On a transitive transformation group of a compact group manifold

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ON A TRANSITIVE TRANSFORMATION GROUP OF
A COMPACT GROUP MANIFOLD

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1. Introduction. Let $K$ be a connected compact Lie group and $H$ a closed subgroup of $K$. Suppose a connected Lie subgroup $G$ of $K$ acts simply transitively on the coset space $K/H$ by the left translation. Then the composition mapping

$$F: G \times H \to K$$

defined by $F(g, h) = gh$ ($g \in G$, $h \in H$) gives rise to a diffeomorphism of the product manifold $G \times H$ onto $K$. Consequently, for their Lie algebras, we have

$$\mathfrak{f} = \mathfrak{g} + \mathfrak{h} \quad \text{(direct sum of vector spaces).}$$

We shall prove in this paper the following:

**Theorem 1.** Let $\mathfrak{f}$ be a compact Lie algebra. Suppose there exist two subalgebras $\mathfrak{g}$ and $\mathfrak{h}$ of $\mathfrak{f}$ such that

$$\mathfrak{f} = \mathfrak{g} + \mathfrak{h} \quad \text{(direct sum of vector spaces).}$$

Then there exist a direct sum decomposition

$$\mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

of Lie algebras and Lie algebra homomorphisms

$$\varphi: \mathfrak{g}_1 \to \mathfrak{h}_1 \quad \text{and} \quad \psi: \mathfrak{h}_1 \to \mathfrak{g}_1$$

with the following properties:

(i) $\mathfrak{g} = \{(X, \varphi(X)) | X \in \mathfrak{g}_1\}$.

(ii) $\mathfrak{h} = \{\psi(Y), Y \in \mathfrak{h}_1\}$.

(iii) $\psi \circ \varphi$ has no non-zero fixed vector.

As a result we see that the Lie algebra $\mathfrak{f}$ is isomorphic with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lie algebras. This theorem gives us an infinitesimal characterization of a homogeneous space of the type mentioned in the above. Some
application and remarks will be added after its proof.

Such a homogeneous space is related with a study of isometries of a compact group manifold. Let $G$ be a connected compact Lie group and choose a left invariant Riemannian metric $d\sigma$ on $G$. Denote by $K$ the identity component of the isometry group of $(G, d\sigma)$. We identify an element $g$ of $G$ with its left translation $L_g$ on $G$. Ochiai-Takahashi [2] proved that if $G$ is simple then $G$ is normal in $K$. Their theorem follows immediately from our Theorem 1. The conclusion of their theorem does not hold in general if $G$ is not simple, as our example shows. However, our Theorem 3 asserts that if $G$ is simply connected then we have a similar conclusion by a suitable change of the action of $G$ on the space.

2. Recall that a Lie algebra $\mathfrak{f}$ is said to be compact if it can be represented as a Lie algebra of a compact Lie group. For a compact Lie algebra $\mathfrak{f}$, we denote by $c(\mathfrak{f})$ its center and by $s(\mathfrak{f})$ its maximal semi-simple ideal, so that we have $s(\mathfrak{f}) = [\mathfrak{f}, \mathfrak{f}]$ and

\[
\mathfrak{f} = s(\mathfrak{f}) \oplus c(\mathfrak{f})
\]

(direct sum of Lie algebras). The same notation will be used for a connected Lie group $K$ when the Lie algebra $\mathfrak{k}$ of $K$ is compact. $c(K)$ and $s(k)$ are the connected Lie subgroups of $K$ corresponding to Lie subalgebras $c(\mathfrak{f})$ and $s(\mathfrak{f})$ respectively.

Note that a connected Lie group $K$ has a compact Lie algebra if and only if $K$ has a bi-invariant Riemannian metric and also that any subalgebra of a compact Lie algebra is compact. In the sequel, for a Lie group homomorphism, the induced Lie algebra homomorphism is denoted by the same symbol.

**Lemma 1.** Let $K$, $G$ and $H$ be connected Lie groups with Lie algebras $\mathfrak{k}$, $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Suppose $\mathfrak{k}$ is compact. Let $\phi: G \to K$ and $\psi: H \to K$ be Lie group homomorphisms such that the induced homomorphisms $\phi: \mathfrak{g} \to \mathfrak{k}$ and $\psi: \mathfrak{h} \to \mathfrak{k}$ are both injective and

\[
\mathfrak{k} = \phi(\mathfrak{g}) + \psi(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).
\]

Then the composition mapping

\[
F: G \times H \to K
\]

defined by $F(g, h) = \phi(g) \cdot \psi(h)$ is a covering map.

