



Title	On a transitive transformation group of a compact group manifold
Author(s)	Ozeki, Hideki
Citation	Osaka Journal of Mathematics. 1977, 14(3), p. 519-531
Version Type	VoR
URL	<a href="https://doi.org/10.18910/7892">https://doi.org/10.18910/7892</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

Ozeki, H.  
Osaka J. Math.  
14 (1977), 519–531

## ON A TRANSITIVE TRANSFORMATION GROUP OF A COMPACT GROUP MANIFOLD

HIDEKI OZEKI

(Received June 10, 1976)

**1. Introduction.** Let  $K$  be a connected compact Lie group and  $H$  a closed subgroup of  $K$ . Suppose a connected Lie subgroup  $G$  of  $K$  acts simply transitively on the coset space  $K/H$  by the left translation. Then the composition mapping

$$F: G \times H \rightarrow K$$

defined by  $F(g, h) = gh$  ( $g \in G, h \in H$ ) gives rise to a diffeomorphism of the product manifold  $G \times H$  onto  $K$ . Consequently, for their Lie algebras, we have

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

We shall prove in this paper the following:

**Theorem 1.** *Let  $\mathfrak{k}$  be a compact Lie algebra. Suppose there exist two subalgebras  $\mathfrak{g}$  and  $\mathfrak{h}$  of  $\mathfrak{k}$  such that*

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

*Then there exist a direct sum decomposition*

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

*of Lie algebras and Lie algebra homomorphisms*

$$\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1 \quad \text{and} \quad \psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$$

*with the following properties:*

- (i)  $\mathfrak{g} = \{(X, \varphi(X)) \mid X \in \mathfrak{g}_1\}$ .
- (ii)  $\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}$ .
- (iii)  $\psi \circ \varphi$  has no non-zero fixed vector.

As a result we see that the Lie algebra  $\mathfrak{k}$  is isomorphic with the direct sum  $\mathfrak{g} \oplus \mathfrak{h}$  of Lie algebras. This theorem gives us an infinitesimal characterization of a homogeneous space of the type mentioned in the above. Some

application and remarks will be added after its proof.

Such a homogeneous space is related with a study of isometries of a compact group manifold. Let  $G$  be a connected compact Lie group and choose a left invariant Riemannian metric  $ds^2$  on  $G$ . Denote by  $K$  the identity component of the isometry group of  $(G, ds^2)$ . We identify an element  $g$  of  $G$  with its left translation  $L_g$  on  $G$ . Ochiai-Takahashi [2] proved that if  $G$  is simple then  $G$  is normal in  $K$ . Their theorem follows immediately from our Theorem 1. The conclusion of their theorem does not hold in general if  $G$  is not simple, as our example shows. However, our Theorem 3 asserts that if  $G$  is simply connected then we have a similar conclusion by a suitable change of the action of  $G$  on the space.

**2.** Recall that a Lie algebra  $\mathfrak{k}$  is said to be *compact* if it can be represented as a Lie algebra of a compact Lie group. For a compact Lie algebra  $\mathfrak{k}$ , we denote by  $c(\mathfrak{k})$  its center and by  $s(\mathfrak{k})$  its maximal semi-simple ideal, so that we have  $s(\mathfrak{k}) = [\mathfrak{k}, \mathfrak{k}]$  and

$$\mathfrak{k} = s(\mathfrak{k}) \oplus c(\mathfrak{k})$$

(direct sum of Lie algebras). The same notation will be used for a connected Lie group  $K$  when the Lie algebra  $\mathfrak{k}$  of  $K$  is compact.  $c(K)$  and  $s(k)$  are the connected Lie subgroups of  $K$  corresponding to Lie subalgebras  $c(\mathfrak{k})$  and  $s(\mathfrak{k})$  respectively.

Note that a connected Lie group  $K$  has a compact Lie algebra if and only if  $K$  has a bi-invariant Riemannian metric and also that any subalgebra of a compact Lie algebra is compact. In the sequel, for a Lie group homomorphism, the induced Lie algebra homomorphism is denoted by the same symbol.

**Lemma 1.** *Let  $K, G$  and  $H$  be connected Lie groups with Lie algebras  $\mathfrak{k}, \mathfrak{g}$  and  $\mathfrak{h}$  respectively. Suppose  $\mathfrak{k}$  is compact. Let  $\phi: G \rightarrow K$  and  $\psi: H \rightarrow K$  be Lie group homomorphisms such that the induced homomorphisms  $\phi: \mathfrak{g} \rightarrow \mathfrak{k}$  and  $\psi: \mathfrak{h} \rightarrow \mathfrak{k}$  are both injective and*

$$\mathfrak{k} = \phi(\mathfrak{g}) + \psi(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

*Then the composition mapping*

$$F: G \times H \rightarrow K$$

*defined by  $F(g, h) = \phi(g) \cdot \psi(h)$  is a covering map.*

**Proof.** In general, we denote the left translation and the right translation of a group induced by an element  $x$  in it by  $L_x$  and  $R_x$  respectively. Then, for the mapping  $F$ , we have the following commutative diagram:

$$\begin{array}{ccc}
 & F & \\
 G \times H & \xrightarrow{\quad} & K \\
 \downarrow (L_g, R_h) & & \downarrow L_{\phi(g)} \circ R_{\varphi(h)} \\
 G \times H & \xrightarrow{\quad F \quad} & K
 \end{array}$$

for  $(g, h) \in G \times H$ . This gives an identity

$$F = (L_{\phi(g)} \circ R_{\varphi(h)}) \circ F \circ (L_{g^{-1}}, R_{h^{-1}}).$$

