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REAL HYPERSURFACES WITH KILLING STRUCTURE JACOBI OPERATOR IN THE COMPLEX HYPERBOLIC QUADRIC

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Abstract

First we introduce the notion of Killing structure Jacobi operator for real hypersurfaces in the complex hyperbolic quadric $Q^{m*} = SO_{2,m}^0/SO_2SO_m$. Next we give a complete classification of real hypersurfaces in $Q^{m*} = SO_{2,m}^0/SO_2SO_m$ with Killing structure Jacobi operator.

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1. Introduction

In case of Hermitian symmetric space of rank 1, we say a complex projective space $\mathbb{C}P^m$ and a complex hyperbolic space $\mathbb{C}H^m$. In the complex projective space $\mathbb{C}P^m$, a full classification of real hypersurfaces with isometric Reeb flow was obtained by Okumura in [16]. He proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_mU_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. Moreover, Takagi [41] gave a complete classification of homogeneous hypersurfaces in $\mathbb{C}P^m$ and Kimura and etc., [7] considered the notion GTW Reeb parallel shape operator. In the complex hyperbolic space $\mathbb{C}H^m$, Montiel and Romero [13] have given a complete classification of real hypersurface with isometric Reeb flow.

As another kind of Hermitian symmetric space with rank 2 of non-compact type different from the above ones, we can give the example of complex hyperbolic quadric $Q^{m*} = SO_{2,m}^0/SO_2SO_m$. By using the method given in Kobayashi and Nomizu [12], Chapter XI, Example 10.6, the complex hyperbolic quadric $Q^{m*} = SO_{2,m}^0/SO_2SO_m$ can be immersed in indefinite complex hyperbolic space $\mathbb{C}H_1^{m+1}$ as a space-like complex hypersurface (see Montiel and Romero [15] and Suh [34]). The complex hyperbolic quadric Q^{m*} is the non-compact Hermitian symmetric space $SO_{2,m}^0/SO_2SO_m$ of rank 2 and also can be regarded as a kind of real Grassmann manifold of all oriented space-like 2-dimensional subspaces in indefinite flat Riemannian space \mathbb{R}_2^{m+2} (see Montiel and Romero [14] and [15]). Accordingly, the complex hyperbolic quadric admits both a complex conjugation structure A and a Kähler structure J , which anti-commutes with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^{m*}, J, g) is a Hermitian symmetric space of noncompact type with rank 2 and its minimal sectional curvature is equal to -4 (see Klein [8] and Reckziegel [22]).

Now let us consider a real hypersurface in the complex hyperbolic quadric Q^{m*} with isometric Reeb flow. Then from the view of the previous results a natural expectation might be the totally geodesic $Q^{m-1*} \subset Q^{m*}$. But, surprisingly, in the complex hyperbolic quadric Q^{m*} the situation is quite different from the above ones. Recently, Suh [34] has introduced the following result:

Theorem A. *Let M be a real hypersurface of the complex hyperbolic quadric $Q^{m*} = SO_{m,2}^o/SO_m SO_2$, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$ or a horosphere whose center at infinity is \mathfrak{A} -isotropic singular.*

Jacobi fields along geodesics of a given Riemannian manifold (M, g) satisfy a well known differential equation. This equation naturally inspires the so-called Jacobi operator. That is, if R denotes the curvature operator of M , and X is tangent vector field to M , then the Jacobi operator $R_X \in \text{End}(T_x M)$ with respect to X at $x \in M$, defined by $(R_X Y)(x) = (R(Y, X)X)(x)$ for any $Y \in T_x M$, becomes a self adjoint endomorphism of the tangent bundle TM of M . Thus, each tangent vector field X to M provides a Jacobi operator R_X with respect to X . In particular, for the Reeb vector field ξ , the Jacobi operator R_ξ is said to be a *structure Jacobi operator*.

Recently Ki, Pérez, Santos and Suh [5] have investigated the Reeb parallel structure Jacobi operator in the complex space form $M_m(c)$, $c \neq 0$ and have used it to study some principal curvatures for a tube over a totally geodesic submanifold. In particular, Pérez, Jeong and Suh [20] have investigated real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field X on M . Jeong, Suh and Woo [4] and Pérez and Santos [18] have generalized such a notion to the recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for a certain 1-form β and any vector fields X, Y on M in $G_2(\mathbb{C}^{m+2})$. Moreover, Pérez, Santos and Suh [19] have further investigated the property of the Lie ξ -parallel structure Jacobi operator in complex projective space $\mathbb{C}P^m$, that is, $\mathcal{L}_\xi R_\xi = 0$.

The Reeb vector field ξ is *Killing* on M in Q^{m*} if and only if $g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$ for any vector fields X and Y on M . As a generalization of such a Killing vector field first Yano [42] defined the notion of *Killing tensor* as follows:

A skew symmetric tensor $T_{i_1 \dots i_r}$ is called a *Killing tensor* of order r if it satisfies

$$\nabla_{i_1} T_{i_2 \dots i_{r+1}} + \nabla_{i_2} T_{i_1 \dots i_{r+1}} = 0.$$

Next Blair [2] has applied the notion of Killing tensor to a tensor field of T type $(1, 1)$ on a Riemannian manifold and a geodesic γ on M . If we denote by γ' the tangent vector of the geodesic γ , then $T\gamma'$ is parallel along the geodesic γ for the Killing tensor field T . Geometrically, this means that $(\nabla_{\gamma'} T)\gamma' = 0$ along a geodesic γ on M . If this is the case for any geodesic on M , we have

$$(\nabla_X T)X = 0 \quad \text{or equivalently} \quad (\nabla_X T)Y + (\nabla_Y T)X = 0$$

for any vector fields X and Y on M . In this case we say that the tensor T is a *Killing tensor field of type $(1, 1)$* .

Now we consider such a situation to the structure Jacobi operator R_ξ , which is a tensor field of type $(1, 1)$ on a real hypersurface M in Q^{m*} . The structure Jacobi operator R_ξ of M in Q^m is said to be *Killing* if the structure Jacobi operator R_ξ satisfies

$$(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$$

for any $X, Y \in T_z M$, $z \in M$. The equation is equivalent to $(\nabla_X R_\xi)X = 0$ for any $X \in T_z M$, $z \in M$, because of polarization. Moreover, we can give the geometric meaning of the Killing Jacobi operator as follows:

When we consider a geodesic γ with initial conditions such that $\gamma(0) = z$ and $\dot{\gamma}(0) = X$. Then the transformed vector field $R_\xi \dot{\gamma}$ is Levi-Civita *parallel* along the geodesic γ of the vector field X (see Blair [2] and Tachibana [40]).

In addition to the complex structure J there is another distinguished geometric structure on Q^{m*} , namely a parallel rank two vector bundle \mathfrak{A} which contains an S^1 -bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^{m*} . This geometric structure determines a maximal \mathfrak{A} -invariant subbundle \mathcal{Q} of the tangent bundle TM of a real hypersurface M in Q^{m*} as follows:

$$\mathcal{Q} = \{X \in T_z M \mid AX \in T_z M \text{ for all } A \in \mathfrak{A}\}.$$

Recall that a nonzero tangent vector $W \in T_{[z]} Q^{m*}$ is called singular if it is tangent to more than one maximal flat in Q^{m*} . There are two types of singular tangent vectors for the complex hyperbolic quadric Q^{m*} :

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called \mathfrak{A} -principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called \mathfrak{A} -isotropic

where $V(A) = \{X \in T_{[z]} Q^{m*} \mid AX = X\}$ and $JV(A) = \{X \in T_{[z]} Q^{m*} \mid AX = -X\}$, $[z] \in Q^{m*}$, are the $(+1)$ -eigenspace and (-1) -eigenspace for the involution A on $T_{[z]} Q^{m*}$, $[z] \in Q^{m*}$.

In the study of real hypersurfaces in the complex quadric Q^m we considered the notion of parallel Ricci tensor, that is, $\nabla \text{Ric} = 0$ (see Suh [31]). But from the assumption of Ricci parallel, it was difficult for us to derive the fact that either the unit normal N is \mathfrak{A} -isotropic or \mathfrak{A} -principal. So in [31] we gave a classification with the further assumption of \mathfrak{A} -isotropic. But fortunately, if we consider a Hopf real hypersurfaces, which is defined by $S\xi = \alpha\xi$ for the Reeb function $\alpha = g(S\xi, \xi)$ and the shape operator S , in the complex hyperbolic quadric Q^{m*} with Killing structure Jacobi operator, we can assert that the unit normal vector field N becomes either \mathfrak{A} -isotropic or \mathfrak{A} -principal as follows:

Main Theorem 1. *Let M be a Hopf real hypersurface in Q^{m*} , $m \geq 3$, with Killing structure Jacobi operator. Then the unit normal vector field N is singular, that is, N is \mathfrak{A} -isotropic or \mathfrak{A} -principal.*

When we consider a hypersurface M in the complex hyperbolic quadric Q^{m*} , the unit normal vector field N of M in Q^{m*} can be divided into two cases : N is \mathfrak{A} -isotropic or

\mathfrak{A} -principal (see [34], [35] and [27]). In the first case where M has an \mathfrak{A} -isotropic unit normal N , we have asserted in [34] and [35] that M is locally congruent to a tube over a totally geodesic complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k*} or a horosphere with \mathfrak{A} -isotropic unit normal vector field centered at the infinity. In the second case when N is \mathfrak{A} -principal we have proved that M is locally congruent to a tube over a totally geodesic and totally real submanifold Q^{m-1*} in Q^{m*} (see [34], [36] and [38]).

