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ON A TRANSITIVE TRANSFORMATION GROUP OF A COMPACT GROUP MANIFOLD

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1. Introduction. Let K be a connected compact Lie group and H a closed subgroup of K . Suppose a connected Lie subgroup G of K acts simply transitively on the coset space K/H by the left translation. Then the composition mapping

$$F: G \times H \rightarrow K$$

defined by $F(g, h) = gh$ ($g \in G, h \in H$) gives rise to a diffeomorphism of the product manifold $G \times H$ onto K . Consequently, for their Lie algebras, we have

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

We shall prove in this paper the following:

Theorem 1. *Let \mathfrak{k} be a compact Lie algebra. Suppose there exist two subalgebras \mathfrak{g} and \mathfrak{h} of \mathfrak{k} such that*

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

Then there exist a direct sum decomposition

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

of Lie algebras and Lie algebra homomorphisms

$$\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1 \quad \text{and} \quad \psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$$

with the following properties:

- (i) $\mathfrak{g} = \{(X, \varphi(X)) \mid X \in \mathfrak{g}_1\}$.
- (ii) $\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}$.
- (iii) $\psi \circ \varphi$ has no non-zero fixed vector.

As a result we see that the Lie algebra \mathfrak{k} is isomorphic with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lie algebras. This theorem gives us an infinitesimal characterization of a homogeneous space of the type mentioned in the above. Some

application and remarks will be added after its proof.

Such a homogeneous space is related with a study of isometries of a compact group manifold. Let G be a connected compact Lie group and choose a left invariant Riemannian metric ds^2 on G . Denote by K the identity component of the isometry group of (G, ds^2) . We identify an element g of G with its left translation L_g on G . Ochiai-Takahashi [2] proved that if G is simple then G is normal in K . Their theorem follows immediately from our Theorem 1. The conclusion of their theorem does not hold in general if G is not simple, as our example shows. However, our Theorem 3 asserts that if G is simply connected then we have a similar conclusion by a suitable change of the action of G on the space.

2. Recall that a Lie algebra \mathfrak{k} is said to be *compact* if it can be represented as a Lie algebra of a compact Lie group. For a compact Lie algebra \mathfrak{k} , we denote by $\mathfrak{c}(\mathfrak{k})$ its center and by $\mathfrak{s}(\mathfrak{k})$ its maximal semi-simple ideal, so that we have $\mathfrak{s}(\mathfrak{k}) = [\mathfrak{k}, \mathfrak{k}]$ and

$$\mathfrak{k} = \mathfrak{s}(\mathfrak{k}) \oplus \mathfrak{c}(\mathfrak{k})$$

(direct sum of Lie algebras). The same notation will be used for a connected Lie group K when the Lie algebra \mathfrak{k} of K is compact. $\mathfrak{c}(K)$ and $\mathfrak{s}(K)$ are the connected Lie subgroups of K corresponding to Lie subalgebras $\mathfrak{c}(\mathfrak{k})$ and $\mathfrak{s}(\mathfrak{k})$ respectively.

Note that a connected Lie group K has a compact Lie algebra if and only if K has a bi-invariant Riemannian metric and also that any subalgebra of a compact Lie algebra is compact. In the sequel, for a Lie group homomorphism, the induced Lie algebra homomorphism is denoted by the same symbol.

Lemma 1. *Let K, G and H be connected Lie groups with Lie algebras $\mathfrak{k}, \mathfrak{g}$ and \mathfrak{h} respectively. Suppose \mathfrak{k} is compact. Let $\phi: G \rightarrow K$ and $\psi: H \rightarrow K$ be Lie group homomorphisms such that the induced homomorphisms $\phi: \mathfrak{g} \rightarrow \mathfrak{k}$ and $\psi: \mathfrak{h} \rightarrow \mathfrak{k}$ are both injective and*

$$\mathfrak{k} = \phi(\mathfrak{g}) + \psi(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Then the composition mapping

$$F: G \times H \rightarrow K$$

defined by $F(g, h) = \phi(g) \cdot \psi(h)$ is a covering map.

Proof. In general, we denote the left translation and the right translation of a group induced by an element x in it by L_x and R_x respectively. Then, for the mapping F , we have the following commutative diagram:

$$\begin{array}{ccc}
 G \times H & \xrightarrow{F} & K \\
 (L_g, R_h) \downarrow & & \downarrow L_{\phi(g)} \circ R_{\varphi(h)} \\
 G \times H & \xrightarrow{F} & K
 \end{array}$$

for $(g, h) \in G \times H$. This gives an identity

$$F = (L_{\phi(g)} \circ R_{\varphi(h)}) \circ F \circ (L_g^{-1}, R_h^{-1}).$$

Taking the differentials, we have

$$(dF)_{(g,h)} = (d(L_{\phi(g)} \circ R_{\varphi(h)}))_e \circ (dF)_{(e,e)} \circ (d(L_g^{-1}, R_h^{-1}))_{(g,h)}.$$