Taking the differentials, we have

$$(dF)_{(g, h)} = (d(L_{\phi(g)} \circ R_{\varphi(h)}))_e \circ (dF)_{(e, e)} \circ (d(L_{g^{-1}}, R_{h^{-1}}))_{(g, h)}.$$

We identify  $T_{(e, e)}(G \times H)$  with  $T_e(G) + T_e(H) = \mathfrak{g} + \mathfrak{h}$  (direct sum of vector spaces). Since  $(dF)_{(e, e)}|T_e(G) = \phi$  and  $(dF)_{(e, e)}|T_e(H) = \psi$ , our assumption in the lemma implies that  $(dF)_{(e, e)}$  gives an isomorphism of  $T_{(e, e)}(G \times H)$  onto  $T_e(K) = \mathfrak{k}$ . By the above identity, we see that  $(dF)_{(g, h)}$  is isomorphic at each point  $(g, h)$  of  $G \times H$ . Since  $\mathfrak{k}$  is compact, we can choose a bi-invariant Riemannian metric  $ds^2$  on  $K$ . Then  $ds^2 = F^*(ds^2)$  gives a Riemannian metric on the manifold  $G \times H$ , which is locally isometric with  $(K, ds^2)$  via  $F$ . In virtue of the first commutative diagram, the Riemannian metric  $ds^2$  on  $G \times H$  is  $L(G)$  and  $R(H)$ -invariant, and hence it is complete. Thus we see that  $F$  is a locally isometric mapping of a complete Riemannian manifold  $(G \times H, ds^2)$  into  $(K, ds^2)$ . This proves that  $F$  is a covering map. q.e.d.

**Lemma 2.** *Let  $\mathfrak{k}$  be a compact Lie algebra, and let  $\mathfrak{g}$  and  $\mathfrak{h}$  be two subalgebras of  $\mathfrak{k}$  such that*

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

*Then,  $\mathfrak{k}$  is isomorphic with the direct sum  $\mathfrak{g} \oplus \mathfrak{h}$  of Lie algebras. consequently, we have*

$$\dim c(\mathfrak{k}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h}).$$

**Proof.** For  $\mathfrak{g}$ ,  $\mathfrak{h}$  and  $\mathfrak{k}$ , choose simply connected Lie groups  $G$ ,  $H$  and  $K$  with the corresponding Lie algebras respectively. Let

$$\phi: \mathfrak{g} \rightarrow \mathfrak{k} \quad \text{and} \quad \psi: \mathfrak{h} \rightarrow \mathfrak{k}$$

be the inclusion mappings. They induce Lie group homomorphisms

$$\phi: G \rightarrow K \quad \text{and} \quad \psi: H \rightarrow K.$$

The composition mapping  $F$  of the product manifold  $G \times H$  into  $K$  defined by

$$F(g, h) = \phi(g)\psi(h)$$

is a covering map by Lemma 1. Since  $K$  is assumed to be simply connected, we

have a diffeomorphism of  $G \times H$  onto  $K$ .  $\mathfrak{k}$  is compact and hence  $\mathfrak{g}$  and  $\mathfrak{h}$  are compact. Since  $G$ ,  $H$  and  $K$  are simply connected and their Lie algebras are compact, we see  $G = s(G) \times c(G)$ ,  $H = s(H) \times c(H)$  and  $K = s(K) \times c(K)$ . Since  $F$  is a diffeomorphism of the product manifold  $G \times H$  onto  $K$  we see

$$\dim c(K) = \dim c(G) + \dim c(H)$$

and hence

$$\dim c(\mathfrak{k}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h}).$$

Note that  $s(K)$  is a maximal compact subgroup of  $K$ . Also we see that  $F$  induces a homotopy equivalence between  $s(G) \times s(H)$  and  $s(K)$ .

A theorem in homotopy theory ([3], [4]) states that if two simply connected compact Lie groups are homotopically equivalent then they are isomorphic as Lie groups. Thus, we see that the Lie group  $s(K)$  is isomorphic with the direct product  $s(G) \times s(H)$  of Lie groups. Finally we can conclude that the Lie algebra  $\mathfrak{k}$  is isomorphic with the direct sum  $\mathfrak{g} \oplus \mathfrak{h}$  of Lie algebras. q.e.d.

**Corollary 1.** *Under the same assumption as above, we have*

$$s(\mathfrak{k}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

**Proof.** Since  $\mathfrak{k}$  and  $\mathfrak{g} \oplus \mathfrak{h}$  are isomorphic,  $s(\mathfrak{k})$  and  $s(\mathfrak{g}) \oplus s(\mathfrak{h})$  are isomorphic. Especially, we have

$$\dim s(\mathfrak{k}) = \dim s(\mathfrak{g}) + \dim s(\mathfrak{h}).$$

On the other hand, we know

$$s(\mathfrak{k}) = [\mathfrak{k}, \mathfrak{k}], \quad s(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}], \quad s(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{h}].$$

Thus, we have

$$s(\mathfrak{k}) \supset s(\mathfrak{g}) \quad \text{and} \quad s(\mathfrak{k}) \supset s(\mathfrak{h}).$$

The assumption  $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$  (direct sum of vector spaces) shows that  $s(\mathfrak{g}) + s(\mathfrak{h})$  is a direct sum of vector spaces in  $s(\mathfrak{k})$ . The first equality on dimension proves our corollary. q.e.d.