Proof. In general, we denote the left translation and the right translation of a group induced by an element $x$ in it by $L_x$ and $R_x$ respectively. Then, for the mapping $F$, we have the following commutative diagram:
for $(g, h) \in G \times H$. This gives an identity

$$F = (L_{\phi(g)} \circ R_{\psi(h)}) \circ F \circ (L_{g^{-1}} \circ R_{h^{-1}}).$$

Taking the differentials, we have

$$(dF)_{(g, h)} = (d(L_{\phi(g)} \circ R_{\psi(h)}))_e \circ (dF)_{(g, h)} \circ (d(L_{g^{-1}} \circ R_{h^{-1}}))_{(g, h)}.$$

We identify $T_{(g, h)}(G \times H)$ with $\Gamma_0(G) + \Gamma_0(H)$ (direct sum of vector spaces). Since $(dF)_{(g, h)}| T_{(g)}(G) = \phi$ and $(dF)_{(g, h)}| T_{(h)}(H) = \psi$, our assumption in the lemma implies that $(dF)_{(g, h)}$ gives an isomorphism of $T_{(g, h)}(G \times H)$ onto $T_{(g, h)}(K)$. By the above identity, we see that $(dF)_{(g, h)}$ is isomorphic at each point $(g, h)$ of $G \times H$. Since $\mathfrak{t}$ is compact, we can choose a bi-invariant Riemannian metric $d\mathfrak{s}^2$ on $K$. Then $d\mathfrak{s}^2 = F^*(d\mathfrak{s}^2)$ gives a Riemannian metric on the manifold $G \times H$, which is locally isometric with $(K, d\mathfrak{s}^2)$ via $F$. In virtue of the first commutative diagram, the Riemannian metric $d\mathfrak{s}^2$ on $G \times H$ is $L(G)$ and $R(H)$-invariant, and hence it is complete. Thus we see that $F$ is a locally isometric mapping of a complete Riemannian manifold $(G \times H, d\mathfrak{s}^2)$ into $(K, d\mathfrak{s}^2)$. This proves that $F$ is a covering map.

**Lemma 2.** Let $\mathfrak{g}$ be a compact Lie algebra, and let $\mathfrak{g}$ and $\mathfrak{h}$ be two subalgebras of $\mathfrak{k}$ such that

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

Then, $\mathfrak{k}$ is isomorphic with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lie algebras. Consequently, we have

$$\dim \mathfrak{c}(\mathfrak{k}) = \dim \mathfrak{c}(\mathfrak{g}) + \dim \mathfrak{c}(\mathfrak{h}).$$

**Proof.** For $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$, choose simply connected Lie groups $G$, $H$ and $K$ with the corresponding Lie algebras respectively. Let

$$\phi: \mathfrak{g} \to \mathfrak{k} \quad \text{and} \quad \psi: \mathfrak{h} \to \mathfrak{k}$$

be the inclusion mappings. They induce Lie group homomorphisms

$$\phi: G \to K \quad \text{and} \quad \psi: H \to K.$$ 

The composition mapping $F$ of the product manifold $G \times H$ into $K$ defined by

$$F(g, h) = \phi(g) \psi(h)$$

is a covering map by Lemma 1. Since $K$ is assumed to be simply connected, we
have a diffeomorphism of $G \times H$ onto $K$. $\mathfrak{f}$ is compact and hence $\mathfrak{g}$ and $\mathfrak{h}$ are compact. Since $G$, $H$ and $K$ are simply connected and their Lie algebras are compact, we see $G = s(G) \times c(G)$, $H = s(H) \times c(H)$ and $K = s(K) \times c(K)$. Since $F$ is a diffeomorphism of the product manifold $G \times H$ onto $K$ we see

$$\dim c(K) = \dim c(G) + \dim c(H)$$

and hence

$$\dim c(\mathfrak{f}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h}).$$

Note that $s(K)$ is a maximal compact subgroup of $K$. Also we see that $F$ induces a homotopy equivalence between $s(G) \times s(H)$ and $s(K)$.

A theorem in homotopy theory ([3], [4]) states that if two simply connected compact Lie groups are homotopically equivalent then they are isomorphic as Lie groups. Thus, we see that the Lie group $s(K)$ is isomorphic with the direct product $s(G) \times s(H)$ of Lie groups. Finally we can conclude that the Lie algebra $\mathfrak{f}$ is isomorphic with the direct sum $s(\mathfrak{g}) \oplus s(\mathfrak{h})$ of Lie algebras.

$q.e.d.$

**Corollary 1.** Under the same assumption as above, we have

$$s(\mathfrak{f}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

**Proof.** Since $\mathfrak{f}$ and $\mathfrak{g} \oplus \mathfrak{h}$ are isomorphic, $s(\mathfrak{f})$ and $s(\mathfrak{g}) \oplus s(\mathfrak{h})$ are isomorphic. Especially, we have

$$\dim s(\mathfrak{f}) = \dim s(\mathfrak{g}) + \dim s(\mathfrak{h}).$$

On the other hand, we know

$$s(\mathfrak{f}) = [\mathfrak{f}, \mathfrak{f}], \quad s(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}], \quad s(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{h}].$$

Thus, we have

$$s(\mathfrak{f}) \supset s(\mathfrak{g}) \quad \text{and} \quad s(\mathfrak{f}) \supset s(\mathfrak{h}).$$

The assumption $\mathfrak{f} = \mathfrak{g} + \mathfrak{h}$ (direct sum of vector spaces) shows that $s(\mathfrak{g}) + s(\mathfrak{h})$ is a direct sum of vector spaces in $s(\mathfrak{f})$. The first equality on dimension proves our corollary. $q.e.d.$

3. Theorem 1 will follow easily from the following:

**Proposition 1.** Let $\mathfrak{t}$ be a compact Lie algebra and let $\mathfrak{g}$ and $\mathfrak{h}$ be its subalgebras such that

$$\mathfrak{t} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

Then $\mathfrak{t}$ has a direct sum decomposition of Lie algebras
\( \mathfrak{t} = \mathfrak{l} \oplus \mathfrak{l}' \)

with the following properties:

(i) The projection \( \pi \) of \( \mathfrak{t} \) onto \( \mathfrak{l} \) with respect to the above decomposition induces an isomorphism of \( \mathfrak{g} \) onto \( \mathfrak{l} \).

