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BOUNDARY LIMITS OF MONOTONE SOBOLEV FUNCTIONS ON METRIC SPACES

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Abstract

In this paper we are concerned with weighted boundary limits of monotone Sobolev functions in Orlicz spaces on bounded (η, ψ)-John domains in a metric space. We also deal with Lindelöf type theorems for monotone Sobolev functions on uniform domains in a metric space.

1. Introduction

A continuous function u on an open set Ω in $\mathbf{R}^n$ is called monotone in the sense of Lebesgue (see [11]) if the equalities

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u$$

hold whenever G is a domain with compact closure $\overline{G} \subset \Omega$. A function $u \in W^{1,p}_{loc}(\Omega)$ is $\mathcal{A}$-harmonic if it is a weak solution of equation

$$\operatorname{div}(\mathcal{A}(x, \nabla u)) = 0,$$

where $\mathcal{A}(x, \xi) \cdot \xi \approx |\xi|^p$ for some fixed $p \in (1, \infty)$, $\xi \in \mathbf{R}^n$ (see [9]). Harmonic functions are monotone, $\mathcal{A}$-harmonic functions and hence coordinate functions of quasiregular mappings are monotone (see [9] and [28]), and thus the class of monotone functions is considerably wide. If u is a monotone Sobolev function on Ω and $p > n - 1$, then

$$|u(x) - u(y)| \le C(n, p) r^{1-n/p} \left(\int_{2B} |\nabla u(z)|^p \, dz \right)^{1/p} \tag{1.1}$$

whenever $y \in B = B(x, r)$ with $2B \subset \Omega$, where $C(n, p)$ is a positive constant depending only on n and p (see [9], [22, Chap. 8] and [31, Section 16]). In [6], [13], [14], [15] and [21], boundary behavior of monotone Sobolev functions were studied using the inequality like (1.1). For harmonic functions and polyharmonic functions, see [17, 18, 19, 20, 26, 27]. We refer to [10] for $\mathcal{A}$-harmonic functions and [30] for quasiregular mappings.

We consider a positive nondecreasing function φ on the interval $[0, \infty)$ such that φ is of log-type, that is, there exists a positive constant C satisfying

$$\varphi(r^2) \le C \varphi(r) \qquad \text{for all } r \ge 0. \tag{1.2}$$

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Set $\Phi_p(r) = r^p\varphi(r)$ for $p > 1$. For properties for functions of log-type, see e.g., [24].

Mizuta [19] studied weighted boundary limits of harmonic functions when D is a bounded Lipschitz domain in $\mathbf{R}^n$ and a weight is a nonnegative monotone function on the interval $(0, \infty)$ satisfying the doubling condition, as an extension of [18]. In fact, he proved the following:

Theorem A ([19, Theorem 1])**.** *Let ω be a nonnegative monotone function on the interval $(0, \infty)$ satisfying the doubling condition, that is, there exists a positive constant C such that*

$$C^{-1}\omega(r) \le \omega(2r) \le C\omega(r) \text{ for all } r > 0. \tag{1.3}$$

Set

$$\tilde{\kappa}(r) = \left(\int_r^1 t^{(1-n)/(p-1)}\omega(t)^{-1/(p-1)}(\varphi(t^{-1}))^{-1/(p-1)}dt\right)^{1-1/p}.$$

Suppose u is a harmonic function in a bounded Lipschitz domain D in $\mathbf{R}^n$ and satisfies

$$\int_D \Phi_p(|\nabla u(z)|)\omega(\delta_D(z))dz < \infty,$$

where $\delta_D(z) = \mathrm{dist}(z, D^c)$.

(1) *If $\tilde{\kappa}(0) = \infty$, then*

$$\lim_{x\to\partial D} \tilde{\kappa}(\delta_D(x))^{-1}u(x) = 0.$$

(2) *If $\tilde{\kappa}(0) < \infty$, then u has a finite limit at each boundary point of D.*

For the existence of boundary limits of harmonic functions in the case when $\Phi_p(r) = r^p$ and $\omega(r) = r^\alpha$, see Carleson [2], Mizuta [17] and Wallin [32], etc.