We identify $T_{(e,e)}(G \times H)$ with $T_e(G) + T_e(H) = \mathfrak{g} + \mathfrak{h}$ (direct sum of vector spaces). Since $(dF)_{(e,e)}|_{T_e(G)} = \phi$ and $(dF)_{(e,e)}|_{T_e(H)} = \psi$, our assumption in the lemma implies that $(dF)_{(e,e)}$ gives an isomorphism of $T_{(e,e)}(G \times H)$ onto $T_e(K) = \mathfrak{k}$. By the above identity, we see that $(dF)_{(g,h)}$ is isomorphic at each point (g, h) of $G \times H$. Since \mathfrak{k} is compact, we can choose a bi-invariant Riemannian metric ds^2 on K . Then $ds^2 = F^*(ds^2)$ gives a Riemannian metric on the manifold $G \times H$, which is locally isometric with (K, ds^2) via F . In virtue of the first commutative diagram, the Riemannian metric ds^2 on $G \times H$ is $L(G)$ and $R(H)$ -invariant, and hence it is complete. Thus we see that F is a locally isometric mapping of a complete Riemannian manifold $(G \times H, ds^2)$ into (K, ds^2) . This proves that F is a covering map. q.e.d.

Lemma 2. *Let \mathfrak{k} be a compact Lie algebra, and let \mathfrak{g} and \mathfrak{h} be two subalgebras of \mathfrak{k} such that*

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

Then, \mathfrak{k} is isomorphic with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lie algebras. consequently, we have

$$\dim \mathfrak{c}(\mathfrak{k}) = \dim \mathfrak{c}(\mathfrak{g}) + \dim \mathfrak{c}(\mathfrak{h}).$$

Proof. For \mathfrak{g} , \mathfrak{h} and \mathfrak{k} , choose simply connected Lie groups G , H and K with the corresponding Lie algebras respectively. Let

$$\phi: \mathfrak{g} \rightarrow \mathfrak{k} \quad \text{and} \quad \psi: \mathfrak{h} \rightarrow \mathfrak{k}$$

be the inclusion mappings. They induce Lie group homomorphisms

$$\phi: G \rightarrow K \quad \text{and} \quad \psi: H \rightarrow K.$$

The composition mapping F of the product manifold $G \times H$ into K defined by

$$F(g, h) = \phi(g)\psi(h)$$

is a covering map by Lemma 1. Since K is assumed to be simply connected, we

have a diffeomorphism of $G \times H$ onto K . \mathfrak{k} is compact and hence \mathfrak{g} and \mathfrak{h} are compact. Since G, H and K are simply connected and their Lie algebras are compact, we see $G = s(G) \times c(G)$, $H = s(H) \times c(H)$ and $K = s(K) \times c(K)$. Since F is a diffeomorphism of the product manifold $G \times H$ onto K we see

$$\dim c(K) = \dim c(G) + \dim c(H)$$

and hence

$$\dim c(\mathfrak{k}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h}).$$

Note that $s(K)$ is a maximal compact subgroup of K . Also we see that F induces a homotopy equivalence between $s(G) \times s(H)$ and $s(K)$.

A theorem in homotopy theory ([3], [4]) states that if two simply connected compact Lie groups are homotopically equivalent then they are isomorphic as Lie groups. Thus, we see that the Lie group $s(K)$ is isomorphic with the direct product $s(G) \times s(H)$ of Lie groups. Finally we can conclude that the Lie algebra \mathfrak{k} is isomorphic with the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of Lie algebras. q.e.d.

Corollary 1. *Under the same assumption as above, we have*

$$s(\mathfrak{k}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Proof. Since \mathfrak{k} and $\mathfrak{g} \oplus \mathfrak{h}$ are isomorphic, $s(\mathfrak{k})$ and $s(\mathfrak{g}) \oplus s(\mathfrak{h})$ are isomorphic. Especially, we have

$$\dim s(\mathfrak{k}) = \dim s(\mathfrak{g}) + \dim s(\mathfrak{h}).$$

On the other hand, we know

$$s(\mathfrak{k}) = [\mathfrak{k}, \mathfrak{k}], \quad s(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}], \quad s(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{h}].$$

Thus, we have

$$s(\mathfrak{k}) \supset s(\mathfrak{g}) \quad \text{and} \quad s(\mathfrak{k}) \supset s(\mathfrak{h}).$$

The assumption $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ (direct sum of vector spaces) shows that $s(\mathfrak{g}) + s(\mathfrak{h})$ is a direct sum of vector spaces in $s(\mathfrak{k})$. The first equality on dimension proves our corollary. q.e.d.

