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ON THE K-THEORY OF PE_7

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0. Introduction

Let E_7 be the compact, connected, simply-connected, simple Lie group of type E_7 and let PE_7 be the projective group associated with E_7. The purpose of this paper is to determine the algebras $K^*(PE_7)$ and $KO^*(PE_7)$ (Theorems 3.1 and 4.1) where K and KO denote respectively the complex and real K-theories. $K^*(PE_7)$ is already computed in [7, 9]. We study, however, it here again by the similar argument used to calculate $K^*(SO(n))$ and $KO^*(SO(n))$ in [11, 12]. Also in the same fashion we calculate $KO^*(PE_7)$ using certain results obtained in course of computation of $K^*(PE_7)$ as well as the results on $K^*(PE_7)$.

An outline of our method is as follows. Since the centre of E_7 is isomorphic to $\boldsymbol{Z}_2$, we regard E_7 as a $\boldsymbol{Z}_2$-space with the action of the centre as a subgroup. And we show that there exists a $\boldsymbol{Z}_2$-equivariant map $S^{4,0}\to E_7$, which is a homomorphism of groups, where $S^{4,0}$ is the unit quaternions S^3 with antipodal involution. This map yields a homeomorphism

$$S^{4,0}\times_{\boldsymbol{Z}_2} E_7 \approx P^3\times E_7$$

where P^3 is the real projective 3-space. Let $h=K$ or KO and let $h_{\boldsymbol{Z}_2}$ denote the $\boldsymbol{Z}_2$-equivariant h-theory. Then we have a canonical isomorphism $h^*_{\boldsymbol{Z}_2}(E_7)\cong h^*(PE_7)$ and furthermore $h^*_{\boldsymbol{Z}_2}(S^{4,0}\times E_7)\cong h^*(P^3\times E_7)$ induced by the above homeomorphism. Moreover we have a Künneth isomorphism $h^*(P^3\times E_7)\cong h^*(P^3)\otimes_{h^*(+)}h^*(E_7)$ since $h^*(E_7)$ is a free $h^*(+)$-module as mentioned below (here $+$ denotes a point). Making use of these isomorphisms and the Thom isomorphism in equivariant h-theory we carry out the calculation of $h^*(PE_7)$ by reducing to that of $h^*(P^3)\otimes_{h^*(+)}h^*(E_7)$ as in [11, 12]. For the algebras $h^*(P^3)$ and $h^*(E_7)$ we refer to [2, 5, 12] and [8, 13] respectively.

We use also the square formulas of [4, 12] (see (1.10) and (1.11) below). But we leave the 2nd exterior or exteriorlike power of the representation inserted into the functor $\beta(\quad)$ uncalculated since it is complicated.

§1 is devoted to recalling some basic facts needed for our computation and also §2 to collecting the results on the K-groups of E_7 and P^n (for small n needed in the sequel). In §3 we compute $K^*(PE_7)$ and in §§4, 5 we determine

$KO^*(PE_7)$.

Throughout this paper we write $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ respectively for the real, complex and quaternion number-fields, and denote by F any one of these fields.

1. Preliminaries

a) Let $Z(E_7)$ be the centre of E_7. Then $PE_7=E_7/Z(E_7)$, the projective group, and

$$Z(E_7) \cong \boldsymbol{Z}_2 .$$

Let G be the multiplicative group consisting of ± 1. Let $R^{p,q}$ denote $\boldsymbol{R}^{p+q}$ with non-trivial G-action on the first p coordinates and $S^{p,q}$ the unit sphere in $R^{p,q}$. We regard E_7 as a G-space with $Z(E_7)$-action as a subgroup. Moreover, since $Z(Sp(1))\cong\boldsymbol{Z}_2$ we can regard $Sp(1)$ as a G-space in the similar manner. So we identify $Sp(1)$, viewed as a G-space, with $S^{4,0}$. Then we have

Lemma 1.1. *There exists a G-map ι: $S^{4,0}\rightarrow E_7$, which is, in fact, a monomorphism of $Sp(1)$ into E_7 such that $\iota(Z(Sp(1)))=Z(E_7)$.*

Proof. For the proof it is enough to find a subgroup of E_7 isomorphic to $Sp(1)$ containing the generator of $Z(E_7)$. We consider the symmetric pair (E_7, K) such that the corresponding compact symmetric space is EVI in Cartan's notation. Here $K=\mathrm{Spin}(12)\times_C Sp(1)$ where C is a central subgroup of $\mathrm{Spin}(12)\times Sp(1)$ generated by an element of the form $(g, -1)$ with $1\neq g\in Z(\mathrm{Spin}(12))$. It is known that $Z(\mathrm{Spin}(12))=\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$ which is multiplicatively generated by -1 and $e_1 e_2\cdots e_{12}$ in the Clifford algebra (see [10]). So we see that $Z(K)$ is generated also by -1 and $e_1 e_2\cdots e_{12}$ under the identification $\mathrm{Spin}(12)\times_C 1=\mathrm{Spin}(12)$, so that the generator of $Z(E_7)$ is one of -1, $e_1 e_2\cdots e_{12}$ and $-e_1 e_1\cdots e_{12}$.

Now $\mathrm{Spin}(4)$ can be diagonally embedded into $\mathrm{Spin}(12)$ as a subgroup such that $e_1 e_2 e_3 e_4$ corresponds to $e_1 e_2\cdots e_{12}$ and moreover it is known that $\mathrm{Spin}(4)=Sp(1)\times Sp(1)$. Identifying these two groups, it therefore suffices to choose the diagonal subgroup of $Sp(1)\times Sp(1)$, $Sp(1)\times 1$ or $1\times Sp(1)$ as the required subgroup of E_7 according as whether the generator of $Z(E_7)$ is -1, $e_1 e_2\cdots e_{12}$ or $-e_1 e_2\cdots e_{12}$.

REMARK. The above map ι can be no more extended to $S^{5,0}$. Because, if ι can be extended to a G-map ι_5: $S^{5,0}\rightarrow E_7$, then in virtue of $\pi_k(E_7)=0$ for $4\leq k\leq 10$ we see that this can be done inductively to a G-map ι_{12}: $S^{12,0}\rightarrow E_7$ in the natural way. We now show that this leads to a contradiction. Let ξ' be the canonical complex line bundle over PE_7 associated with the principal G-bundle $E_7\rightarrow PE_7$ and let $\tilde{\iota}_{12}$: $P^{11}\rightarrow PE_7$ be the map covered with ι_{12}. Then $\tilde{\iota}_{12}^*(\xi')$ is just the canonical complex line bundle over P^{11}. Now, as stated in §§2, 3 below, the order of $\tilde{\iota}_{12}^*(\xi')-1\in K(P^{11})$ is 32, and also that of ξ'-$1\in K(PE_7)$ is 8, so that $8(\tilde{\iota}_{12}^*(\xi')-1)=0$ (here 1's denote the trivial line bundles). Obviously this is a contradiction.

Let $P^n=S^{n+1,0}/G$, the real projective n-space. Denote the projections $E_7\to PE_7$ and $S^{n+1,0}\to P^n$ by the same letter π. Let ι: $S^{4,0}\to E_7$ be the map as in Lemma 1.1 and $\tilde{\iota}$: $P^3\to PE_7$ the map covered with ι. Then it follows that

$$\pi\iota=\tilde{\iota}\pi .$$

The map $S^{4,0}\times E_7\to P^3\times E_7$ defined by

$$(x,g)\mapsto(\pi(x),\iota(x)g)$$

yields a homeomorphism

$$S^{4,0}\times_G E_7\approx P^3\times E_7 \tag{1.2}$$

where let G act diagonally on the product $S^{4,0}\times E_7$.

We consider the inclusion $S^{p,0}\subset S^{q,0}$ for $p\leq q$. Let $B^{p,q}$ be the unit ball in $R^{p,q}$. By [1], Lemma 3.11 we then have the G-equivariant homeomorphism

$$S^{q,0}-S^{p,0}\approx \operatorname{Int} B^{p,0}\times S^{q-p,0}$$

given by

$$(x_1,\cdots,x_q)\mapsto((x_1,\cdots,x_p),(x_{p+1}\lambda,\cdots,x_q\lambda))$$

where $\lambda(x_{p+1}^2+\cdots+x_q^2)^{1/2}=1$ and Int A denotes the interior of A. Let us put $\Sigma^{p,q}=B^{p,q}/S^{p,q}$ with the collapsed $S^{p,q}$ as base-point. Then the above homeomorphism induces a G-equivariant homeomorphism

$$S^{q,0}/S^{p,0}\approx\Sigma^{p,0}\wedge S_+^{q-p,0} \tag{1.3}$$

where $X_+=X\cup\{+\}$, the disjoint sum of X and a point $+$, with $+$ as base-point.

Let h denote either K or KO and let h_G be the equivariant h-theory associated with G. Using (1.3) we have a cofibre sequence

$$S^{p,0}\times X\xrightarrow{i}S^{q,0}\times X\xrightarrow{j}\Sigma^{p,0}\wedge S_+^{q-p,0}\wedge X_+$$

for a compact G-space X where i is a canonical inclusion and j is the composite of a canonical projection with the homeomorphism of (1.3). Applying h_G to this sequence, we obtain an exact sequence

(1.4)

$$\cdots\to h_G^{i-1}(S^{p,0}\times X)\xrightarrow{\delta'}\tilde{h}_G^i(\Sigma^{p,0}\wedge S_+^{q-p}\wedge X_+)\xrightarrow{j^*}h_G^i(S^{q,0}\times X)\xrightarrow{i^*}h_G^i(S^{p,0}\times X)\to\cdots$$

where δ' is the coboundary homomorphism and moreover we have

$$\delta'(xi^*(y))=\delta'(x)y \tag{1.5}$$

for $x\in h_G^i(S^{p,0}\times X)$ and $y\in h_G^j(S^{q,0}\times X)$.

Here we note that if X is a free G-space then there is a canonical isomorphism $h_G^*(X)\cong h^*(X/G)$.

b) From [3, 11, 12] we recall the Thom isomorphism theorem in G-equivariant h-theory. We view $\Sigma^{n,0}$ as a union $B_+^{n,0}\cup B_-^{n,0}$ of two copies of $B^{n,0}$ intersecting on their boundaries, that is, $B_+^{n,0}\cap B_-^{n,0}=S^{n,0}$ and also view the centre of $B_-^{n,0}$ as base-point of $\Sigma^{n,0}$. Moreover we regard G as a subgroup of Spin(n) consisting of ± 1 and $S^{n,0}$ as a canonical G-invariant subspace of Spin(n) as in [6].

We first consider the complex case. Let Δ_{2n}^+: Spin$(2n)\to GL(2^{n-1}, \boldsymbol{C})$ be the even half-spin representation of Spin$(2n)$. Let us set $L=R^{1,0}\otimes\boldsymbol{C}$ and denote $R^{k,0}\otimes\boldsymbol{C}$ by kL. Define E to be the quotient of the disjoint union $B_+^{2n,0}\times\boldsymbol{C}^{2^{n-1}}\cup B_-^{2n,0}\times 2^{n-1}L$ by the equivalence relation which identifies (x, v) with $(x, \Delta_{2n}^+(x)\,v)$ for $x\in S^{2n,0}$ and $v\in\boldsymbol{C}^{2^{n-1}}$. Then E becomes a G-vector bundle over $\Sigma^{2n,0}$. So we write $\tau_{2n,0}=[E]-2^{n-1}[\underline{L}]\in\tilde{K}_G(\Sigma^{2n,0})$. Here $\underline{L}$ denotes the bundle with fibre L and $[F]$ the isomorphism class of F. Then $x\mapsto\tau_{2n,0}\wedge x$ defines an isomorphism

$$\phi_{2n,0}\colon \tilde{K}_G(X)\xrightarrow{\cong}\tilde{K}_G(\Sigma^{2n,0}\wedge X)$$

where X is a compact G-space with base-point.

Let $\psi\colon h_G\to h$ be the forgetful functor and $j\colon B^{p,q}\to\Sigma^{p,q}$ the projection. Then we have

$$\psi(\tau_{2n,0})=\mu^n \quad \text{(up to sign) and} \quad j^*(\tau_{2n,0})=2^{n-1}(1-L) \tag{1.6}$$

where $\mu\in\tilde{K}(S^2)\cong\boldsymbol{Z}$ denotes the Bott class and let us view $K_G(B^{2n,0})=R(G)$, the complex representation ring of G. (We may assume that $\psi(\tau_{2n,0})=\mu^n$. Because it suffices to replace $\tau_{2n,0}$ by $-L\tau_{2n,0}$, if necessary.) In particular, because Δ_{8n}^+ is real, by using $H=R^{1,0}$ instead of L in the above we can define a similar element $\tau_{8n,0}^R\in\widetilde{KO}_G(\Sigma^{8n,0})$ satisfying

$$\psi(\tau_{8n,0}^R)=\eta_8^n \quad \text{(up to sign) and} \quad j^*(\tau_{8n,0}^R)=2^{4n-1}(1-H) \tag{1.7}$$

where $\eta_8\in\widetilde{KO}(S^8)\cong\boldsymbol{Z}$ denotes the Bott class and let $KO_G(B^{8n,0})=RO(G)$, the real representation ring of G. (We may also assume that $\psi(\tau_{8n,0}^R)=\eta_8^n$ for the reason similar to $\tau_{2n,0}$). And also multiplication by $\tau_{8n,0}^R$ induces an isomorphism

$$\psi_{8n,0}\colon \widetilde{KO}_G(X)\xrightarrow{\cong}\widetilde{KO}_G(\Sigma^{8n,0}\wedge X)$$

where X is a compact G-space with base-point.

To explain one more type of the Thom isomorphism in the real case we make some preparations. Suppose that X is a Real space with trivial Real structure in the sense of Atiyah [1]. By a Real (resp. quaternionic) vector bundle over X we mean a complex vector bundle $E\to X$ together with a conjugate linear involution (resp. anti-involution) $J_E\colon E\to E$ preserving fibre. Let KR (resp. KSp)

denote the K-functor of Real (resp. quaternionic) vector bundles. Since X is a trivial Real space, the assignment $E \mapsto \boldsymbol{C} \otimes E$ defines an isomorphism

$$KO(X) \cong KR(X)$$

where $\boldsymbol{C}$ has the standard Real structure by complex conjugation. So we identify $KO(X)$ with $KR(X)$ via this isomorphism henceforth.

The argument parallel to the above can be done in G-equivariant theories. Namely we have a similar isomorphism

$$KO_G(X) \cong KR_G(X)$$

for a G-space X with trivial Real structure. And so we analogously identify $KO_G(X)$ with $KR_G(X)$ via this isomorphism.

We assume that the quaternionic structure on $\boldsymbol{H}$ is right multiplication by j. Define a bundle isomorphism

$$\alpha \colon S^{0,4} \times \boldsymbol{H} \overset{\cong}{\to} S^{0,4} \times \boldsymbol{H}$$

by $\alpha(v, w) = (v, vw)$ for $v \in S^{0,4}$(=the unit quaternions) and $w \in \boldsymbol{H}$. Let $\underline{\underline{\boldsymbol{H}}}$ denote the trivial bundle over $B^{0,4}$ with fibre $\boldsymbol{H}$. Then the triple $(\underline{\underline{\boldsymbol{H}}}, \underline{\underline{\boldsymbol{H}}}, \alpha)$ defines an element of $\widetilde{KSp}(\Sigma^{0,4})$ which we denote by σ. The construction of this element is similar to that of $\tau_{2n,0}$ as stated below. View $\Sigma^{0,4}$ as a union $B_+^{0,4} \cup B_-^{0,4}$ of two distinguished balls $B^{0,4}$'s intersecting on the boundaries $S^{0,4}$'s, with the centre of $B_-^{0,4}$ as base-point. And define E to be the quotient of the disjoint union $B_+^{0,4} \times \boldsymbol{H} \cup B_-^{0,4} \times \boldsymbol{H}$ by the equivalence relation which identifies (v, w) with (v, vw) for $v \in S^{0,4}$ and $w \in \boldsymbol{H}$. Then σ is given by $\sigma = [E] - [\underline{\underline{\boldsymbol{H}}}] \in \widetilde{KSp}(\Sigma^{0,4})$. It is known that σ has the following properties.