In this paper we consider the case that the structure Jacobi operator R_ξ of M in Q^{m*} is Killing, that is, $(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$ for any tangent vector field X and Y on M , and we prove the following

Main Theorem 2. *There does not exist a Hopf hypersurface in Q^{m*} , $m \geq 3$ with Killing structure Jacobi operator and \mathfrak{A} -principal unit normal vector field.*

Now it remains to prove the case that the unit normal vector field is \mathfrak{A} -isotropic. Then by our Main Theorems 1 and 2, we give a classification of real hypersurfaces in Q^{m*} with Killing structure Jacobi operator as follows:

Main Theorem 3. *Let M be a Hopf hypersurface in Q^{m*} , $m \geq 3$ with Killing structure Jacobi operator. If the Reeb function is constant along the Reeb direction, then M has 4 distinct constant principal curvatures*

$$\alpha, \quad \beta = 0, \quad \lambda_1 = \lambda_2.$$

Here the corresponding eigen spaces $\xi \in T_\alpha$, $T_\beta = Q^\perp$, and $T_{\lambda_1} \oplus T_{\lambda_2} = Q$, where the principal curvatures λ_1 and λ_2 are two distinct constants given by

$$\lambda_1 = \frac{\alpha(\alpha^2 - 1) + \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}$$

and

$$\lambda_2 = \frac{\alpha(\alpha^2 - 1) - \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}.$$

with multiplicities $(m - 2)$ respectively and $\alpha^2 > 2\sqrt{2} + 1$.

REMARK 1.1. In [29] Suh has proved that the Reeb function $\alpha = g(S\xi, \xi)$ is constant for real hypersurfaces with singular normal vector field in the complex quadric Q^m . But in the complex hyperbolic quadric Q^{m*} the Reeb function α is constant only if the unit normal vector field N is \mathfrak{A} -principal (see Suh, Pérez and Woo [39]). Until now it does not known to us whether the Reeb function α is constant for real hypersurfaces in the complex hyperbolic quadric Q^{m*} with \mathfrak{A} -isotropic unit normal vector field.

The subbundle Q mentioned in Main Theorem 3 is the maximal invariant subbundle of $T_z M$, $z \in M$, such that $Q \oplus Q^\perp = [\xi]^\perp$, where $Q^\perp = \text{Span}\{A\xi, AN\}$ and $[\xi]^\perp$ denotes the orthogonal complement of the Reeb vector field ξ in $T_z M$, $z \in M$, in Q^{m*} .

When we consider a parallel structure Jacobi operator on M in Q^{m*} , we know that $(\nabla_X R_\xi)Y = 0$ for any vector fields X and Y on M . This gives a condition stronger than

the notion of Killing structure Jacobi operator. So naturally it satisfies the assumptions of Killing in Main Theorems 1, 2 and 3. For the case of isotropic unit normal N , it can be easily checked that the results in our Main Theorem 3 do not satisfy the strong assumption of parallel structure Jacobi operator. So we also conclude the following

Corollary (see [39]). *There does not exist a Hopf hypersurface in the complex hyperbolic quadric Q^{m*} , $m \geq 3$, with parallel structure Jacobi operator.*

2. The complex hyperbolic quadric

In this section, let us introduce a new known result of the complex hyperbolic quadric Q^{m*} different from the complex quadric Q^m . This section is due to Klein and Suh [10].

The m -dimensional complex hyperbolic quadric Q^{m*} is the non-compact dual of the m -dimensional complex quadric Q^m , which is a kind of Hermitian symmetric space of non-compact type with rank 2 (see Besse [1], and Helgason [3]).

The complex hyperbolic quadric Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$. In fact, Smyth [24, Theorem 3(ii)] has shown that every homogeneous complex hypersurface in $\mathbb{C}H^{m+1}$ is totally geodesic. This is in marked contrast to the situation for the complex quadric Q^m , which can be realized as a homogeneous complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$ in such a way that the shape operator for any unit normal vector to Q^m is a real structure on the corresponding tangent space of Q^m , see [8] and [22]. Another related result by Smyth, [24, Theorem 1], which states that any complex hypersurface $\mathbb{C}H^{m+1}$ for which the square of the shape operator has constant eigenvalues (counted with multiplicity) is totally geodesic, also precludes the possibility of a model of Q^{m*} as a complex hypersurface of $\mathbb{C}H^{m+1}$ with the analogous property for the shape operator.

Therefore we realize the complex hyperbolic quadric Q^{m*} as the quotient manifold $SO_{2,m}^0/SO_2SO_m$. As Q^{1*} is isomorphic to the real hyperbolic space $\mathbb{R}H^2 = SO_{1,2}^0/SO_2$, and Q^{2*} is isomorphic to the Hermitian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$, we suppose $m \geq 3$ in the sequel and throughout this paper. Let $G := SO_{2,m}^0$ be the transvection group of Q^{m*} and $K := SO_2SO_m$ be the isotropy group of Q^{m*} at the “origin” $p_0 := eK \in Q^{m*}$. Then

$$\sigma : G \rightarrow G, \quad g \mapsto sgs^{-1} \quad \text{with} \quad s := \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

is an involutive Lie group automorphism of G with $\text{Fix}(\sigma)_0 = K$, and therefore $Q^{m*} = G/K$ is a Riemannian symmetric space. The center of the isotropy group K is isomorphic to SO_2 , and therefore Q^{m*} is in fact a Hermitian symmetric space.

The Lie algebra $\mathfrak{g} := \mathfrak{so}_{2,m}$ of G is given by

$$\mathfrak{g} = \{X \in \mathfrak{gl}(m+2, \mathbb{R}) \mid X^T \cdot s = -s \cdot X\}$$

(see [11, p. 59]). In the sequel we will write members of \mathfrak{g} as block matrices with respect to the decomposition $\mathbb{R}^{m+2} = \mathbb{R}^2 \oplus \mathbb{R}^m$, i.e. in the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where $X_{11}, X_{12}, X_{21}, X_{22}$ are real matrices of the dimension 2×2 , $2 \times m$, $m \times 2$ and $m \times m$, respectively. Then

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \mid X_{11}^t = -X_{11}, X_{12}^t = X_{21}, X_{22}^t = -X_{22} \right\}.$$

The linearisation $\sigma_L = \text{Ad}(s) : \mathfrak{g} \rightarrow \mathfrak{g}$ of the involutive Lie group automorphism σ induces the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where the Lie subalgebra

$$\begin{aligned} \mathfrak{k} &= \text{Eig}(\sigma_*, 1) = \{X \in \mathfrak{g} \mid sXs^{-1} = X\} \\ &= \left\{ \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \mid X_{11}^t = -X_{11}, X_{22}^t = -X_{22} \right\} \\ &\cong \mathfrak{so}_2 \oplus \mathfrak{so}_m \end{aligned}$$

is the Lie algebra of the isotropy group K , and the $2m$ -dimensional linear subspace

$$\mathfrak{m} = \text{Eig}(\sigma_*, -1) = \{X \in \mathfrak{g} \mid sXs^{-1} = -X\} = \left\{ \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \mid X_{12}^t = X_{21} \right\}$$

is canonically isomorphic to the tangent space $T_{p_0}Q^{m*}$. Under the identification $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the Riemannian metric g of Q^{m*} (where the constant factor of the metric is chosen so that the formulae become as simple as possible) is given by

$$g(X, Y) = \frac{1}{2} \text{tr}(Y^t \cdot X) = \text{tr}(Y_{12} \cdot X_{21}) \quad \text{for } X, Y \in \mathfrak{m}.$$

g is clearly $\text{Ad}(K)$ -invariant, and therefore corresponds to an $\text{Ad}(G)$ -invariant Riemannian metric on Q^{m*} . The complex structure J of the Hermitian symmetric space is given by

$$JX = \text{Ad}(j)X \quad \text{for } X \in \mathfrak{m}, \quad \text{where } j := \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \in K.$$

Because j is in the center of K , the orthogonal linear map J is $\text{Ad}(K)$ -invariant, and thus defines an $\text{Ad}(G)$ -invariant Hermitian structure on Q^{m*} . By identifying the multiplication with the unit complex number i with the application of the linear map J , the tangent spaces of Q^{m*} thus become m -dimensional complex linear spaces, and we will adopt this point of view in the sequel.

As mentioned for the complex quadric (again compare [8], [9], and [22]), there is another important structure on the tangent bundle of the complex quadric besides the Riemannian metric and the complex structure, namely an S^1 -bundle \mathfrak{A} of real structures. The situation here differs from that of the complex quadric in that for Q^{m*} , the real structures in \mathfrak{A} cannot be interpreted as the shape operator of a complex hypersurface in a complex space form, but as the following considerations will show, \mathfrak{A} still plays an important role in the description of the geometry of Q^{m*} .

Let

$$a_0 := \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}.$$

Note that we have $a_0 \notin K$, but only $a_0 \in O_2 SO_m$. However, $\text{Ad}(a_0)$ still leaves \mathfrak{m} invariant, and therefore defines an \mathbb{R} -linear map A_0 on the tangent space $\mathfrak{m} \cong T_{p_0} Q^{m*}$. A_0 turns out to be an involutive orthogonal map with $A_0 \circ J = -J \circ A_0$ (i.e. A_0 is anti-linear with respect to the complex structure of $T_{p_0} Q^{m*}$), and hence a real structure on $T_{p_0} Q^{m*}$. But A_0 commutes with $\text{Ad}(g)$ not for all $g \in K$, but only for $g \in SO_m \subset K$. More specifically, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_m$, say $g_1 = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ with $t \in \mathbb{R}$ (so that $\text{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu := e^{it}$), we have

$$A_0 \circ \text{Ad}(g) = \mu^{-2} \cdot \text{Ad}(g) \circ A_0.$$

This equation shows that the object which is $\text{Ad}(K)$ -invariant and therefore geometrically relevant is not the real structure A_0 by itself, but rather the “circle of real structures”

$$\mathfrak{A}_{p_0} := \{\lambda A_0 | \lambda \in S^1\}.$$

\mathfrak{A}_{p_0} is $\text{Ad}(K)$ -invariant, and therefore generates an $\text{Ad}(G)$ -invariant S^1 -subbundle \mathfrak{A} of the endomorphism bundle $\text{End}(TQ^{m*})$, consisting of real structures on the tangent spaces of Q^{m*} . For any $A \in \mathfrak{A}$, the tangent line to the fibre of \mathfrak{A} through A is spanned by JA .