**3.** Theorem 1 will follow easily from the following:

**Proposition 1.** *Let  $\mathfrak{k}$  be a compact Lie algebra and let  $\mathfrak{g}$  and  $\mathfrak{h}$  be its subalgebras such that*

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

*Then  $\mathfrak{k}$  has a direct sum decomposition of Lie algebras*

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

with the following properties:

- (i) The projection  $\pi$  of  $\mathfrak{k}$  onto  $\mathfrak{l}$  with respect to the above decomposition induces an isomorphism of  $\mathfrak{g}$  onto  $\mathfrak{l}$ .
- (ii)  $\mathfrak{k} = \mathfrak{l} + \mathfrak{h}$  (direct sum of vector spaces).

Proof. We prove the proposition by induction on  $\dim \mathfrak{k}$ . When  $\dim \mathfrak{k} = 1$ , the proposition holds since  $\mathfrak{k} = \mathfrak{g}$  or  $\mathfrak{k} = \mathfrak{h}$ . Now assume that the proposition holds when  $\dim \mathfrak{k} < N$ . Let  $\dim \mathfrak{k} = N$ . To simplify the argument we prepare the following:

**Sublemma.** Suppose  $\mathfrak{k}$  has a non-trivial proper ideal  $\mathfrak{k}_1$  such that

$$\mathfrak{k}_1 = (\mathfrak{g} \cap \mathfrak{k}_1) + (\mathfrak{h} \cap \mathfrak{k}_1) \quad (\text{direct sum of vector spaces}).$$

Then the assertion of Proposition 1 holds for  $\mathfrak{k}$ ,  $\mathfrak{g}$  and  $\mathfrak{h}$ .

Proof. For  $\mathfrak{k}_1$ , we choose a complementary ideal  $\mathfrak{k}_2$  so that we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2.$$

Let  $\pi_2$  be the projection of  $\mathfrak{k}$  onto  $\mathfrak{k}_2$ . We have

$$\begin{aligned} \dim \mathfrak{g} &= \dim \mathfrak{g} \cap \mathfrak{k}_1 + \dim \pi_2(\mathfrak{g}), \\ \dim \mathfrak{h} &= \dim \mathfrak{h} \cap \mathfrak{k}_1 + \dim \pi_2(\mathfrak{h}). \end{aligned}$$

Thus,

$$\begin{aligned} \dim \mathfrak{k}_2 &= \dim \mathfrak{k} - \dim \mathfrak{k}_1 \\ &= \dim \pi_2(\mathfrak{g}) + \dim \pi_2(\mathfrak{h}). \end{aligned}$$

Since  $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ ,  $\mathfrak{k}_2 = \pi_2(\mathfrak{k})$  is spanned by  $\pi_2(\mathfrak{g})$  and  $\pi_2(\mathfrak{h})$ , and hence we have

$$\mathfrak{k}_2 = \pi_2(\mathfrak{g}) + \pi_2(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Consider  $\mathfrak{k}_1$  and its subalgebras  $\mathfrak{g} \cap \mathfrak{k}_1$  and  $\mathfrak{h} \cap \mathfrak{k}_1$  and also  $\mathfrak{k}_2$  and its subalgebras  $\pi_2(\mathfrak{g})$  and  $\pi_2(\mathfrak{h})$ . By the inductive hypothesis, we have direct sum decompositions

$$\mathfrak{k}_1 = \mathfrak{l}_1 \oplus \mathfrak{l}'_1 \quad \text{and} \quad \mathfrak{k}_2 = \mathfrak{l}_2 \oplus \mathfrak{l}'_2$$

with the properties:

- i. The projections  $\mathfrak{g} \cap \mathfrak{k}_1 \rightarrow \mathfrak{l}_1$  and  $\pi_2(\mathfrak{g}) \rightarrow \mathfrak{l}_2$  are isomorphisms.
- ii.  $\mathfrak{k}_1 = \mathfrak{l}_1 + \mathfrak{h} \cap \mathfrak{k}_1$ ,  $\mathfrak{k}_2 = \mathfrak{l}_2 + \pi_2(\mathfrak{h})$  (direct sums of vector spaces).

Let

$$\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \quad \text{and} \quad \mathfrak{l}' = \mathfrak{l}'_1 \oplus \mathfrak{l}'_2.$$

We claim that the direct sum decomposition

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

satisfies the required properties.