(1.8) $$s(\sigma) = \mu^2, \ \sigma \wedge_C \underline{\underline{\boldsymbol{H}}} = \eta_4 \quad \text{and} \quad \sigma \wedge_C \sigma = \eta_8 .$$

Here s denote the complexification functor $KSp \to K$, $\wedge_C$ the smash product induced by the exterior tensor product over $\boldsymbol{C}$ and η_4 a generator of $\widetilde{KO}(S^4) \cong \boldsymbol{Z}$. In general, the exterior tensor product $E \hat{\otimes}_{\boldsymbol{C}} F$ of two quaternionic vector bundles $E \to X$ and $F \to Y$ becomes a Real vector bundle over $X \times Y$ with Real structure $J_E \hat{\otimes}_{\boldsymbol{C}} J_F$. Hence we see that the functor $\wedge_C \colon \widetilde{KSp}(X) \otimes \widetilde{KSp}(Y) \to \widetilde{KO}(X \wedge Y)$ can be defined.

Now we are ready to state another Thom isomorphism in the real case. Regard $S^{4,0}$ as ahe unit quaternions and define a G-equivariant bundle isomorphism

$$\alpha_G \colon S^{4,0} \times \boldsymbol{H} \overset{\cong}{\to} S^{4,0} \times \boldsymbol{H} \otimes H$$

by $\alpha_G(v, w) = (v, vw)$ for $v \in S^{4,0}$ and $w \in \boldsymbol{H}$. Then the triple $(\underline{\underline{\boldsymbol{H}}}, \underline{\underline{\boldsymbol{H}}} \otimes H, \alpha_G)$

defines an element of $\widetilde{KSp}_G(\Sigma^{4,0})$, denoted by σ_G, analogously to σ where $\underline{\underline{H}}$ is the trivial vector bundle over $B^{4,0}$ with fibre $\boldsymbol{H}$. Let

$$\tau^R_{4,4} = \sigma_G \wedge_C \sigma \in \widetilde{KO}_G(\Sigma^{4,4}) .$$

Then $\tau^R_{4,4}$ satisfies

$$(1.9) \qquad \psi(\tau^R_{4,4}) = \eta_8,\ c(\tau^R_{4,4}) = \tau_{4,0} \wedge \mu^2 \quad \text{and} \quad i^*(\tau_{4,4}) = (1-H)\,\eta_4$$

where $i\colon \Sigma^{0,4} \subset \Sigma^{4,4}$ is the inclusion, $c\colon KO_G \rightarrow K_G$ is the complexification functor and let $KO_G(\Sigma^{0,4}) = RO(G) \cdot \eta_4$. And moreover it is known that multiplication by $\tau_{4,4}$ induces an isomorphism

$$\psi_{4,4}\colon \widetilde{KO}_G(X) \overset{\cong}{\rightarrow} \widetilde{KO}_G(\Sigma^{4,4} \wedge X)$$

for a compact G-space with base-point.

c) We state here the construction of the elements of dgeree -1 and -5 and also recall the square formula for these elements. Let X be a compact space with $x_0 \in X$ as base-point, and let $f\colon X \rightarrow GL(n, F)$ be a base-point preserving map where we regard the unit matrix I_n as a base-point of $GL(n, F)$. Let $\alpha\colon S^{0,1} \times X \times F^n \rightarrow S^{0,1} \times X \times F^n$ be a bundle isomorphism given by $\alpha(1, x, v) = (1, x, v)$ and $\alpha(-1, x, v) = (-1, x, f(x)\, v)$ for $x \in X$ and $v \in F^n$. Then the triple $(\underline{\underline{F}}^n, \underline{\underline{F}}^n, \alpha)$ defines an element of $\tilde{h}^{-1}(X)$ in the way mentioned in b) above where $\underline{\underline{F}}^n$ is the trivial bundle over $B^{0,1} \times X$ with fibre F^n and $h = KO$, K or KSp according as $F = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. We denote this element by

$$\beta(f) \in \tilde{h}^{-1}(X) .$$

The construction of this element is similar to that of σ. But we explain it simply again for use in the following. Decompose $\Sigma^{0,1} \wedge X$ as a union $C_+X \cup C_-X$ of two cones $C_+X = [0, 1] \times X/1 \times X \cup [0, 1] \times x_0$ and $C_-X = [-1, 0] \times X/(-1) \times X \cup [-1, 0] \times x_0$, and define E to be the quotient of the disjoint union $C_+X \times F^n \cup C_-X \times F^n$ by the equivalence relation which identifies $(0, x, v)$ with $(0, x, f(x)\, v)$ for $x \in X$ and $v \in F^n$. Then $\beta(f) = [E] - [\underline{\underline{F}}^n] \in \tilde{h}(\Sigma^{0,1} \wedge X) = \tilde{h}^{-1}(X)$.

Also we need a G-equivariant version of this construction.

Now we mention the square formula. In case of $F = \boldsymbol{C}$, that is, $\beta(f) \in \tilde{K}^{-1}(X)$, it is well-known that $\beta(f)^2 = 0$. In case of $F = \boldsymbol{R}$, that is, $\beta(f) \in \widetilde{KO}^{-1}(X)$, by [4] we have

$$(1.10) \qquad \beta(f)^2 = \eta_1(\beta(\lambda^2 f) + n\beta(f))$$

where $\lambda^2 f\colon X \rightarrow GL(\binom{n}{2}, \boldsymbol{R})$ is the map given by $\lambda^2 f(x) = f(x) \wedge f(x)\colon \boldsymbol{R}^n \wedge \boldsymbol{R}^n \rightarrow \boldsymbol{R}^n \wedge \boldsymbol{R}^n$ for $x \in X$ and η_1 denotes a generator of $\widetilde{KO}(S^1) \cong \boldsymbol{Z}_2$. In case of $F = \boldsymbol{H}$, that is, $\beta(f) \in \widetilde{KSp}^{-1}(X)$, it follows that $\sigma \wedge_C \beta(f) \in \widetilde{KR}^{-5}(X) = \widetilde{KO}^{-5}(X)$. So we write for this element

$$\bar{\beta}(f)=\sigma\wedge_C\beta(f)\in\widetilde{KO}^{-5}(X).$$

Then by [12] we have

$$\bar{\beta}(f)^2=\eta_1(\beta(\lambda_C^2 f)+n\beta(1\wedge_C f)) \tag{1.11}$$

where $\lambda_C^2 f\colon X\to GL(\binom{2n}{2}, \boldsymbol{C})$ and $1\wedge_C f\colon X\to GL(2n, \boldsymbol{C})$ are the maps respectively given by Real maps $(\lambda_C^2 f)(x)=f(x)\wedge_C f(x)\colon \boldsymbol{H}^n\wedge_C\boldsymbol{H}^n\to\boldsymbol{H}^n\wedge_C\boldsymbol{H}^n$ and $(1\wedge_C f)(x)=1\wedge_C f(x)\colon \boldsymbol{H}\wedge_C\boldsymbol{H}^n\to\boldsymbol{H}\wedge_C\boldsymbol{H}^n$ for $x\in X$.

Finally we make a remark about K- and KO-theories. As usual $K^*(X)$ is a $\boldsymbol{Z}_2$-graded algebra with the coefficients $K^0(+)\cong\boldsymbol{Z}$ and $K^{-1}(+)=0$. However, when we deal with $KO^*(PE_7)$, we regard for convenience $K^*(X)$ as a Z_8-graded cohomology theory with the coefficient ring

$$K^*(+)=\boldsymbol{Z}[\mu]/(\mu^4-1)$$

where μ is as above. Of course $KO^*(X)$ is a Z_8-graded algebra with the coefficient ring

$$KO^*(+)=\boldsymbol{Z}[\eta_1,\eta_4]/(2\eta_1,\eta_1^3,\eta_1\eta_4,\eta_4^2-4)$$

where η_1, η_4 are also as above.

2. The K-groups of E_7 and P^n

By [8] we have

Proposition 2.1. *$K^*(E_7)=\Lambda_{\boldsymbol{Z}}(\beta(\rho_1),\cdots,\beta(\rho_7))$ as an algebra where $\rho_1,\cdots,\rho_7$ are the fundamental irreducible representations $\rho_i\colon E_7\to GL(d_i,\boldsymbol{C})$ of E_7.*

According to [14], using the same notation, we can consider that ρ_2, ρ_4, ρ_5, ρ_6 are real, that is, continuous homomorphisms such that

$$\rho_i\colon E_7\to GL(d_i,\boldsymbol{R})\quad\text{for}\quad i=2,4,5,6$$

and ρ_1, ρ_3, ρ_7 are quaternionic, that is, continuous homomorphisms such that

$$\rho_i\colon E_7\to GL(d_i/2,\boldsymbol{H})\quad\text{for}\quad i=1,3,7$$

where $d_1=56$, $d_2=1539$, $d_3=27664$, $d_4=365750$, $d_5=8645$, $d_6=133$ and $d_7=912$. Therefore, by [13] ,together with (1.10) and (1.11), we have

Proposition 2.2.

$$KO^*(E_7)=\Lambda_{KO^*(+)}(\beta(\rho_i)\ (i=2,4,5,6),\ \bar{\beta}(\rho_j)\ (j=1,3,7))$$

as a $KO^*(+)$-module, where the generators are subject to the relations

$$\beta(\rho_i)^2 = \eta_1(\beta(\lambda^2 \rho_i)+d_i \beta(\rho_i)) \quad \textit{for} \quad i = 2, 4, 5, 6 \quad \textit{and}$$
$$\bar{\beta}(\rho_i)^2 = \eta_1 \beta(\lambda_C^2 \rho_i) \quad \textit{for} \quad i = 1, 3, 7 .$$

REMARK. It is immediate by definition that

$$\eta_4 \bar{\beta}(\rho_i) = \beta(r(\rho_i)) \quad \text{for} \quad i = 1, 3, 7$$

where r is the realification functor.

By γ'_n we denote the real line bundle over P^n

$$S^{n+1,0} \times_G H \to P^n$$

and set $\gamma_n = \gamma'_n - 1 \in \widetilde{KO}(P^n)$ where 1 denotes the trivial real line bundle over P^n. Also if there is no confusion, then by the same symbol γ'_n and γ_n respectively we denote $c(\gamma'_n)$ and $c(\gamma_n)$ where c denotes the complexification functor $KO \to K$.

Denote again by Δ^+_{2n} the restriction of Δ^+_{2n}: $\mathrm{Spin}(2n) \to GL(2^{n-1}, F)$ to $S^{2n,0}$ where $F=\boldsymbol{C}$ and F may be taken to be $\boldsymbol{R}$ if $n \equiv 0 \bmod 4$. Let f: $P^{2n-1} \to GL(2^{n-1}, F)$ be a map given by $f(\pi(x)) = \Delta^+_{2n}(x)^2$ for $x \in S^{2n,0}$. Then $\beta(f) \in \tilde{K}^{-1}(P^{2n-1})$ or $\widetilde{KO}^{-1}(P^{2n-1})$ according as $F=\boldsymbol{C}$ or $\boldsymbol{R}$. Write $\nu_{2n-1} = \beta(f)$. Then we have

Proposition 2.3. [2]. 1) $\tilde{K}^0(P^{2n-1}) = \boldsymbol{Z}_{2^{n-1}} \cdot \gamma_{2n-1}$, $\tilde{K}^{-1}(P^{2n-1}) = \boldsymbol{Z} \cdot \nu_{2n-1}$ *where the generators are subject to the relations* $\gamma^2_{2n-1} = -2\gamma_{2n-1}$, $\nu^2_{2n-1} = \gamma_{2n-1} \nu_{2n-1} = 0$.

2) $\tilde{K}^0(P^{2n}) = \boldsymbol{Z}_{2^n} \cdot \gamma_{2n}$, $\tilde{K}^{-1}(P^{2n}) = 0$ *where* γ_{2n} *is subject to the relation* $\gamma^2_{2n} = -2\gamma_{2n}$.

From [12, 11] we quote data on the algebra structure of $KO^*(P^n)$ for $n=3, 4, 8$ needed in §§4, 5. To describe the results we recall some generators. The bundle automorphism of $S^{4,0} \times \boldsymbol{H}$, given by $(u, v) \mapsto (u, u^2 v)$ for $u \in S^{4,0}$ (=the quaternions) and $v \in \boldsymbol{H}$, defines an element $\bar{\nu}'_3 \in \widetilde{KSp}^{-1}(P^3)$. Let us put $\bar{\nu}_3 = \sigma \wedge_C \bar{\nu}'_3 \in \widetilde{KO}^{-5}(P^3)$. Moreover we denote by μ_3 an element of $KO^{-6}(P^3)$ satisfying the formula $c(\mu_3) = \mu^3 \gamma_3$, by μ_4 an element of $\widetilde{KO}^{-6}(P^4)$ satisfying the formula $c(\mu_4) = 2\mu^3 \gamma_3$ and by $\bar{\nu}_8$ an element of $\widetilde{KO}^{-2}(P^8)$ satisfying $i^*(\bar{\nu}_8) = \eta_1 \nu_7$ where i: $P^7 \subset P^8$ is the inclusion. Then we have

Proposition 2.4. 1) $\widetilde{KO}^0(P^3) = \boldsymbol{Z}_4 \cdot \gamma_3$,

$$\widetilde{KO}^{-1}(P^3) = \boldsymbol{Z} \cdot \eta_4 \bar{\nu}_3 \oplus \boldsymbol{Z}_2 \cdot \eta_1 \gamma_3 ,$$
$$\widetilde{KO}^{-2}(P^3) = \boldsymbol{Z}_2 \cdot \eta_1^2 \gamma_3 ,$$
$$\widetilde{KO}^{-3}(P^3) = \widetilde{KO}^{-4}(P^3) = 0 ,$$
$$\widetilde{KO}^{-5}(P^3) = \boldsymbol{Z} \cdot \bar{\nu}_3 ,$$
$$\widetilde{KO}^{-6}(P^3) = \boldsymbol{Z}_2 \cdot \eta_1 \bar{\nu}_3 \oplus \boldsymbol{Z}_2 \cdot \mu_3 \quad \textit{and}$$
$$\widetilde{KO}^{-7}(P^3) = \boldsymbol{Z}_2 \cdot \eta_1^2 \bar{\nu}_3 \oplus \boldsymbol{Z}_2 \cdot \eta_1 \mu_3$$

where the generators are subject to the relations

$\gamma_3^2=-2\gamma_3,\ \bar{\nu}_3^2=\eta_4\,\mu_3=\gamma_3\,\bar{\nu}_3=0,\ \eta_1\,\bar{\nu}_3=\gamma_3\,\mu_3$ *and* $\eta_1^2\,\mu_3=2\gamma_3$.