3. Theorem 1 will follow easily from the following:

Proposition 1. *Let \mathfrak{k} be a compact Lie algebra and let \mathfrak{g} and \mathfrak{h} be its subalgebras such that*

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

Then \mathfrak{k} has a direct sum decomposition of Lie algebras

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

with the following properties:

(i) The projection π of \mathfrak{k} onto \mathfrak{l} with respect to the above decomposition induces an isomorphism of \mathfrak{g} onto \mathfrak{l} .

(ii) $\mathfrak{k} = \mathfrak{k} + \mathfrak{h}$ (direct sum of vector spaces).

Proof. We prove the proposition by induction on $\dim \mathfrak{k}$. When $\dim \mathfrak{k} = 1$, the proposition holds since $\mathfrak{k} = \mathfrak{g}$ or $\mathfrak{k} = \mathfrak{h}$. Now assume that the proposition holds when $\dim \mathfrak{k} < N$. Let $\dim \mathfrak{k} = N$. To simplify the argument we prepare the following:

Sublemma. Suppose \mathfrak{k} has a non-trivial proper ideal \mathfrak{k}_1 such that

$$\mathfrak{k}_1 = (\mathfrak{g} \cap \mathfrak{k}_1) + (\mathfrak{h} \cap \mathfrak{k}_1) \quad (\text{direct sum of vector spaces}).$$

Then the assertion of Proposition 1 holds for \mathfrak{k} , \mathfrak{g} and \mathfrak{h} .

Proof. For \mathfrak{k}_1 , we choose a complementary ideal \mathfrak{k}_2 so that we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2.$$

Let π_2 be the projection of \mathfrak{k} onto \mathfrak{k}_2 . We have

$$\dim \mathfrak{g} = \dim \mathfrak{g} \cap \mathfrak{k}_1 + \dim \pi_2(\mathfrak{g}),$$

$$\dim \mathfrak{h} = \dim \mathfrak{h} \cap \mathfrak{k}_1 + \dim \pi_2(\mathfrak{h}).$$

Thus,

$$\begin{aligned} \dim \mathfrak{k}_2 &= \dim \mathfrak{k} - \dim \mathfrak{k}_1 \\ &= \dim \pi_2(\mathfrak{g}) + \dim \pi_2(\mathfrak{h}). \end{aligned}$$

Since $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$, $\mathfrak{k}_2 = \pi_2(\mathfrak{k})$ is spanned by $\pi_2(\mathfrak{g})$ and $\pi_2(\mathfrak{h})$, and hence we have

$$\mathfrak{k}_2 = \pi_2(\mathfrak{g}) + \pi_2(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Consider \mathfrak{k}_1 and its subalgebras $\mathfrak{g} \cap \mathfrak{k}_1$ and $\mathfrak{h} \cap \mathfrak{k}_1$ and also \mathfrak{k}_2 and its subalgebras $\pi_2(\mathfrak{g})$ and $\pi_2(\mathfrak{h})$. By the inductive hypothesis, we have direct sum decompositions

$$\mathfrak{k}_1 = \mathfrak{l}_1 \oplus \mathfrak{l}_1' \quad \text{and} \quad \mathfrak{k}_2 = \mathfrak{l}_2 \oplus \mathfrak{l}_2'$$

with the properties:

i. The projections $\mathfrak{g} \cap \mathfrak{k}_1 \rightarrow \mathfrak{l}_1$ and $\pi_2(\mathfrak{g}) \rightarrow \mathfrak{l}_2$ are isomorphisms.

ii. $\mathfrak{k}_1 = \mathfrak{l}_1 + \mathfrak{h} \cap \mathfrak{k}_1$, $\mathfrak{k}_2 = \mathfrak{l}_2 + \pi_2(\mathfrak{h})$ (direct sums of vector spaces).

Let

$$\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \quad \text{and} \quad \mathfrak{l}' = \mathfrak{l}_1' \oplus \mathfrak{l}_2'.$$

We claim that the direct sum decomposition

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

satisfies the required properties.

First suppose $X \in \mathfrak{g} \cap \mathfrak{l}'$. Then $\pi_2(X) \in \pi_2(\mathfrak{g}) \cap \mathfrak{l}_2'$. However $\pi_2(\mathfrak{g}) \cap \mathfrak{l}_2' = \{0\}$ from the assumption. Thus $\pi_2(X) = 0$, and hence $X \in \mathfrak{k}_1$. Then $X \in (\mathfrak{g} \cap \mathfrak{k}_1) \cap \mathfrak{l}_1' = \{0\}$. Consequently we have $\mathfrak{g} \cap \mathfrak{l}' = \{0\}$. This shows that the projection of \mathfrak{g} into \mathfrak{l} with respect to $\mathfrak{l} \oplus \mathfrak{l}'$ is injective. Since they have the same dimension, we have the property (i). Next suppose $\mathfrak{l} \cap \mathfrak{h} \in X$. $\pi_2(X) \in \pi_2(\mathfrak{h}) \cap \mathfrak{l}_2 = \{0\}$, and hence $X \in \mathfrak{k}_1$. We see that $X \in \mathfrak{l}_1 \cap (\mathfrak{h} \cap \mathfrak{k}_1) = \{0\}$. Thus, we have $\mathfrak{l} \cap \mathfrak{h} = \{0\}$. Since $\dim \mathfrak{k} = \dim \mathfrak{l} + \dim \mathfrak{h}$, we see $\mathfrak{k} = \mathfrak{l} + \mathfrak{h}$ (direct sum of vector spaces). Thus we have the property (ii) also. q.e.d.