Let X be a metric space with a metric d and μ be a Borel measure on X which is positive and finite on balls. We denote by $B(x, r)$ the open ball centered at $x \in X$ with radius $r > 0$ and set $\lambda B = B(x, \lambda r)$ for each ball $B = B(x, r)$ and $\lambda > 0$. Let μ be a Borel measure on X satisfying the doubling condition:

$$\mu(2B) \le C_\mu \mu(B) \tag{1.4}$$

for every ball $B \subset X$. We further assume that

$$\frac{\mu(B(x', r'))}{\mu(B(x, r))} \ge C\left(\frac{r'}{r}\right)^Q \tag{1.5}$$

for all $x, x' \in \overline{D}$ with $x' \in B(x, r)$ and $0 < r' \le r$, where $Q > 1$. Here note that if μ satisfies (1.4), then (1.5) with $Q = \log_2 C_\mu$ holds (see e.g. [1, Lemma 3.3] and [8]).

In this paper, we are concerned with boundary limits of functions u on a domain $D \subset X$ for which there exists a nonnegative function $g \in L^p_{loc}(D; \mu)$ such that

$$|u(x) - u(x')| \le Cr\left(⨍_{\sigma B} g(z)^p d\mu(z)\right)^{1/p} \tag{1.6}$$

for every $x, x' \in B$ with $\sigma B \subset D$, where $\sigma > 1$, $B = B(y, r)$ and

$$\int_D \Phi_p(g(z))\omega(\delta_D(z))d\mu(z) < \infty, \tag{1.7}$$

where ω is a nonnegative monotone function on $(0, \infty)$ satisfying (1.3). Here we used the standard notation

$$\fint_E u(z)d\mu(z) = \frac{1}{\mu(E)} \int_E u(z)d\mu(z)$$

for a measurable set E with $0 < \mu(E) < \infty$. Note that (1.6) is a stronger property than p-Poincaré inequality. Also, if the pair of u and g satisfies a p-Poincaré inequality in D with $p > Q - 1$ and u satisfies

$$\sup_{x,x' \in B} |u(x) - u(x')| \leq \sup_{x,x' \in \partial B} |u(x) - u(x')|$$

for each ball B with $\sigma B \subset D$, then the pair of u and g satisfies (1.6) (see [8, Section 7]). We refer to [1] and [8, Section 11] for Poincaré inequalities of Carnot group with the Carnot-Carathéodory metric and [3] and [12] for recent studies about Orlicz spaces in a general metric setting.

When $\Phi_p(r) = r^p$ and $\omega(r) = r^\alpha$, the boundary limits of monotone Sobolev functions on bounded John domains in a metric space were studied in [6]. Recently, this result has been extended by the authors [7] to the Orlicz case, that is, $\Phi_p(r) = r^p\varphi(r)$ and $\omega(r) = r^\alpha$. For harmonic functions, polyharmonic functions and monotone functions on the upper half space $\mathbf{R}^n_+$, see [18, 20].

Our first aim in this paper is to find a positive function $k(r)$ such that $k(\delta_D(x))u(x)$ tends to zero as x tends to the boundary ∂D when u is a function on an (η, ψ)-John domain D satisfying (1.6) and (1.7) (Theorem 2.1), as an extension of Theorem A (see also [6, 7, 18, 20]). See Section 2 for the definition of (η, ψ)-John domain. The key lemma for our results is Lemma 3.3 below.

On the other hand, using the inequality (1.1), Lindelöf theorems for monotone Sobolev functions on the half space of $\mathbf{R}^n$ were studied in the L^p case when $\omega(r) = r^\alpha$ ([5]), as an extension of Mizuta [21, Theorem 2] and Manfredi-Villamor [14, 15]. This result was extended to a uniform domain in [4].

Our second aim in this paper is to establish Lindelöf theorems when u is a function on a uniform domain $D \subset X$ satisfying (1.6) and

$$\int_D g(z)^p \omega(\delta_D(z))d\mu(z) < \infty \tag{1.8}$$

(Theorem 5.1), as an extension of [4, 5, 14, 21] (see Section 5 for the definition of uniform domain). We discuss the size of the exceptional set in Theorem 5.1 (Remark 5.2).

Throughout this paper, let C denote various constants independent of the variables in question.