2) $\widetilde{KO}^0(P^4)=\boldsymbol{Z}_8\cdot\gamma_4$,
$\widetilde{KO}^{-1}(P^4)=\boldsymbol{Z}_2\cdot\eta_1\,\gamma_4$,
$\widetilde{KO}^{-2}(P^4)=\boldsymbol{Z}_2\cdot\eta_1^2\,\gamma_4$,
$\widetilde{KO}^{-3}(P^4)=0$,
$\widetilde{KO}^{-4}(P^4)=\boldsymbol{Z}_2\cdot\eta_4\,\gamma_4$,
$\widetilde{KO}^{-5}(P^4)=0$,
$\widetilde{KO}^{-6}(P^4)=\boldsymbol{Z}_2\cdot\mu_4$ *and*
$\widetilde{KO}^{-7}(P^4)=\boldsymbol{Z}_2\cdot\eta_1\,\mu_4$

where the generators are subject to the relations

$$\gamma_4^2=-2\gamma_4,\ \mu_4^2=\eta_4\,\mu_4=\gamma_4\,\mu_4=0 \quad \textit{and} \quad \eta_1^2\,\mu_4=4\gamma_4\,.$$

3) $\widetilde{KO}^0(P^8)=\boldsymbol{Z}_{16}\cdot\gamma_8$,
$\widetilde{KO}^{-1}(P^8)=\boldsymbol{Z}_2\cdot\eta_1\,\gamma_8$,
$\widetilde{KO}^{-2}(P^8)=\boldsymbol{Z}_2\cdot\bar{\nu}_8\oplus\boldsymbol{Z}_2\cdot\eta_1^2\,\gamma_8$,
$\widetilde{KO}^{-3}(P^8)=\boldsymbol{Z}_2\cdot\eta_1\,\bar{\nu}_8$,
$\widetilde{KO}^{-4}(P^8)=\boldsymbol{Z}_{16}\cdot\eta_4\,\gamma_8$ *and*
$\widetilde{KO}^{-5}(P^8)=\widetilde{KO}^{-6}(P^8)=\widetilde{KO}^{-7}(P^8)=0$

where the generators are subject to the relations

$$\gamma_8^2=-2\gamma_8,\ \gamma_8\,\bar{\nu}_8=\bar{\nu}_8^2=0 \quad \textit{and} \quad \eta_1^2\,\bar{\nu}_8=8\eta_4\,\gamma_4\,.$$

In the following sections we use freely the results of Propositions 2.1-2.4 without making reference to these propositions.

To state the main theorems we now provide certain of the generators of $K^*(PE_7)$ and $KO^*(PE_7)$. Let ξ' be the canonical real line bundle over PE_7 associated with the principal G-bundle $E_7\to PE_7$ and let us put $\xi=\xi'-1\in\widetilde{KO}(PE_7)$. We denote $c(\xi')$ and $c(\xi)$ respectively also by the same letters ξ' and ξ unless there is confusion.

Since ρ_1, ρ_3 and ρ_7 are not trivial on $Z(E_7)$ and the greatest common measure of d_1, d_3 and d_7 is 8, we get

$$(2.5) \qquad 8c(\xi)=0\,, \quad \text{so that} \quad 16\xi=8\eta_4\,\xi=0\,.$$

As the map f in §1, c) we take the maps $PE_7\to GL(d_i,\boldsymbol{C})$ $(i=1,3,7)$, $PE_7\to GL(114d_1,\boldsymbol{C})$, $PE_7\to GL(494d_1,\boldsymbol{C})$ and $PE_7\to GL(57d_3,\boldsymbol{C})$ given by $\pi(g)\mapsto\rho_i(g)^2$

($i=1, 3, 7$), $\pi(g)\mapsto 114\rho_1(g)\,(7\rho_7(g))^{-1}$, $\pi(g)\mapsto 494\rho_1(g)\,\rho_3(g)^{-1}$ and $\pi(g)\mapsto 57\rho_3(g)$ $(1729\rho_7(g))^{-1}$ for $g\in E_7$ respectively where $k\rho_i$ denotes the k-times direct sum of ρ_i. Then for $\beta(f)$'s we write

$$\beta(\rho_1^2)\ (i=1,3,7)\,,\quad \beta(114\rho_1-7\rho_7)\,,\quad \beta(494\rho_1-\rho_3)\,,$$
$$\beta(57\rho_3-1729\rho_7)\in\tilde{K}^{-1}(PE_7)$$

in the order defined above. Using ρ_i ($i=1, 3, 7$) viwed as a quaternionic representation we get

$$\beta'(\rho_1^2)\ (i=1,3,7)\,,\quad \beta'(114\rho_1-7\rho_7)\,,\quad \beta'(494\rho_1-\rho_3)\,,$$
$$\beta'(57\rho_3-1729\rho_7)\in\widetilde{KSp}^{-1}(PE_7)$$

quite similarly and so multiplying them by σ (using multiplication $\wedge_C$) we have

$$\bar{\beta}(\rho_i^2)\ (i=1,3,7)\,,\quad \bar{\beta}(114\rho_1-7\rho_7)\,,\quad \bar{\beta}(494\rho_1-\rho_3)\,,$$
$$\bar{\beta}(57\rho_3-1729\rho_7)\in\widetilde{KO}^{-5}(PE_7)\,.$$

REMARK. By definition it follows that

$$\beta(57\rho_3-1729\rho_7)=247\beta(114\rho_1-7\rho_7)-57\beta(494\rho_1-\rho_3)\quad\text{and}$$
$$\bar{\beta}(57\rho_3-1729\rho_7)=247\bar{\beta}(114\rho_1-7\rho_7)-57\bar{\beta}(494\rho_1-\rho_3)\,.$$

Because ρ_2, ρ_4, ρ_5 and ρ_6 are trivial on $Z(E_7)$, they factor through PE_7 and so they can be regarded as complex and real representations of PE_7. Hence they defines

$$\beta(\rho_i)\in\tilde{K}^{-1}(PE_7)\quad\text{and}\quad \beta(\rho_i)\in\widetilde{KO}^{-1}(PE_7)$$

for $i=2, 4, 5, 6$.

3. The complex K-groups of PE_7

One of the main theorems is the following.

Theorem 3.1 [7, 9]. *With the notation as in* §2

$$K^*(PE_7)=\Lambda_{\boldsymbol{Z}}(\beta(\rho_i)\ (i=2,4,5,6),\ \beta(114\rho_1-7\rho_7),\ \beta(494\rho_1-\rho_3),\ \tau)$$
$$\otimes(\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}_8\cdot\xi)$$

as an algebra where ξ and τ are subject to the relations

$$\xi^2=-2\xi\quad\textit{and}\quad \xi\tau=0$$

and τ is defined in (3.19) *below. Here* (*and henceforth*) *we write xy for $x\times y$.*

The proof is divided into some steps. By (1.4) in case of $h=K$, $p=q-4=4$ and $X=E_7$ we have an exact sequence with Thom isomorphism

$$\cdots \to K_G^*(S^{4,0}\times E_7) \xrightarrow{\delta'} \tilde{K}_G^*(\Sigma^{4,0}\wedge S_+^{4,0}\wedge E_{7+}) \xrightarrow{j^*} K_G^*(S^{8,0}\times E_7) \xrightarrow{i^*} K_G^*(S^{4,0}\times E_7) \to \cdots$$
$$\cong \uparrow \phi_{4,0}$$
$$K_G^*(S^{4,0}\times E_7)$$

Applying (1.2) to this we get an exact sequence

$$(3.2) \qquad \cdots \to K^*(P^3\times E_7) \xrightarrow{\delta} K^*(P^3\times E_7) \xrightarrow{J} K_G^*(S^{8,0}\times E_7) \xrightarrow{I} K^*(P^3\times E_7) \to \cdots$$

and also by (1.5) we get

$$(3.3) \qquad \delta(xI(y)) = \delta(x)y$$

for $x\in K^*(P^3\times E_7)$ and $y\in K_G^*(S^{8,0}\times E_7)$.

Using (3.2), (3.3) we first deduce the structure of $K_G^*(S^{8,0}\times E_7)$ from Propositions 2.1 and 2.3. For this we determine the coboundary homomorphism δ of (3.2) on each additive generator of $K^*(P^3\times E_7)$. Here we notice that $K^*(P^3\times E_7)$ is isomorphic to $K^*(P^3)\otimes K^*(E_7)$ since $K^*(E_7)$ is torsion-free.

Observe the exact sequence for the pair $(B^{4,0}, S^{4,0})$

$$\begin{array}{ccccccc} \cdots \to & K_G^{-1}(S^{4,0}) & \xrightarrow{\delta'} & \tilde{K}_G(\Sigma^{4,0}) & \xrightarrow{j^*} & K_G(B^{4,0}) & \xrightarrow{i^*} \cdots \\ & \| & & \| & & \| & \\ & K^{-1}(P^3) & & R(G)\cdot\tau_{4,0} & & R(G) & \end{array}$$

where i: $S^{4,0}\subset B^{4,0}$ and j: $B^{4,0}\to\Sigma^{4,0}$ are the obvious maps and δ' the coboundary homomorphism. By the exactness we then see that $\delta'(\nu_3)=k(1+L)\tau_{4,0}$ for some $k\in \boldsymbol{Z}$. So by forgetting the G-action we get $k=1$. Hence we have

$$(3.4) \qquad \delta(\nu_3\times 1) = (\gamma_3+2)\times 1\,.$$

Let p_1; $S^{8,0}\times E_7\to S^{8,0}$ be the first projection and p_1^*: $K^*(P^7)\cong K_G^*(S^{8,0})\to K_G^*(S^{8,0}\times E_7)$ the homomorphism induced by p_1. Write

$$\xi = p_1^*(\gamma_7)\in K_G(S^{8,0}\times E_7)\,,$$

then clearly by definition

$$(3.5) \qquad I(\xi) = \gamma_3\times 1\,, \quad \text{so that} \quad \delta(\gamma_3\times 1) = 0\,.$$

Let p_2: $S^{8,0}\times E_7\to E_7$ be the second projection and p_2^*: $K^*(PE_7)\cong K_G^*(E_7)\to K_G^*(S^{8,0}\times E_7)$ the induced homomorphism. We write

$$\tilde{\beta}(\rho_i) = p_2^*(\beta(\rho_i))\in K_G^{-1}(S^{8,0}\times E_7) \quad \text{for} \quad i = 2, 4, 5, 6\,.$$

Let

$$f_i\colon P^3\times E_7 \to GL(d_i, \boldsymbol{C}) \quad (i = 2, 4, 5, 6)$$

be the map given by $f_i(\pi(x), g)=\rho_i(\iota(x)^{-1}g)$ for $x\in S^{4,0}$ and $g\in E_7$ where ι is as in Lemma 1.1. Then by definition it follows immediately that

$$I(\tilde{\beta}(\rho_i))=\beta(f_i)$$

and also

$$\beta(f_i)=1\times\beta(\rho_i)-\beta(\rho_i\tilde{\iota})\times 1$$

where $\rho_i\tilde{\iota}$ is the composite $P^3\xrightarrow{\tilde{\iota}}PE_7\xrightarrow{\rho_i}GL(d_i, \boldsymbol{C})$.

To determine $\beta(\rho_i\tilde{\iota})$ we view ρ_i as a real representation and observe $\beta(\rho_i\tilde{\iota})$ in $\widetilde{KO}^{-1}(P^3)$. According to Lemma 1.1, ι is a homomorphism and $\rho_i\iota(-1)=I_{d_i}$. Therefore $\rho_i\tilde{\iota}$ can be regarded as a real representation of $SO(3)$. Furthermore the restriction $RO(SO(5))\to RO(SO(3))$ is surjective and $P^3\approx SO(3)$. It therefore follows that $\beta(\rho_i\tilde{\iota})$ belongs to the image of the composite $\widetilde{KO}^{-1}(SO(5))\to\widetilde{KO}^{-1}(P^4)\to\widetilde{KO}^{-1}(P^3)$ of the homomorphisms induced by two canonical inclusions $P^3\subset P^4\subset SO(5)$. So we get owing to $\widetilde{KO}^{-1}(P^4)=\boldsymbol{Z}_2\cdot\eta_1\gamma_4$

$$\beta(\rho_i\tilde{\iota})=\eta_1\gamma_3 \text{ or } 0 \text{ in } \widetilde{KO}^{-1}(P^3) \quad\text{for}\quad i=2, 4, 5, 6 \tag{3.6}$$

for ρ_i viewd as a real representation, so that

$$\beta(\rho_i\tilde{\iota})=0 \text{ in } \tilde{K}^{-1}(P^3)$$

for ρ_i viewed as a complex representation. Consequenctly we get

$$I(\tilde{\beta}(\rho_i))=1\times\beta(\rho_i), \text{ so that } \delta(1\times\beta(\rho_i))=0 \quad\text{for}\quad i=2, 4, 5, 6\,. \tag{3.7}$$

Since $d_i(=\dim_{\boldsymbol{C}}\rho_i)$ $(i=1, 3, 7)$ are divisible by 8, we write

$$d_i=8l_i \quad\text{for}\quad i=1, 3, 7$$

(here note that l_1 is odd and l_3, l_7 even).

Define a map

$$f_i\colon S^{8,0}\times E_7\to GL(8l_i, \boldsymbol{C}) \quad\text{for}\quad i=1, 3, 7$$

by

$$f_i(x, g)=(\Delta_8^+(x)\otimes I_{l_i})\,\rho_i(g) \quad\text{for}\quad x\in S^{8,0}, g\in E_7$$

where we consider $S^{8,0}\subset\mathrm{Spin}(8)$ as stated in §1, a) and I_l is the unit matrix of degree l. Then if follows that $f_i(-x, -g)=f_i(x, g)$ because of $\rho_i(-1)=-I_{8l_i}$ where $-1\in Z(E_7)$. This implies that f_i is a G-equivariant map. Analogously to $\beta(f)$ as in §1, c) this map therefore defines an element $\beta(f_i)$ of $K_G^{-1}(S^{8,0}\times E_7)$, which we denote by

$$\beta_{\rho_i}\in K_G^{-1}(S^{8,0}\times E_7) \quad\text{for}\quad i=1, 3, 7\,.$$

We now consider $I(\beta_{\rho_i})$. By definition we see that $I(\beta_{\rho_i})$ can be written

$$I(\beta_{\rho_i}) = 1\times\beta(\rho_i)+\beta(g_i)\times 1 \quad \text{for} \quad i = 1, 3, 7$$

where g_i: $P^3 \to GL(8l_i, \boldsymbol{C})$ is the map given by

$$g_i(\pi(x)) = (\Delta_8^+(x)\otimes I_{l_i})(\rho_i\,\tilde{\iota})(\pi(x))^{-1}$$

for $x\in S^{4,0}(\subset S^{8,0})$. So it suffices to determine $\beta(g_i)$. For this consider the exact sequence stated preceding (3.4).

Since Im $\delta'=\boldsymbol{Z}\cdot(1+L)\,\tau_{4,0}$, we can write

$$\delta'(\beta(g_i)) = n_i(1+L)\,\tau_{4,0} \tag{i}$$

for some $n_i\in\boldsymbol{Z}$. We now determine n_i. To begin with it follows that

$$\pi^*(\beta(g_i)) = -\beta(\rho_i\,\iota) \tag{ii}$$

Because ρ_i in this case is quaternionic, we may assume that $\rho_i\,\iota$ factors through $T_{\rho_i}\subset GL(4l_i, \boldsymbol{H})$ as follows:

$$\rho_i\,\iota\colon Sp(1)\,(=S^{4,0}) \to T_{\rho_i}\subset GL(4l_i, \boldsymbol{H})\subset GL(8l_i, \boldsymbol{C})$$

where T_{ρ_ι} is the sympletic maximal torus of $GL(4l_i, \boldsymbol{H})$. Since $\rho_i\,\iota(-1)=-I_{4\iota_i}$ by Lemma 1.1 we can put

$$\rho_i\,\iota(x) = (x^{s(1)}, \cdots, x^{s(4l_i)}) \quad \text{for} \quad x\in Sp(1)$$

where ths $s(i)$ are odd integers. Hence we see that $\beta(\rho_i\,\iota)$ can be represented

$$\beta(\rho_i\,\iota) = (2k_i+4l_i)\,\mu^2 \quad \text{for some} \quad k_i\in\boldsymbol{Z}$$

under the identification $\tilde{K}^{-1}(S^3)=\tilde{K}(S^4)$. Therefore from (i) and (ii) we get $n_i=-(k_i+2l_i)$ since $\psi(\tau_{4,0})=\mu^2$ by (1.6). Consequently we have

$$\delta'(\beta(g_i)) = -(k_i+2l_i)\,(1+L)\,\tau_{4,0}\,.$$

On the other hand

$$\delta'(\nu_3) = (1+L)\,\tau_{4,0}$$

as was verified above. But, since δ' is injective, we get

$$\beta(g_i) = -(k_i+2l_i)\,\nu_3$$

by comparing with these two equalities. Thus we see that

$$I(\beta_{\rho_i}) = 1\times\beta(\rho_i)-(k_i+2l_i)\,\nu_3\times 1 \quad \text{for} \quad i = 1, 3, 7 \tag{3.8}$$

and so, together with (3.4), we obtain

$$\delta(1\times\beta(\rho_i)) = (k_i+2l_i)\,(\gamma_3+2)\times 1\,.$$

Because of $2\gamma_3=0$, from this it follows that

$$\delta(1\times\beta(\rho_i)) = (k_i\gamma_3+2k_i+4l_i)\times 1 \quad \text{for} \quad i=1,3,7\,. \tag{3.9}$$

Using (3.3) it follows from (3.4) that

$$\delta((\nu_3\times 1)\,I(\beta_{\rho_i})) = (\gamma_3+2)\,\beta_{\rho_i}\,.$$

Since $\nu_3^2=0$, by (3.8) and this we get

$$\delta(\nu_3\times\beta(\rho_i)) = (\gamma_3+2)\,\beta_{\rho_i}\,.$$

We need to analyze $(\gamma_3+2)\,\beta_{\rho_i}$ on the right-hand side. But from (3.8) it is immediate that

$$(\gamma_3+2)\,\beta_{\rho_i} = (\gamma_3+2)\times\beta(\rho_i)-(2k_i+4l_i)\,\nu_3\times 1$$

because of $\gamma_3\nu_3=0$. Therefore we have

$$\delta(\nu_3\times\beta(\rho_i)) = (\gamma_3+2)\times\beta(\rho_i)-(2k_i+4l_i)\,\nu_3\times 1 \quad \text{for} \quad i=1,3,7\,. \tag{3.10}$$

Furthermore, (3.3) combined with (3.4), (3.5) and (3.7)—(3.10) yields the following formulas.