We continue our proof of Proposition 1. First consider easy cases.

- (1) *Suppose \mathfrak{k} is abelian.*

Then $\mathfrak{l} = \mathfrak{g}$ and $\mathfrak{l}' = \mathfrak{h}$ satisfy the required properties.

- (2) *Suppose \mathfrak{k} is simple.*

Then, by Lemma 2 we see $\mathfrak{k} = \mathfrak{g}$ or $\mathfrak{k} = \mathfrak{h}$. Thus our assertion holds trivially.

- (3) *Suppose \mathfrak{g} contains a non-trivial proper ideal, say \mathfrak{k}_1 , of \mathfrak{k} .*

Then choose a complementary ideal \mathfrak{k}_2 of \mathfrak{k}_1 in \mathfrak{k} , so that we have

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2.$$

Clearly, $\mathfrak{k}_1 \cap \mathfrak{g} = \mathfrak{k}_1$, $\mathfrak{k}_1 \cap \mathfrak{h} \subset \mathfrak{g} \cap \mathfrak{h} = \{0\}$. Applying the above sublemma we see that Proposition 1 holds in this case.

- (4) *Suppose \mathfrak{h} contains a non-trivial proper ideal, say \mathfrak{k}_1 , of \mathfrak{k} .*

Then again we have $\mathfrak{k}_1 \cap \mathfrak{g} = \{0\}$, and $\mathfrak{k}_1 \cap \mathfrak{h} = \mathfrak{k}_1$. Thus we can apply the sublemma in this case also.

- (5) *Suppose \mathfrak{k} is not semi-simple.*

We may suppose \mathfrak{k} is not abelian. Then the semi-simple part $s(\mathfrak{k})$ is a non-trivial proper ideal of \mathfrak{k} . By Corollary 1, we have

$$s(\mathfrak{k}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Since $s(\mathfrak{g}) \subset \mathfrak{g} \cap s(\mathfrak{k})$, $s(\mathfrak{h}) \subset \mathfrak{h} \cap s(\mathfrak{k})$ and $(\mathfrak{g} \cap s(\mathfrak{k})) \cap (\mathfrak{h} \cap s(\mathfrak{k})) = \{0\}$, we have $s(\mathfrak{g}) = \mathfrak{g} \cap s(\mathfrak{k})$ and $s(\mathfrak{h}) = \mathfrak{h} \cap s(\mathfrak{k})$. Thus

$$s(\mathfrak{k}) = \mathfrak{g} \cap s(\mathfrak{k}) + \mathfrak{h} \cap s(\mathfrak{k})$$

is a direct sum of vector spaces, and hence we can apply our sublemma.

The above argument shows that we may suppose \mathfrak{k} is semi-simple and not simple.

(6) Suppose \mathfrak{k} is semi-simple and all simple factors of \mathfrak{k} are mutually isomorphic with each other.

In this case we shall show that either \mathfrak{g} or \mathfrak{h} contains a proper ideal of \mathfrak{k} , so that the proposition holds by (3) or (4). Suppose neither \mathfrak{g} nor \mathfrak{h} contains a non trivial proper ideal of \mathfrak{k} . Let

$$\mathfrak{k} = \sum_{i \in I} \mathfrak{k}_i, \quad \mathfrak{g} = \sum_{j \in J} \mathfrak{g}_j, \quad \text{and} \quad \mathfrak{h} = \sum_{k \in K} \mathfrak{h}_k$$

be the decompositions of \mathfrak{k} , \mathfrak{g} and \mathfrak{h} into simple factors. By the present assumption, all \mathfrak{k}_i 's are mutually isomorphic. By Lemma 2, we see also that all \mathfrak{g}_j 's, \mathfrak{h}_k 's and \mathfrak{k}_i 's are mutually isomorphic, and that

$$|I| = |J| + |K|$$

where $||$ indicates the number of elements.