(ii) \( \mathfrak{t} = \mathfrak{t} + \mathfrak{h} \) (direct sum of vector spaces).

Proof. We prove the proposition by induction on \( \dim \mathfrak{t} \). When \( \dim \mathfrak{t} = 1 \), the proposition holds since \( \mathfrak{t} = \mathfrak{g} \) or \( \mathfrak{t} = \mathfrak{h} \). Now assume that the proposition holds when \( \dim \mathfrak{t} < N \). Let \( \dim \mathfrak{t} = N \). To simplify the argument we prepare the following:

Sublemma. Suppose \( \mathfrak{t} \) has a non-trivial proper ideal \( \mathfrak{t}_1 \) such that

\[ \mathfrak{t}_1 = (\mathfrak{g} \cap \mathfrak{t}_1) + (\mathfrak{h} \cap \mathfrak{t}_1) \] (direct sum of vector spaces).

Then the assertion of Proposition 1 holds for \( \mathfrak{t}, \mathfrak{g} \) and \( \mathfrak{h} \).

Proof. For \( \mathfrak{t}_1 \), we choose a complementary ideal \( \mathfrak{t}_2 \) so that we have a direct sum decomposition

\[ \mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2 . \]

Let \( \pi_2 \) be the projection of \( \mathfrak{t} \) onto \( \mathfrak{t}_2 \). We have

\[
\begin{align*}
\dim \mathfrak{g} & = \dim (\mathfrak{g} \cap \mathfrak{t}_1) + \dim \pi_2(\mathfrak{g}) , \\
\dim \mathfrak{h} & = \dim (\mathfrak{h} \cap \mathfrak{t}_1) + \dim \pi_2(\mathfrak{g}) .
\end{align*}
\]

Thus,

\[
\dim \mathfrak{t}_2 = \dim \mathfrak{t} - \dim \mathfrak{t}_1 = \dim \pi_2(\mathfrak{g}) + \dim \pi_2(\mathfrak{h}) .
\]

Since \( \mathfrak{t} = \mathfrak{g} + \mathfrak{h} \), \( \mathfrak{t}_2 = \pi_2(\mathfrak{t}) \) is spanned by \( \pi_2(\mathfrak{g}) \) and \( \pi_2(\mathfrak{h}) \), and hence we have

\[ \mathfrak{t}_2 = \pi_2(\mathfrak{g}) + \pi_2(\mathfrak{h}) \] (direct sum of vector spaces).

Consider \( \mathfrak{t}_1 \) and its subalgebras \( \mathfrak{g} \cap \mathfrak{t}_1 \) and \( \mathfrak{h} \cap \mathfrak{t}_1 \) and also \( \mathfrak{t}_2 \) and its subalgebras \( \pi_2(\mathfrak{g}) \) and \( \pi_2(\mathfrak{h}) \). By the inductive hypothesis, we have direct sum decompositions

\[ \mathfrak{t}_1 = I_1 \oplus I_1' \quad \text{and} \quad \mathfrak{t}_2 = I_2 \oplus I_2' \]

with the properties:

i. The projections \( \mathfrak{g} \cap \mathfrak{t}_1 \rightarrow I_1 \) and \( \pi_2(\mathfrak{g}) \rightarrow I_2 \) are isomorphisms.

ii. \( \mathfrak{t}_1 = \mathfrak{t}_1 + \mathfrak{h} \cap \mathfrak{t}_1, \mathfrak{t}_2 = \mathfrak{t}_2 + \pi_2(\mathfrak{h}) \) (direct sums of vector spaces).

Let

\[ I = I_1 \oplus I_2 \quad \text{and} \quad I' = I_1' \oplus I_2' . \]
We claim that the direct sum decomposition

$$\mathfrak{I} = \mathfrak{I} \oplus \mathfrak{I}'$$

satisfies the required properties.