(3,11) $\delta(1\times x) = \delta(\gamma_3\times x) = 0\,,\quad \delta(\nu_3\times x) = (\gamma_3+2)\times x\,,$

$\delta(1\times\beta(\rho_i)\,\beta(\rho_j)\,x) = (k_i\gamma_3+2k_i+4l_i)\times\beta(\rho_j)\,x-(k_j\gamma_3+2k_j+4l_j)\times\beta(\rho_i)\,x$ $(i,j=1,3,7)\,,$

$\delta(\gamma_3\times\beta(\rho_i)\,\beta(\rho_j)\,x) = 0 \quad (i,j=1,3,7)\,,$

$\delta(\nu_3\times\beta(\rho_i)\,\beta(\rho_j)\,x) = -(2k_i+4l_i)\,\nu_3\times\beta(\rho_j)\,x+(2k_j+4l_j)\,\nu_3\times\beta(\rho_i)\,x$ $+(\gamma_3+2)\times\beta(\rho_i)\,\beta(\rho_j)\,x \quad (i,j=1,3,7)\,,$

$\delta(1\times\beta(\rho_1)\,\beta(\rho_3)\,\beta(\rho_7)\,x) = (k_1\gamma_3+2k_1+4l_1)\times\beta(\rho_3)\,\beta(\rho_7)\,x$ $+(k_3\gamma_3+2k_3+4l_3)\times\beta(\rho_7)\,\beta(\rho_1)\,x+(k_7\gamma_3+2k_7+4l_7)\times\beta(\rho_1)\,\beta(\rho_3)\,x,$

$\delta(\gamma_3\times\beta(\rho_1)\,\beta(\rho_3)\,\beta(\rho_7)\,x) = 0$ and

$\delta(\nu_3\times\beta(\rho_1)\,\beta(\rho_3)\,\beta(\rho_7)\,x) = -\,(2k_7+4l_7)\,\nu_3\times\beta(\rho_1)\,\beta(\rho_3)\,x-(2k_1+4l_1)\,\nu_3$ $\times\beta(\rho_3)\,\beta(\rho_7)\,x-(2k_3+4l_3)\,\nu_3\times\beta(\rho_7)\,\beta(\rho_1)\,x+(\gamma_3+2)\times\beta(\rho_1)\,\beta(\rho_3)\,\beta(\rho_7)\,x$

for $x=\beta(\rho_{i_1})\cdots\beta(\rho_{i_s})$ $(i_1,\cdots,i_s=2,4,5,6)$.

The behaivior of δ can be completely described by using (3.4), (3.5) and (3.7)—(3.11). And so from these formulas and Propositions 2.1, 2.3 we get

Lemma 3.12.

$\operatorname{Ker}\delta = \boldsymbol{Z}_2\cdot(\gamma_3\times 1)\otimes\Lambda_{\boldsymbol{Z}}(1\times\beta(\rho_i)\ (1\leq i\leq 7))\oplus\Lambda_{\boldsymbol{Z}}(1\times\beta(\rho_\cdot)\ (i=2,4,5,6))$ $\otimes\Lambda_{\boldsymbol{Z}}(I(\beta_{\rho_i})\ (i=1,3,7))$

where γ_3 is subject to the relation $\gamma_3^2=-2\gamma_3$ and

$$\operatorname{Coker}\delta=\Lambda_{Z_4}(1\times\beta(\rho_i)\ (i=2,4,5,6))\oplus Z\cdot(\nu_3\times 1)\otimes\Lambda_Z(1\times\beta(\rho_i)\ (i=2,4,5,6))\oplus(Z\cdot 1\oplus Z\cdot(\nu_3\times 1))\otimes\Lambda_Z(1\times\beta(\rho_i)\ (i=1,3,7))$$

where the generators are subject to the following relations.

$$\nu_3^2=0,\ 4\times\beta(\rho_i)-4(k_i+2l_i)\,\nu_3\times 1=0\ \ (i=1,3,7),$$
$$(4k_i\,l_j-4k_j\,l_i)\,\nu_3\times 1+4l_i\times\beta(\rho_j)-4l_j\times\beta(\rho_i)=0\ \ (i,j=1,3,7),$$
$$-4\times\beta(\rho_i)\,\beta(\rho_j)-(4k_j+8l_j)\,\nu_3\times\beta(\rho_i)+(4k_i+8l_i)\,\nu_3\times\beta(\rho_j)=0\quad (i,j=1,3,7),$$
$$(4k_3\,l_7-4k_7\,l_3)\,\nu_3\times\beta(\rho_1)+(4k_7\,l_1-4k_1\,l_7)\,\nu_3\times\beta(\rho_3)+(4k_1\,l_3-4k_3\,l_1)\,\nu_3\times\beta(\rho_7)+4l_1\times\beta(\rho_3)\,\beta(\rho_7)+4l_3\times\beta(\rho_7)\,\beta(\rho_1)+4l_7\times\beta(\rho_1)\,\beta(\rho_3)=0\quad\textit{and}$$
$$-4\times\beta(\rho_1)\,\beta(\rho_3)\,\beta(\rho_7)+(4k_7+8l_7)\,\nu_3\times\beta(\rho_1)\,\beta(\rho_3)+(4k_1+8l_1)\,\nu_3\times\beta(\rho_3)\,\beta(\rho_7)+(4k_3+8l_3)\,\nu_3\times\beta(\rho_7)\,\beta(\rho_3)=0\,.$$

Clearly the map $J\colon K^*(P^3\times E_7)\to K_G^*(S^{8,0}\times E_7)$ of (3.2) factors through Coker δ as follows:

$$K^*(P^3\times E_7)\to\operatorname{Coker}\delta\to K_G^*(S^{8,0}\times E_7)$$

where the first arrow denotes the canonical projection. Again by J we denote the second arrow Coker $\delta\to K_G^*(S^{8,0}\times E_7)$ which we consider now. Let us put

$$\tilde{\nu}=p_1^*(\nu_7)\in K_G^{-1}(S^{8,0}\times E_7)\,.$$

Then by observing the exact sequence of (3.2) with a point instead of E_7 we can readily check that

$$J(\nu_3\times 1)=\tilde{\nu}\quad\text{(up to sign)}\,. \tag{3.13}$$

(For brevity we assume the sign to be plus in the following.) And moreover by definition, together with (3.5), (3.7) and (3.8), we easily obtain

(3.14) $J(1)=-2\xi\,,$

$$J(1\times\beta(\rho_{i_1})\cdots\beta(\rho_{i_s}))=-2\xi\tilde{\beta}(\rho_{i_1})\cdots\tilde{\beta}(\rho_{i_s})\ \ (i_1,\cdots,i_s=2,4,5,6)\,,$$
$$J(\nu_3\times\beta(\rho_{i_1})\cdots\beta(\rho_{i_s}))=\tilde{\nu}\tilde{\xi}(\rho_{i_1})\cdots\tilde{\beta}(\rho_{i_s})\ \ (i_1,\cdots,i_s=2,4,5,6)\,,$$
$$J(1\times\beta(\rho_i))=-2\xi\beta_{\rho_i}+(k_i+2l_i)\,\tilde{\nu}\ \ (i=1,3,7)\,,$$
$$J(\nu_3\times\beta(\rho_i))=\tilde{\nu}\beta_{\rho_i}\ \ (i=1,3,7)\,,$$
$$J(1\times\beta(\rho_i)\,\beta(\rho_j))=-2\xi\beta_{\rho_i}\,\beta_{\rho_j}+(k_i+2l_i)\,\tilde{\nu}\beta_{\rho_j}-(k_j+2l_j)\,\tilde{\nu}\beta_{\rho_i}\ \ (i,j=1,3,7)\,,$$
$$J(\nu_3\times\beta(\rho_i)\,\beta(\rho_j))=\tilde{\nu}\beta_{\rho_i}\,\beta_{\rho_j}\ \ (i,j=1,3,7)\,,$$
$$J(1\times\beta(\rho_1)\,\beta(\rho_3)\,\beta(\rho_7))=-2\xi\beta_{\rho_1}\,\beta_{\rho_3}\,\beta_{\rho_7}+(k_1+2l_1)\,\tilde{\nu}\beta_{\rho_3}\,\beta_{\rho_7}+(k_3+2l_3)\,\tilde{\nu}\beta_{\rho_7}\,\beta_{\rho_1}+(k_7+2l_7)\,\tilde{\nu}\beta_{\rho_1}\,\beta_{\rho_3}\quad\text{and}$$
$$J(\nu_3\times\beta(\rho_1)\,\beta(\rho_3)\,\beta(\rho_7))=\tilde{\nu}\beta_{\rho_1}\,\beta_{\rho_3}\,\beta_{\rho_7}\,.$$

From Lemma 3.12, (3.5), (3.7), (3.8) and (3.14) we get

Lemma 3.15.

$K_G^*(S^{8,0}\times E_7)=(\boldsymbol{Z}\cdot 1\oplus \boldsymbol{Z}_8\cdot\tilde{\xi})\otimes\Lambda_{\boldsymbol{Z}}(\tilde{\nu})\otimes\Lambda_{\boldsymbol{Z}}(\tilde{\beta}(\rho_i)\ (i=2,4,5,6),\ \beta_{\rho_j}\ (j=1,3,7))$

where $\tilde{\xi}$ *and* $\tilde{\nu}$ *are subject to the relations*

$$\tilde{\xi}^2=-2\tilde{\xi},\ \tilde{\xi}\tilde{\nu}=0\,.$$

Proof of Theorem 3.1. Observe the following exact sequence for the cofibration $S^{8,0}\times E_7\xrightarrow{i}B^{8,0}\times E_7\xrightarrow{j}\Sigma^{8,0}\wedge E_{7+}$ with Thom isomorphism

$$\begin{array}{ccccccc}
\cdots\to \tilde{K}_G^*(\Sigma^{8,0}\wedge E_{7+}) & \xrightarrow{j^*} & K_G^*(B^{8,0}\times E_7) & \xrightarrow{i^*} & K_G^*(S^{8,0}\times E_7) & \xrightarrow{\delta'} & \cdots \\
\cong\uparrow\phi_{8,0} & & \| & & & & \\
K_G^*(E_7) & & K^*(PE_7) & & & & \\
\| & & & & & & \\
K^*(PE_7) & & & & & &
\end{array}$$

where i, j are the obvious maps and δ' the coboundary homomorphism. Now by (1.6) $j^*(\tau_{8,0}\wedge 1)=-8\tilde{\xi}$ and so by Proposition 2.3, 1) or (2.5) we get $j^*(\tau_{8,0}\wedge 1)=0$. This implies that the composite $j^*\phi_{8,0}$ is zero. Hence the above sequence becomes a short exact sequence

$$0\to K^*(PE_7)\xrightarrow{I}K_G^*(S^{8,0}\times E_7)\xrightarrow{\delta}K^*(PE_7)\to 0\,,\tag{3.16}$$

provided with the formula $\delta(xI(y))=\delta(x)y\ (x\in K_G^*(S^{8,0}\times E_7), y\in K^*(PE_7))$ where I and δ denote the homomorphisms induced by i^* and δ'.

Let

$$\tilde{\beta}(\rho_i^2)=p_2^*(\beta(\rho_i^2))\quad\text{for}\quad i=1,3,7\,.$$

For these elements and the ones of $K^*(PE_7)$ given above we can check easily by definition that there holds the following formulas.

(3.17) $I(\xi)=\tilde{\xi},\ I(\beta(\rho_i))=\tilde{\xi}(\rho_i)\ \ (i=2,4,5,6)\,,$

$I(\beta(114\rho_1-7\rho_7))=(\tilde{\xi}+1)\,(114\beta_{\rho_1}-7\beta_{\rho_7})\,,$

$I(\beta(494\rho_1-\rho_3))=(\tilde{\xi}+1)\,(494\beta_{\rho_1}-\beta_{\rho_3})\,,$

$I(\beta(57\rho_3-1729\rho_7))=(\tilde{\xi}+1)\,(57\beta_{\rho_3}-1729\beta_{\rho_7})$, so that $\beta(57\rho_3-1729\rho_7)=247\beta(114\rho_1-7\rho_7)-57\beta(494\rho_1-\rho_3)$ (cf. Remark in §2 after (2.5)) and

$$\begin{aligned} I(\beta(\rho_i^2)) &= \tilde{\beta}(\rho_i^2)\\ &=(\tilde{\xi}+1)\,\beta_{\rho_i}-l_i\,\tilde{\nu}\ \ (i=1,3,7)\,. \end{aligned}$$

Moreover from definition we obtain

(3.18) $$\delta(\beta_{\rho_i}) = l_i(\bar{\xi}+1) \ (i=1, 3, 7) \quad \text{and} \quad \delta(\mathfrak{v}) = \bar{\xi}+2\,.$$

From this it follows that

$$\delta((\bar{\xi}+2)(49\beta_{\rho_1}-3\beta_{\rho_7})-\mathfrak{v}) = 0\,.$$

Therefore we see that

(3.19) there exists an element τ of $K^{-1}(PE_7)$ such that
$$I(\tau) = (\bar{\xi}+2)(49\beta_{\rho_1}-3\beta_{\rho_7})-\mathfrak{v}\,.$$

Note that τ satisfies $I(\xi\tau)=I(\tau^2)=0$, so that $\xi\tau=\tau^2=0$.

Denote by R^* the algebra on the right-hand side of the equality in Theorem 3.1. Since I is injective by (3.16), we then see by (3.17), (3.19) and Lemma 3.15 that $K^*(PE_7)$ contains R^* as a subalgebra.