First suppose  $X \in \mathfrak{g} \cap \mathfrak{l}'$ . Then  $\pi_2(X) \in \pi_2(\mathfrak{g}) \cap \mathfrak{l}'_2$ . However  $\pi_2(\mathfrak{g}) \cap \mathfrak{l}'_2 = \{0\}$  from the assumption. Thus  $\pi_2(X) = 0$ , and hence  $X \in \mathfrak{l}_1$ . Then  $X \in (\mathfrak{g} \cap \mathfrak{l}_1) \cap \mathfrak{l}'_1 = \{0\}$ . Consequently we have  $\mathfrak{g} \cap \mathfrak{l}' = \{0\}$ . This shows that the projection of  $\mathfrak{g}$  into  $\mathfrak{l}$  with respect to  $\mathfrak{l} \oplus \mathfrak{l}'$  is injective. Since they have the same dimension, we have the property (i). Next suppose  $\mathfrak{l} \cap \mathfrak{h} \in X$ .  $\pi_2(X) \in \pi_2(\mathfrak{h}) \cap \mathfrak{l}_2 = \{0\}$ , and hence  $X \in \mathfrak{l}_1$ . We see that  $X \in \mathfrak{l}_1 \cap (\mathfrak{h} \cap \mathfrak{l}_1) = \{0\}$ . Thus, we have  $\mathfrak{l} \cap \mathfrak{h} = \{0\}$ . Since  $\dim \mathfrak{k} = \dim \mathfrak{l} + \dim \mathfrak{h}$ , we see  $\mathfrak{k} = \mathfrak{l} + \mathfrak{h}$  (direct sum of vector spaces). Thus we have the property (ii) also. q.e.d.

We continue our proof of Proposition 1. First consider easy cases.

(1) *Suppose  $\mathfrak{k}$  is abelian.*

Then  $\mathfrak{l} = \mathfrak{g}$  and  $\mathfrak{l}' = \mathfrak{h}$  satisfy the required properties.

(2) *Suppose  $\mathfrak{k}$  is simple.*

Then, by Lemma 2 we see  $\mathfrak{k} = \mathfrak{g}$  or  $\mathfrak{k} = \mathfrak{h}$ . Thus our assertion holds trivially.

(3) *Suppose  $\mathfrak{g}$  contains a non-trivial proper ideal, say  $\mathfrak{k}_1$ , of  $\mathfrak{k}$ .*

Then choose a complementary ideal  $\mathfrak{k}_2$  of  $\mathfrak{k}_1$  in  $\mathfrak{k}$ , so that we have

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2.$$

Clearly,  $\mathfrak{k}_1 \cap \mathfrak{g} = \mathfrak{k}_1$ ,  $\mathfrak{k}_1 \cap \mathfrak{h} \subset \mathfrak{g} \cap \mathfrak{h} = \{0\}$ . Applying the above sublemma we see that Proposition 1 holds in this case.

(4) *Suppose  $\mathfrak{h}$  contains a non-trivial proper ideal, say  $\mathfrak{k}_1$ , of  $\mathfrak{k}$ .*

Then again we have  $\mathfrak{k}_1 \cap \mathfrak{g} = \{0\}$ , and  $\mathfrak{k}_1 \cap \mathfrak{h} = \mathfrak{k}_1$ . Thus we can apply the sublemma in this case also.

(5) *Suppose  $\mathfrak{k}$  is not semi-simple.*

We may suppose  $\mathfrak{k}$  is not abelian. Then the semi-simple part  $s(\mathfrak{k})$  is a non-trivial proper ideal of  $\mathfrak{k}$ . By Corollary 1, we have

$$s(\mathfrak{k}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Since  $s(\mathfrak{g}) \subset \mathfrak{g} \cap s(\mathfrak{k})$ ,  $s(\mathfrak{h}) \subset \mathfrak{h} \cap s(\mathfrak{k})$  and  $(\mathfrak{g} \cap s(\mathfrak{k})) \cap (\mathfrak{h} \cap s(\mathfrak{k})) = \{0\}$ , we have  $s(\mathfrak{g}) = \mathfrak{g} \cap s(\mathfrak{k})$  and  $s(\mathfrak{h}) = \mathfrak{h} \cap s(\mathfrak{k})$ . Thus

$$s(\mathfrak{k}) = \mathfrak{g} \cap s(\mathfrak{k}) + \mathfrak{h} \cap s(\mathfrak{k})$$

is a direct sum of vector spaces, and hence we can apply our sublemma.

The above argument shows that we may suppose  $\mathfrak{k}$  is semi-simple and not simple.

(6) Suppose  $\mathfrak{k}$  is semi-simple and all simple factors of  $\mathfrak{k}$  are mutually isomorphic with each other.

In this case we shall show that either  $\mathfrak{g}$  or  $\mathfrak{h}$  contains a proper ideal of  $\mathfrak{k}$ , so that the proposition holds by (3) or (4). Suppose neither  $\mathfrak{g}$  nor  $\mathfrak{h}$  contains a non trivial proper ideal of  $\mathfrak{k}$ . Let

$$\mathfrak{k} = \sum_{i \in I} \mathfrak{k}_i, \mathfrak{g} = \sum_{j \in J} \mathfrak{g}_j, \text{ and } \mathfrak{h} = \sum_{k \in K} \mathfrak{h}_k$$

be the decompositions of  $\mathfrak{k}$ ,  $\mathfrak{g}$  and  $\mathfrak{h}$  into simple factors. By the present assumption, all  $\mathfrak{k}_i$ 's are mutually isomorphic. By Lemma 2, we see also that all  $\mathfrak{g}_j$ 's,  $\mathfrak{h}_k$ 's and  $\mathfrak{k}_i$ 's are mutually isomorphic, and that

$$|I| = |J| + |K|$$

where  $||$  indicates the number of elements.