First suppose \( X \in \mathfrak{g} \cap \mathfrak{I} \). Then \( \pi_\mathfrak{I}(X) \in \pi_\mathfrak{I}(\mathfrak{g}) \cap \mathfrak{I}' \). However \( \pi_\mathfrak{I}(\mathfrak{g}) \cap \mathfrak{I}' = \{0\} \) from the assumption. Thus \( \pi_\mathfrak{I}(X) = 0 \), and hence \( X \in \mathfrak{I} \). Then \( X \in (\mathfrak{g} \cap \mathfrak{I}) \cap \mathfrak{I}' = \{0\} \). Consequently we have \( \mathfrak{g} \cap \mathfrak{I}' = \{0\} \). This shows that the projection of \( \mathfrak{g} \) into \( \mathfrak{I} \) with respect to \( \mathfrak{I} \oplus \mathfrak{I}' \) is injective. Since they have the same dimension, we have the property (i). Next suppose \( \mathfrak{I} \cap \mathfrak{h} \in X \). \( \pi_\mathfrak{I}(X) \in \pi_\mathfrak{I}(\mathfrak{h}) \cap \mathfrak{I}' = \{0\} \), and hence \( X \in \mathfrak{I} \). We see that \( X \in \mathfrak{I} \cap (\mathfrak{h} \cap \mathfrak{I}) = \{0\} \). Thus, we have \( \mathfrak{I} \cap \mathfrak{h} = \{0\} \). Since \( \dim \mathfrak{I} = \dim \mathfrak{I} + \dim \mathfrak{h} \), we see \( \mathfrak{I} = \mathfrak{I} + \mathfrak{h} \) (direct sum of vector spaces). Thus we have the property (ii) also. q.e.d.

We continue our proof of Proposition 1. First consider easy cases.

(1) \textit{Suppose} \( \mathfrak{I} \) is abelian.

Then \( \mathfrak{I} = \mathfrak{g} \) and \( \mathfrak{I}' = \mathfrak{h} \) satisfy the required properties.

(2) \textit{Suppose} \( \mathfrak{I} \) is simple.

Then, by Lemma 2 we see \( \mathfrak{I} = \mathfrak{g} \) or \( \mathfrak{I} = \mathfrak{h} \). Thus our assertion holds trivially.

(3) \textit{Suppose} \( \mathfrak{g} \) contains a non-trivial proper ideal, say \( \mathfrak{i}_1 \), of \( \mathfrak{I} \).

Then choose a complementary ideal \( \mathfrak{i}_2 \) of \( \mathfrak{i}_1 \) in \( \mathfrak{I} \), so that we have

$$\mathfrak{I} = \mathfrak{i}_1 \oplus \mathfrak{i}_2.$$ 

Clearly, \( \mathfrak{i}_1 \cap \mathfrak{g} = \mathfrak{i}_1, \mathfrak{i}_1 \cap \mathfrak{h} \subset \mathfrak{g} \cap \mathfrak{h} = \{0\} \). Applying the above sublemma we see that Proposition 1 holds in this case.

(4) \textit{Suppose} \( \mathfrak{h} \) contains a non-trivial proper ideal, say \( \mathfrak{i}_1 \), of \( \mathfrak{I} \).

Then again we have \( \mathfrak{i}_1 \cap \mathfrak{g} = \{0\} \), and \( \mathfrak{i}_1 \cap \mathfrak{h} = \mathfrak{i}_1 \). Thus we can apply the sublemma in this case also.

(5) \textit{Suppose} \( \mathfrak{I} \) is not semi-simple.

We may suppose \( \mathfrak{I} \) is not abelian. Then the semi-simple part \( s(\mathfrak{I}) \) is a non-trivial proper ideal of \( \mathfrak{I} \). By Corollary 1, we have

$$s(\mathfrak{I}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad \text{(direct sum of vector spaces)}.$$ 

Since \( s(\mathfrak{g}) \subset \mathfrak{g} \cap s(\mathfrak{I}), s(\mathfrak{h}) \subset \mathfrak{h} \cap s(\mathfrak{I}) \) and \( \mathfrak{g} \cap s(\mathfrak{I}) = \{0\} \), we have \( s(\mathfrak{g}) = \mathfrak{g} \cap s(\mathfrak{I}) \) and \( s(\mathfrak{h}) = \mathfrak{h} \cap s(\mathfrak{I}) \). Thus

$$s(\mathfrak{I}) = \mathfrak{g} \cap s(\mathfrak{I}) + \mathfrak{h} \cap s(\mathfrak{I})$$

is a direct sum of vector spaces, and hence we can apply our sublemma.
The above argument shows that we may suppose \( f \) is semi-simple and not simple.

(6) Suppose \( f \) is semi-simple and all simple factors of \( f \) are mutually isomorphic with each other.

In this case we shall show that either \( g \) or \( h \) contains a proper ideal of \( f \), so that the proposition holds by (3) or (4). Suppose neither \( g \) nor \( h \) contains a non trivial proper ideal of \( f \). Let

\[
\begin{align*}
\mathfrak{f} &= \sum_{i \in I} \mathfrak{i}_i, \quad g = \sum_{j \in \mathcal{S}} g_j \quad \text{and} \quad h = \sum_{k \in K} h_k
\end{align*}
\]

be the decompositions of \( \mathfrak{f} \), \( g \) and \( h \) into simple factors. By the present assumption, all \( \mathfrak{i}_i \)'s are mutually isomorphic. By Lemma 2, we see also that all \( g_j \)'s, \( h_k \)'s and \( \mathfrak{i}_i \)'s are mutually isomorphic, and that

\[
|I| = |J| + |K|
\]

where \(|\cdot|\) indicates the number of elements.