By (3.17) and by using the injectivity of I again we get

$$\begin{aligned}
&494\beta(\rho_1^2)-\beta(\rho_3^2) = (\xi+2)\,\beta(494\rho_1-\rho_3)\,,\\
&114\beta(\rho_1^2)-\beta(\rho_7^2) = (\xi+2)\,\beta(114\rho_1-7\rho_7)\,,\\
&57\beta(\rho_3^2)-1729\beta(\rho_7^2) = (\xi+2)\,\beta(57\rho_3-1729\rho_7)
\end{aligned}$$

in $K^*(PE_7)$ and hence also by (3.19) we have

(3.20) $$\begin{aligned}
\beta(\rho_1^2) &= 7\tau-3(\xi+2)\,\beta(114\rho_1-7\rho_7)\,,\\
\beta(\rho_3^2) &= 3458\tau-1482(\xi+2)\,\beta(114\rho_1-7\rho_7)-(\xi+2)\,\beta(494\rho_1-\rho_3)\,,\\
\beta(\rho_7^2) &= 114\tau-49(\xi+2)\,\beta(114\rho_1-7\rho_7)\,.
\end{aligned}$$

Furthermore, using the formula of (3.16), together with (3.17) and (3.18), yields

(3.21) $$\begin{aligned}
&\delta(\beta_{\rho_1}\beta_{\rho_3}) = -7(\xi+1)\,\beta(494\rho_1-\rho_3)\,,\\
&\delta(\beta_{\rho_1}\beta_{\rho_7}) = -(\xi+1)\,\beta(114\rho_1-7\rho_7)\,,\\
&\delta(\beta_{\rho_3}\beta_{\rho_7}) = (\xi+1)\,(114\beta(494\rho_2-\rho_3)-494\beta(114\rho_1-7\rho_3))\\
&\qquad\qquad = -2(\xi+1)\,\beta(57\rho_3-1729\rho_7)\,,\\
&\delta(\beta_{\rho_1}\beta_{\rho_3}\beta_{\rho_7}) = -(\xi+1)\,\beta(114\rho_1-7\rho_7)\,\beta(494\rho_1-\rho_3)\,,\\
&\delta(\mathfrak{v}\beta_{\rho_i}) = \beta(\rho_i^2) \ (i=1, 3, 7)\,,\\
&\delta(\mathfrak{v}\beta_{\rho_1}\beta_{\rho_3}) = -\beta(\rho_1^2)\,\beta(494\rho_1-\rho_3)\,,\\
&\delta(\mathfrak{v}\beta_{\rho_1}\beta_{\rho_7}) = -\tau\beta(114\rho_1-7\rho_7)\,,\\
&\delta(\mathfrak{v}\beta_{\rho_3}\beta_{\rho_7}) = -494\beta(114\rho_1-\rho_7)+\beta(\rho_7^2)\,\beta(494\rho_1-\rho_3) \quad \text{and}\\
&\delta(\mathfrak{v}\beta_{\rho_1}\beta_{\rho_3}\beta_{\rho_7}) = -\tau\beta(114\rho_1-7\rho_7)\,\beta(494\rho_1-\rho_3)\,.
\end{aligned}$$

Since δ of (3.16) is surjective, from (3.17), (3.18), (3.20), (3.21) and Lemma 3.15 we infer that R^* fills $K^*(PE_7)$, so that

$$K^*(PE_7) = R^*$$

which completes the proof of Theorem 3.1.

4. The real K-group of PE_7

In this and the following sections we prove the following theorem.

Theorem 4.1. *With the notation as in* §2

$$KO^*(PE_7)=\Lambda_{KO^*(+)}(\beta(\rho_i)\ (i=2,4,5,6),\ \bar{\beta}(114\rho_1-7\rho_7),\ \bar{\beta}(494\rho_1-\rho_3),\ \bar{\beta}(\rho_1^2))$$
$$\otimes(\boldsymbol{Z}\cdot 1\oplus \boldsymbol{Z}_{16}\cdot\xi\oplus \boldsymbol{Z}_2\cdot\alpha\oplus \boldsymbol{Z}_2\cdot\alpha\xi)$$

as a $KO^*(+)$-module where the generators are subject to the relations

$$\begin{aligned}
&\xi^2=-2\xi\,,\ 4\eta_4\,\xi=0\\
&\beta(\rho_i)^2=\eta_1(\beta(\lambda^2\rho_i)+d_i\,\beta(\rho_i))\quad(i=2,4,5,6)\\
&\bar{\beta}(114\rho_1-7\rho_7)^2=\eta_1\,\beta(\lambda_C^2\,\rho_7)\,,\\
&\bar{\beta}(494\rho_1-\rho_3)^2=\eta_1\,\beta(\lambda_C^2\,\rho_3)\,,\\
&\bar{\beta}(\rho_1^2)^2=\xi\bar{\beta}(\rho_1^2)=0\,,\quad \alpha\bar{\beta}(\rho_1^2)=\eta_1^2\,\xi\beta(\rho_2)\,,\\
&\alpha^2=\eta_4\,\alpha=\eta_1^2\,\alpha=0
\end{aligned}$$

and α is defined in (4.17) *below.*

The proof is quite similar to that of the complex case, though it is more complicated. In this section, using Propositions 2.2 and 2.4, we deduce the structures of $KO_G^*(S^{5,0}\times E_7)$ and $KO_G^*(S^{9,0}\times E_G^*)$. We begin with the case of $KO_G^*(S^{5,0}\times E_7)$. By (1.4) in case of $h=KO$, $p=q-1=4$ and $X=E_7$ we have an exact sequence with Thom isomorphism

$$\begin{tikzcd}
\cdots\arrow[r] & KO_G^*(S^{4,0}\times E_7)\arrow[r,"\delta'"] & \widetilde{KO}_G^*(\Sigma^{4,0}\wedge S_+^{1,0}\wedge E_{7+})\arrow[r,"j^*"] & KO_G^*(S^{5,0}\times E_7)\arrow[r,"i^*"] & KO_G^*(S^{4,0}\times E_7)\arrow[r] & \cdots\\
 & & KO_G^*(S^{1,0}\times E_7)\arrow[u,"\cong","\psi_{4,4}"'] \arrow[d,equal] & & & \\
 & & KO^*(E_7) & & &
\end{tikzcd}$$

Using (1.2) this yields an exact sequence

$$\cdots\to KO^*(P^3\times E_7)\xrightarrow{\delta}KO^*(E_7)\xrightarrow{J}KO_G^*(S^{5,0}\times E_7)\xrightarrow{I}KO^*(P^3\times E_7)\to\cdots\tag{4.2}$$

and by (1.5) we have

$$\delta(xI(y))=\delta(x)\,y\tag{4.3}$$

for $x\in KO^*(P^3\times E_7)$ and $y\in KO_G^*(S^{5,0}\times E_7)$.

Similarly to the complex case we first investigate the coboundary homomorphism δ of (4.2). Here we also note that $KO^*(P^3\times E_7)\cong KO^*(P^3)\otimes_{KO^*(+)}KO^*$

(E_7) since $KO^*(E_7)$ is a free $KO^*(+)$-module. Let $p_1\colon S^{5,0}\times E_7\to S^{5,0}$ and $p_2\colon S^{5,0}\times E_7\to E_7$ be the obvious projections. And let $p_1^*\colon KO^*(P^4)\cong KO_G^*(S^{5,0})\to K_G^*(S^{5,0}\times E_7)$ and $p_2^*\colon KO^*(PE_7)\cong KO_G^*(E_7)\to K_G^*(S^{5,0}\times E_7)$ denote the homomorphisms induced by p_1 and p_2. Define

$$\begin{aligned}
&\xi = p_1^*(\gamma_4)\in KO_G(S^{5,0}\times E_7)\,,\\
&\tilde\beta(\rho_i) = p_2^*(\beta(\rho_i))\in KO_G^{-1}(S^{5,0}\times E_7)\quad (i=2,4,5,6)\,,\\
&\tilde{\bar\beta}(\rho_i^2) = p_2^*(\bar\beta(\rho_i^2))\quad (i=1,3,7)\,,\\
&\tilde{\bar\beta}(114\rho_1-7\rho_7) = p_2^*(\bar\beta(114\rho_1-7\rho_7)),\ \tilde{\bar\beta}(494\rho_1-\rho_3) = p_2^*(\bar\beta(494\rho_1-\rho_3))\,,\\
&\tilde{\bar\beta}(57\rho_3-1729\rho_7)) = p_2^*(\bar\beta(57\rho_3-1729\rho_7))\in KO_G^{-5}(S^{5,0}\times E_7)\,.
\end{aligned}$$

Then

$$\begin{aligned}
(4.4)\qquad & I(\xi) = \gamma_3\times 1,\ \text{so that}\ \delta(\gamma_3\times 1)=0\quad\text{and}\\
& I(\tilde\beta(\rho_i)) = 1\times\beta(\rho_i)+\eta_1\,\gamma_3\times 1,\ \text{so that}\ \delta(1\times\beta(\rho_i))=0\\
& \text{for}\quad i=2,4,5,6.
\end{aligned}$$

Here the first formula is immediate from definition and the proof of the second can be found in that of (3.7) (in particular, (3.6) is essential). Also the argument similar to that of (3.4) shows that

$$(4.5)\qquad \delta(\tilde\nu_3\times 1)=2\,.$$

Let $\Delta_5\colon S^{5,0}\subset \mathrm{Spin}(5)\to GL(2,\boldsymbol{H})$ be the composite of the canonical inclusion $S^{5,0}\subset\mathrm{Spin}(5)$ with the spin representation Δ_5 of $\mathrm{Spin}(5)$ [10]. We denote by

$$\beta'_{\rho_i}\in KSp_G^{-1}(S^{5,0}\times E_7)\quad\text{for}\quad i=1,3,7$$

the element $\beta(f_i)$ of $KSp_G^{-1}(S^{5,0}\times E_7)$ for the map

$$f_i\colon S^{5,0}\times E_7\to GL(4l_i,\boldsymbol{H})$$

given by $f_i(x,g)=(\Delta_5(x)\otimes I_{2l_i})\,\rho_i(g)$ for $x\in S^{5,0}$ and $g\in E_7$, which is a G-equivariant map in the sense of $f_i(-x,-g)=-f_i(x,g)$. And let us put

$$\beta_{\rho_i}=\sigma\wedge_C\beta'_{\rho_i}\in KO_G^{-5}(S^{5,0}\times E_7)\quad\text{for}\quad i=1,3,7\,.$$

Then by the argument similar to (3.8) we have

$$(4.6)\qquad I(\beta_{\rho_i}) = 1\times\bar\beta(\rho_i)-(k_i+2l_i)\,\tilde\nu_3\times 1\quad\text{for}\quad i=1,3,7$$

and so, together with (4.5), this shows

$$(4.7)\qquad \delta(1\times\bar\beta(\rho_i)) = (k_i\,\gamma_3+2k_i+4l_i)\times 1\quad\text{for}\quad i=1,3,7\,.$$

Consider the exact sequence for the pair (P^4, P^3)

$$\begin{array}{ccccccc} \cdots\to KO^{-6}(P^4) & \xrightarrow{i^*} & KO^{-6}(P^3) & \xrightarrow{\delta'} & KO^{-5}(S^4) & \to 0 \\ \| & & \| & & \| & \\ \boldsymbol{Z}_2\cdot\mu_4 & & \boldsymbol{Z}_2\cdot\eta_1\,\bar{\nu}_3\oplus\boldsymbol{Z}_2\cdot\mu_3 & & \boldsymbol{Z}_2\cdot\eta_1 & \end{array}$$

where i is an inclusion $P^3\subset P^4$ and δ' the coboundary homomorphism. Then using the facts such that $c(\mu_3)=\mu^3\gamma_3$, $c(\mu_4)=2\mu^3\gamma_4$ we get $i^*(\mu_4)=\eta_1\bar{\nu}_3$ and hence $\delta'(\mu_3)=\eta_1$ which implies that

$$\delta(\mu_3\times 1)=\eta_1\,. \tag{4.8}$$

Using (4.3) and the fact such that $\bar{\nu}_3^2=\bar{\nu}_2\,\mu_3=0$, from (4.5), (4.8) and (4.6) it follows that

$$\delta(\bar{\nu}_3\times\bar{\beta}(\rho_i))=2\bar{\beta}(\rho_i) \text{ and } \delta(\mu_3\times\bar{\beta}(\rho_i))=\eta_1\bar{\beta}(\rho_i) \quad \text{for} \quad i=1,3,7\,. \tag{4.9}$$

Using (4.3)—(4.9), the argument analogous to that of (3.11) proves the following.

$$\begin{aligned}(4.10)\quad & \delta(1\times\bar{\beta}(\rho_i)\,\bar{\beta}(\rho_j))=(2k_i+4l_i)\,\bar{\beta}(\rho_j)-(2k_j+4l_j)\,\bar{\beta}(\rho_i)\ (i,j=1,3,7)\,,\\ & \delta(\gamma_3\times\bar{\beta}(\rho_i)\,\bar{\beta}(\rho_j))=0\ (i,j=1,3,7)\,,\\ & \delta(\bar{\nu}_3\times\bar{\beta}(\rho_i)\,\bar{\beta}(\rho_j))=2\bar{\beta}(\rho_i)\,\bar{\beta}(\rho_j)\ (i,j=1,3,7)\,,\\ & \delta(\mu_3\times\bar{\beta}(\rho_i)\,\bar{\beta}(\rho_j))=\eta_1\,\bar{\beta}(\rho_i)\,\bar{\beta}(\rho_j)\ (i,j=1,3,7)\,,\\ & \delta(1\times\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7))=(2k_1+4l_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7)+(2k_3+4l_3)\,\bar{\beta}(\rho_7)\,\bar{\beta}(\rho_1)\\ & \qquad +(2k_7+4l_7)\,\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,,\\ & \delta(\gamma_3\times\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7))=0\,,\\ & \delta(\bar{\nu}_3\times\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7)=2\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7) \quad\text{and}\\ & \delta(\mu_3\times\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7))=\eta_1\,\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7)\,.\end{aligned}$$

By (4.2) we have a short exact sequence

$$0\to \operatorname{Coker}\delta \xrightarrow{J} KO_G^*(S^{5,0}\times E_7)\xrightarrow{I}\operatorname{Ker}\delta\to 0$$

where again by J we denote the homomorphism induced by J of (4.2).

Combining (4.3)-(4.10) and Propositions 2.2, 2.4 we get

Lemma 4.11.

$$\operatorname{Coker}\delta=\Lambda_{KO^*(+)}\,(\beta(\rho_i)\ (i=2,4,5,6),\ \bar{\beta}(\rho_j)\ (j=1,3,7))/(2,\eta_1)$$

as a $KO^*(+)$-module where the generators are subject to the relations

$$\begin{aligned}&\beta(\rho_i)^2=\eta_1(\beta(\lambda^2\rho_i)+d_i\,\beta(\rho_i))\ (i=2,4,5,6)\,,\\ &\bar{\beta}(\rho_i)^2=\eta_1\,\beta(\lambda_C^2\,\rho_i)\ (i=1,3,7)\,,\\ &(2k_i+4l_i)\,\bar{\beta}(\rho_j)-(2k_j+4l_j)\,\bar{\beta}(\rho_i)=0\ (i,j=1,3,7)\,,\end{aligned}$$

$$(2k_1+4l_1)\,\beta(\bar{\rho}_3)\,\beta(\bar{\rho}_7)+(2k_3+4l_3)\,\beta(\bar{\rho}_7)\,\beta(\bar{\rho}_1)+(2k_7+4l_7)\,\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)=0$$

and

$$\begin{aligned}\operatorname{Ker}\,\delta=\;&\Lambda_{KO^*(+)}\,(1\times\beta(\rho_i)\ (i=2,4,5,6),\ I(\beta_{\rho_j})\ (j=1,3,7))\\&\otimes(\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}_2\cdot(\eta_1\,\bar{\nu}_3\times 1)\oplus\boldsymbol{Z}_4\cdot(\gamma_3\times 1))\end{aligned}$$

as a $KO^*(+)$-module where the generators are subject to the relations

$$\begin{aligned}&(\eta_1\,\bar{\nu}_3)^2\times 1=\gamma_3\,\bar{\nu}_3\times 1=0\,,\\&\beta(\rho_i)^2\times 1=\eta_1(\beta(\lambda^2\,\rho_i)+d_i\,\beta(\rho_i))\times 1\ \ (i=2,4,5,6)\,,\\&I(\beta_{\rho_i})^2=1\times\eta_1\,\beta(\lambda_C^2\,\rho_i)\ \ (i=1,3,7)\,.\end{aligned}$$

As for the homomorphism J we have by definition

$$J(\beta(\rho_{i_1})\cdots\beta(\rho_{i_s})\,\bar{\beta}(\rho_{j_1})\cdots\bar{\beta}(\rho_{j_t}))=-\eta_4\,\tilde{\xi}\tilde{\beta}(\rho_{i_1})\cdots\tilde{\beta}(\rho_{i_s})\,\beta_{\rho_{j_1}}\cdots\beta_{\rho_{j_t}}$$