For any $p \in Q^{m*}$ and $A \in \mathfrak{A}_p$, the real structure A induces a splitting

$$T_p Q^{m*} = V(A) \oplus JV(A)$$

into two orthogonal, maximal totally real subspaces of the tangent space $T_p Q^{m*}$. Here $V(A)$ resp. $JV(A)$ are the $(+1)$ -eigenspace resp. the (-1) -eigenspace of A . For every unit vector $W \in T_p Q^{m*}$ there exist $t \in [0, \frac{\pi}{4}]$, $A \in \mathfrak{A}_p$ and orthonormal vectors $X, Y \in V(A)$ so that

$$W = \cos(t) \cdot X + \sin(t) \cdot JY$$

holds; see [22, Proposition 3]. Here t is uniquely determined by W . The vector W is singular, i.e. contained in more than one Cartan subalgebra of \mathfrak{m} , if and only if either $t = 0$ or $t = \frac{\pi}{4}$ holds. The vectors with $t = 0$ are called \mathfrak{A} -principal, whereas the vectors with $t = \frac{\pi}{4}$ are called \mathfrak{A} -isotropic. If W is regular, i.e. $0 < t < \frac{\pi}{4}$ holds, then also A and X, Y are uniquely determined by W .

The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$. Later we will need the eigenvalues and eigenspaces of the Jacobi operator $R_W = R(\cdot, W)W$ for a singular unit tangent vector W .

1. If W is an \mathfrak{A} -principal singular unit tangent vector with respect to $A \in \mathfrak{A}$, then the eigenvalues of R_W are 0 and 2 and the corresponding eigenspaces are $\mathbb{R}W \oplus J(V(A) \ominus \mathbb{R}W)$ and $(V(A) \ominus \mathbb{R}W) \oplus \mathbb{R}JW$, respectively.
2. If W is an \mathfrak{A} -isotropic singular unit tangent vector with respect to $A \in \mathfrak{A}$ and $X, Y \in V(A)$, then the eigenvalues of R_W are 0, 1 and 4 and the corresponding eigenspaces are $\mathbb{R}W \oplus \mathbb{C}(JX + Y)$, $T_z Q^m \ominus (\mathbb{C}X \oplus \mathbb{C}Y)$ and $\mathbb{R}JW$, respectively.

Like for the complex quadric, the Riemannian curvature tensor \bar{R} of Q^{m*} can be fully described in terms of the “fundamental geometric structures” g , J and \mathfrak{A} . In fact, under the correspondence $T_{p_0}Q^{m*} \cong \mathfrak{m}$, the curvature $\bar{R}(X, Y)Z$ corresponds to $-[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, see [12, Chapter XI, Theorem 3.2(1)]. By evaluating the latter expression explicitly, one can show that one has

$$\begin{aligned}\bar{R}(X, Y)Z &= -g(Y, Z)X + g(X, Z)Y \\ &\quad -g(JY, Z)JX + g(JX, Z)JY + 2g(JX, Y)JZ \\ &\quad -g(AY, Z)AX + g(AX, Z)AY \\ &\quad -g(JAY, Z)JAX + g(JAX, Z)JAY\end{aligned}$$

for arbitrary $A \in \mathfrak{A}_{p_0}$. Therefore the curvature of Q^{m*} is the negative of that of the complex quadric Q^m , compare [22, Theorem 1]. This confirms that the symmetric space Q^{m*} which we have constructed here is indeed the non-compact dual of the complex quadric.

3. Some general equations

Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M . The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM . The structure tensor field ϕ restricted to \mathcal{C} coincides with the complex structure J restricted to \mathcal{C} , and $\phi\xi = 0$.

At each point $z \in M$ we define the maximal \mathfrak{A} -invariant subspace of $T_z M$, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_z M \mid AX \in T_z M \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1 (see [29]). *For each $z \in M$ we have*

- (i) *If N_z is \mathfrak{A} -principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
- (ii) *If N_z is not \mathfrak{A} -principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then for the Reeb vector field ξ the shape operator S becomes

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi, \xi)$ on M . When we consider a transform JX of the Kaehler structure J on the complex hyperbolic quadric Q^{m*} for any vector field X on M in Q^{m*} , we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M .

Then we now consider the Codazzi equation

$$\begin{aligned}
(3.1) \quad g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) \\
&\quad - g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z) \\
&\quad - g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z).
\end{aligned}$$

Putting $Z = \xi$ we get

$$\begin{aligned}
g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= 2g(\phi X, Y) \\
&\quad - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\
&\quad + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\
&= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\
&= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi S X, Y).
\end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$(3.2) \quad Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\
&= 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\
&\quad - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X) \\
&\quad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi S X, Y).
\end{aligned}$$

Altogether this implies

$$\begin{aligned}
0 &= 2g(S\phi S X, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\
&\quad - g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\
&\quad + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\
&\quad - 2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\
&\quad + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$ (see Proposition 3 in [22]). Note that t is a function on M . First of all, since $\xi = -JN$, we have

$$\begin{aligned}
AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\
\xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\
A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.
\end{aligned}$$

This implies $g(\xi, AN) = 0$ and hence

$$0 = 2g(S\phi S X, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y)$$

$$\begin{aligned}
& -g(X, AN)g(Y, A\xi) + g(Y, AN)g(X, A\xi) \\
& + g(X, A\xi)g(JY, A\xi) - g(Y, A\xi)g(JX, A\xi) \\
& + 2g(X, AN)g(\xi, A\xi)\eta(Y) - 2g(Y, AN)g(\xi, A\xi)\eta(X).
\end{aligned}$$

We have $JA\xi = -AJ\xi = -AN$, and inserting this into the previous equation implies

Lemma 3.2. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*} with (local) unit normal vector field N . For each point $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Then*

$$\begin{aligned}
0 = & 2g(S\phi S X, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y) \\
& - 2g(X, AN)g(Y, A\xi) + 2g(Y, AN)g(X, A\xi) \\
& - 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\}
\end{aligned}$$

holds for all vector fields X and Y on M .

We can write for any vector field Y on M in Q^{m*}

$$AY = BY + \rho(Y)N,$$

where BY denotes the tangential component of AY and $\rho(Y) = g(AY, N)$.

If N is \mathfrak{A} -principal, that is, $AN = N$, we have $\rho = 0$, because $\rho(Y) = g(Y, AN) = g(Y, N) = 0$ for any tangent vector field Y on M in Q^{m*} . So we have $AY = BY$ for any tangent vector field Y on M in Q^{m*} . Otherwise we can use Lemma 3.1 to calculate $\rho(Y) = g(Y, AN) = g(Y, AJ\xi) = -g(Y, JA\xi) = -g(Y, JB\xi) = -g(Y, \phi B\xi)$ for any tangent vector field Y on M in Q^{m*} . From this, together with Lemma 3.2, we have proved

Lemma 3.3. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*} , $m \geq 3$. Then we have*

$$(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi - \beta\xi) + 2g(X, B\xi - \beta\xi)\phi B\xi,$$

where the function β is given by $\beta = g(\xi, A\xi) = -g(N, AN)$.

If the unit normal vector field N is \mathfrak{A} -principal, we can choose a real structure $A \in \mathfrak{A}$ such that $AN = N$. Then we have $\rho = 0$ and $\phi B\xi = -\phi\xi = 0$, and therefore

$$(3.3) \quad 2S\phi S - \alpha(\phi S + S\phi) = -2\phi.$$

If N is not \mathfrak{A} -principal, we can choose a real structure $A \in \mathfrak{A}$ as in Lemma 3.1 and get

$$\begin{aligned}
(3.4) \quad & \rho(X)(B\xi - \beta\xi) + g(X, B\xi - \beta\xi)\phi B\xi \\
& = -g(X, \phi(B\xi - \beta\xi))(B\xi - \beta\xi) + g(X, B\xi - \beta\xi)\phi(B\xi - \beta\xi) \\
& = \|B\xi - \beta\xi\|^2\{g(X, U)\phi U - g(X, \phi U)U\} \\
& = \sin^2(2t)\{g(X, U)\phi U - g(X, \phi U)U\},
\end{aligned}$$

which is equal to 0 on Q and equal to $\sin^2(2t)\phi X$ on $C \ominus Q$. Altogether we have proved:

Lemma 3.4. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*} , $m \geq 3$. Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

leaves \mathcal{Q} and $\mathcal{C} \ominus \mathcal{Q}$ invariant and we have

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

and

$$2S\phi S - \alpha(\phi S + S\phi) = -2\beta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},$$

where $\beta = g(A\xi, \xi) = -\cos 2t$ as in section 3.

Then from the equation of Gauss the curvature tensor R of M in complex quadric Q^{m*} is defined so that

$$\begin{aligned} R(X, Y)Z &= -g(Y, Z)X + g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z \\ &\quad -g(AY, Z)(AX)^T + g(AX, Z)(AY)^T - g(JAY, Z)(JAX)^T \\ &\quad + g(JAX, Z)(JAY)^T + g(SY, Z)SX - g(SX, Z)SY, \end{aligned}$$

where $(AX)^T$ and S denote the tangential component of the vector field AX and the shape operator of M in Q^{m*} respectively.