Denote by π_i the projection of \mathfrak{k} onto \mathfrak{k}_i . One sees that

$$\begin{aligned} \pi_i(\mathfrak{g}_j) &= \mathfrak{k}_i \quad \text{or} \quad \{0\}, \\ \pi_i(\mathfrak{h}_k) &= \mathfrak{k}_i \quad \text{or} \quad \{0\} \end{aligned}$$

for all i, j, k . Put

$$\begin{aligned} A_j &= \{i \in I \mid \pi_i(\mathfrak{g}_j) \neq \{0\}\}, \\ B_k &= \{i \in I \mid \pi_i(\mathfrak{h}_k) \neq \{0\}\} \end{aligned}$$

for each $j \in J$ and $k \in K$. Let $j_1, j_2 \in J$, and $j_1 \neq j_2$. Then $[\pi_i(\mathfrak{g}_{j_1}), \pi_i(\mathfrak{g}_{j_2})] = 0$. Thus $A_{j_1} \cap A_{j_2} = \emptyset$. Hence A_j 's are mutually disjoint and so are B_k 's.

Suppose A_j consists of exactly one element, say i . Then we see $\mathfrak{g}_j = \mathfrak{k}_i$ and hence \mathfrak{g} contains a non trivial proper ideal. This is a contradiction. Thus each A_j contains at least two elements. Similarly we have $|B_k| \geq 2$. Thus we have

$$\sum |A_j| + \sum |B_k| \geq 2(|J| + |K|) = 2|I|$$

On the other hand,

$$\sum |A_j| \leq |I| \quad \text{and} \quad \sum |B_k| \leq |I|.$$

Combining together, we see

$$|A_j| = |B_k| = 2$$

for every $j \in J$ and $k \in K$.

By an elementary combinatorial argument one can decompose the index set I into two disjoint subsets I_1 and I_2 such that, for every j, k , the sets $A_j \cap I_1$,

$A_j \cap I_2$, $B_k \cap I_1$ and $B_k \cap I_2$ are all non empty. Let

$$\alpha_1 = \sum_{i \in I_1} \mathfrak{k}_i \quad \text{and} \quad \alpha_2 = \sum_{i \in I_2} \mathfrak{k}_i,$$

so that we have $\mathfrak{k} = \alpha_1 \oplus \alpha_2$. Denote by p_i the projection of \mathfrak{k} onto α_i (for $i=1, 2$). It follows from our construction that the homomorphisms $p_1|_{\mathfrak{g}}$, $p_2|_{\mathfrak{g}}$, $p_1|_{\mathfrak{h}}$ and $p_2|_{\mathfrak{h}}$ are all onto isomorphisms. Using the decomposition $\mathfrak{k} = \alpha_1 \oplus \alpha_2$, we can write

$$\mathfrak{g} = \{(X, \phi(X)) \mid X \in \alpha_1\}$$

and

$$\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \alpha_2\}$$

by suitable onto isomorphisms $\phi: \alpha_1 \rightarrow \alpha_2$ and $\psi: \alpha_2 \rightarrow \alpha_1$. Consider an automorphism $\psi \circ \phi$ of α_1 . By a result due to Borel and Mostow [1], every automorphism of a semi-simple Lie algebra has a non-zero fixed vector. Thus, we have an element X in α_1 such that $X \neq 0$ and $\psi(\phi(X)) = X$. Then we have

$$(X, \phi(X)) = (\psi(\phi(X)), \phi(X)) \in \mathfrak{g} \cap \mathfrak{h} = \{0\}.$$

This is a contradiction. Thus, in this case, either \mathfrak{g} or \mathfrak{h} contains a proper ideal of \mathfrak{k} .

(7) Suppose \mathfrak{k} is semi-simple and \mathfrak{k} contains at least two simple ideals which are not isomorphic.

Choose a simple ideal α of \mathfrak{k} such that $\dim \alpha$ is minimal among the simple ideals of \mathfrak{k} . Let \mathfrak{k}_0 be the direct sum of all simple ideals isomorphic to α , and \mathfrak{k}_1 the complementary ideal, so that we have

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1.$$

Similarly, decompose \mathfrak{g} and \mathfrak{h} as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1,$$

where \mathfrak{g}_0 (resp. \mathfrak{h}_0) is the direct sum of all simple ideals in \mathfrak{g} (resp. \mathfrak{h}) isomorphic to α .

