alpha_1. \text{ Consider an automorphism } \phi \circ \psi \text{ of } \alpha_1. \text{ By a result due to Borel and Mostow } [1], \text{ every automorphism of a semi-simple Lie algebra has a non-zero fixed vector. Thus, we have an element } X \text{ in } \alpha_1 \text{ such that } X \neq 0 \text{ and } \psi(\phi(X)) = X. \text{ Then we have}
(X, \phi(X)) = (\psi(\phi(X)), \phi(X)) \in \mathfrak{g} \cap \mathfrak{h} = \{0\}.

This is a contradiction. Thus, in this case, either \mathfrak{g} \text{ or } \mathfrak{h} \text{ contains a proper ideal of } \mathfrak{f}.

(7) \text{ Suppose } \mathfrak{f} \text{ is semi-simple and } \mathfrak{f} \text{ contains at least two simple ideals which are not isomorphic.}

Choose a simple ideal } \alpha \text{ of } \mathfrak{f} \text{ such that dim } \alpha \text{ is minimal among the simple ideals of } \mathfrak{f}. \text{ Let } \mathfrak{f}_0 \text{ be the direct sum of all simple ideals isomorphic to } \alpha, \text{ and } \mathfrak{f}_1 \text{ the complementary ideal, so that we have}
\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1.

Similarly, decompose \mathfrak{g} \text{ and } \mathfrak{h} \text{ as}
\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \text{ and } \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1,

where \mathfrak{g}_0 \text{ (resp. } \mathfrak{h}_0) \text{ is the direct sum of all simple ideals in } \mathfrak{g} \text{ (resp. } \mathfrak{h}) \text{ isomorphic to } \alpha.

In virtue of Lemma 2, we see that \mathfrak{f}_0 \text{ and } \mathfrak{f}_1 \text{ are isomorphic with } \mathfrak{g}_0 \oplus \mathfrak{h}_0 \text{ and } \mathfrak{g}_1 \oplus \mathfrak{h}_1 \text{ respectively. We claim that the ideal } \mathfrak{f}_1 \text{ satisfies the required condition in the sublemma. Let } \pi_0 \text{ and } \pi_1 \text{ be the projections of } \mathfrak{f} \text{ onto } \mathfrak{f}_0 \text{ and } \mathfrak{f}_1 \text{ respectively. Consider } \pi_0: \mathfrak{g}_1 \rightarrow \mathfrak{f}_0. \text{ From the definitions of } \mathfrak{g}_1 \text{ and } \mathfrak{f}_0, \text{ we see } \pi_0|\mathfrak{g}_1 = \{0\}. \text{ Thus, } \mathfrak{g}_1 \subseteq \mathfrak{f}_1. \text{ Similarly we have } \mathfrak{h}_1 \subseteq \mathfrak{f}_1. \text{ Thus, } \mathfrak{f}_1 \supseteq \mathfrak{g}_1 + \mathfrak{h}_1. \text{ Since } \mathfrak{g}_1 \cap \mathfrak{h}_1 = \{0\} \text{ and dim } \mathfrak{f}_1 = \text{dim } \mathfrak{g}_1 + \text{dim } \mathfrak{h}_1, \text{ we conclude that}
\mathfrak{f}_1 = \mathfrak{g}_1 + \mathfrak{h}_1 \text{ (direct sum of vector spaces).}
Since $\mathfrak{t} = \mathfrak{g} + \mathfrak{h}$ is a direct sum of vector spaces, we see that $(\mathfrak{t} \cap \mathfrak{g}) \cap (\mathfrak{t} \cup \mathfrak{h}) = \{0\}$. On the other hand, $\mathfrak{t} \cap \mathfrak{g} \supseteq \mathfrak{g}_i$ and $\mathfrak{t} \cap \mathfrak{h} \supseteq \mathfrak{h}_i$, and also $\mathfrak{t} = \mathfrak{g}_i + \mathfrak{h}_i$ (direct sum of vector spaces). It follows that $\mathfrak{g}_i = \mathfrak{g} \cap \mathfrak{t}_i$ and $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{t}_i$ and hence

$$\mathfrak{t}_i = (\mathfrak{g} \cap \mathfrak{t}_i) + (\mathfrak{h} \cap \mathfrak{t}_i) \quad \text{(direct sum of vector spaces)}.$$

This proves our claim.

Thus we have completed the proof of Proposition 1.