From this, putting $Y = Z = \xi$ and using $g(A\xi, N) = 0$, the structure Jacobi operator is defined by

$$\begin{aligned} R_\xi(X) &= R(X, \xi)\xi \\ &= -X + \eta(X)\xi - g(A\xi, \xi)(AX)^T + g(AX, \xi)A\xi \\ &\quad + g(X, AN)(AN)^T + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \end{aligned}$$

Then we may put the following

$$(AY)^T = AY - g(AY, N)N.$$

Now let us denote by ∇ and $\bar{\nabla}$ the covariant derivative of M and the covariant derivative of Q^{m*} respectively. Then by using the Gauss and Weingarten formulas we can assert the following

Lemma 3.5. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} . Then*

$$\begin{aligned} (3.5) \quad \nabla_X(AY)^T &= q(X)JAY + A\nabla_X Y + g(SX, Y)AN \\ &\quad - g(\{q(X)JAY + A\nabla_X Y + g(SX, Y)AN\}, N)N \\ &\quad + g(AY, N)SX. \end{aligned}$$

Proof. First let us use the Gauss formula to $(AY)^T = AY - g(AY, N)N$. Then it follows that

$$\begin{aligned}\nabla_X(AY)^T &= \bar{\nabla}_X(AY)^T - \sigma(X, (AY)^T) \\ &= \bar{\nabla}_X\{AY - g(AY, N)N\} - g(SX, (AY)^T)N \\ &= (\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y - g((\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y, N)N - g(AY, \bar{\nabla}_X N)N \\ &\quad - g(AY, N)\bar{\nabla}_X N - g(SX, (AY)^T)N,\end{aligned}$$

where σ denotes the second fundamental form and N the unit normal vector field on M in Q^{m*} . Then from this, if we use Weingarten formula $\bar{\nabla}_X N = -SX$, then we get the above formula. \square

By putting $Y = \xi$ and using $g(A\xi, N) = 0$, we have

$$\begin{aligned}(3.6) \quad \nabla_X(A\xi) &= q(X)JA\xi + A\phi SX + \alpha\eta(X)AN \\ &\quad - \{q(X)g(JA\xi, N) + g(A\phi SX, N) + \alpha\eta(X)g(AN, N)\}N.\end{aligned}$$

Moreover, let us also use Gauss and Weingarten formula to $(AN)^T = AN - g(AN, N)N$. Then it follows that

$$\begin{aligned}(3.7) \quad \nabla_X(AN)^T &= \bar{\nabla}_X(AN)^T - \sigma(X, (AN)^T) \\ &= \bar{\nabla}_X\{AN - g(AN, N)N\} - \sigma(X, (AN)^T) \\ &= (\bar{\nabla}_X A)N + A\bar{\nabla}_X N - g((\bar{\nabla}_X A)N + A\bar{\nabla}_X N, N) \\ &\quad - g(AN, \bar{\nabla}_X N)N - g(AN, N)\bar{\nabla}_X N - \sigma(X, (AN)^T) \\ &= q(X)JAN - ASX - g(q(X)JAN - ASX, N)N + g(AN, N)SX.\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}(3.8) \quad X\beta &= X(g(A\xi, \xi)) \\ &= g((\bar{\nabla}_X A)\xi + A\bar{\nabla}_X \xi, \xi) + g(A\xi, \bar{\nabla}_X \xi) \\ &= g(q(X)JA\xi + A\phi SX + g(SX, \xi)AN, \xi) + g(A\xi, \phi SX + g(SX, \xi)N) \\ &= 2g(A\phi SX, \xi).\end{aligned}$$

4. Some Important Lemmas and Proof of Theorem 1

The curvature tensor $R(X, Y)Z$ for a Hopf real hypersurface M in the complex hyperbolic quadric Q^{m*} induced from the curvature tensor of Q^{m*} is given in section 3. Now the structure Jacobi operator R_ξ can be rewritten as follows:

$$\begin{aligned}(4.1) \quad R_\xi(X) &= R(X, \xi)\xi \\ &= -X + \eta(X)\xi - \beta(AX)^T + g(AX, \xi)A\xi + g(AX, N)(AN)^T \\ &\quad + \alpha SX - g(SX, \xi)S\xi,\end{aligned}$$

where we have put $\alpha = g(S\xi, \xi)$ and $\beta = g(A\xi, \xi)$, because we assume that M is Hopf. The Reeb vector field $\xi = -JN$ and the anti-commuting property $AJ = -JA$ gives that the function β becomes $\beta = -g(AN, N)$. When this function $\beta = g(A\xi, \xi)$ identically vanishes,

we say that a real hypersurface M in Q^{m*} is \mathfrak{A} -isotropic as in section 1.

Here let us differentiate the structure Jacobi operator R_ξ along any direction X on M in the complex hyperbolic quadric Q^{m*} . Then (4.1), together with (3.5), (3.6), (3.7), give that

$$\begin{aligned}
(4.2) \quad \nabla_X R_\xi(Y) &= \nabla_X(R_\xi(Y)) - R_\xi(\nabla_X Y) \\
&= g(\phi S X, Y)\xi + \eta(Y)\phi S X - (X\beta)(AY)^T \\
&\quad - \beta[q(X)JAY + A\nabla_X Y + g(S X, Y)AN \\
&\quad - g(\{q(X)JAY + A\nabla_X Y + g(S X, Y)AN\}, N)N \\
&\quad + g(AY, N)SX] \\
&\quad + g(q(X)JA\xi + A\phi S X + \alpha\eta(X)AN, Y)A\xi \\
&\quad + g(AY, \xi)[g(q(X)JA\xi + A\phi S X + \alpha\eta(X)AN \\
&\quad - \{q(X)g(JA\xi, N) + g(A\phi S X, N) + \alpha\eta(X)g(AN, N)\}N] \\
&\quad + [g(q(X)JAN - AS X + g(AN, N)SX, Y)(AN)^T \\
&\quad + g(Y, (AN)^T)\{q(X)JAN - AS X + g(AN, N)SX \\
&\quad - g(q(X)JAN - AS X, N)N\}] \\
&\quad + (X\alpha)SY + \alpha(\nabla_X S)Y - X(\alpha^2)\eta(Y)\xi \\
&\quad - \alpha^2(\nabla_X \eta)(Y)\xi - \alpha^2\eta(Y)\nabla_X \xi,
\end{aligned}$$

where we have used $g(A\xi, N) = 0$, and N the unit normal to M in Q^{m*} .

Here let us assume that the structure Jacobi operator is Killing, that is, $(\nabla_X R_\xi)Y + (\nabla_Y R_\xi)X = 0$ for any tangent vector fields X and Y on M in Q^{m*} . Then from this, together with (4.1), we have the following

$$\begin{aligned}
(4.3) \quad 0 &= \nabla_X R_\xi(Y) + \nabla_Y R_\xi(X) \\
&= \{g(\phi S X, Y) + g(\phi S Y, X)\}\xi + \eta(Y)\phi S X + \eta(X)\phi S Y \\
&\quad - (X\beta)(AY)^T - (Y\beta)(AX)^T \\
&\quad - \beta[q(X)JAY + q(Y)JAX + A(\nabla_X Y + \nabla_Y X) + 2g(S X, Y)AN \\
&\quad - g(\{q(X)JAY + q(Y)JAX + A(\nabla_X Y + \nabla_Y X) + 2g(S X, Y)AN\}, N)N \\
&\quad + g(AY, N)SX + g(AX, N)SY] \\
&\quad + [g(q(X)JA\xi + A\phi S X + \alpha\eta(X)AN, Y) \\
&\quad + g(q(Y)JA\xi + A\phi S Y + \alpha\eta(Y)AN, X)]A\xi \\
&\quad + g(AY, \xi)[q(X)JA\xi + A\phi S X + \alpha\eta(X)AN \\
&\quad - \{q(X)g(JA\xi, N) + g(A\phi S X, N) + \alpha\eta(X)g(AN, N)\}N] \\
&\quad + g(AX, \xi)[q(Y)JA\xi + A\phi S Y + \alpha\eta(Y)AN \\
&\quad - \{q(Y)g(JA\xi, N) + g(A\phi S Y, N) + \alpha\eta(Y)g(AN, N)\}N] \\
&\quad + [g(q(X)JAN - AS X + g(AN, N)SX, Y)
\end{aligned}$$

$$\begin{aligned}
& + g(q(Y)JAN - ASY + g(AN, N)SY, X)(AN)^T \\
& + g(Y, (AN)^T)\{q(X)JAN - ASX - g(q(X)JAN - ASX, N)N \\
& + g(AN, N)SX\} \\
& + g(X, (AN)^T)\{q(Y)JAN - ASY - g(q(Y)JAN - ASY, N)N \\
& + g(AN, N)SY\} \\
& + (X\alpha)SY + (Y\alpha)SX + \alpha\{(\nabla_X S)Y + (\nabla_Y S)X\} \\
& - X(\alpha^2)\eta(Y)\xi - (Y\alpha^2)\eta(X)\xi - \alpha^2\{(\nabla_X \eta)(Y)\xi + (\nabla_Y \eta)(X)\xi\} \\
& - \alpha^2\{\eta(Y)\nabla_X \xi + \eta(X)\nabla_Y \xi\}.
\end{aligned}$$

From this, by taking the inner product of (4.3) with the Reeb vector field ξ , we have

$$\begin{aligned}
0 = & g((\phi S - S\phi)X, Y) - (X\beta)g(AY, \xi) - (Y\beta)g(AX, \xi) \\
& - \beta\{q(X)g(JAY, \xi) + q(Y)g(JAX, \xi) + g(A(\nabla_X Y + \nabla_Y X), \xi) \\
& + g(AY, N)g(SX, \xi) + g(AX, N)g(SY, \xi)\} \\
& + \{g(q(X)JA\xi + A\phi S X + \alpha\eta(X)AN, Y) \\
& + g(q(Y)JA\xi + A\phi S Y + \alpha\eta(Y)AN, X)\}g(A\xi, \xi) \\
& + g(AY, \xi)g(A\phi S X, \xi) + g(AX, \xi)g(A\phi S Y, \xi) \\
& + g(Y, (AN)^T)\{g(q(X)JAN, \xi) - g(ASX, \xi) + g(AN, N)g(SX, \xi)\} \\
& + g(X, (AN)^T)\{g(q(Y)JAN, \xi) - g(ASY, \xi) + g(AN, N)g(SY, \xi)\} \\
& + \alpha(X\alpha)\eta(Y) + \alpha(Y\alpha)\eta(X) \\
& + \alpha\{g((\nabla_X S)Y, \xi) + g((\nabla_Y S)X, \xi)\} \\
& - X(\alpha^2)\eta(Y) - Y(\alpha^2)\eta(X) - \alpha^2(\nabla_X \eta)(Y) - \alpha^2(\nabla_Y \eta)(X).
\end{aligned}$$