$(i_1,\cdots,i_s=2,4,5,6;\,j_1,\cdots,j_t=1,3,7)$. Moreover let us put

$$\tilde{\mu}=p_1^*(\mu_4)\in KO_G^{-6}(S^{5,0}\times E_7)\,.$$

Then we have

$$I(\tilde{\mu})=\eta_1\,\bar{\nu}_3\times 1$$

which can be obtained by observing the proof of (4.8). From these formulas, (4.4), (4.6), Lemma 4.11 and Proposition 2.4 it follows that

Lemma 4.12.

$$\begin{aligned}KO_G^*(S^{5,0}\times E_7)=\;&\Lambda_{KO^*(+)}\,(\tilde{\beta}(\rho_i)\ (i=2,4,5,6),\ \beta_{\rho_j}\ (j=1,3,7))\\&\otimes(\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}_8\cdot\tilde{\xi}\oplus\boldsymbol{Z}_2\cdot\tilde{\mu})\end{aligned}$$

as a $KO^*(+)$-module where the generators are subject to the relations

$$\begin{aligned}&\tilde{\xi}^2=-2\tilde{\xi},\ 2\eta_4\,\tilde{\xi}=0,\ \tilde{\mu}^2=\eta_4\,\tilde{\mu}=\tilde{\xi}\tilde{\mu}=0,\ \eta_1^2\,\tilde{\mu}=4\tilde{\xi}\,,\\&\tilde{\beta}(\rho_i)^2=\eta_1(\tilde{\beta}(\lambda^2\,\rho_i)+d_i\,\tilde{\beta}(\rho_i))\ \ (i=2,4,5,6)\,,\\&\beta_{\rho_i}^2=\eta_1\,\tilde{\beta}(\lambda_C^2\,\rho_i)\ \ (i=1,3,7)\,.\end{aligned}$$

REMARK. By definition it follows readily that there holds

$$\begin{aligned}&\tilde{\beta}(\rho_i^2)=(\tilde{\xi}+2)\,\beta_{\rho_i}\ \ (i=1,3,7)\,,\\&\tilde{\beta}(114\rho_1-7\rho_7)=(\tilde{\xi}+1)\,(114\beta_{\rho_1}-7\beta_{\rho_7})\,,\\&\tilde{\beta}(494\rho_1-\rho_3)=(\tilde{\xi}+1)\,(494\beta_{\rho_1}-\beta_{\rho_3})\,,\\&\tilde{\beta}(57\rho_3-1729\rho_7)=(\tilde{\xi}+1)\,(57\beta_{\rho_3}-1729\beta_{\rho_7})\end{aligned}$$

in $KO_G^{-5}(S^{5,0}\times E_7)$.

Next we deduce the structure of $KO_G^*(S^{9,0}\times E_7)$ from information about $KO_G^*(S^{5,0}\times E_7)$ and $KO^*(P^3\times E_7)$. For simplicity, for the generators of $KO_G^*(S^{9,0}\times E_7)$ we use the same notation as in $KO_G^*(S^{5,0}\times E_7)$. Let $p_1\colon S^{9,0}\times E_7\to S^{9,0}$, $p_2\colon S^{9,0}\times E_7\to E_7$ be the projections and let $p_1^*\colon KO^*(P^8)\cong KO_G^*(S^{9,0})\to KO_G^*(S^{9,0}\times E_7)$, $p_2^*\colon KO^*(PE_7)\cong KO_G^*(E_7)\to KO_G^*(S^{9,0}\times E_7)$ be the homomorphisms induced by p_1, p_2 respectively. Write

$$
\begin{aligned}
&\xi = p_1^*(\gamma_7)\in KO_G^0(S^{9,0}\times E_7),\ \tilde{\nu}_1 = p_1^*(\nu_8)\in KO_G^{-2}(S^{9,0}\times E_7)\,,\\
&\tilde{\beta}(\rho_i) = p_2^*(\beta(\rho_i))\in KO_G^{-1}(S^{9,0}\times E_7)\ \ (i=2,4,5,6)\,,\\
&\tilde{\beta}(\rho_i^2) = p_2^*(\beta(\rho_i^2))\ \ (i=1,3,7)\,,\\
&\tilde{\beta}(114\rho_1-7\rho_7) = p_2^*(\beta(114\rho_1-7\rho_7))\,,\\
&\tilde{\beta}(494\rho_1-\rho_3) = p_2^*(\beta(494\rho_1-\rho_3))\,,\\
&\tilde{\beta}(57\rho_3-1729\rho_7) = p_2^*(\beta(57\rho_3-1729\rho_7))\in KO_G^{-5}(S^{9,0}\times E_7)\,.
\end{aligned}
$$

Consider the following exact sequence of (1.4) with Thom isomorphism

$$
\begin{array}{ccccccc}
\cdots\xrightarrow{\delta'} & \widetilde{KO}_G^*(\Sigma^{4,0}\wedge S_+^{5,0}\wedge E_{7+}) & \xrightarrow{j^*} & KO_G^*(S^{9,0}\times E_7) & \xrightarrow{i^*} & KO_G^*(S^{4,0}\times E_7) & \to\cdots\\
& \cong\uparrow\psi_{4,4} & & & & \| & \\
& KO_G^*(S^{5,0}\times E_7) & & & & KO^*(P^3\times E_7) &
\end{array}
$$

This induces an exact sequence

$$
\cdots\xrightarrow{\delta} KO_G^*(S^{5,0}\times E_7)\xrightarrow{J} KO_G^*(S^{9,0}\times E_7)\xrightarrow{I} KO^*(P^3\times E_7)\to\cdots
$$

and by (1.5) we have

$$
\delta(xI(y)) = \delta(x)\,y
$$

($x\in KO^*(P^3\times E_7)$, $y\in KO_G^*(S^{9,0}\times E_7)$) which is used freely in the following.

From definition the direct computation yields

$$
\begin{aligned}
(4.13)\quad & I(\xi) = \gamma_3\times 1\,,\\
& I(\tilde{\beta}(\rho_i)) = 1\times\beta(\rho_i)+\varepsilon\eta_1\gamma_3\times 1\ \ (i=2,4,5,6;\ \varepsilon=0\text{ or }1)\,,\\
& I(\tilde{\beta}(\rho_i^2)) = (\gamma_3+2)\times\bar{\beta}(\rho_i)-(2k_i+4l_i)\,\nu_3\times 1\ \ (i=1,3,7)\,,\\
& I(\tilde{\beta}(114\rho_1-7\rho_7)) = 114\times\bar{\beta}(\rho_1)-7\times\bar{\beta}(\rho_7)+(114\cdot(k_1+2l_1)-7\cdot(k_7+2l_7))\,\nu_3\times 1\,,\\
& I(\tilde{\beta}(494\rho_1-\rho_3)) = 494\times\bar{\beta}(\rho_1)-1\times\bar{\beta}(\rho_3)+(494\cdot(k_1+2l_1)-(k_3+2l_3))\,\nu_3\times 1\,,
\end{aligned}
$$

($\tilde{\beta}(57\rho_3-1729\rho_7)=247\tilde{\beta}(114\rho_1-7\rho_7)-57\tilde{\beta}(494\rho_1-\rho_3)$ by Remark in §2) after (2.5)).

$$
\begin{aligned}
& \delta(\gamma_3\times 1) = 0\,,\quad \delta(\nu_3\times 1) = \xi+2\,,\quad \delta(\mu_3\times 1) = \eta_1\,,\\
& \delta(1\times\beta(\rho_i)) = 0\ \ (i=2,4,5,6)\,,\\
& \delta(1\times\bar{\beta}(\rho_i)) = k_i(\xi+2)+4l_i(\xi+1)\ \ (i=1,3,7)\,,
\end{aligned}
$$

$\delta(\gamma_3\times\bar\beta(\rho_i)) = -4l_i(\xi+1)$ $(i = 1, 3, 7)$,
$\delta(\bar\nu_3\times\bar\beta(\rho_i)) = (\xi+2)\,\beta_{\rho_i}$ $(i = 1, 3, 7)$,
$\delta(\mu_3\times\bar\beta(\rho_i)) = \eta_1\,\beta_{\rho_i}$ $(i = 1, 3, 7)$,
$\delta(1\times\bar\beta(\rho_i)\,\bar\beta(\rho_j)) = (\xi+1)\,((k_i(\xi+2)+4l_i)\,\beta_{\rho_j}-(k_j(\xi+2)+4l_j)\,\beta_{\rho_i})$
$(i, j = 1, 3, 7)$,
$\delta(\gamma_3\times\bar\beta(\rho_i)\,\bar\beta(\rho_j)) = \xi\delta(1\times\bar\beta(\rho_i)\,\bar\beta(\rho_j))$ $(i, j = 1, 3, 7)$,
$\delta(\bar\nu_3\times\bar\beta(\rho_i)\,\bar\beta(\rho_j)) = (\xi+2)\,\beta_{\rho_i}\,\beta_{\rho_j}$ $(i, j = 1, 3, 7)$,
$\delta(\mu_3\times\bar\beta(\rho_i)\,\bar\beta(\rho_j)) = \eta_1\,\beta_{\rho_i}\,\beta_{\rho_j}$ $(i, j = 1, 3, 7)$,
$\delta(1\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)) = (\xi+1)\,((k_1(\xi+2)+4l_1)\,\beta_{\rho_3}\,\beta_{\rho_7}+(k_3(\xi+2)+4l_3)\,\beta_{\rho_7}\,\beta_{\rho_1}$
$+(k_7(\xi+2)+4l_7)\,\beta_{\rho_1}\,\beta_{\rho_3})$,
$\delta(\gamma_3\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)) = \xi\delta(1\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)$,
$\delta(\bar\nu_3\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)) = (\xi+2)\,\beta_{\rho_1}\,\beta_{\rho_3}\,\beta_{\rho_7}$ and
$\delta(\mu_3\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)) = \eta_1\,\beta_{\rho_1}\,\beta_{\rho_3}\,\beta_{\rho_7}$.

From (4.13) and Propositions 2.2, 2.4 it follows that

Lemma 4.14.

Coker $\delta = \Lambda_{KO^*(+)}(\tilde\beta(\rho_i)\ (i = 2, 4, 5, 6),\ \beta_{\rho_j}(j = 1, 3, 7))/(\eta_1)\otimes(\boldsymbol{Z}_4\cdot 1\oplus\boldsymbol{Z}_2\cdot\tilde\mu)$ *as a $KO^*(+)$-module where $\tilde\mu$ is subject to the relations*

$$\tilde\mu^2 = \eta_4\,\tilde\mu = \eta_1^2\,\tilde\mu = 0$$

and Ker δ *is a submodule of $KO^*(P^3\times E_7)$ generated by the elements of*

$\Lambda_{KO^*(+)}(1\times\beta(\rho_i)$ $(i = 2, 4, 5, 6)$, $1\times\eta_4\,\bar\beta(\rho_j)-(k_j+2l_j)\,\eta_4\,\bar\nu_3\times 1$ $(j = 1, 3, 7)$,
$I(\tilde\beta(114\rho_1-7\rho_7))$, $I(\tilde\beta(494\rho_1-\rho_3))$, $I(\tilde\beta(\rho_1^2))\otimes M$

where M is a submodule of $KO^*(P^3\times E_7)$ generated by

1, $\gamma_3\times 1$, $2\gamma_3\times\bar\beta(\rho_1)$, $\gamma_3\times\bar\beta(\rho_3)$, $\gamma_3\times\bar\beta(\rho_7)$, $2\gamma_3\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)$,
$2\gamma_3\times\bar\beta(\rho_1)\,\bar\beta(\rho_7)$, $\gamma_3\times\bar\beta(\rho_3)\,\bar\beta(\rho_7)$, $2\gamma_3\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)$, $\eta_1\times\bar\beta(\rho_i)$,
$\eta_1\,\gamma_3\times\bar\beta(\rho_i)$, $\eta_1\times\bar\beta(\rho_i)\,\bar\beta(\rho_j)$, $\eta_1\,\gamma_3\times\bar\beta(\rho_i)\,\bar\beta(\rho_j)$ $(i, j = 1, 3, 7)$,
$\eta_1\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)$, $\eta_1\,\gamma_3\times\bar\beta(\rho_1)\,\bar\beta(\rho_3)\,\bar\beta(\rho_7)$.

Denote by the same letter J the homomorphism Coker $\delta\to KO_G^*(S^{9,0}\times E_7)$ induced by J. We now study this J.

By (1.9) we see readily that

$$J(1) = -\eta_4\,\xi \tag{4.15}$$

and by observing the exact sequence preceding (4.13) with a point instead of E_7, we get

$$J(\tilde{\mu}) = \tilde{\nu}\,. \tag{4.16}$$

Here we construct some new elements of $KO_G^*(S^{9,0}\times E_7)$. Define a G-equivariant bundle isomorphism

$$E_7\times \boldsymbol{H}^{4l_1} \cong E_7\times \boldsymbol{H}^{4l_1}\otimes H$$

by $(g, v)\mapsto(g, \rho_1(g)\, v)$ for $g\in E_7$ and $v\in \boldsymbol{H}^{4l_1}$. This isomorphism defines an element $\alpha'\in \widetilde{KSp}_G(\Sigma^{0,1}\wedge E_7)$ in the way similar to that of σ_G in §1, b) and so $\sigma\wedge_C \alpha'\in \widetilde{KO}_G^{-4}(\Sigma^{0,1}\wedge E_7)$. Since we have canonical isomorphisms $\widetilde{KO}_G^{-5}(\Sigma^{0,1}\wedge E_7)\cong \widetilde{KO}_G^{-5}(\Sigma^{0,1}\wedge E_7)\cong \widetilde{KO}_G^{-6}(PE_7)$, we can consider $\eta_1\, \sigma\wedge_C \alpha'\in \widetilde{KO}^{-6}(PE_7)$. We write

$$\alpha = \eta_1\, \sigma\wedge_C \alpha'\in \widetilde{KO}^{-6}(PE_7) \tag{4.17}$$

and also

$$\tilde{\alpha} = p_2^*(\alpha)\in KO_G^{-6}(S^{9,0}\times E_7)\,.$$

Let us view ρ_i $(i=1, 3, 7)$ and Δ_9 respectively as a quaternionic representation ρ_i: $E_7\to GL(4l_i, \boldsymbol{H})$ $(i=1, 3, 7)$ and a real representation Δ_9: Spin(9)$\to GL(16, \boldsymbol{R})$ [10]. Then we define a G-equivariant bundle isomorphism

$$S^{9,0}\times E_7\times \boldsymbol{H}^{4l_i}\otimes_{\boldsymbol{C}} \boldsymbol{H} \cong S^{9,0}\times E_7\times \boldsymbol{H}^{4l_i}\otimes_{\boldsymbol{C}} \boldsymbol{H}$$

by $(x, g, v)\mapsto(x, g, (\Delta_9(x)\otimes I_{l_i})\, (\rho_i(g)\otimes 1)\, (v))$ for $x\in S^{9,0}$, $g\in E_7$ and $v\in \boldsymbol{H}^{4l_i}\otimes_{\boldsymbol{C}} \boldsymbol{H}$ where $S^{9,0}$ is regarded as a canonical subspace of Spin(9) also as stated in §1. In the way similar to that of β'_{ρ_i} this defines an element of $KO_G^{-1}(S^{9,0}\times E_7)$ which we denote by

$$\beta_{\rho_i}\in KO_G^{-1}(S^{9,0}\times E_7) \quad \text{for} \quad i = 1, 3, 7\,.$$

And we write

$$\tilde{\beta}_\mu = J(\tilde{\mu}\beta_{\rho_1})\in KO_G^{-7}(S^{9,0}\times E_7)\,.$$