4. Now we can prove Theorem 1

Proof of Theorem 1. First assume that $\mathfrak{t}$ is semi-simple. Apply Proposition 1 to $\mathfrak{t}$, $\mathfrak{g}$ and $\mathfrak{h}$. We get a direct sum decomposition

$$\mathfrak{t} = \mathfrak{I} \oplus \mathfrak{I}'$$

with the properties:

(i) The projection of $\mathfrak{t}$ onto $\mathfrak{I}$ with respect to the above decomposition induces an isomorphism of $\mathfrak{g}$ onto $\mathfrak{I}$.

(ii) $\mathfrak{t} = \mathfrak{I} + \mathfrak{h}$ (direct sum of vector spaces).

Again apply Proposition 1 to $\mathfrak{t}$, $\mathfrak{h}$ and $\mathfrak{I}$. We have a direct sum decomposition

$$\mathfrak{t} = \mathfrak{m} \oplus \mathfrak{m}'$$

with the properties:

(i') The projection of $\mathfrak{t}$ onto $\mathfrak{m}$ with respect to this decomposition induces an isomorphisms of $\mathfrak{h}$ onto $\mathfrak{m}$.

(ii') $\mathfrak{t} = \mathfrak{m} + \mathfrak{I}$ (direct sum of vector spaces).

Since $\mathfrak{m}$ and $\mathfrak{I}$ are both ideals of $\mathfrak{t}$, we have a direct sum

$$\mathfrak{t} = \mathfrak{m} \oplus \mathfrak{I}$$

of Lie algebras. The assumption that $\mathfrak{t}$ is semi-simple implies $\mathfrak{m} = \mathfrak{I}'$. Thus, with respect to the direct sum

$$\mathfrak{t} = \mathfrak{I} \oplus \mathfrak{I}'$$

we see that the projections of $\mathfrak{t}$ onto $\mathfrak{I}$ and $\mathfrak{I}'$ induce isomorphisms of $\mathfrak{g}$ and $\mathfrak{h}$ onto $\mathfrak{I}$ and $\mathfrak{I}'$ respectively. Setting $\mathfrak{g}_i = \mathfrak{I}_i$ and $\mathfrak{h}_i = \mathfrak{I}'_i$, we see that the decomposition

$$\mathfrak{t} = \mathfrak{g}_i \oplus \mathfrak{h}_i$$

satisfies the first two properties. The third property follows from $\mathfrak{g} \cap \mathfrak{h} = \{0\}$.

In fact, suppose $\psi(\phi(\mathfrak{X})) = \mathfrak{X}$ for $\mathfrak{X} \in \mathfrak{g}_i$. Then, $(\mathfrak{X}, \phi(\mathfrak{X})) = (\psi(\phi(\mathfrak{X})), \psi(\phi(\mathfrak{X})))$.
\( \phi(X) \subseteq g \cap h = \{0\} \). Thus \( X = 0 \).

Consider the general case. By Corollary 1, we have

\[ s(\mathfrak{f}) = s(g) + s(h) \quad (\text{direct sum of vector spaces}). \]

Also by Lemma 1, \( \dim c(\mathfrak{f}) = \dim c(g) + \dim c(h) \). It is easily seen that the projection \( \pi \) of \( \mathfrak{f} \) onto \( c(\mathfrak{f}) \) induces

\[ c(\mathfrak{f}) = \pi(c(g)) + \pi(c(h)) \quad (\text{direct sum of vector spaces}). \]

From the first argument, we can choose a direct sum decomposition

\[ s(\mathfrak{f}) = g' \oplus h', \]

such that the projections of \( s(\mathfrak{f}) \) onto \( g' \) and \( h' \) induce isomorphisms of \( s(g) \) and \( s(h) \) onto \( g' \) and \( h' \) respectively. Now put

\[ g_1 = g' + \pi(c(g)) \]

and

\[ h_1 = h' + \pi(c(h)). \]

we have a direct sum decomposition

\[ \mathfrak{f} = g_1 \oplus h_1. \]

We claim that this decomposition satisfies the required properties in Theorem 1. The first two are easy. The last one follows from the first two and \( g \cap h = \{0\} \).

Q.E.D.