In virtue of Lemma 2, we see that \mathfrak{k}_0 and \mathfrak{k}_1 are isomorphic with $\mathfrak{g}_0 \oplus \mathfrak{h}_0$ and $\mathfrak{g}_1 \oplus \mathfrak{h}_1$ respectively. We claim that the ideal \mathfrak{k}_1 satisfies the required condition in the sublemma. Let π_0 and π_1 be the projections of \mathfrak{k} onto \mathfrak{k}_0 and \mathfrak{k}_1 respectively. Consider $\pi_0: \mathfrak{g}_1 \rightarrow \mathfrak{k}_0$. From the definitions of \mathfrak{g}_1 and \mathfrak{k}_0 , we see $\pi_0|_{\mathfrak{g}_1} = \{0\}$. Thus, $\mathfrak{g}_1 \subset \mathfrak{k}_1$. Similarly we have $\mathfrak{h}_1 \subset \mathfrak{k}_1$. Thus, $\mathfrak{k}_1 \supset \mathfrak{g}_1 + \mathfrak{h}_1$. Since $\mathfrak{g}_1 \cap \mathfrak{h}_1 = \{0\}$ and $\dim \mathfrak{k}_1 = \dim \mathfrak{g}_1 + \dim \mathfrak{h}_1$, we conclude that

$$\mathfrak{k}_1 = \mathfrak{g}_1 + \mathfrak{h}_1 \quad (\text{direct sum of vector spaces}).$$

Since $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ is a direct sum of vector spaces, we see that $(\mathfrak{k}_1 \cap \mathfrak{g}) \cap (\mathfrak{k}_1 \cup \mathfrak{h}) = \{0\}$. On the other hand, $\mathfrak{k}_1 \cap \mathfrak{g} \supset \mathfrak{g}_1$ and $\mathfrak{k}_1 \cap \mathfrak{h} \supset \mathfrak{h}_1$, and also $\mathfrak{k}_1 = \mathfrak{g}_1 + \mathfrak{h}_1$ (direct sum of vector spaces). It follows that $\mathfrak{g}_1 = \mathfrak{g} \cap \mathfrak{k}_1$ and $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{k}_1$ and hence

$$\mathfrak{k}_1 = (\mathfrak{g} \cap \mathfrak{k}_1) + (\mathfrak{h} \cap \mathfrak{k}_1) \quad (\text{direct sum of vector spaces}).$$

This proves our claim.

Thus we have completed the proof of Proposition 1.

4. Now we can prove Theorem 1

Proof of Theorem 1. First assume that \mathfrak{k} is semi-simple. Apply Proposition 1 to \mathfrak{k} , \mathfrak{g} and \mathfrak{h} . We get a direct sum decomposition

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

with the properties:

(i) The projection of \mathfrak{k} onto \mathfrak{l} with respect to the above decomposition induces an isomorphism of \mathfrak{g} onto \mathfrak{l} .

(ii) $\mathfrak{k} = \mathfrak{l} + \mathfrak{h}$ (direct sum of vector spaces).

Again apply Proposition 1 to \mathfrak{k} , \mathfrak{h} and \mathfrak{l} . We have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{m}'$$

with the properties:

(i') The projection of \mathfrak{k} onto \mathfrak{m} with respect to this decomposition induces an isomorphism of \mathfrak{h} onto \mathfrak{m} .

(ii') $\mathfrak{k} = \mathfrak{m} + \mathfrak{l}$ (direct sum of vector spaces).

Since \mathfrak{m} and \mathfrak{l} are both ideals of \mathfrak{k} , we have a direct sum

$$\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{l}$$

of Lie algebras. The assumption that \mathfrak{k} is semi-simple implies $\mathfrak{m} = \mathfrak{l}'$. Thus, with respect to the direct sum

$$\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{l}'$$

we see that the projections of \mathfrak{k} onto \mathfrak{l} and \mathfrak{l}' induce isomorphisms of \mathfrak{g} and \mathfrak{h} onto \mathfrak{l} and \mathfrak{l}' respectively. Setting $\mathfrak{g}_1 = \mathfrak{l}$, and $\mathfrak{h}_1 = \mathfrak{l}'$, we see that the decomposition

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

satisfies the first two properties. The third property follows from $\mathfrak{g} \cap \mathfrak{h} = \{0\}$.

In fact, suppose $\psi(\phi(X)) = X$ for $X \in \mathfrak{g}_1$. Then, $(X, \phi(X)) = (\psi(\phi(X)),$

$\phi(X) \in \mathfrak{g} \cap \mathfrak{h} = \{0\}$. Thus $X=0$.

Consider the general case. By Corollary 1, we have

$$s(\mathfrak{k}) = s(\mathfrak{g}) + s(\mathfrak{h}) \quad (\text{direct sum of vector spaces}).$$

Also by Lemma 1, $\dim c(\mathfrak{k}) = \dim c(\mathfrak{g}) + \dim c(\mathfrak{h})$. It is easily seen that the projection π of \mathfrak{k} onto $c(\mathfrak{k})$ induces

$$c(\mathfrak{k}) = \pi(c(\mathfrak{g})) + \pi(c(\mathfrak{h})) \quad (\text{direct sum of vector spaces}).$$

From the first argument, we can choose a direct sum decomposition

$$s(\mathfrak{k}) = \mathfrak{g}_1' \oplus \mathfrak{h}_1'$$

such that the projections of $s(\mathfrak{k})$ onto \mathfrak{g}_1' and \mathfrak{h}_1' induce isomorphisms of $s(\mathfrak{g})$ and $s(\mathfrak{h})$ onto \mathfrak{g}_1' and \mathfrak{h}_1' respectively. Now put

$$\mathfrak{g}_1 = \mathfrak{g}_1' \oplus \pi(c(\mathfrak{g}))$$

and

$$\mathfrak{h}_1 = \mathfrak{h}_1' \oplus \pi(c(\mathfrak{h})).$$

we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1.$$

We claim that this decomposition satisfies the required properties in Theorem 1. The first two are easy. The last one follows from the first two and $\mathfrak{g} \cap \mathfrak{h} = \{0\}$.
Q.E.D.