Then a short computation proves that

$$\begin{aligned}
&\eta_4\, \tilde{\beta}(\rho_i^2) = (\tilde{\xi}+2)\, \beta_{\rho_i} \ (i = 1, 3, 7)\\
&\eta_4\, \tilde{\beta}(494\rho_1-\rho_4) = (\tilde{\xi}+1)\, (494\beta_{\rho_1}-\beta_{\rho_3})\,,\\
&\eta_4\, \tilde{\beta}(114\rho_1-7\rho_7) = (\tilde{\xi}+1)\, (114\beta_{\rho_1}-7\beta_{\rho_7})\,,\\
&\eta_4\, \tilde{\beta}(57\rho_3-\rho_7) = (\tilde{\xi}+1)\, (57\beta_{\rho_3}-1729\beta_{\rho_7})\,,\\
&494\tilde{\beta}(\rho_1^2)-\tilde{\beta}(\rho_3^2) = (\tilde{\xi}+2)\, \tilde{\beta}(494\rho_1-\rho_3)\,,\\
&114\tilde{\beta}(\rho_1^2)-7\tilde{\beta}(\rho_7^2) = (\tilde{\xi}+2)\, \tilde{\beta}(114\rho_1-7\rho_7)\,,\\
&57\tilde{\beta}(\rho_3^2)-1729\tilde{\beta}(\rho_7^2) = (\tilde{\xi}+2)\, \tilde{\beta}(57\rho_3-1729\rho_7)\,,
\end{aligned} \tag{4.18}$$

so that

$$\tilde{\beta}(\rho_3^2) = 494\tilde{\beta}(\rho_1^2) - (\xi+2)\,\tilde{\beta}(494\rho_1-\rho_3)\,,$$
$$\tilde{\beta}(\rho_7^2) = (2\xi+4)\,\eta_4\,\beta_{\rho_7} - 114\tilde{\beta}(\rho_1^2) + (\xi+2)\,\tilde{\beta}(114\rho_1-7\rho_7)\,.$$

And moreover, further computation shows that

(4.19) $$I(\tilde{\alpha}) = 1\times\eta_1\,\tilde{\beta}(\rho_1)\,,$$
$$I(\beta_{\rho_i}) = 1\times\eta_4\,\tilde{\beta}(\rho_i) - (k_i+2l_i)\,\eta_4\,\tilde{\nu}_3\times 1 \quad (i = 1, 3, 7)$$

and

$$J(\beta_{\rho_i}) = -\xi\beta_{\rho_i} \quad (i = 1, 3, 7)\,,$$
$$J(\beta_{\rho_1}\beta_{\rho_3}) = -\xi\beta_{\rho_1}\tilde{\beta}(494\rho_1-\rho_3)\,,$$
$$J(\beta_{\rho_1}\beta_{\rho_7}) = -7\xi\beta_{\rho_1}\tilde{\beta}(114\rho_1-7\rho_7)\,,$$
$$J(\beta_{\rho_3}\beta_{\rho_7}) = 14J(\beta_{\rho_1}\beta_{\rho_7}) - 14J(\beta_{\rho_1}\beta_{\rho_3})$$
$$-7\eta_4\,\xi\tilde{\beta}(494\rho_1-\rho_3)\,\tilde{\beta}(114\rho_1-7\rho_7) - 8\xi\beta_{\rho_3}\,\beta(\rho_7^2)\,,$$
$$J(\beta_{\rho_1}\beta_{\rho_3}\beta_{\rho_7}) = \xi\beta_{\rho_1}\beta_{\rho_7}\tilde{\beta}(494\rho_1-\rho_3)\,,$$
$$J(\tilde{\mu}\beta_{\rho_3}) = \tilde{\nu}\tilde{\beta}(494\rho_1-\rho_3)\,,$$
$$J(\tilde{\mu}\beta_{\rho_7}) = \tilde{\nu}\tilde{\beta}(114\rho_1-7\rho_7)\,,$$
$$J(\tilde{\mu}\beta_{\rho_1}\beta_{\rho_3}) = \tilde{\beta}_\mu\,\tilde{\beta}(494\rho_1-\rho_3)\,,$$
$$J(\tilde{\mu}\beta_{\rho_1}\beta_{\rho_7}) = \tilde{\beta}_\mu\,\tilde{\beta}(114\rho_1-7\rho_7)\,,$$
$$J(\tilde{\mu}\beta_{\rho_3}\beta_{\rho_7}) = \tilde{\nu}\tilde{\beta}(494\rho_1-\rho_3)\,\tilde{\beta}(114\rho_1-7\rho_7)\,,$$
$$J(\tilde{\mu}\beta_{\rho_1}\beta_{\rho_3}\beta_{\rho_7}) = \tilde{\beta}_\mu\,\tilde{\beta}(494\rho_1-\rho_3)\,\tilde{\beta}(114\rho_1-7\rho_7)\,.$$

By applying Lemma 4.14, (4.13), (4.15), (4.16) and (4.19) to the exact sequence

$$0 \to \mathrm{Coker}\,\delta \xrightarrow{J} KO_G^*(S^{9,0}\times E_7) \xrightarrow{I} \mathrm{Ker}\,\delta \to 0$$

we obtain

Lemma 4.20. *$KO_G^*(S^{9,0}\times E_7)$ is a factor module of*

$$\Lambda_{KO^*(+)}(\tilde{\beta}(\rho_i)\ (i = 2, 4, 5, 6),\ \beta_{\rho_1}, \beta_{\rho_7}, \tilde{\beta}(\rho_1^2), \tilde{\beta}(114\rho_1-7\rho_7), \tilde{\beta}(494\rho_1-\rho_3))$$
$$\otimes(\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}_2\cdot\tilde{\beta}_\mu)\otimes(\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}_2\cdot\tilde{\alpha})\otimes(\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}_2\cdot\tilde{\nu})\otimes(\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}_{16}\cdot\xi)$$

where there hold at least the following relations.

$$\xi^2 = -2\xi,\ 4\eta_4\,\xi = 0,\ \tilde{\beta}(\rho_i)^2 = \eta_1(\tilde{\beta}(\lambda^2\rho_i) + d_i\,\tilde{\beta}(\rho_i))\ (i = 2, 4, 5, 6)\,,$$
$$\tilde{\nu}^2 = \xi\tilde{\nu} = \eta_4\,\tilde{\nu} = \eta_1^2\,\tilde{\nu} = 0,\ \tilde{\alpha}^2 = 2\tilde{\alpha} = \eta_1^2\,\tilde{\alpha} = \eta_4\,\tilde{\alpha} = 0\,,$$
$$\eta_1^2\,\tilde{\beta}_\mu = 8\xi\beta_{\rho_1},\ \xi\tilde{\beta}_\mu = \eta_4\,\tilde{\beta}_\mu = \tilde{\beta}_\mu^2 = 0,\ \tilde{\alpha}\tilde{\beta}(\rho_1^2) = \eta_1^2\,\xi\tilde{\beta}(\rho_2)\,,$$
$$114\beta_{\rho_1} - 7\beta_{\rho_7} = \eta_4(\xi+1)\,\tilde{\beta}(114\rho_1-7\rho_7),\ \eta_4\,\tilde{\beta}(\rho_1^2) = (\xi+1)\,\beta_{\rho_1}\,,$$

$$\tilde{\beta}(114\rho_1-7\rho_7)^2=\eta_1\tilde{\beta}(\lambda_C^2\rho_7),\ \tilde{\beta}(494\rho_1-\rho_3)=\eta_1\tilde{\beta}(\lambda_C^2\rho_3),$$
$$\tilde{\beta}(\rho_1^2)=\xi\tilde{\beta}(\rho_1^2)=0,\ \beta_{\rho_1}^2=\eta_1\tilde{\beta}(\lambda^2(r\rho_1)),\ \beta_{\rho_7}^2=\eta_1\tilde{\beta}(\lambda^2(r\rho_7))$$

(here r denotes the realification functor).

In particular, $\tilde{\beta}(\rho_i)$ $(i=2,4,5,6)$, $\tilde{\beta}(114\rho_1-7\rho_7)$, $\tilde{\beta}(494\rho_1-\rho_3)$, $\tilde{\beta}(\rho_1^2)$, $\tilde{\alpha}$ and ξ are subject to the relations above.

In the next section we give some additional relations.

The greater part of the relations above is already mentioned, or can be readily shown by the facts mentioned preceding Lemma 4.20. We explain how to get the remains. We begin with showing that $\beta(114\rho_1-7\rho_7)^2=\eta_1\beta(\lambda_C^2\rho_7)$, $\beta(494\rho_1-\rho_3)^2=\eta_1\beta(\lambda_C^2\rho_3)$ and $\beta(\rho_1^2)^2=0$. Applying p_2^* to these we get immediately the required formulas above.

Let f, g: $E_7\to GL(n, \boldsymbol{H})$ be G-maps in the sense of $f(-x)=-f(x)$, $g(-x)=-g(x)$ $(x\in E_7)$ and fg: $E_7\to GL(n, \boldsymbol{H})$ the map given by $(fg)(x)=f(x)g(x)$ for $x\in E_7$. Then it follows that

$$\beta(fg\wedge_C fg)=\beta(f\wedge_C f)+\beta(g\wedge_C g)\ \text{ in } KO^*(E_7).$$

Using this we have $\beta(\rho_1^2\wedge_C\rho_1^2)=2\beta(\lambda_C^2\rho_1)$ and by (1.11) we get $\beta(\rho_1^2)^2=\eta_1\beta(\rho_1^2\wedge_C\rho_1^2)$. Hence we have $\beta(\rho_1^2)^2=0$. To consider $\beta(494\rho_1-\rho_3)^2$ we take f, g to be $f(x)=494\rho_1(x)$, $g(x)=\rho_3(x)^{-1}$ for $x\in E_7$. Then $\beta(494\rho_1-\rho_3)^2=\eta_1\beta(fg\wedge_C fg)$ $=\eta_1\beta(g\wedge_C g)$ and so $\beta(494\rho_1-\rho_3)^2=\eta_1\beta(\lambda_C^2\rho_3)$. Similarly another one follows.

The last two formulas are immediate from (1.10).

To prove that $\tilde{\alpha}^2=0$ we observe the following exact sequence associated with the cofibration $S^{1,0}\times E_7\xrightarrow{i}B^{1,0}\times E_7\xrightarrow{j}\Sigma^{1,0}\wedge E_{7+}$ where i, j are the obvious maps.

$$\text{(4.21)}\qquad \cdots\xrightarrow{\delta'}\widetilde{KO}_G^*(\Sigma^{1,0}\wedge E_{7+})\xrightarrow{\chi}KO^*(PE_7)\xrightarrow{\pi^*}KO^*(E_7)\to\cdots$$

where δ' is the coboundary homomorphism and χ the map induced by j. In particular, if we take a point instead of E_7 in (4.21), then we see that there exists an element $\tau_{1,0}^R$ of $\widetilde{KO}_G(\Sigma^{1,0})$ such that $\chi(\tau_{1,0}^R)=1-H$ and then it follows that

$$\widetilde{KO}_G^*(\Sigma^{1,0})=KO^*(+)\cdot\tau_{1,0}^R$$

with relations $H\tau_{1,0}^R=-\tau_{1,0}^R$, $(\tau_{1,0}^R)^2=2\tau_{1,0}^R$. Moreover, as is seen below (see (5.6)) any element of $\widetilde{KO}_G^*(\Sigma^{1,0}\wedge E_{7+})$ takes the form of $\tau_{1,0}\wedge x$ where $x\in\widetilde{KO}_G^*(E_{7+})=KO^*(PE_7)$.

So by definition $\pi^*(\alpha^2)=0$, so that $\alpha^2\in\operatorname{Im}\chi$. Hence we see that α^2 is divisible by ξ because of $\chi(\tau_{1,0}^R\wedge 1)=\xi$. Next consider the complexification of α, then by definition we see that $c(\alpha)=n\mu^6c(\xi)$ for some $n\in\boldsymbol{Z}$. But, since the orders of α and $c(\xi)$ are respectively 2 and 8, n must be divisible by 4. This implies $c(\alpha^2)=0$. Therefore, from Atiyah's exact sequence ([1], Proposition 3.2) it follows that α^2 is divisible by η_1. Consequently we conclude that $\tilde{\alpha}^2$ is divisible by $\tilde{\xi}$ and η_1. On the other hand, by definition we have $I(\tilde{\alpha}^2)=0$, so that $\tilde{\alpha}^2 \in \operatorname{Im} J$. According to (4.15), (4,16) and (4.19), however, there is no such element in $\operatorname{Im} J$ except for zero. Namely $\tilde{\alpha}^2=0$.

We show that $\xi\beta(\rho_1^2)=0$. From (3.20), (3.19) it follows that $c(\xi\beta(\rho_1^2))=0$. So $\xi\beta(\rho_1^2)$ is divisible by η_1. Hence by the same reason as above we see that $\xi\beta(\rho_1^2)=0$, so that $\tilde{\xi}\tilde{\beta}(\rho_1^2)=0$.

Here we note that $\tilde{\alpha}\tilde{\beta}(\rho_1^2)\neq 0$. By dimension reason $\lambda^2\rho_1=\rho_2+1$ as complex representations and ρ_2 is real. So by (4.13) and (4.19) it follows that $I(\tilde{\alpha}\tilde{\beta}(\rho_1^2))=\eta_1^2\,\gamma_3\times\beta(\rho_2)\neq 0$ which shows the above inequality.

5. Proof of Theorem 4.1.

We study the following exact sequence with Thom isomorphism

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\delta'} & \widetilde{KO}{}_G^*(\Sigma^{9,0}\wedge E_{7+}) & \xrightarrow{j^*} & KO_G^*(B^{9,0}\times E_7) & \xrightarrow{i^*} & KO_G^*(S^{9,0}\times E_7) & \rightarrow\cdots \\
& \cong\uparrow \psi_{8,0} & & \| & & & \\
& \widetilde{KO}{}_G^*(\Sigma^{1,0}\wedge E_{7+}) & & KO^*(PE_7) & & &
\end{array}$$

associated with the cofibration $S^{9,0}\times E_7 \xrightarrow{i} B^{9,0}\times E_7 \xrightarrow{j} \Sigma^{9,0}\wedge E_{7+}$ where i, j are the obvious maps and δ' the coboundary homomorphism.

For any $x\in\widetilde{KO}{}_G^*(\Sigma^{1,0}\wedge E_{7+})$

$$\begin{aligned} j^*\psi_{8,0}(x) &= j^*(\tau_{8,0}^R\wedge x) \\ &= -8\xi\chi(x) \qquad \text{by (1.7)} \end{aligned}$$

where χ is as in (4.21). We first prove that $8\xi\chi(x)=0$, equivalently $j^*\psi_{8,0}=0$. Consider the complex version of (4.21)

$$\cdots\rightarrow \tilde{K}_G^*(\Sigma^{1,0}\wedge E_{7+}) \xrightarrow{\chi} K^*(PE_7) \xrightarrow{\pi^*} K^*(E_7)\rightarrow\cdots$$

and investigate the images of the multiplicative generators of $K^*(PE_7)$ (given in Theorem 3.1) by π^*. Then by definition we have

$$\begin{aligned}
&\pi^*(\tau) = 98\beta(\rho_1)-6\beta(\rho_7)\,, \\
&\pi^*(\beta(\rho_i)) = \beta(\rho_i)\ \ (i=2, 4, 5, 6)\,, \\
&\pi^*(\beta(114\rho_1-7\rho_7)) = 114\beta(\rho_1)-7\beta(\rho_7)\,, \\
&\pi^*(\beta(494\rho_1-\rho_3)) = 494\beta(\rho_1)-\beta(\rho_3) \quad \text{and} \\
&\pi^*(c(\xi)) = 0\,.
\end{aligned}$$

From this we see that any element of $\operatorname{Im}\chi$ takes the form of $c(\xi)y$ where $y\in K^*(PE_7)$. So we have

$$\chi(c(x)) = c(\xi)y \quad \text{for some} \quad y \in K^*(PE_7),$$

so that by operating r we get

$$2\chi(x) = \xi r(y).$$

Hence multiplying by 4ξ gives

$$8\xi\chi(x) = -8\xi r(y) \quad \text{for some} \quad y \in K^*(PE)_7.$$

To consider $r(y)$ we here view $K^*(PE_7)$ as a Z_8-graded module over $K^*(+) = \boldsymbol{Z}[\mu]/(\mu^4-1)$. Using the relations of r with c and the values of r (resp. c) on $K^*(+)$ (resp. on $KO^*(+)$) (see e.g. [13]) we can easily verify that the value of r on each (additive) generator of $KO^*(PE_7)$, which does not contains τ as a multiplicative component, is 0, or divisible by 2, η_1^2 or η_4. So if y is such a generator, then $8\xi r(y)=0$ because $16\xi=8\eta_4\xi=0$ by (2.5).