**Remark 1.** The converse of Theorem 1 holds. Let \( g_1 \) and \( h_1 \) be Lie algebras, and let \( \phi: g_1 \to h_1 \) and \( \psi: h_1 \to g_1 \) be Lie algebra homomorphisms such that \( \psi \circ \phi \) has no non-zero fixed vector. In the direct sum \( g_1 \oplus h_1 = \mathfrak{f} \) of Lie algebras, define \( g \) and \( h \) by (i) and (ii). Then \( g \) and \( h \) are subalgebras and we have

\[ \mathfrak{f} = g + h \quad (\text{direct sum of vector spaces}). \]

**Remark 2.** Suppose \( M = K/H \) is a homogeneous space of the type mentioned in the introduction. Then the action of \( K \) on \( K/H \) is almost effective if and only if \( \psi \) is injective.

**Remark 3.** Let \( M = K/H \) be as above. By the theorem of Borel-Mostow cited before, the Lie algebra homomorphism \( \psi \circ \phi = 0 \) if \( g \) is simple. Thus we see that if \( G \) is simple and the \( K \)-action on \( K/H \) is almost effective then \( G \) is normal. Thus, Ochiai-Takahashi’s theorem follows from Theorem 1.

5. Now we consider a homogeneous space of the type mentioned in the introduction. Let \( M = K/H \) be a homogeneous space of a connected compact
Lie group $K$. We assume that a connected Lie subgroup $G$ acts simply transitively on $K/H$. Since $K/H$ is compact, $G$ is necessarily compact. The composition mapping

$$F: G \times H \rightarrow K$$

is a diffeomorphism, so that we have

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{h} \quad \text{(direct sum of vector spaces)},$$

for their Lie algebras. Applying Theorem 1, we have a direct sum decomposition

$$\mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

and homomorphisms $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$ and $\psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$ such that we have

$$\begin{align*}
\mathfrak{g} &= \{(X, \phi(X)) \mid X \in \mathfrak{g}_1\}, \\
\mathfrak{h} &= \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}.
\end{align*}$$

Further, as we see from the proof of Theorem 1, we can assume that

$$\mathfrak{c}(\mathfrak{g}_1) = \pi(\mathfrak{c}(\mathfrak{g})),$$

where $\pi$ denotes the projection of $\mathfrak{f}$ onto its center.

Let $G_1$ be the connected Lie subgroup of $K$ corresponding to the subalgebra $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is an ideal of $\mathfrak{f}$, $G_1$ is a normal subgroup of $K$. Next we claim that $G_1$ is compact. $s(G_1)$ is closed in $K$ since it is semi-simple. Thus it suffices to show that $\mathfrak{c}(G_1)$ is compact. However, from our construction, $\mathfrak{c}(G_1) = \tau(\mathfrak{c}(\mathfrak{g}))$.

Consider the Lie group homomorphism $\pi: K \rightarrow K/\mathfrak{s}(K)$. $\pi|\mathfrak{c}(K)$ is a finite covering map. Thus $\mathfrak{c}(G_1)$ is closed in $\mathfrak{c}(K)$ if and only if $\pi(\mathfrak{c}(G_1))$ is closed. On the other hand, $\mathfrak{c}(G)$ is compact, and hence $\pi(\mathfrak{c}(G))$ is compact. $\mathfrak{c}(G_1) = \pi(\mathfrak{c}(G))$ implies that $\pi(\mathfrak{c}(G_1)) = \pi(\mathfrak{c}(G))$. Thus, $\mathfrak{c}(G_1)$ is closed, and hence $G_1$ is compact. From the property that $\mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$ and $\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}$, we have

$$\mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1 \quad \text{(direct sum of vector spaces)}.$$

By Lemma 1, the composition mapping

$$G_1 \times H \rightarrow K$$

defines a covering map. Consequently, $G_1$ acts transitively on the coset space $K/H$. Furthermore, fix a point $p$ in $K/H$. Then the mapping

$$G_1 \rightarrow K/H$$

defined by $g \mapsto g(p)$ is a covering map. Thus, if $G(\approx K/H)$ is simply connected, then $G_1$ is also simply connected. Thus we have proved the following:
Theorem 2. Let $K$ be a connected compact Lie group and $H$ a closed subgroup of $K$. Assume that a connected Lie subgroup $G$ acts simply transitively on the homogeneous space $K/H$ by the left translation. Then there exists a connected closed normal subgroup $G_1$ of $K$ such that $G_1$ acts transitively on $K/H$ and $G_1$ is locally isomorphic with $G$ as Lie groups.