Denote by \( \pi_i \), the projection of \( f \) onto \( \mathfrak{i}_i \). One sees that

\[
\begin{align*}
\pi_i(g_j) &= \mathfrak{i}_i \quad \text{or} \quad \{0\}, \\
\pi_i(h_k) &= \mathfrak{i}_i \quad \text{or} \quad \{0\}
\end{align*}
\]

for all \( i, j, k \). Put

\[
\begin{align*}
A_j &= \{i \in I | \pi_i(g_j) \neq \{0\}\}, \\
B_k &= \{i \in I | \pi_i(h_k) \neq \{0\}\}
\end{align*}
\]

for each \( j \in J \) and \( k \in K \). Let \( j_1, j_2 \in J \), and \( j_1 \neq j_2 \). Then \([\pi_i(g_{j_1}), \pi_i(g_{j_2})] = 0\). Thus \( A_{j_1} \cap A_{j_2} = \emptyset \). Hence \( A_j \)'s are mutually disjoint and so are \( B_k \)'s.

Suppose \( A_j \) consists of exactly one element, say \( i \). Then we see \( g_i = \mathfrak{i}_i \) and hence \( g \) contains a non trivial proper ideal. This is a contradiction. Thus each \( A_j \) contains at least two elements. Similarly we have \(|B_k| \geq 2\). Thus we have

\[
\sum |A_j| + \sum |B_k| \geq 2(|J| + |K|) = 2|I|
\]

On the other hand,

\[
\sum |A_j| \leq |I| \quad \text{and} \quad \sum |B_k| \leq |I|.
\]

Combining together, we see

\[
|A_j| = |B_k| = 2
\]

for every \( j \in J \) and \( k \in K \).

By an elementary combinatorial argument one can decompose the index set \( I \) into two disjoint subsets \( I_1 \) and \( I_2 \) such that, for every \( j, k \), the sets \( A_j \cap I_i \),
$A_1 \cap I_2, B_1 \cap I_1$ and $B_1 \cap I_2$ are all non empty. Let

$$\alpha_1 = \sum_{i \in I_1} \mathfrak{g} \text{ and } \alpha_2 = \sum_{j \in I_2} \mathfrak{g},$$

so that we have $\mathfrak{g} = \alpha_1 \oplus \alpha_2$. Denote by $p_i$ the projection of $\mathfrak{g}$ onto $\alpha_i$ (for $i=1, 2$). It follows from our construction that the homomorphisms $p_1|\mathfrak{g}, p_2|\mathfrak{g}, p_1|\mathfrak{h}$ and $p_2|\mathfrak{h}$ are all onto isomorphisms. Using the decomposition $\mathfrak{g} = \alpha_1 \oplus \alpha_2$, we can write

$$\mathfrak{g} = \{(X, \phi(X)) | X \in \alpha_1\}$$

and

$$\mathfrak{h} = \{(\psi(Y), Y) | Y \in \alpha_2\}$$

by suitable onto isomorphisms $\phi: \alpha_1 \rightarrow \alpha_2$ and $\psi: \alpha_2 \rightarrow \alpha_1$. Consider an automorphism $\psi \circ \phi$ of $\alpha_1$. By a result due to Borel and Mostow [1], every automorphism of a semi-simple Lie algebra has a non-zero fixed vector. Thus, we have an element $X$ in $\alpha_1$ such that $X \neq 0$ and $\psi(\phi(X)) = X$. Then we have

$$(X, \phi(X)) = (\psi(\phi(X)), \phi(X)) \in \mathfrak{g} \cap \mathfrak{h} = \{0\} .$$

This is a contradiction. Thus, in this case, either $\mathfrak{g}$ or $\mathfrak{h}$ contains a proper ideal of $\mathfrak{g}$.

(7) Suppose $\mathfrak{g}$ is semi-simple and $\mathfrak{g}$ contains at least two simple ideals which are not isomorphic.

Choose a simple ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that $\dim \mathfrak{a}$ is minimal among the simple ideals of $\mathfrak{g}$. Let $\mathfrak{g}_0$ be the direct sum of all simple ideals isomorphic to $\mathfrak{a}$, and $\mathfrak{g}_1$ the complementary ideal, so that we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 .$$

Similarly, decompose $\mathfrak{g}$ and $\mathfrak{h}$ as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \text{ and } \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 ,$$

where $\mathfrak{g}_0$ (resp. $\mathfrak{h}_0$) is the direct sum of all simple ideals in $\mathfrak{g}$ (resp. $\mathfrak{h}$) isomorphic to $\mathfrak{a}$.