Denote by  $\pi_i$  the projection of  $\mathfrak{k}$  onto  $\mathfrak{k}_i$ . One sees that

$$\begin{aligned} \pi_i(\mathfrak{g}_j) &= \mathfrak{k}_i \text{ or } \{0\}, \\ \pi_i(\mathfrak{h}_k) &= \mathfrak{k}_i \text{ or } \{0\} \end{aligned}$$

for all  $i, j, k$ . Put

$$\begin{aligned} A_j &= \{i \in I \mid \pi_i(\mathfrak{g}_j) \neq \{0\}\}, \\ B_k &= \{i \in I \mid \pi_i(\mathfrak{h}_k) \neq \{0\}\} \end{aligned}$$

for each  $j \in J$  and  $k \in K$ . Let  $j_1, j_2 \in J$ , and  $j_1 \neq j_2$ . Then  $[\pi_{j_1}(\mathfrak{g}_{j_1}), \pi_{j_2}(\mathfrak{g}_{j_2})] = 0$ . Thus  $A_{j_1} \cap A_{j_2} = \emptyset$ . Hence  $A_j$ 's are mutually disjoint and so are  $B_k$ 's.

Suppose  $A_j$  consists of exactly one element, say  $i$ . Then we see  $\mathfrak{g}_j = \mathfrak{k}_i$  and hence  $\mathfrak{g}$  contains a non trivial proper ideal. This is a contradiction. Thus each  $A_j$  contains at least two elements. Similarly we have  $|B_k| \geq 2$ . Thus we have

$$\sum |A_j| + \sum |B_k| \geq 2(|J| + |K|) = 2|I|$$

On the other hand,

$$\sum |A_j| \leq |I| \text{ and } \sum |B_k| \leq |I|.$$

Combining together, we see

$$|A_j| = |B_k| = 2$$

for every  $j \in J$  and  $k \in K$ .

By an elementary combinatorial argument one can decompose the index set  $I$  into two disjoint subsets  $I_1$  and  $I_2$  such that, for every  $j, k$ , the sets  $A_j \cap I_1$ ,

$A_j \cap I_2$ ,  $B_k \cap I_1$  and  $B_k \cap I_2$  are all non empty. Let

$$\alpha_1 = \sum_{i \in I_1} \mathfrak{k}_i \quad \text{and} \quad \alpha_2 = \sum_{i \in I_2} \mathfrak{k}_i,$$

so that we have  $\mathfrak{k} = \alpha_1 \oplus \alpha_2$ . Denote by  $p_i$  the projection of  $\mathfrak{k}$  onto  $\alpha_i$  (for  $i=1, 2$ ). It follows from our construction that the homomorphisms  $p_1|_{\mathfrak{g}}$ ,  $p_2|_{\mathfrak{g}}$ ,  $p_1|_{\mathfrak{h}}$  and  $p_2|_{\mathfrak{h}}$  are all onto isomorphisms. Using the decomposition  $\mathfrak{k} = \alpha_1 \oplus \alpha_2$ , we can write

$$\mathfrak{g} = \{(X, \phi(X)) \mid X \in \alpha_1\}$$

and

$$\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \alpha_2\}$$

by suitable onto isomorphisms  $\phi: \alpha_1 \rightarrow \alpha_2$  and  $\psi: \alpha_2 \rightarrow \alpha_1$ . Consider an automorphism  $\psi \circ \phi$  of  $\alpha_1$ . By a result due to Borel and Mostow [1], every automorphism of a semi-simple Lie algebra has a non-zero fixed vector. Thus, we have an element  $X$  in  $\alpha_1$  such that  $X \neq 0$  and  $\psi(\phi(X)) = X$ . Then we have

$$(X, \phi(X)) = (\psi(\phi(X)), \phi(X)) \in \mathfrak{g} \cap \mathfrak{h} = \{0\}.$$

This is a contradiction. Thus, in this case, either  $\mathfrak{g}$  or  $\mathfrak{h}$  contains a proper ideal of  $\mathfrak{k}$ .

(7) Suppose  $\mathfrak{k}$  is semi-simple and  $\mathfrak{k}$  contains at least two simple ideals which are not isomorphic.

Choose a simple ideal  $\alpha$  of  $\mathfrak{k}$  such that  $\dim \alpha$  is minimal among the simple ideals of  $\mathfrak{k}$ . Let  $\mathfrak{k}_0$  be the direct sum of all simple ideals isomorphic to  $\alpha$ , and  $\mathfrak{k}_1$  the complementary ideal, so that we have

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1.$$

Similarly, decompose  $\mathfrak{g}$  and  $\mathfrak{h}$  as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1,$$

where  $\mathfrak{g}_0$  (resp.  $\mathfrak{h}_0$ ) is the direct sum of all simple ideals in  $\mathfrak{g}$  (resp.  $\mathfrak{h}$ ) isomorphic to  $\alpha$ .