Then, first, by putting $Y = \xi$ and using $g(A\xi, N) = 0$, we have

$$\begin{aligned}
(4.4) \quad 0 = & - (X\beta)g(A\xi, \xi) - \beta g(A\phi S X, \xi) + \beta g(A\phi S X, \xi) + \beta g(A\phi S X, \xi) \\
& - (\xi\beta)g(AX, \xi) - \beta\{q(\xi)g(JAX, \xi) + g(A\nabla_\xi X, \xi) + \alpha g(AX, N)\} \\
& + \{g(q(\xi)JA\xi + A\phi S \xi + \alpha AN, X)\}g(A\xi, \xi) \\
& + g(X, AN)(q(\xi) - 2\alpha)\beta \\
& - \beta\{g(A\phi S X, \xi) + g(A\nabla_\xi X, \xi) - (q(\xi) - 2\alpha)g(X, AN)\}.
\end{aligned}$$

Here if the function $\beta = g(A\xi, \xi) = -\cos 2t = 0$, we have $t = \frac{\pi}{4}$. Then the unit normal vector field N becomes

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$ as in section 3, that is, the unit normal N is \mathfrak{A} -isotropic.

Now hereafter, from (4.4) let us consider the following case

$$(4.5) \quad 0 = \{g(A\phi S X, \xi) + g(A\nabla_\xi X, \xi) - (q(\xi) - 2\alpha)g(X, AN)\}.$$

On the other hand, by using (3.1) for any tangent vector field $X \perp A\xi$, we have

$$(4.6) \quad \begin{aligned} g(A\nabla_\xi X, \xi) &= g(\nabla_\xi X, A\xi) = -g(X, \nabla_\xi(A\xi)) \\ &= -g(q(\xi)JA\xi + \alpha AN, X) = (q(\xi) - \alpha)g(AN, X). \end{aligned}$$

Then from (4.5) and (4.6) we have the following for any tangent vector field X orthogonal to $A\xi$

$$(4.7) \quad \begin{aligned} 0 &= g(A\phi S X, \xi) + (q(\xi) - \alpha)g(AN, X) - (q(\xi) - 2\alpha)g(AN, X) \\ &= g(A\phi S X, \xi) + \alpha g(AN, X) \\ &= g(SAN + \alpha AN, X). \end{aligned}$$

So it follows that

$$(4.8) \quad g(S(AN)^T, (AN)^T) = -\alpha(1 - \beta^2),$$

where $g((AN)^T, (AN)^T) = g(AN - g(AN, N)N, AN - g(AN, N)N) = 1 - g(AN, N)^2 = 1 - \beta^2$.

On the other hand, by using (3.3) to the second term of (4.5) for $X = (AN)^T$, we have

$$(4.9) \quad \begin{aligned} g(A\nabla_\xi(AN)^T, \xi) &= g(q(\xi)\xi - S\xi + \alpha g(AN, N)A\xi, \xi) \\ &= q(\xi) - \alpha - \alpha\beta^2, \end{aligned}$$

where we have used $A^2 = I$ and $g(AN, N) = -g(A\xi, \xi) = -\beta$.

Then by putting $X = (AN)^T$ in (4.5) and using (4.8) and (4.9), we have

$$(4.10) \quad \begin{aligned} 0 &= g(A\phi S(AN)^T, \xi) + g(A\nabla_\xi(AN)^T, \xi) - (q(\xi) - 2\alpha)g((AN)^T, (AN)^T) \\ &= -\alpha(1 - \beta^2) + q(\xi) - \alpha - \alpha\beta^2 - (q(\xi) - 2\alpha)(1 - \beta^2) \\ &= (q(\xi) - 2\alpha)\beta^2, \end{aligned}$$

where we have used $g(A\phi S(AN)^T, \xi) = g(S(AN)^T, (AN)^T) = -\alpha(1 - \beta^2)$. Here we note that $\xi\beta = 0$, because we can calculate the following

$$\begin{aligned} \xi\beta &= \xi g(A\xi, \xi) \\ &= g((\bar{\nabla}_\xi A)\xi + A\bar{\nabla}_\xi\xi, \xi) + g(A\xi, \bar{\nabla}_\xi\xi) \\ &= g(q(\xi)JA\xi, \xi) \\ &= -q(\xi)g(A\xi, N) \\ &= 0. \end{aligned}$$

Now we consider an open subset $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$ in M . Then by (4.10), we have

Lemma 4.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} , $m \geq 3$. Then*

$$q(\xi) = 2\alpha$$

holds on \mathcal{U} on M in Q^{m*} .

Now hereafter unless otherwise stated, on such an open subset \mathcal{U} let us prove that the unit vector field N in the complex hyperbolic quadric Q^{m*} is \mathfrak{A} -principal. Then by Lemma 4.1 and (4.4), we have the following for any tangent vector field X on M

$$g(A\phi S X, \xi) + g(A\nabla_\xi X, \xi) = 0.$$

From this, by putting $X = A\xi$ and using $g(A\xi, A\xi) = 1$, we know that

$$(4.11) \quad 0 = g(A\phi S A\xi, \xi) = g(SA\xi, (AN)^T).$$

Moreover, for any $X \perp A\xi$ the second term in the left side of the above equation becomes

$$g(A\nabla_\xi X, \xi) = -g(X, \nabla_\xi A\xi) = \alpha g((AN)^T, X),$$

where in the third equality we have used Lemma 4.1. Consequently, for any tangent vector field $X \perp A\xi$ we conclude

$$\begin{aligned} 0 &= g(A\phi S X, \xi) + g(A\nabla_\xi X, \xi) \\ &= g(X, S(AN)^T) + \alpha g((AN)^T, X) \\ &= g(S(AN)^T + \alpha(AN)^T, X). \end{aligned}$$

Moreover, by (4.11) we also know that

$$g(S(AN)^T + \alpha(AN)^T, A\xi) = 0.$$

So these two equations give the following

Lemma 4.2. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} , $m \geq 3$. Then*

$$S(AN)^T = -\alpha(AN)^T$$

holds on \mathcal{U} on M in Q^{m*} .

Now let us differentiate the equation in Lemma 4.2. Then it follows that

$$(\nabla_X S)(AN)^T + S\nabla_X(AN)^T = -(X\alpha)(AN)^T - \alpha\nabla_X(AN)^T.$$

From this, by taking the inner product with the Reeb vector field ξ and using the formulas (3.3), we have

$$\begin{aligned} 0 &= g((AN)^T, (\nabla_X S)\xi) \\ &\quad + 2\alpha g(q(X)JAN - ASX - g(q(X)JAN - ASX, N)N, \xi) \\ &\quad + 2\alpha g(AN, N)g(SX, \xi) \\ &= g((AN)^T, \alpha\phi S X - S\phi S X) \\ &\quad + 2\alpha\{q(X)g(A\xi, \xi) - g(SX, A\xi) + g(AN, N)g(SX, \xi)\}. \end{aligned}$$

Then by putting $X = (AN)^T$ and using Lemma 4.2, we have $\alpha q((AN)^T) = 0$. When the function $\alpha = 0$, in section 3, $\beta g(Y, AN) = 0$ for any tangent vector field Y on M . Then on the open subset $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$ in M we conclude

Lemma 4.3. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} , $m \geq 3$. Then either*

$$q((AN)^T) = 0$$

or the unit normal vector field N is \mathfrak{A} -principal.

On the other hand, by putting $X = \xi$ in (3.3) and using Lemma 4.1, we have

$$(4.12) \quad \begin{aligned} \nabla_\xi(AN)^T &= (q(\xi) - \alpha)A\xi + \alpha g(AN, N)\xi \\ &= \alpha(A\xi - \beta\xi). \end{aligned}$$

Differentiating the equation in Lemma 4.2 along the Reeb direction ξ and using (4.12) implies

$$(4.13) \quad \begin{aligned} (\nabla_\xi S)(AN)^T &= -S\nabla_\xi(AN)^T - (\xi\alpha)(AN)^T - \alpha\nabla_\xi(AN)^T \\ &= -\alpha(SA\xi - \alpha\beta\xi) - (\xi\alpha)(AN)^T - \alpha^2(A\xi - \beta\xi). \end{aligned}$$

Moreover, differentiating $S\xi = \alpha\xi$ and using Lemma 4.2, we get the following

$$(4.14) \quad \begin{aligned} (\nabla_{(AN)^T} S)\xi &= \{(AN)^T\alpha\}\xi + \alpha\phi S(AN)^T - S\phi S(AN)^T \\ &= \{(AN)^T\alpha\}\xi - \alpha^2\phi(AN)^T + \alpha S\phi(AN)^T. \end{aligned}$$

Then subtracting (4.14) from (4.13) and Lemma 4.2 give

$$(4.15) \quad \begin{aligned} g((\nabla_\xi S)(AN)^T - (\nabla_{(AN)^T} S)\xi, (AN)^T) &= -(\xi\alpha)(1 - \beta^2) \\ &= -g(\phi(AN)^T, (AN)^T) - g(\xi, A\xi)g(JA(AN)^T, (AN)^T) \\ &= 0, \end{aligned}$$

where in the second equality we have used the equation of Codazzi (3.1) in section 3. Then it follows that

$$\xi\alpha = 0 \quad \text{or} \quad \beta^2 = 1.$$

When the latter part $\beta = \pm 1$ occurs on \mathcal{U} , then $AN = \pm N$. So we know that the unit normal vector field N is \mathfrak{A} -principal. When $\xi\alpha = 0$, if we use the derivative formula $Y\alpha$ and $g(\xi, AN) = 0$ in section 3, we have the following

Lemma 4.4. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*} , $m \geq 3$. Then either*

$$\text{grad } \alpha = 2\beta(AN)^T$$

or the unit normal vector field N is \mathfrak{A} -principal.