REMARK 1. The converse of Theorem 1 holds. Let \mathfrak{g}_1 and \mathfrak{h}_1 be Lie algebras, and let $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$ and $\psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$ be Lie algebra homomorphisms such that $\psi \circ \phi$ has no non-zero fixed vector. In the direct sum $\mathfrak{g}_1 \oplus \mathfrak{h}_1 = \mathfrak{k}$ of Lie algebras, define \mathfrak{g} and \mathfrak{h} by (i) and (ii). Then \mathfrak{g} and \mathfrak{h} are subalgebras and we have

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

REMARK 2. Suppose $M = K/H$ is a homogeneous space of the type mentioned in the introduction. Then the action of K on K/H is almost effective if and only if ψ is injective.

REMARK 3. Let $M = K/H$ be as above. By the theorem of Borel-Mostow cited before, the Lie algebra homomorphism $\psi \circ \phi = 0$ if \mathfrak{g} is simple. Thus we see that if G is simple and the K -action on K/H is almost effective then G is normal. Thus, Ochiai-Takahashi's theorem follows from Theorem 1.

5. Now we consider a homogeneous space of the type mentioned in the introduction. Let $M = K/H$ be a homogeneous space of a connected compact

Lie group K . We assume that a connected Lie subgroup G acts simply transitively on K/H . Since K/H is compact, G is necessarily compact. The composition mapping

$$F: G \times H \rightarrow K$$

is a diffeomorphism, so that we have

$$\mathfrak{k} = \mathfrak{g} + \mathfrak{h} \quad (\text{direct sum of vector spaces}),$$

for their Lie algebras. Applying Theorem 1, we have a direct sum decomposition

$$\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$$

and homomorphisms $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$ and $\psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$ such that we have

$$\begin{aligned} \mathfrak{g} &= \{(X, \phi(X)) \mid X \in \mathfrak{g}_1\}, \\ \mathfrak{h} &= \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}. \end{aligned}$$

Further, as we see from the proof of Theorem 1, we can assume that

$$c(\mathfrak{g}_1) = \pi(c(\mathfrak{g})),$$

where π denotes the projection of \mathfrak{k} onto its center.

Let G_1 be the connected Lie subgroup of K corresponding to the subalgebra \mathfrak{g}_1 . Since \mathfrak{g}_1 is an ideal of \mathfrak{k} , G_1 is a normal subgroup of K . Next we claim that G_1 is compact. $s(G_1)$ is closed in K since it is semi-simple. Thus it suffices to show that $c(G_1)$ is compact. However, from our construction, $c(\mathfrak{g}_1) = \pi(c(\mathfrak{g}))$. Consider the Lie group homomorphism $\tilde{\pi}: K \rightarrow K/s(K)$. $\tilde{\pi}|_{c(K)}$ is a finite covering map. Thus $c(G_1)$ is closed in $c(K)$ if and only if $\tilde{\pi}(c(G_1))$ is closed. On the other hand, $c(G)$ is compact, and hence $\tilde{\pi}(c(G))$ is compact. $c(\mathfrak{g}_1) = \pi(c(\mathfrak{g}))$ implies that $\tilde{\pi}(c(G_1)) = \tilde{\pi}(c(G))$. Thus, $c(G_1)$ is closed, and hence G_1 is compact. From the property that $\mathfrak{k} = \mathfrak{g}_1 \oplus \mathfrak{h}_1$ and $\mathfrak{h} = \{(\psi(Y), Y) \mid Y \in \mathfrak{h}_1\}$, we have

$$\mathfrak{k} = \mathfrak{g}_1 + \mathfrak{h} \quad (\text{direct sum of vector spaces}).$$

By Lemma 1, the composition mapping

$$G_1 \times H \rightarrow K$$

defines a covering map. Consequently, G_1 acts transitively on the coset space K/H . Furthermore, fix a point p in K/H . Then the mapping

$$G_1 \rightarrow K/H$$

defined by $g \rightarrow g(p)$ is a covering map. Thus, if $G(\cong K/H)$ is simply connected, then G_1 is also simply connected. Thus we have proved the following:

Theorem 2. *Let K be a connected compact Lie group and H a closed subgroup of K . Assume that a connected Lie subgroup G acts simply transitively on the homogeneous space K/H by the left translation. Then there exists a connected closed normal subgroup G_1 of K such that G_1 acts transitively on K/H and G_1 is locally isomorphic with G as Lie groups.*

Theorem 3. *Under the same assumption as in Theorem 2, assume further that G is simply connected. Then there exists a connected closed normal subgroup G_1 of K such that G_1 is isomorphic with G as Lie groups and G_1 acts simply transitively on K/H .*

6. We give here two examples. The first one shows that the conclusion of Ochiai-Takahashi's theorem does not hold any more if G is not simple.