In virtue of Lemma 2, we see that  $\mathfrak{k}_0$  and  $\mathfrak{k}_1$  are isomorphic with  $\mathfrak{g}_0 \oplus \mathfrak{h}_0$  and  $\mathfrak{g}_1 \oplus \mathfrak{h}_1$  respectively. We claim that the ideal  $\mathfrak{k}_1$  satisfies the required condition in the sublemma. Let  $\pi_0$  and  $\pi_1$  be the projections of  $\mathfrak{k}$  onto  $\mathfrak{k}_0$  and  $\mathfrak{k}_1$  respectively. Consider  $\pi_0: \mathfrak{g}_1 \rightarrow \mathfrak{k}_0$ . From the definitions of  $\mathfrak{g}_1$  and  $\mathfrak{k}_0$ , we see  $\pi_0|_{\mathfrak{g}_1} = \{0\}$ . Thus,  $\mathfrak{g}_1 \subset \mathfrak{k}_1$ . Similarly we have  $\mathfrak{h}_1 \subset \mathfrak{k}_1$ . Thus,  $\mathfrak{k}_1 \supset \mathfrak{g}_1 + \mathfrak{h}_1$ . Since  $\mathfrak{g}_1 \cap \mathfrak{h}_1 = \{0\}$  and  $\dim \mathfrak{k}_1 = \dim \mathfrak{g}_1 + \dim \mathfrak{h}_1$ , we conclude that

$$\mathfrak{k}_1 = \mathfrak{g}_1 + \mathfrak{h}_1 \quad (\text{direct sum of vector spaces}).$$

Since  $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$  is a direct sum of vector spaces, we see that  $(\mathfrak{k}_1 \cap \mathfrak{g}) \cap (\mathfrak{k}_1 \cup \mathfrak{h}) = \{0\}$ . On the other hand,  $\mathfrak{k}_1 \cap \mathfrak{g} \supset \mathfrak{g}_1$  and  $\mathfrak{k}_1 \cap \mathfrak{h} \supset \mathfrak{h}_1$ , and also  $\mathfrak{k}_1 = \mathfrak{g}_1 + \mathfrak{h}_1$  (direct sum of vector spaces). It follows that  $\mathfrak{g}_1 = \mathfrak{g} \cap \mathfrak{k}_1$  and  $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{k}_1$  and hence

$$\mathfrak{k}_1 = (\mathfrak{g} \cap \mathfrak{k}_1) + (\mathfrak{h} \cap \mathfrak{k}_1) \quad (\text{direct sum of vector spaces}).$$

This proves our claim.

Thus we have completed the proof of Proposition 1.

#### 4. Now we can prove Theorem 1

Proof of Theorem 1. First assume that  $\mathfrak{k}$  is semi-simple. Apply Proposition 1 to  $\mathfrak{k}$ ,  $\mathfrak{g}$  and  $\mathfrak{h}$ . We get a direct sum decomposition

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

with the properties:

- (i) The projection of  $\mathfrak{k}$  onto  $\mathfrak{l}$  with respect to the above decomposition induces an isomorphism of  $\mathfrak{g}$  onto  $\mathfrak{l}$ .
- (ii)  $\mathfrak{k} = \mathfrak{l} + \mathfrak{h}$  (direct sum of vector spaces).

Again apply Proposition 1 to  $\mathfrak{k}$ ,  $\mathfrak{h}$  and  $\mathfrak{l}$ . We have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{m}'$$

with the properties:

- (i') The projection of  $\mathfrak{k}$  onto  $\mathfrak{m}$  with respect to this decomposition induces an isomorphism of  $\mathfrak{h}$  onto  $\mathfrak{m}$ .
- (ii')  $\mathfrak{k} = \mathfrak{m} + \mathfrak{l}$  (direct sum of vector spaces).

Since  $\mathfrak{m}$  and  $\mathfrak{l}$  are both ideals of  $\mathfrak{k}$ , we have a direct sum

$$\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{l}$$

of Lie algebras. The assumption that  $\mathfrak{k}$  is semi-simple implies  $\mathfrak{m} = \mathfrak{l}'$ . Thus, with respect to the direct sum

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

we see that the projections of  $\mathfrak{k}$  onto  $\mathfrak{l}$  and  $\mathfrak{l}'$  induce isomorphisms of  $\mathfrak{g}$  and  $\mathfrak{h}$  onto  $\mathfrak{l}$  and  $\mathfrak{l}'$  respectively. Setting  $\mathfrak{g}_1 = \mathfrak{l}$ , and  $\mathfrak{h}_1 = \mathfrak{l}'$ , we see that the decomposition

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

satisfies the first two properties. The third property follows from  $\mathfrak{g} \cap \mathfrak{h} = \{0\}$ .

In fact, suppose  $\psi(\phi(X)) = X$  for  $X \in \mathfrak{g}_1$ . Then,  $(X, \phi(X)) = (\psi(\phi(X)),$

$\phi(X)) \in \mathfrak{g} \cap \mathfrak{h} = \{0\}$ . Thus  $X = 0$ .

Consider the general case. By Corollary 1, we have

$$s(\mathfrak{k}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Also by Lemma 1,  $\dim c(\mathfrak{k}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h})$ . It is easily seen that the projection  $\pi$  of  $\mathfrak{k}$  onto  $c(\mathfrak{k})$  induces

$$c(\mathfrak{k}) = \pi(c(\mathfrak{g})) + \pi(c(\mathfrak{h})) \quad (\text{direct sum of vector spaces}).$$

From the first argument, we can choose a direct sum decomposition

$$s(\mathfrak{k}) = \mathfrak{g}_1' \oplus \mathfrak{h}_1'$$

such that the projections of  $s(\mathfrak{k})$  onto  $\mathfrak{g}_1'$  and  $\mathfrak{h}_1'$  induce isomorphisms of  $s(\mathfrak{g})$  and  $s(\mathfrak{h})$  onto  $\mathfrak{g}_1'$  and  $\mathfrak{h}_1'$  respectively. Now put

$$\mathfrak{g}_1 = \mathfrak{g}_1' \oplus \pi(c(\mathfrak{g}))$$

and

$$\mathfrak{h}_1 = \mathfrak{h}_1' \oplus \pi(c(\mathfrak{h})).$$

we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1.$$

We claim that this decomposition satisfies the required properties in Theorem 1. The first two are easy. The last one follows from the first two and  $\mathfrak{g} \cap \mathfrak{h} = \{0\}$ .