Now let us consider the first formula in Lemma 4.4. Then by differentiating the above formula it follows that

$$(4.16) \quad \begin{aligned} \nabla_X \text{grad } \alpha &= 2(X\beta)(AN)^T + 2\beta\nabla_X(AN)^T \\ &= 4g(A\phi S X, \xi)(AN)^T + 2\beta\{q(X)JAN - AS X \\ &\quad - g(q(X)JAN - AS X, N)N + g(AN, N)SX\}. \end{aligned}$$

Then we have

$$(4.17) \quad \begin{aligned} g(\nabla_X \text{grad } \alpha, Y) &= 4g(A\phi S X, \xi)g((AN)^T, Y) + 2\beta\{q(X)g(JAN, Y) - g(AS X, Y)\} \\ &\quad + 2\beta g(AN, N)g(S X, Y). \end{aligned}$$

Since $g(\nabla_X \text{grad } \alpha, Y) = g(\nabla_Y \text{grad } \alpha, X)$ and Lemma 4.2, we have

$$(4.18) \quad 0 = 2\beta\{q(X)g(JAN, Y) - q(Y)g(JAN, X)\} - 2\beta\{g(AS X, Y) - g(AS Y, X)\}.$$

So on the open subset $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$ in M it follows that

$$q(X)g(JAN, Y) - q(Y)g(JAN, X) = g(AS X, Y) - g(AS Y, X).$$

From this, by putting $X = \xi$, we know that

$$SA\xi = -\alpha A\xi + \beta \text{grad } q.$$

Then differentiating this formula gives

$$(4.19) \quad (\nabla_X S)A\xi + S \nabla_X A\xi = -(X\alpha)A\xi - \alpha \nabla_X A\xi + (X\beta) \text{grad } q + \beta \nabla_X \text{grad } q.$$

First let us take the inner product of (4.19) with Y and make the skew-symmetric part with respect X and Y . Next we use $g(\nabla_X \text{grad } q, Y) = g(\nabla_Y \text{grad } q, X)$ to the obtained equation. Then finally by putting $X = \xi$, we have

$$(4.20) \quad \begin{aligned} g((\nabla_\xi S)A\xi, Y) - g((\nabla_Y S)A\xi, \xi) + g(S(\nabla_\xi A\xi), Y) - g(S(\nabla_Y A\xi), \xi) \\ = -(\xi\alpha)g(A\xi, Y) + (Y\alpha)g(A\xi, \xi) \\ - \alpha\{g(\nabla_\xi A\xi, Y) - g(\nabla_Y A\xi, \xi)\} + (\xi\beta)q(Y) - (Y\beta)q(\xi). \end{aligned}$$

In this equation (4.20), we want to use the following formulas

$$q(\xi) = 2\alpha, \quad \xi\alpha = 0, \quad \xi\beta = 0,$$

$$(4.21) \quad \begin{aligned} \nabla_\xi(A\xi) &= 2\alpha JA\xi + \alpha AN - \{2\alpha g(JA\xi, N) + \alpha g(AN, N)\}N \\ &= -\alpha AN - \alpha\beta N \\ &= -\alpha(AN)^T, \end{aligned}$$

and

$$(4.22) \quad \begin{aligned} g(\nabla_Y(A\xi), \xi) &= q(Y)g(JA\xi, \xi) + g(A\phi S Y, \xi) \\ &= g(S Y, AN) = -\alpha g((AN)^T, Y). \end{aligned}$$

Then by the help of (4.21) and (4.22), the equation (4.20) can be reformed as

$$(4.23) \quad \begin{aligned} g((\nabla_\xi S)A\xi, Y) - g((\nabla_Y S)A\xi, \xi) + 2\alpha^2 g((AN)^T, Y) \\ = (Y\alpha)\beta - 2\alpha(Y\beta). \end{aligned}$$

On the other hand, if we use the equation of Codazzi (3.1) in the first term of (4.23), we have

$$(4.24) \quad \begin{aligned} g((\nabla_\xi S)A\xi, Y) &= g((\nabla_\xi S)Y, A\xi) = g((\nabla_Y S)\xi, A\xi) \\ &\quad - g(\phi Y, A\xi) + g(Y, AN)g(A\xi, A\xi) - g(\xi, A\xi)g(JAY, A\xi). \end{aligned}$$

Then substituting (4.24) into the first term of (4.23) gives

$$(4.25) \quad \begin{aligned} -g(\phi Y, A\xi) + g(Y, AN)g(A\xi, A\xi) - g(\xi, A\xi)g(JAY, A\xi) + 2\alpha^2 g((AN)^T, Y) \\ = (Y\alpha)\beta - 2\alpha(Y\beta) \end{aligned}$$

$$= 2\beta^2 g(Y, AN) + 4\alpha^2 g(Y, (AN)^T),$$

where in the second equality we have used $\xi\alpha = 0$ in (3.2) of section 3, Lemma 4.2 and (3.8) in the following formula

$$\begin{aligned} Y\beta &= 2g(A\phi SY, \xi) = 2g(SY, AJ\xi) \\ &= 2g(SY, (AN)^T) = -2\alpha g(Y, (AN)^T). \end{aligned}$$

In (4.25) the first two terms of the left side cancelled out each other and the third term vanishes identically. The fourth term $2\alpha^2 g((AN)^T, Y)$ can be deleted with the second term in the right side of (4.25). So (4.25) implies $2(\alpha^2 + \beta^2)g(Y, AN) = 0$ for any tangent vector field Y on M , which means that on the open subset $\mathcal{U} = \{p \in M \mid \beta(p) \neq 0\}$ the unit normal vector field N is \mathfrak{A} -principal $AN = g(AN, N)N$.

Summing up the above discussions, we can prove our Main Theorem 1 in the introduction.

By virtue of Main Theorem 1, we can distinguish two classes of real hypersurfaces in the complex hyperbolic quadric Q^{m*} with Killing structure Jacobi operator : those that have \mathfrak{A} -principal unit normal, and those that have \mathfrak{A} -isotropic unit normal vector field N . We treat the respective cases in sections 5 and 6.

5. Killing structure Jacobi operator with \mathfrak{A} -principal normal

In this section we consider a real hypersurface M in the complex hyperbolic quadric Q^{m*} with \mathfrak{A} -principal unit normal vector field. Then the unit normal vector field N satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$. Naturally, we have also the following

$$A\xi = -\xi, \quad \text{and} \quad JA\xi = -J\xi = -N.$$

Then the structure Jacobi operator R_ξ is given by

$$(5.1) \quad R_\xi(X) = -X + 2\eta(X)\xi + AX + g(S\xi, \xi)SX - g(SX, \xi)S\xi.$$

Since we assume that M is Hopf, (5.1) becomes

$$(5.2) \quad R_\xi(X) = -X + 2\eta(X)\xi + AX + \alpha SX - \alpha^2\eta(X)\xi.$$

By the assumption of the Killing structure Jacobi operator R_ξ , the derivative of R_ξ along any tangent vector field Y on M is given by

$$\begin{aligned} (5.3) \quad (\nabla_Y R_\xi)(X) &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= 2\{(\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi\} + (\nabla_Y A)X + (Y\alpha)SX \\ &\quad + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\ &\quad - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi. \end{aligned}$$

We can write

$$AY = BY + \rho(Y)N,$$

where BY denotes the tangential component of AY and $\rho(Y) = g(AY, N) = g(Y, AN) = g(Y, N) = 0$. So for any tangent vector field Y on M the vector field $AY (= BY)$ also becomes

a tangent vector field on M in Q^{m*} . Then it follows

$$\begin{aligned}
 (5.4) \quad (\nabla_Y A)X &= \nabla_Y(AX) - A\nabla_Y X \\
 &= \bar{\nabla}_Y(AX) - \sigma(Y, AX) - A\nabla_Y X \\
 &= (\bar{\nabla}_Y A)X + A\bar{\nabla}_Y X - \sigma(Y, AX) - A\nabla_Y X \\
 &= q(Y)JAX + A\sigma(Y, X) - \sigma(Y, AX) \\
 &= q(Y)JAX + g(SX, Y)AN - g(SY, AX)N,
 \end{aligned}$$

where we have used the equation of Gauss in the second equality and the Weingarten formula in the fifth equality. From this, together with (5.3) and using that \mathfrak{A} -principal, the Killing structure Jacobi operator gives

$$\begin{aligned}
 (5.5) \quad 0 &= (\nabla_Y R_\xi)(X) + (\nabla_X R_\xi)(Y) \\
 &= (2 + \alpha^2)\{(\nabla_Y \eta)(X)\xi + \eta(X)\nabla_Y \xi\} \\
 &\quad + (2 + \alpha^2)\{(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi\} \\
 &\quad + \{q(Y)JAX + g(SX, Y)N - g(SY, AX)N\} \\
 &\quad + \{q(X)JAY + g(SY, X)N - g(SX, AY)N\} \\
 &\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\
 &\quad + (X\alpha)SY + \alpha(\nabla_X S)Y - (X\alpha^2)\eta(Y)\xi.
 \end{aligned}$$

From this, taking the inner product of (5.5) with the Reeb vector field ξ , and using the constancy of the Reeb function α in Lemma 3.2, we have

$$\begin{aligned}
 (5.6) \quad 0 &= (2 + \alpha^2)\{g(\phi SY, X) + g(\phi SX, Y)\} + \alpha g((\nabla_Y S)X + (\nabla_X S)Y, \xi) \\
 &= 2g((\phi S - S\phi)Y, X)
 \end{aligned}$$

where we have used $g(JAX, \xi) = -g(AX, N) = -g(X, AN) = -g(X, N) = 0$ for any tangent vector field X on M in Q^{m*} and $(\nabla_X S)\xi = \alpha\phi SX - S\phi SX$. The formula (5.6) means that the shape operator S commutes with the structure tensor ϕ . Then by Theorem A in the introduction, M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$ or a horosphere whose center at infinity is \mathfrak{A} -isotropic singular. That is, the Reeb flow on M is isometric.