EXAMPLE 1. Let A be a connected compact semi-simple Lie group and α its Lie algebra. We put

$$\begin{aligned} K &= A \times A \times A, \\ G &= \{(x, y, x) \mid x, y \in A\}, \\ H &= \{(e, z, z) \mid z \in A\}. \end{aligned}$$

H is a closed subgroup of K . Consider the homogeneous space K/H . We see easily that G acts simply transitively on K/H . G is compact semi-simple and not simple. Choose a K -invariant Riemannian metric ds^2 on K/H . Since K/H can be identified with G , ds^2 is a left -invariant Riemannian metric on G . From the definition, K is contained in the identity-component of isometries of $(K/H=G, ds^2)$. G is not normal in K , thus G is not normal in the identity-component of isometries.

For this example, an explicit description of Theorem 1 is as follows:

$$\mathfrak{k} = \alpha \oplus \alpha \oplus \alpha.$$

Let
$$\begin{aligned} \mathfrak{g}_1 &= \{(X, Y, 0) \mid X, Y \in \alpha\}, \\ \mathfrak{h}_1 &= \{(0, 0, Z) \mid Z \in \alpha\}. \end{aligned}$$

Define $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{h}_1$ by

$$\phi((X, Y, 0)) = (0, 0, X)$$

and $\psi: \mathfrak{h}_1 \rightarrow \mathfrak{g}_1$ by

$$\psi((0, 0, Z)) = (0, Z, 0).$$

Then we have

$$\begin{aligned} \mathfrak{g} &= \{(X, \phi(X)) \in \mathfrak{g}_1 \oplus \mathfrak{h}_1 \mid X \in \mathfrak{g}_1\}, \\ \mathfrak{g} &= \{(\psi(Y), Y) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1 \mid Y \in \mathfrak{h}_1\}. \end{aligned}$$

The next example shows that the conclusion of Theorem 3 does not hold if G is not simply connected.

EXAMPLE 2. We choose two simply connected compact Lie groups A and B with the following properties:

1. There exists an injective homomorphism j of A into B .
2. The center $Z(A)$ of A is non-trivial and

$$j(Z(A)) \cap Z(B) = \{e\}.$$

For instance, choose positive integers m and n such that $n > m > 2$. Then $A = SU(m)$, $B = SU(n)$ and the canonical injection of $SU(m)$ into $SU(n)$ satisfy the required properties.

Let

$$\begin{aligned} K &= A \times B \times A, \\ G_1 &= A \times B \times \{e\}, \\ G &= \{(a, b, a) \mid a \in A, b \in B\}, \\ H &= \{(e, j(a), a) \mid a \in A\}, \\ \Gamma &= \{(x, e, x) \mid x \in Z(A)\}. \end{aligned}$$

The Lie algebras of A and B are denoted by \mathfrak{a} and \mathfrak{b} respectively. Γ is a finite group contained in the center of K . We consider the quotient group $\bar{K} = K/\Gamma$, and denote by π the canonical projection of K onto \bar{K} . $\bar{H} = \pi(H)$ is a closed subgroup of \bar{K} . Consider \bar{K}/\bar{H} . One can easily show that the group $\bar{G} = \pi(G)$ acts simply transitively on \bar{K}/\bar{H} . We claim that no normal subgroup of \bar{K} acts simply transitively on \bar{K}/\bar{H} . Suppose a normal subgroup G_1' of \bar{K} acts simply transitively on \bar{K}/\bar{H} . Then its Lie algebra \mathfrak{g}_1' satisfies

$$\mathfrak{k} = \mathfrak{g}_1' + \mathfrak{g} \quad (\text{direct sum of vector spaces}),$$

where $\mathfrak{h} = \{(0, j(X), X) \mid X \in \mathfrak{a}\}$. Since \mathfrak{g}_1' is an ideal of \mathfrak{k} , we see $\mathfrak{g}_1' = \mathfrak{g}_1 = \{(X, Y, 0) \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}$. It follows that $\pi(G_1) = G_1'$. However, $\pi(G_1)$ is simply connected because $\pi(G_1) = G_1/(G_1 \cap \Gamma) \cong G_1$. This is a contradiction.

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