Q.E.D.

**REMARK 1.** The converse of Theorem 1 holds. Let  $\mathfrak{g}_1$  and  $\mathfrak{h}_1$  be Lie algebras, and let  $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$  and  $\psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$  be Lie algebra homomorphisms such that  $\psi \circ \phi$  has no non-zero fixed vector. In the direct sum  $\mathfrak{g}_1 \oplus \mathfrak{h}_1 = \mathfrak{k}$  of Lie algebras, define  $\mathfrak{g}$  and  $\mathfrak{h}$  by (i) and (ii). Then  $\mathfrak{g}$  and  $\mathfrak{h}$  are subalgebras and we have

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

**REMARK 2.** Suppose  $M = K/H$  is a homogeneous space of the type mentioned in the introduction. Then the action of  $K$  on  $K/H$  is almost effective if and only if  $\psi$  is injective.

**REMARK 3.** Let  $M = K/H$  be as above. By the theorem of Borel-Mostow cited before, the Lie algebra homomorphism  $\psi \circ \phi = 0$  if  $\mathfrak{g}$  is simple. Thus we see that if  $G$  is simple and the  $K$ -action on  $K/H$  is almost effective then  $G$  is normal. Thus, Ochiai-Takahashi's theorem follows from Theorem 1.

**5.** Now we consider a homogeneous space of the type mentioned in the introduction. Let  $M = K/H$  be a homogeneous space of a connected compact

Lie group  $K$ . We assume that a connected Lie subgroup  $G$  acts simply transitively on  $K/H$ . Since  $K/H$  is compact,  $G$  is necessarily compact. The composition mapping

$$F: G \times H \rightarrow K$$

is a diffeomorphism, so that we have

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}),$$

for their Lie algebras. Applying Theorem 1, we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

and homomorphisms  $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$  and  $\psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$  such that we have

$$\mathfrak{g} = \{(X, \phi(X)) \mid X \in \mathfrak{g}_1\},$$

$$\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}.$$

Further, as we see from the proof of Theorem 1, we can assume that

$$c(\mathfrak{g}_1) = \pi(c(\mathfrak{g})),$$

where  $\pi$  denotes the projection of  $\mathfrak{k}$  onto its center.

Let  $G_1$  be the connected Lie subgroup of  $K$  corresponding to the subalgebra  $\mathfrak{g}_1$ . Since  $\mathfrak{g}_1$  is an ideal of  $\mathfrak{k}$ ,  $G_1$  is a normal subgroup of  $K$ . Next we claim that  $G_1$  is compact.  $s(G_1)$  is closed in  $K$  since it is semi-simple. Thus it suffices to show that  $c(G_1)$  is compact. However, from our construction,  $c(\mathfrak{g}_1) = \pi(c(\mathfrak{g}))$ . Consider the Lie group homomorphism  $\tilde{\pi}: K \rightarrow K/s(K)$ .  $\tilde{\pi}|c(K)$  is a finite covering map. Thus  $c(G_1)$  is closed in  $c(K)$  if and only if  $\tilde{\pi}(c(G_1))$  is closed. On the other hand,  $c(G)$  is compact, and hence  $\tilde{\pi}(c(G))$  is compact.  $c(\mathfrak{g}_1) = \pi(c(\mathfrak{g}))$  implies that  $\tilde{\pi}(c(G_1)) = \tilde{\pi}(c(G))$ . Thus,  $c(G_1)$  is closed, and hence  $G_1$  is compact. From the property that  $\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$  and  $\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}$ , we have

$$\mathfrak{k} = \mathfrak{g}_1 + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

By Lemma 1, the composition mapping

$$G_1 \times H \rightarrow K$$

defines a covering map. Consequently,  $G_1$  acts transitively on the coset space  $K/H$ . Furthermore, fix a point  $p$  in  $K/H$ . Then the mapping

$$G_1 \rightarrow K/H$$

defined by  $g \rightarrow g(p)$  is a covering map. Thus, if  $G \cong K/H$  is simply connected, then  $G_1$  is also simply connected. Thus we have proved the following:

**Theorem 2.** *Let  $K$  be a connected compact Lie group and  $H$  a closed subgroup of  $K$ . Assume that a connected Lie subgroup  $G$  acts simply transitively on the homogeneous space  $K/H$  by the left translation. Then there exists a connected closed normal subgroup  $G_1$  of  $K$  such that  $G_1$  acts transitively on  $K/H$  and  $G_1$  is locally isomorphic with  $G$  as Lie groups.*

**Theorem 3.** *Under the same assumption as in Theorem 2, assume further that  $G$  is simply connected. Then there exists a connected closed normal subgroup  $G_1$  of  $K$  such that  $G_1$  is isomorphic with  $G$  as Lie groups and  $G_1$  acts simply transitively on  $K/H$ .*

6. We give here two examples. The first one shows that the conclusion of Ochiai-Takahashi's theorem does not hold any more if  $G$  is not simple.