On the other hand, we want to introduce the following proposition (see Suh [34]).

Proposition 5.1. *Let M be a real hypersurface in Q^{m*} , $m \geq 3$, with isometric Reeb flow. Then the unit normal vector field N is \mathfrak{A} -isotropic everywhere.*

By Proposition 5.1, we know that the unit normal vector field N of M is \mathfrak{A} -isotropic, not \mathfrak{A} -principal. This rules out the existence of an \mathfrak{A} -principal unit normal N together with Killing structure Jacobi operator. So we give the proof of our Main Theorem 2 with \mathfrak{A} -principal unit normal N .

6. Killing structure Jacobi operator with \mathfrak{A} -isotropic normal

In this section we assume that the unit normal vector field N is \mathfrak{A} -isotropic and the Reeb

function $\alpha = g(S\xi, \xi)$ is constant along the Reeb direction ξ , that is, $\xi\alpha = 0$. Then the normal vector field N can be written as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a $(+1)$ -eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), \quad AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \quad \text{and} \quad JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, \quad g(\xi, AN) = 0 \quad \text{and} \quad g(AN, N) = 0.$$

By virtue of these formulas for \mathfrak{A} -isotropic unit normal, the structure Jacobi operator can be given as follows:

$$\begin{aligned} (6.1) \quad R_\xi(X) &= R(X, \xi)\xi \\ &= -X + \eta(X)\xi + g(AX, \xi)A\xi + g(JAX, \xi)JA\xi \\ &\quad + g(S\xi, \xi)SX - g(SX, \xi)S\xi. \end{aligned}$$

On the other hand, we know that $JA\xi = -JAJN = AJ^2N = -AN$, and $g(JAX, \xi) = -g(AX, J\xi) = -g(AX, N)$. Then the structure Jacobi operator R_ξ can be rearranged as follows:

$$\begin{aligned} (6.2) \quad R_\xi(X) &= -X + \eta(X)\xi + g(AX, \xi)A\xi + g(X, AN)AN \\ &\quad + \alpha SX - \alpha^2\eta(X)\xi. \end{aligned}$$

Then by differentiating (6.2), we obtain

$$\begin{aligned} (6.3) \quad \nabla_Y R_\xi(X) &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= (\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi + g(X, \nabla_Y(A\xi))A\xi \\ &\quad + g(X, A\xi)\nabla_Y(A\xi) + g(X, \nabla_Y(AN))AN + g(X, AN)\nabla_Y(AN) \\ &\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\ &\quad - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi. \end{aligned}$$

Here let us consider the equation of Gauss. It is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$$

for any vector fields X and Y on M in Q^{m*} , where $\nabla_X Y = (\bar{\nabla}_X Y)^T$ and $\sigma(X, Y)$ respectively denote the tangential and normal component on $T_z M$ of $\bar{\nabla}_X Y$ in $T_z Q^{m*}$, $z \in M$. The Weingarten formula is given by

$$\bar{\nabla}_X N = -SX$$

for an \mathfrak{A} -isotropic unit normal vector field N . Here S denotes the shape operator of M in the complex hyperbolic quadric Q^{m*} derived from the unit normal N . Then by using these two equations to some terms in (6.3), we have the following :

$$\begin{aligned}
\nabla_Y(A\xi) &= \bar{\nabla}_Y(A\xi) - \sigma(Y, A\xi) \\
&= (\bar{\nabla}_Y A)\xi + A\bar{\nabla}_Y \xi - \sigma(Y, A\xi) \\
&= q(Y)JA\xi + A\{\phi S Y + \eta(S Y)N\} - g(S Y, A\xi)N
\end{aligned}$$

and

$$\begin{aligned}
\nabla_Y(AN) &= \bar{\nabla}_Y(AN) - \sigma(Y, AN) \\
&= (\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N - \sigma(Y, AN) \\
&= q(Y)JAN - AS Y - g(S Y, AN)N.
\end{aligned}$$

Substituting these formulas into (6.3) and using the assumption of Killing structure Jacobi operator, we have

$$\begin{aligned}
(6.4) \quad 0 &= \nabla_Y R_\xi(X) + \nabla_X R_\xi(Y) \\
&= g(\phi S Y, X)\xi + \eta(X)\phi S Y \\
&\quad + g(\phi S X, Y)\xi + \eta(Y)\phi S X \\
&\quad + \{q(Y)g(A\xi, X) + g(A\phi S Y, X) + g(S Y, \xi)g(AN, X)\}A\xi \\
&\quad + \{q(X)g(A\xi, Y) + g(A\phi S X, Y) + g(S X, \xi)g(AN, Y)\}A\xi \\
&\quad + g(X, A\xi)\{q(Y)JA\xi + A\phi S Y + g(S Y, \xi)AN - g(S Y, A\xi)N\} \\
&\quad + g(Y, A\xi)\{q(X)JA\xi + A\phi S X + g(S X, \xi)AN - g(S X, A\xi)N\} \\
&\quad + \{q(Y)g(X, AN) - g(X, AS Y)\}AN \\
&\quad + \{q(X)g(Y, AN) - g(Y, AS X)\}AN \\
&\quad + g(X, AN)\{q(Y)JAN - AS Y - g(S Y, AN)N\} \\
&\quad + g(Y, AN)\{q(X)JAN - AS X - g(S X, AN)N\} \\
&\quad + (Y\alpha)S X + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\
&\quad + (X\alpha)S Y + \alpha(\nabla_X S)Y - (X\alpha^2)\eta(Y)\xi \\
&\quad - \alpha^2 g(\phi S Y, X)\xi - \alpha^2 \eta(X)\phi S Y \\
&\quad - \alpha^2 g(\phi S X, Y)\xi - \alpha^2 \eta(Y)\phi S X.
\end{aligned}$$

Taking the inner product of (6.4) with the unit normal N and using the properties of \mathfrak{A} -isotropic, that is, $g(A\xi, \xi) = 0$, $g(AN, N) = 0$, it follows that

$$\begin{aligned}
(6.5) \quad 0 &= g(X, A\xi)g(A\phi S Y, N) - g(X, A\xi)g(S Y, A\xi) \\
&\quad + g(Y, A\xi)g(A\phi S X, N) - g(Y, A\xi)g(S X, A\xi) \\
&\quad - g(X, AN)g(AS Y, N) - g(X, AN)g(S Y, AN) \\
&\quad - g(Y, AN)g(AS X, N) - g(Y, AN)g(S X, AN).
\end{aligned}$$

From this, putting $X = AN$ and using that N is \mathfrak{A} -isotropic and $A\xi = \phi AN$, we have

$$0 = -2g(AS Y, N) - 2g(Y, AN)g(S AN, AN) + 2g(Y, A\xi)g(A\phi S AN, N).$$

By putting $Y = AN$, we get $g(S AN, AN) = 0$. Then the above equation reduces to

$$g(AS Y, N) = g(Y, A\xi)g(A\phi S AN, N).$$

So it follows that

$$\begin{aligned} SAN &= g(A\phi SAN, N)A\xi \\ &= -g(SAN, \phi AN)A\xi \\ &= -g(SAN, A\xi)A\xi, \end{aligned}$$

where we have used $A\xi = \phi AN$. Then this gives that $g(SAN, A\xi) = 0$, which implies

$$(6.6) \quad SAN = 0 \quad \text{and} \quad S\phi A\xi = 0.$$

Then (6.5) reduces to the following

$$\begin{aligned} (6.7) \quad 0 &= g(X, A\xi)g(A\phi SY, N) - g(X, A\xi)g(SY, A\xi) \\ &\quad + g(Y, A\xi)g(A\phi SX, N) - g(Y, A\xi)g(SX, A\xi). \end{aligned}$$

By putting $X = A\xi$ in (6.7) and using $A\xi = \phi AN$, it follows that

$$g(SY, A\xi) + g(Y, A\xi)g(SA\xi, A\xi) = 0$$

for any vector field Y on M in Q^{m*} . This gives

$$SA\xi = -g(SA\xi, A\xi)A\xi.$$

Then by taking the inner product with $A\xi$, we know $g(SA\xi, A\xi) = 0$. From this, together with the above equation, we have

$$(6.8) \quad SA\xi = 0 \quad \text{and} \quad S\phi AN = 0.$$

Putting $X = \xi$ into (6.4), and using (6.8) and the \mathfrak{A} -isotropic property $g(A\xi, \xi) = 0$, we have

$$\begin{aligned} (6.9) \quad 0 &= \phi SY + \{q(\xi)g(A\xi, Y) + \alpha g(AN, Y)\}A\xi \\ &\quad + g(Y, A\xi)\{q(\xi)A\xi + \alpha AN - g(S\xi, A\xi)N\} \\ &\quad + \{q(\xi)g(Y, AN) - \alpha g(Y, A\xi)\}AN + g(Y, AN)\{q(\xi)AN - \alpha A\xi\} \\ &\quad + (Y\alpha)\alpha\xi + \alpha(\nabla_Y S)\xi - (Y\alpha^2)\xi - \alpha^2\phi SY \\ &\quad + (\xi\alpha)SY + \alpha(\nabla_\xi S)Y - (\xi\alpha^2)\eta(Y)\xi \\ &= \phi SY + 2q(\xi)g(A\xi, Y)A\xi + 2q(\xi)g(Y, AN)AN \\ &\quad - \alpha S\phi SY + (\xi\alpha)SY - (\xi\alpha^2)\eta(Y)\xi + \alpha(\nabla_\xi S)Y. \end{aligned}$$