Theorem 3. Under the same assumption as in Theorem 2, assume further that $G$ is simply connected. Then there exists a connected closed normal subgroup $G_1$ of $K$ such that $G_1$ is isomorphic with $G$ as Lie groups and $G_1$ acts simply transitively on $K/H$.

6. We give here two examples. The first one shows that the conclusion of Ochiai-Takahashi's theorem does not hold any more if $G$ is not simple.

Example 1. Let $A$ be a connected compact semi-simple Lie group and $\alpha$ its Lie algebra. We put

$$K = A \times A \times A,$$

$$G = \{(x, y, x)| x, y \in A\},$$

$$H = \{(e, z, z)| z \in A\}.$$

$H$ is a closed subgroup of $K$. Consider the homogeneous space $K/H$. We see easily that $G$ acts simply transitively on $K/H$. $G$ is compact semi-simple and not simple. Choose a $K$-invariant Riemannian metric $ds^2$ on $K/H$. Since $K/H$ can be identified with $G$, $ds^2$ is a left -invariant Riemannian metric on $G$. From the definition, $K$ is contained in the identity-component of isometries of $(K/H = G, ds^2)$. $G$ is not normal in $K$, thus $G$ is not normal in the identity-component of isometries.

For this example, an explicit description of Theorem 1 is as follows:

Let

$$g_1 = \{(X, Y, 0)| X, Y \in \alpha\},$$

$$h_1 = \{(0, 0, Z)| Z \in \alpha\}.$$

Define $\phi: g_1 \to h_1$ by

$$\phi((X, Y, 0)) = (0, 0, X)$$

and $\psi: h_1 \to g_1$ by

$$\psi((0, 0, Z)) = (0, Z, 0).$$

Then we have

$$g = \{(X, \phi(X)) \in g_1 \oplus h_1| X \in g_1\},$$

$$g = \{\psi(Y), Y) \in g_1 \oplus g_1\{Y \in h_1\}.$$
The next example shows that the conclusion of Theorem 3 does not hold if $G$ is not simply connected.

**Example 2.** We choose two simply connected compact Lie groups $A$ and $B$ with the following properties:
1. There exists an injective homomorphism $j$ of $A$ into $B$.
2. The center $Z(A)$ of $A$ is non-trivial and
   
   $j(Z(A)) \cap Z(B) = \{e\}$.

For instance, choose positive integers $m$ and $n$ such that $n > m > 2$. Then $A = SU(m)$, $B = SU(n)$ and the canonical injection of $SU(m)$ into $SU(n)$ satisfy the required properties.

Let

$K = A \times B \times A$,  
$G_1 = A \times B \times \{e\}$,  
$G = \{(a, b, a) | a \in A, b \in B\}$,  
$H = \{(e, j(a), a) | a \in A\}$,  
$\Gamma = \{(x, e, x) | x \in Z(A)\}$.

The Lie algebras of $A$ and $B$ are denoted by $\mathfrak{a}$ and $\mathfrak{b}$ respectively. $\Gamma$ is a finite group contained in the center of $K$. We consider the quotient group $K = K/\Gamma$, and denote by $\pi$ the canonical projection of $K$ onto $\tilde{K}$. $\tilde{H} = \pi(H)$ is a closed subgroup of $\tilde{K}$. Consider $\tilde{K}/\tilde{H}$. We claim that no normal subgroup of $\tilde{K}$ acts simply transitively on $\tilde{K}/\tilde{H}$. Suppose a normal subgroup $G_1'$ of $\tilde{K}$ acts simply transitively on $\tilde{K}/\tilde{H}$. Then its Lie algebra $\mathfrak{g}_1'$ satisfies

$\mathfrak{f} = \mathfrak{g}_1' + \mathfrak{g}$ (direct sum of vector spaces),

where $\mathfrak{h} = \{(0, j(X), X) | X \in \mathfrak{a}\}$. Since $\mathfrak{g}_1'$ is an ideal of $\mathfrak{f}$, we see $\mathfrak{g}_1' = \mathfrak{g}_1 = \{(X, Y, 0) | X \in \mathfrak{a}, Y \in \mathfrak{b}\}$. It follows that $\pi(G_1) = G_1'$. However, $\pi(G_1)$ is simply connected because $\pi(G_1) = G_1/(G_1 \cap \Gamma) = G_1$. This is a contradiction.

**References**