EXAMPLE 1. Let  $A$  be a connected compact semi-simple Lie group and  $\mathfrak{a}$  its Lie algebra. We put

$$\begin{aligned} K &= A \times A \times A, \\ G &= \{(x, y, z) | x, y \in A\}, \\ H &= \{(e, z, z) | z \in A\}. \end{aligned}$$

$H$  is a closed subgroup of  $K$ . Consider the homogeneous space  $K/H$ . We see easily that  $G$  acts simply transitively on  $K/H$ .  $G$  is compact semi-simple and not simple. Choose a  $K$ -invariant Riemannian metric  $ds^2$  on  $K/H$ . Since  $K/H$  can be identified with  $G$ ,  $ds^2$  is a left-invariant Riemannian metric on  $G$ . From the definition,  $K$  is contained in the identity-component of isometries of  $(K/H = G, ds^2)$ .  $G$  is not normal in  $K$ , thus  $G$  is not normal in the identity-component of isometries.

For this example, an explicit description of Theorem 1 is as follows:

$$\begin{aligned} \mathfrak{k} &= \mathfrak{a} \oplus \mathfrak{a} \oplus \mathfrak{a}, \\ \text{Let } \mathfrak{g}_1 &= \{(X, Y, 0) | X, Y \in \mathfrak{a}\}, \\ \mathfrak{h}_1 &= \{(0, 0, Z) | Z \in \mathfrak{a}\}. \end{aligned}$$

Define  $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$  by

$$\phi((X, Y, 0)) = (0, 0, X)$$

and  $\psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$  by

$$\psi((0, 0, Z)) = (0, Z, 0).$$

Then we have

$$\begin{aligned} \mathfrak{g} &= \{(X, \phi(X)) \in \mathfrak{g}_1 \oplus \mathfrak{h}_1 | X \in \mathfrak{g}_1\}, \\ \mathfrak{g} &= \{(\psi(Y), Y) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1 | Y \in \mathfrak{h}_1\}. \end{aligned}$$

The next example shows that the conclusion of Theorem 3 does not hold if  $G$  is not simply connected.

EXAMPLE 2. We choose two simply connected compact Lie groups  $A$  and  $B$  with the following properties:

1. There exists an injective homomorphism  $j$  of  $A$  into  $B$ .
2. The center  $Z(A)$  of  $A$  is non-trivial and

$$j(Z(A)) \cap Z(B) = \{e\}.$$

For instance, choose positive integers  $m$  and  $n$  such that  $n > m > 2$ . Then  $A = SU(m)$ ,  $B = SU(n)$  and the canonical injection of  $SU(m)$  into  $SU(n)$  satisfy the required properties.

Let

$$\begin{aligned} K &= A \times B \times A, \\ G_1 &= A \times B \times \{e\}, \\ G &= \{(a, b, a) \mid a \in A, b \in B\}, \\ H &= \{(e, j(a), a) \mid a \in A\}, \\ \Gamma &= \{(x, e, x) \mid x \in Z(A)\}. \end{aligned}$$

The Lie algebras of  $A$  and  $B$  are denoted by  $\mathfrak{a}$  and  $\mathfrak{b}$  respectively.  $\Gamma$  is a finite group contained in the center of  $K$ . We consider the quotient group  $\bar{K} = K/\Gamma$ , and denote by  $\pi$  the canonical projection of  $K$  onto  $\bar{K}$ .  $\bar{H} = \pi(H)$  is a closed subgroup of  $\bar{K}$ . Consider  $\bar{K}/\bar{H}$ . One can easily show that the group  $\bar{G} = \pi(G)$  acts simply transitively on  $\bar{K}/\bar{H}$ . We claim that no normal subgroup of  $\bar{K}$  acts simply transitively on  $\bar{K}/\bar{H}$ . Suppose a normal subgroup  $G_1'$  of  $\bar{K}$  acts simply transitively on  $\bar{K}/\bar{H}$ . Then its Lie algebra  $\mathfrak{g}_1'$  satisfies

$$\mathfrak{k} = \mathfrak{g}_1' + \mathfrak{g} \quad (\text{direct sum of vector spaces}),$$

where  $\mathfrak{h} = \{(0, j(X), X) \mid X \in \mathfrak{a}\}$ . Since  $\mathfrak{g}_1'$  is an ideal of  $\mathfrak{k}$ , we see  $\mathfrak{g}_1' = \mathfrak{g}_1 = \{(X, Y, 0) \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}$ . It follows that  $\pi(G_1) = G_1'$ . However,  $\pi(G_1)$  is simply connected because  $\pi(G_1) = G_1 / (G_1 \cap \Gamma) \cong G_1$ . This is a contradiction.

OSAKA UNIVERSITY

---

### References

- [1] A. Borel and G.D. Mostow: *On semi-simple automorphisms of Lie algebras*, Ann. of Math. **61** (1966), 389–504.
- [2] T. Ochiai and T. Takahashi: *The group of isometries of a left invariant Riemannian metric on a Lie group*, Math. Ann. **223** (1976), 91–96.
- [3] H. Scheerer: *Homotopieäquivalente kompakte Liesche Gruppen*, Topology **7** (1968), 227–232.
- [4] H. Toda: *A note on compact semi-simple Lie groups*, Japan. J. Math. **2** (1976), 354–360.