In virtue of Lemma 2, we see that $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are isomorphic with $\mathfrak{g}_0 \oplus \mathfrak{h}_0$ and $\mathfrak{g}_1 \oplus \mathfrak{h}_1$ respectively. We claim that the ideal $\mathfrak{g}_0$ satisfies the required condition in the sublemma. Let $\pi_0$ and $\pi_1$ be the projections of $\mathfrak{g}$ onto $\mathfrak{g}_0$ and $\mathfrak{g}_1$ respectively. Consider $\pi_0|\mathfrak{g}_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$. From the definitions of $\mathfrak{g}_0$ and $\mathfrak{g}_1$, we see $\pi_0|\mathfrak{g}_1 = \{0\}$. Thus, $\mathfrak{g}_0 \subset \mathfrak{g}_1$. Similarly we have $\mathfrak{h}_1 \subset \mathfrak{g}_1$. Thus, $\mathfrak{g}_1 \supset \mathfrak{g}_1 + \mathfrak{h}_1$. Since $\mathfrak{g}_1 \cap \mathfrak{h}_1 = \{0\}$ and $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_1 + \dim \mathfrak{h}_1$, we conclude that

$$\mathfrak{g}_1 = \mathfrak{g}_1 + \mathfrak{h}_1 \text{ (direct sum of vector spaces).}$$
Since $\mathfrak{f} = \mathfrak{g} + \mathfrak{h}$ is a direct sum of vector spaces, we see that $(\mathfrak{f}_1 \cap \mathfrak{g}) \cap (\mathfrak{f}_1 \cup \mathfrak{h}) = \{0\}$. On the other hand, $\mathfrak{f}_1 \cap \mathfrak{g} \supset \mathfrak{g}_i$ and $\mathfrak{f}_1 \cap \mathfrak{h} \supset \mathfrak{h}_i$, and also $\mathfrak{f}_i = \mathfrak{g}_i + \mathfrak{h}_i$ (direct sum of vector spaces). It follows that $\mathfrak{g}_i = \mathfrak{g} \cap \mathfrak{f}_i$ and $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{f}_i$ and hence

$$\mathfrak{f}_i = (\mathfrak{g}_i \cap \mathfrak{f}_i) + (\mathfrak{h}_i \cap \mathfrak{f}_i) \quad (\text{direct sum of vector spaces}).$$

This proves our claim.

Thus we have completed the proof of Proposition 1.

4. Now we can prove Theorem 1

Proof of Theorem 1. First assume that $\mathfrak{f}$ is semi-simple. Apply Proposition 1 to $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$. We get a direct sum decomposition

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{h}$$

with the properties:

(i) The projection of $\mathfrak{f}$ onto $\mathfrak{g}$ with respect to the above decomposition induces an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}$. 

(ii) $\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{h}$ (direct sum of vector spaces).

Again apply Proposition 1 to $\mathfrak{f}$, $\mathfrak{h}$ and $\mathfrak{I}$. We have a direct sum decomposition

$$\mathfrak{f} = \mathfrak{m} \oplus \mathfrak{m}'$$

with the properties:

(i') The projection of $\mathfrak{f}$ onto $\mathfrak{m}$ with respect to this decomposition induces an isomorphisms of $\mathfrak{m}$ onto $\mathfrak{m}$. 

(ii') $\mathfrak{f} = \mathfrak{m} \oplus \mathfrak{I}$ (direct sum of vector spaces).

Since $\mathfrak{m}$ and $\mathfrak{I}$ are both ideals of $\mathfrak{f}$, we have a direct sum

$$\mathfrak{f} = \mathfrak{m} \oplus \mathfrak{I}$$

of Lie algebras. The assumption that $\mathfrak{f}$ is semi-simple implies $\mathfrak{m} = \mathfrak{I}'$. Thus, with respect to the direct sum

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{h}$$

we see that the projections of $\mathfrak{f}$ onto $\mathfrak{g}$ and $\mathfrak{h}$ induce isomorphisms of $\mathfrak{g}$ and $\mathfrak{h}$ onto $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Setting $\mathfrak{g}_i = \mathfrak{g} \cap \mathfrak{f}_i$ and $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{f}_i$, we see that the decomposition

$$\mathfrak{f} = \mathfrak{g}_i \oplus \mathfrak{h}_i$$

satisfies the first two properties. The third property follows from $\mathfrak{g} \cap \mathfrak{h} = \{0\}$.

In fact, suppose $\psi(\phi(X)) = X$ for $X \in \mathfrak{g}_i$. Then, $(\mathbf{X}, \phi(X)) = (\psi(\phi(X)),$
Consider the general case. By Corollary 1, we have

\[ s(\mathfrak{f}) = s(\mathfrak{g}) + s(\mathfrak{h}) \] (direct sum of vector spaces).

Also by Lemma 1, \( \dim c(\mathfrak{f}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h}) \). It is easily seen that the projection \( \pi \) of \( \mathfrak{f} \) onto \( c(\mathfrak{f}) \) induces

\[ c(\mathfrak{f}) = \pi(c(\mathfrak{g})) + \pi(c(\mathfrak{h})) \] (direct sum of vector spaces).