Since $c(\xi)\tau=0$ by Theorem 3.1 it follows that $\xi r(\mu^i\tau)=0$ for $i=0, 1, 2, 3$. So if y is divisible by τ, then also by using this fact we can check that $\xi r(y)=0$. Consequently we see that $8\xi r(y)=0$ for any $y \in K^*(PE_7)$, namely $j^*\psi_{8,0}=0$. Hence we have a short exact sequence

$$(5.1) \qquad 0 \to KO^*(PE_7) \xrightarrow{I} KO_G^*(S^{9,0}\times E_7) \xrightarrow{\delta} \widetilde{KO}_G^*(\Sigma^{1,0}\wedge E_7) \to 0$$

(here also there holds the equality $\delta(xI(y))=\delta(x)y$ for $x \in KO_G^*(S^{9,0}\times E_7)$ and $y \in KO^*(PE_7)$).

By R^* we denote the $KO^*(+)$-module, with the relations, on the right-hand side of the equality in Theorem 4.1. Then by virtue of the injectivity of I of (5.6) and Lemma 4.20 it follows that $KO^*(PE_7)$ contains R^* as a submodule. Because I sends any multiplicative generator of R^* to the element of the same symbol with tilde as in $KO_G^*(S^{9,0}\times E_7)$.

Next we show that R^* fills $KO^*(PE_7)$. For this we study the image of δ. By

$$\tau_{9,0}^R \in \widetilde{KO}_G(\Sigma^{9,0})$$

we denote the element of $\widetilde{KO}_G(\Sigma^{9,0})$ which Δ_9: Spin(9)$\to GL(16, \boldsymbol{R})$ defines in the manner similar to $\tau_{8,0}^R$. Then we have

$$\tau_{9,0}^R = \tau_{8,0}^R \wedge \tau_{1,0}^R \in \widetilde{KO}_G(\Sigma^{9,0}) = \widetilde{KO}_G(\Sigma^{8,0}\wedge\Sigma^{1,0}).$$

By using this fact and observing the definition we get

$$(5.2) \qquad \delta(\beta_{\rho_j}) = -l_i\,\tau_{1,0}^R\wedge 1 \quad \text{for} \quad j=1, 3, 7 \quad \text{and} \quad \delta(\tilde{\nu}) = \eta_1\,\tau_{1,0}^R\wedge 1.$$

Consider the homomorphism

$$(1\wedge j)^*\colon KO_G^{-3}(S^{5,0}\times E_7) = \widetilde{KO}_G^{-2}(S_+^{5,0}\wedge\Sigma E_{7+}) \to \widetilde{KO}_G^{-2}(S_+^{5,0}\wedge\Sigma E_7)$$

where ΣE_7 denote the unreduced suspension of E_7 and j the projection $\Sigma E_7 \to \Sigma E_{7+}$. Here we can view $\widetilde{KO}_G^{-2}(S_+^{5,0}\wedge \Sigma E_7)$ as a direct summand of $KO_G^{-2}(S^{5,0}\times \Sigma E_7)$ via a canonical map and so $\tilde{\mu}(\sigma\wedge_C \alpha')\in KO_G^{-2}(S^{5,0}\times \Sigma E_7)$ as an element of $KO_G^{-2}(S_+^{5,0}\wedge \Sigma E_7)$. Hence by definition we see that

$$(1\wedge j)^*(\tilde{\mu}\beta_{\rho_1}) = \tilde{\mu}(\sigma\wedge_C \alpha')\,H\,.$$

Since $\widetilde{KO}_G^{-5}(\Sigma^{9,0})=\widetilde{KO}_G^{-6}(\Sigma^{9,0})=0$, $(1\wedge j)^*\colon \widetilde{KO}_G^{-5}(\Sigma^{9,0}\wedge \Sigma E_{7+})\to \widetilde{KO}_G^{-5}(\Sigma^{9,0}\wedge \Sigma E_7)$ is an isomorphism, we observe the commutative diagram with this isomorphism

$$\begin{array}{ccc}
\widetilde{KO}_G^{-5}(\Sigma^{9,0}\wedge \Sigma E_{7+}) & \xrightarrow{\cong} & \widetilde{KO}_G^{-5}(\Sigma^{9,0}\wedge \Sigma E_7)\\
\| & & \|\\
\widetilde{KO}_G^{-1}(\Sigma^{4,4}\wedge \Sigma^{5,0}\wedge \Sigma E_{7+}) & \to & \widetilde{KO}_G^{-1}(\Sigma^{4,4}\wedge \Sigma^{5,0}\wedge \Sigma E_7)\\
\delta_1\uparrow & & \uparrow\delta_1\\
\widetilde{KO}_G^{-2}(\Sigma^{4,4}\wedge S_+^{5,0}\wedge \Sigma E_{7+}) & \to & \widetilde{KO}_G^{-2}(\Sigma^{4,4}\wedge S_+^{5,0}\wedge \Sigma E_7)\\
\psi_{4,4}\uparrow & & \uparrow\psi_{4,4}\\
\widetilde{KO}_G^{-2}(S_+^{5,0}\wedge \Sigma E_{7+}) & \to & \widetilde{KO}_G^{-2}(S_+^{5,0}\wedge \Sigma E_7)
\end{array}$$

where all the unlabelled arrows denote $(1\wedge j)^*$'s and δ_1's the coboundary homomorphisms associated with the cofibration $S_+^{5,0}\to B_+^{5,0}\to \Sigma^{5,0}$. Then also by definition we see that the equality above gives

$$(5.3)\qquad \delta(\tilde{\beta}_\mu) = \tau_{1,0}^R\wedge\alpha$$

together with the second formula of (5.2) and the facts such that $(H+1)\,\tau_{1,0}^R = 2\alpha=0$.

Since $I(\eta_4(\xi+1)\,\bar{\beta}(494\rho_1-\beta\rho_3))=494\beta_{\rho_1}-\beta_{\rho_3}$ by (4.18) and $\beta_{\rho_1}^2$ is divisible by η_1, we have

$$\beta_{\rho_1}\, I(\eta_4(\xi+1)\,\bar{\beta}(494\rho_1-\rho_3)) = -\beta_{\rho_1}\beta_{\rho_3}$$

by multiplying by β_{ρ_1}. So applying δ and using the equality of (5.1) and (5.2) we have

$$(5.4)\qquad \delta(\beta_{\rho_1}\beta_{\rho_3}) = -l_1\,\tau_{1,0}^R\wedge\eta_4\,\bar{\beta}(494\rho_1-\rho_3)\,.$$

Similarly we get

$$\begin{aligned}
(5.5)\qquad \delta(\beta_{\rho_1}\beta_{\rho_7}) &= l_1\,\tau_{1,0}^R\wedge(32\cdot 114\eta_4\,\bar{\beta}(\rho_1^2)+9\eta_4\,\bar{\beta}(114\rho_1-7\rho_7))\,,\\
\delta(\beta_{\rho_3}\beta_{\rho_7}) &= 494\delta(\beta_{\rho_1}\beta_{\rho_7})+l_7\,\tau_{1,0}^R\wedge\eta_4\,\bar{\beta}(494\rho_1-\rho_3)\,,\\
\delta(\beta_{\rho_1}\beta_{\rho_3}\beta_{\rho_7}) &= \delta(\beta_{\rho_1}\beta_{\rho_7})\,\eta_4(\xi+1)\,\bar{\beta}(494\rho_1-\rho_3)\,.
\end{aligned}$$

Now we prepare certain relations in $KO_G^*(S^{9,0}\times E_7)$. Consider the homomorphisms mentioned preceding (4.13)

$$KO_G^*(S^{5,0}\times E_7)\overset{\psi_{4,4}}{\to}\widetilde{KO}_G^*(\Sigma^{4,0}\wedge(S^{5,0}\times E_7)_+)\overset{j^*}{\to}KO_G^*(S^{9,0}\times E_7)\,.$$

Because the restriction of Δ_9 to Spin(5) is the 4-times direct sum of Δ_5 as a complex representation, we then see by definition that

$$J(2\eta_4\tilde{\mu}\beta_{\rho_i})=\tilde{\nu}\beta_{\rho_i}$$

where $J=j^*\psi_{4,4}$ and so

$$\tilde{\nu}\beta_{\rho_i}=0\quad\text{for}\quad i=1,3,7$$

since $2\mu_4=0$. Analogously we get

$$\tilde{\beta}_\mu\tilde{\nu}=0\quad\text{and}\quad\tilde{\beta}_\mu\beta_{\rho_i}=0\quad\text{for}\quad i=1,3,7$$

using the relation $\gamma_4\,\mu_4=0$.

From these formulas and (5.2)-(5.5) we can determine by using the formula of (5.1) the values of δ on all additive generators of $KO_G^*(S^{9,0}\times E_7)$ and consequently we obtain

(5.6) $\widetilde{KO}_G^*(\Sigma^{1,0}\wedge E_{7+})$ is a $KO^*(+)$-module generated by the elements in the form $\tau_{1,0}^R\wedge x$ for $x\in R^*$.

We observe (4.21). From (5.6) it follows immediately that

$$\operatorname{Im}\chi=\xi R^* \tag{5.7}$$

as a $KO^*(+)$-module. Next we study $\operatorname{Im}\pi^*$. According to Proposition 2.2

$$KO^*(E_7)=\Lambda_{KO^*(+)}(\beta(\rho_i)\ (i=2,4,5,6),\ \bar{\beta}(\rho_j)\ (j=1,3,7))$$

as a $KO^*(+)$-module. We can easily check that

$$\begin{aligned}(5.8)\qquad &\pi^*(\beta(\rho_i))=\beta(\rho_i)\ (i=2,4,5,6)\,,\\ &\pi^*(\bar{\beta}(\rho_i^2))=2\bar{\beta}(\rho_i)\ (j=1,3,7)\,,\\ &\pi^*(\bar{\beta}(114\rho_1-7\rho_7))=114\bar{\beta}(\rho_1)-7\bar{\beta}(\rho_7)\,,\\ &\pi^*(\bar{\beta}(494\rho_1-\rho_3))=494\bar{\beta}(\rho_1)-\bar{\beta}(\rho_3)\,,\end{aligned}$$

(so that $\pi^*(247\bar{\beta}(\rho_1^2)-\bar{\beta}(494\rho_1-\rho_3))=\bar{\beta}(\rho_3)$,

$$\pi^*(57\bar{\beta}(\rho_1^2)-\bar{\beta}(114\rho_1-7\rho_7)-3\bar{\beta}(\rho_7^2))=\bar{\beta}(\rho_7))\quad\text{and}$$
$$\pi^*(\alpha)=\eta_1\,\bar{\beta}(\rho_1)\,.$$

Using Theorem 3.1 and Proposition 2.1 we see that $\beta(\rho_1)\notin\operatorname{Im}\{\pi^*: K^*(PE_7)\to K^*(E_7)\}$. Becasue, $\pi^*(\beta(\rho_i))=\beta(\rho_i)$ $(i=2,4,5,6)$, $\pi^*(\beta(\rho_i^2))=2\beta(\rho_i)$ $(i=1,3,7)$, $\pi^*(\beta(114\rho_1-7\rho_7))=114\beta(\rho_1)-7\beta(\rho_7)$, $\pi^*(\beta(494\rho_1-\rho_3))=494\beta(\rho_1)-\beta(\rho_3)$, (so that $\pi^*(247\beta(\rho_1^2)-\beta(494\rho_1-\rho_3))=\beta(\rho_3)$, $\pi^*(57\beta(\rho_1^2)-\beta(114\rho_1-7\rho_7)$

$-3\beta(\rho_7^2))=\beta(\rho_7))$, and $\pi^*(\tau)=2\beta(\rho_1)-6\beta(\rho_7)$. So if we suppose that $\bar{\beta}(\rho_1)\in \operatorname{Im}\pi^*$ then it follows that $\mu^2\beta(\rho_1)\in \operatorname{Im}\pi^*$, that is, $\beta(\rho_1)\in \operatorname{Im}\pi^*$. This is a contradiction to the above. Hence we have

(5.9) $$\bar{\beta}(\rho_1)\notin \operatorname{Im}\pi^*\,.$$

In the similar manner we can prove that

(5.10) $$\bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3),\ \bar{\beta}(\rho_1)\,\bar{\beta}(\rho_7),\ \bar{\beta}(\rho_1)\,\bar{\beta}(\rho_3)\,\bar{\beta}(\rho_7)\notin \operatorname{Im}\pi^*\,.$$

Consider the commutative diagram

$$\begin{array}{ccc} KO_G^*(S^{9,0}\times E_7) & \xrightarrow{\psi} & KO^*(S^8\times E_7) \\ \cup & & \cup \\ KO^*(PE_7) & \xrightarrow{\pi^*} & KO^*(E_7) \end{array}$$

where ψ denotes the forgetful homomorphism and the unlabelled inclusions are induced by the second projections. Then by definition we have

$$\psi(\xi)=0,\ \psi(\beta_{\rho_i})=1\times\eta_4\,\bar{\beta}(\rho_i)\ \ (i=1,3,7)$$

and moreover

$$\psi(\tilde{\nu})=\psi(\tilde{\beta}_\mu)=0$$

because of $\pi^*(\mu_4)=0$ fro $\pi^*\colon KO^{-6}(P^4)\to KO^{-6}(S^4)$. Suppose that $\eta_4\bar{\beta}(\rho_1)\in \operatorname{Im}\pi^*$, that is, there exists an element $a\in KO^*(PE_7)$ such that $\pi^*(a)=-\eta_4\bar{\beta}(\rho_1)$. Then we see by these formulas and (5.8) that a takes the form of

$$a=\beta_{\rho_1}+\xi x+\tilde{\nu}y+\tilde{\beta}_\mu z\in KO^*(PE_7)$$

where $x, y, z\in KO_G^*(S^{9,0}\times E_7)$. So

$$0=2\delta(a)=-2\cdot 7\tau_{1,0}^R\wedge 1+2\xi\delta(x)\,.$$

From (5.6) it follows that $\delta(x)$ takes the form of $\delta(x)=\tau_{1,0}^R\wedge x'(x'\in KO^*(PE_7))$. Hence operating $j^*\,c$ to this equality we obtain

$$2\cdot 7c(\xi)+4c(\xi)\,c(x')=0$$

where $j^*\colon \tilde{K}_G^*(\Sigma^{1,0}\wedge E_{7+})\to K_G^*(B^{1,0}\times E_7)=K^*(PE_7)$ and so we have

$$4c(\xi)=0$$

because of $8c(\xi)=0$. This is a contradiction to the fact that the order of $c(\xi)$ is 8. Consequently we have

(5.11) $$\eta_4\,\bar{\beta}(\rho_1)\notin \operatorname{Im}\pi^*\,.$$

Analogously we have

(5.12) $\eta_4\bar{\beta}(\rho_1),\ \eta_4\bar{\beta}(\rho_1)\bar{\beta}(\rho_3),\ \eta_4\bar{\beta}(\rho_1)\bar{\beta}(\rho_7),\ \eta_4\bar{\beta}(\rho_1)\bar{\beta}(\rho_3)\bar{\beta}(\rho_7) \notin \mathrm{Im}\ \pi^*$.

Combining (5.7)-(5.12) we see that R^* fills $KO^*(PE_7)$, that is,

$$KO^*(PE_7) = R^*$$

which complets the proof of Theorem 4.1.

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