On the other hand, $SA\xi = 0$ implies $(\nabla_\xi S)A\xi + S\nabla_\xi(A\xi) = 0$. By the equation of Gauss, the following holds

$$\begin{aligned} \nabla_\xi(A\xi) &= \bar{\nabla}_\xi(A\xi) - \sigma(\xi, A\xi) \\ &= q(\xi)JA\xi + g(S\xi, \xi)AN - g(S\xi, A\xi)N \\ &= q(\xi)JA\xi + \alpha AN. \end{aligned}$$

This gives $S(\nabla_\xi(A\xi)) = q(\xi)SJA\xi + \alpha SAN = 0$ from (6.6). From this, together with the above formula, we have

$$(6.10) \quad (\nabla_\xi S)A\xi = 0.$$

By taking the inner product of (6.9) with $A\xi$ and AN respectively, and using (6.6), (6.8)

and (6.10), we know that

$$q(\xi)A\xi = 0 \quad \text{and} \quad q(\xi)AN = 0.$$

By virtue of these formulas, (6.9) reduces to the following

$$(6.11) \quad 0 = \phi SY - \alpha S\phi SY + (\xi\alpha)SY - (\xi\alpha^2)\eta(Y)\xi + \alpha(\nabla_\xi S)Y.$$

On the other hand, by using the equation of Codazzi, we have

$$\begin{aligned} (\nabla_\xi S)Y &= (\nabla_Y S)\xi - \phi Y + g(AN, Y)A\xi + g(Y, A\xi)\phi A\xi \\ &= (Y\alpha)\xi + \alpha\phi SY - S\phi SY - \phi Y \\ &\quad + g(AN, Y)A\xi + g(Y, A\xi)\phi A\xi. \end{aligned}$$

Then by the properties of M being Hopf and with \mathfrak{A} -isotropic unit normal vector field, we have

$$Y\alpha = g((\nabla_\xi S)Y, \xi) = g((\nabla_\xi S)\xi, Y) = (\xi\alpha)\eta(Y).$$

From this, together with the assumption of $\xi\alpha = 0$ in section 6, it follows that the Reeb function α is constant for a real hypersurface in Q^{m*} with \mathfrak{A} -isotropic unit normal. Then the derivative of the shape operator S along the Reeb direction ξ is given by

$$\begin{aligned} -\alpha(\nabla_\xi S)Y &= -\alpha^2\phi SY + \alpha S\phi SY \\ &\quad + \alpha\phi Y - \alpha g(AN, Y)A\xi - \alpha g(Y, A\xi)\phi A\xi. \end{aligned}$$

From this, by (6.11) and using the constancy of the Reeb function α , we know that

$$(6.12) \quad \begin{aligned} 0 &= \phi SY - 2\alpha S\phi SY + \alpha^2\phi SY \\ &\quad - \alpha\phi Y + \alpha g(AN, Y)A\xi + \alpha g(Y, A\xi)\phi A\xi. \end{aligned}$$

Then for any $Y \in Q$ such that $SY = \lambda Y$, where Y is orthogonal to the vector fields $A\xi$ and AN , (6.12) reduces to the following

$$(6.13) \quad 2\alpha\lambda S\phi Y = (\lambda\alpha^2 - \alpha + \lambda)\phi Y.$$

Then (6.13) gives $\alpha \neq 0$.

In fact, if the Reeb function $\alpha = 0$, from (6.13) it follows that $\lambda = 0$. From this, together with (6.6) and (6.8), the shape operator S becomes identically vanishing. That is, M is totally geodesic. Then by the equation of Codazzi in section 3, we have a contradiction.

Naturally we should have $2\alpha\lambda \neq 0$. If the function $\lambda = 0$, then (6.13) also implies that the Reeb function α vanishes. So also the contradiction appears. This fact gives

$$S\phi Y = \frac{\alpha\lambda - 2}{2\lambda - \alpha}\phi Y = \frac{\alpha^2\lambda - \alpha + \lambda}{2\alpha\lambda}\phi Y.$$

It can be written as follows:

$$(6.14) \quad 2\lambda^2 + \alpha(1 - \alpha^2)\lambda + \alpha^2 = 0.$$

Then the discriminant of (6.14) is given by

$$D = \alpha^2(1 - \alpha^2)^2 - 8\alpha^2 = \alpha^2\{(\alpha^2 - 1)^2 - 8\}.$$

Then the solution has two roots as follows:

$$\lambda = \frac{-\alpha(1-\alpha^2) \pm \alpha\sqrt{(\alpha^2-1-2\sqrt{2})(\alpha^2-1+2\sqrt{2})}}{4}.$$

When $\alpha^2 > 2\sqrt{2} + 1$, we have two distinct roots λ_1 and λ_2 of the equation (6.14).

Now let us consider the case that $\alpha^2 = 2\sqrt{2} + 1$. Then we may put $\alpha = \sqrt{2\sqrt{2} + 1}$. In this case we have

$$\lambda_1 = \lambda_2 = \frac{-\alpha(1-\alpha^2)}{4} = -\sqrt{2\sqrt{2} + \frac{1}{2}}.$$

Here let us put $\delta = -\sqrt{2\sqrt{2} + \frac{1}{2}}$. Then the shape operator S has three distinct constant principal curvatures such that

$$\alpha = \sqrt{2\sqrt{2} + 1}, \quad \beta = \gamma = 0, \quad \text{and} \quad \delta = -\sqrt{2\sqrt{2} + \frac{1}{2}} = -\sqrt{\frac{2\sqrt{2} + 1}{2}}.$$

The corresponding eigen spaces are given by $\xi \in T_0$, $A\xi, AN \in T_\beta = Q^\perp$ and $T_\delta = Q$ with multiplicities 1, 2 and $2m - 4$ respectively.

On the other hand, on the distribution Q let us introduce an important formula mentioned in section 3 as follows:

$$(6.15) \quad 2S\phi S Y - \alpha(\phi S + S\phi)Y = -2\phi Y$$

for any tangent vector field Y on M in Q^m (see also [29], pages 1350050-11). So if $S Y = \delta Y$ in (6.15), then $(2\delta - \alpha)S\phi Y = (\alpha\delta - 2)\phi Y$, which gives

$$(6.16) \quad S\phi Y = \frac{\alpha\delta - 2}{2\delta - \alpha}\phi Y,$$

because if $2\delta - \alpha = 0$, then $\alpha\delta - 2 = 0$. This implies $\alpha^2 = 4$, then $\alpha = 2$ and $\delta = 1$. In this case M is locally congruent to a horosphere whose center at infinity is \mathfrak{A} -isotropic singular.

On the other hand, let us consider $S\phi Y = \delta\phi Y$ for $2\delta \neq \alpha$, because $T_\delta = Q$. From this, together with the above equation, we have

$$\delta^2 - \alpha\delta + 1 = 0.$$

Then $\delta^2 + 1 = \sqrt{2} + \frac{3}{2}$. But $\delta^2 + 1 = \alpha\delta = -\sqrt{2\sqrt{2} + 1}\sqrt{\frac{2\sqrt{2} + 1}{2}} = -\frac{\sqrt{2}}{2} - 2$. This gives a contradiction. So this case can not be happened.

Accordingly, the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_2 \end{bmatrix}$$

where the principal curvatures are constants and are given by

$$\lambda_1 = \frac{\alpha(\alpha^2 - 1) + \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}$$

and respectively

$$\lambda_2 = \frac{\alpha(\alpha^2 - 1) - \alpha\sqrt{(\alpha^2 - 1 - 2\sqrt{2})(\alpha^2 - 1 + 2\sqrt{2})}}{4}.$$

By virtue of Remark below, we note that the horosphere whose center at infinity is \mathfrak{A} -isotropic singular can not be appeared. Then we give a complete proof of our Main Theorem 3.

REMARK 6.1. Let us check that a tube around the totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ or a horosphere whose center at infinity is \mathfrak{A} -isotropic singular. Then by Theorem A in the introduction, the tube has a commuting shape operator, that is, $S\phi = \phi S$ and the unit normal N is \mathfrak{A} -isotropic and the Reeb curvature $\alpha = g(S\xi, \xi)$ is constant (see Suh [34]). By the \mathfrak{A} -isotropic unit normal, the properties $g(A\xi, \xi) = 0$ and $g(AN, N) = 0$ hold on M . Moreover from the expression of this tube we know that $SA\xi = 0$ and $SAN = 0$, by differentiating we also confirm that $(\nabla_\xi S)A\xi = 0$ and $(\nabla_\xi S)AN = 0$.

Now we assume that the tube admits a Killing structure Jacobi operator. Then by the same process as in the proof of our Main Theorem 2, the principal curvature of the tube should satisfies (6.14), that is,

$$2\lambda^2 + \alpha(1 - \alpha^2)\lambda + \alpha^2 = 0.$$

Then two roots $\coth r$ and $\tanh r$ of the tube should satisfy $1 = \lambda\mu = \coth r \cdot \tanh r = \frac{\alpha^2}{2}$. Then $2 = \alpha^2 = \coth^2 r + \tanh^2 r + 2$ implies $\coth^2 r + \tanh^2 r = 0$. This makes a contradiction. So the tube does not admit a Killing structure Jacobi operator. Then naturally the tube around the totally geodesic $\mathbb{C}H^k \subset Q^{2k^*}$ or the horosphere does not have a parallel structure Jacobi operator, which is more strong condition than Killing structure Jacobi operator.

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