From the first argument, we can choose a direct sum decomposition

\[ s(\mathfrak{f}) = \mathfrak{g}_1' \oplus \mathfrak{h}_1' \]

such that the projections of \( s(\mathfrak{f}) \) onto \( \mathfrak{g}_1' \) and \( \mathfrak{h}_1' \) induce isomorphisms of \( s(\mathfrak{g}) \) and \( s(\mathfrak{h}) \) onto \( \mathfrak{g}_1' \) and \( \mathfrak{h}_1' \) respectively. Now put

\[ \mathfrak{g}_1 = \mathfrak{g}_1' \oplus \pi(c(\mathfrak{g})) \]

and

\[ \mathfrak{h}_1 = \mathfrak{h}_1' \oplus \pi(c(\mathfrak{h})) \] .

we have a direct sum decomposition

\[ \mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1 . \]

We claim that this decomposition satisfies the required properties in Theorem 1. The first two are easy. The last one follows from the first two and \( \mathfrak{g} \cap \mathfrak{h} = \{0\} \).

**Q.E.D.**

**REMARK 1.** The converse of Theorem 1 holds. Let \( \mathfrak{g}_1 \) and \( \mathfrak{h}_1 \) be Lie algebras, and let \( \phi: \mathfrak{g}_1 \to \mathfrak{h}_1 \) and \( \psi: \mathfrak{g}_1 \to \mathfrak{g}_1 \) be Lie algebra homomorphisms such that \( \psi \circ \phi \) has no non-zero fixed vector. In the direct sum \( \mathfrak{g}_1 \oplus \mathfrak{h}_1 = \mathfrak{f} \) of Lie algebras, define \( \mathfrak{g} \) and \( \mathfrak{h} \) by (i) and (ii). Then \( \mathfrak{g} \) and \( \mathfrak{h} \) are subalgebras and we have

\[ \mathfrak{f} = \mathfrak{g} + \mathfrak{h} \] (direct sum of vector spaces).

**REMARK 2.** Suppose \( M = K/H \) is a homogeneous space of the type mentioned in the introduction. Then the action of \( K \) on \( K/H \) is almost effective if and only if \( \psi \) is injective.

**REMARK 3.** Let \( M = K/H \) be as above. By the theorem of Borel-Mostow cited before, the Lie algebra homomorphism \( \psi \circ \phi = 0 \) if \( \mathfrak{g} \) is simple. Thus we see that if \( G \) is simple and the \( K \)-action on \( K/H \) is almost effective then \( G \) is normal. Thus, Ochiai-Takahashi's theorem follows from Theorem 1.

5. Now we consider a homogeneous space of the type mentioned in the introduction. Let \( M = K/H \) be a homogeneous space of a connected compact
Lie group $K$. We assume that a connected Lie subgroup $G$ acts simply transitively on $K/H$. Since $K/H$ is compact, $G$ is necessarily compact. The composition mapping

$$F: G \times H \to K$$

is a diffeomorphism, so that we have

$$\mathfrak{f} = \mathfrak{g} + \mathfrak{h} \quad \text{(direct sum of vector spaces)},$$

for their Lie algebras. Applying Theorem 1, we have a direct sum decomposition

$$\mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

and homomorphisms $\phi: \mathfrak{g}_1 \to \mathfrak{h}_1$ and $\psi: \mathfrak{h}_1 \to \mathfrak{g}_1$ such that we have

$$\mathfrak{g} = \{(X, \phi(X)) | X \in \mathfrak{g}_1\},$$

$$\mathfrak{h} = \{((Y, Y)) | Y \in \mathfrak{h}_1\}.$$  

Further, as we see from the proof of Theorem 1, we can assume that

$$c(\mathfrak{g}_1) = \pi(c(\mathfrak{g})), $$

where $\pi$ denotes the projection of $\mathfrak{f}$ onto its center.

Let $G_1$ be the connected Lie subgroup of $K$ corresponding to the subalgebra $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is an ideal of $\mathfrak{f}$, $G_1$ is a normal subgroup of $K$. Next we claim that $G_1$ is compact. Since $s(G_1)$ is closed in $K$ since it is semi-simple. Thus it suffices to show that $c(G_1)$ is compact. However, from our construction, $c(\mathfrak{g}_1) = \pi(c(\mathfrak{g})).$ Consider the Lie group homomorphism $\pi: K \to K/s(K)$. $\pi|c(K)$ is a finite covering map. Thus $c(G_1)$ is closed in $c(K)$ if and only if $\pi(c(G_1))$ is closed. On the other hand, $c(G)$ is compact, and hence $\pi(c(G))$ is compact. $c(\mathfrak{g}_1) = \pi(c(\mathfrak{g}))$ implies that $\pi(c(G_1)) = \pi(c(G))$. Thus, $c(G_1)$ is closed, and hence $G_1$ is compact. From the property that $\mathfrak{f} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$ and $\mathfrak{h}_1 = \{((\psi(Y), Y)) | Y \in \mathfrak{h}_1\}$, we have

$$\mathfrak{f} = \mathfrak{g}_1 + \mathfrak{h} \quad \text{(direct sum of vector spaces)}.$$  

By Lemma 1, the composition mapping

$$G_1 \times H \to K$$

defines a covering map. Consequently, $G_1$ acts transitively on the coset space $K/H$. Furthermore, fix a point $p$ in $K/H$. Then the mapping

$$G_1 \to K/H$$

defined by $g \to g(p)$ is a covering map. Thus, if $G(\simeq K/H)$ is simply connected, then $G_1$ is also simply connected. Thus we have proved the following:
**Theorem 2.** Let $K$ be a connected compact Lie group and $H$ a closed subgroup of $K$. Assume that a connected Lie subgroup $G$ acts simply transitively on the homogeneous space $K/H$ by the left translation. Then there exists a connected closed normal subgroup $G_1$ of $K$ such that $G_1$ acts transitively on $K/H$ and $G_1$ is locally isomorphic with $G$ as Lie groups.

**Theorem 3.** Under the same assumption as in Theorem 2, assume further that $G$ is simply connected. Then there exists a connected closed normal subgroup $G_1$ of $K$ such that $G_1$ is isomorphic with $G$ as Lie groups and $G_1$ acts simply transitively on $K/H$.

6. We give here two examples. The first one shows that the conclusion of Ochiai-Takahashi's theorem does not hold any more if $G$ is not simple.

**Example 1.** Let $A$ be a connected compact semi-simple Lie group and $\alpha$ its Lie algebra. We put

$$
K = A \times A \times A,
$$

$$
G = \{(x, y, z) | x, y \in A\},
$$

$$
H = \{(e, z, z) | z \in A\}.
$$

$H$ is a closed subgroup of $K$. Consider the homogeneous space $K/H$. We see easily that $G$ acts simply transitively on $K/H$. $G$ is compact semi-simple and not simple. Choose a $K$-invariant Riemannian metric $ds^2$ on $K/H$. Since $K/H$ can be identified with $G$, $ds^2$ is a left-invariant Riemannian metric on $G$. From the definition, $K$ is contained in the identity-component of isometries of $(K/H=G, ds^2)$. $G$ is not normal in $K$, thus $G$ is not normal in the identity-component of isometries.

For this example, an explicit description of Theorem 1 is as follows:

$$
\mathfrak{f} = \mathfrak{a} \oplus \mathfrak{a} \oplus \mathfrak{a}.
$$

Let

$$
g_1 = \{(X, Y, 0) | X, Y \in \mathfrak{a}\},
$$

$$
\mathfrak{h}_1 = \{(0, 0, Z) | Z \in \mathfrak{a}\}.
$$

Define $\phi : g_1 \to \mathfrak{h}_1$ by

$$
\phi((X, Y, 0)) = (0, 0, X)
$$

and $\psi : \mathfrak{h}_1 \to g_1$ by

$$
\psi((0, 0, Z)) = (0, Z, 0).
$$

Then we have

$$
g = \{(X, \phi(X)) \in g_1 \oplus \mathfrak{h}_1 | X \in g_1\},
$$

$$
g = \{\psi(Y) | Y \in g_1 \oplus \mathfrak{h}_1, Y \in \mathfrak{h}_1\}.
The next example shows that the conclusion of Theorem 3 does not hold if $G$ is not simply connected.

**Example 2.** We choose two simply connected compact Lie groups $A$ and $B$ with the following properties:
1. There exists an injective homomorphism $j$ of $A$ into $B$.
2. The center $Z(A)$ of $A$ is non-trivial and $j(Z(A)) \cap Z(B) = \{e\}$.

For instance, choose positive integers $m$ and $n$ such that $n > m > 2$. Then $A = SU(m)$, $B = SU(n)$ and the canonical injection of $SU(m)$ into $SU(n)$ satisfy the required properties.

Let $K = A \times B \times A$, $G_1 = A \times B \times \{e\}$, $G = \{(a, b, a) | a \in A, b \in B\}$, $H = \{(e, j(a), a) | a \in A\}$, $\Gamma = \{(x, e, x) | x \in Z(A)\}$.

The Lie algebras of $A$ and $B$ are denoted by $\mathfrak{a}$ and $\mathfrak{b}$ respectively. $\Gamma$ is a finite group contained in the center of $K$. We consider the quotient group $K = K/\Gamma$, and denote by $\pi$ the canonical projection of $K$ onto $\bar{K}$. $\bar{H} = \pi(H)$ is a closed subgroup of $\bar{K}$. Consider $K/\bar{H}$. We claim that no normal subgroup of $K$ acts simply transitively on $K/\bar{H}$. Suppose a normal subgroup $G_1'$ of $\bar{K}$ acts simply transitively on $K/\bar{H}$. Then its Lie algebra $\mathfrak{g}_1'$ satisfies

$$ \mathfrak{k} = \mathfrak{g}_1' + \mathfrak{g} \quad \text{(direct sum of vector spaces)}, $$

where $\mathfrak{h} = \{(0, j(X), X) | X \in \mathfrak{a}\}$. Since $\mathfrak{g}_1'$ is an ideal of $\mathfrak{k}$, we see $\mathfrak{g}_1' = \mathfrak{g}_1 = \{(X, Y, 0) | X \in \mathfrak{a}, Y \in \mathfrak{b}\}$. It follows that $\pi(G_1) = G_1'$. However, $\pi(G_1)$ is simply connected because $\pi(G_1) = G_1/(G_1 \cap \Gamma) \cong G_1$. This is a contradiction.

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References