2. Weighted boundary limits

Let η be a strictly increasing continuous function on $[0, \infty)$ such that $\eta(0) = 0$,

$$\eta(2t) \leq c_1\eta(t) \qquad \text{for all} \quad t > 0 \tag{2.1}$$

with a constant $c_1 \geq 1$,

$$\limsup_{t \to 0} \; t^{-1}\eta(t) < \infty \tag{2.2}$$

and

$$\int_0^1 t^{-\varepsilon-1}\eta^{-1}(t)\,dt < \infty \tag{2.3}$$

for some small ε. We say that a domain D in X is John with respect to η (simply η-John) if there is a point $x^* \in D$ such that each $x \in D$ can be joined to x^* by a rectifiable curve γ satisfying

$$\delta_D(z) \geq \eta(\ell(\gamma(x,z))) \qquad \text{for all} \quad z \in \gamma, \tag{2.4}$$

where $\gamma(x,z)$ and $\ell(\gamma(x,z))$ denote the subarc of γ connecting x and z and the length of $\gamma(x,z)$, respectively. We refer x^* to the John center. If $\eta(t) = c_J t^s$ with $s \geq 1$ and a positive constant c_J, then an η-John domain is called s-John. Further, we consider a positive nondecreasing function ψ on $(0,\infty)$ such that $\psi(0) = \lim_{r\to 0}\psi(r) = 0$. We say that a domain D is (η,ψ)-John if each $x \in D$ can be joined to x^* by a rectifiable curve γ satisfying (2.4) and

$$\delta_D(z) \geq \psi(\delta_D(x)) \qquad \text{for all} \quad z \in \gamma. \tag{2.5}$$

Here note that every η-John domain is (η,ψ)-John domain if we take $\psi(t) = \min\{\eta(t/2), t/2\}$ (see [6, Example 1.5]). See also [16] for some examples of η-John domains.

Consider the function

$$\kappa(r_1,r_2) = \left(\int_{r_1}^{r_2} t^{(1-Q)/(p-1)-1}\omega(t)^{-1/(p-1)}(\varphi(t^{-1}))^{-1/(p-1)}\eta^{-1}(t)dt\right)^{1-1/p}$$

for $0 \leq r_1 < r_2 < \infty$; set $\kappa(r) = \kappa(r,1)$ for $0 \leq r < 1$.

Our first result is the following theorem, which gives an extension of Theorem A (see also [6, 7, 18, 20]).

Theorem 2.1. *Let D be a bounded (η,ψ)-John domain in X such that $\partial D \neq \emptyset$. Assume that the pair of u and g satisfies* (1.6) *and* (1.7).

(1) *If $\kappa(0) = \infty$, then*

$$\lim_{\delta_D(x)\to 0} \kappa(\psi(\delta_D(x)))^{-1}u(x) = 0.$$

(2) *If $\kappa(0) < \infty$, then u is bounded on D.*

For the best possibility of Theorem 2.1 (1) as to the order of infinity, we refer to [19, Proposition 4] for harmonic functions.

Remark 2.2. Let D be a bounded s-John domain in X, $s \geq 1$, that is, D is η-John with $\eta(t) = c_J t^s$ and a positive constant c_J. Then we see that D is an $(\eta, c_2\eta)$-John domain with $c_2 = \min\{2^{-s}, (2c_J)^{-1}(\mathrm{diam} D)^{1-s}\}$. For the result from Theorem 2.1 in the case when D is a bounded s-John domain, see [7, Corollaries 4.1 and 4.2].

Remark 2.3. For $s \geq 1$, let

$$G_s = \{x = (x', x_n) \in \mathbf{R}^{n-1}\times\mathbf{R} : |x'| < 1, 0 < x_n < 2, |x'| < x_n^s\}.$$

Then we see that G_s is an (η,ψ)-John domain with $\eta(t) = c_3 t^s$ and $\psi(t) = t$, where c_3 is a positive constant. In fact, for $x = (x', x_n) \in G_s$ and $0 < x_n < 1$, we define a rectifiable curve γ joining x and $x^* = (0,\dots,0,3/2)$ by

$$\gamma(t) = \begin{cases} ((1-t)x', x_n) & \text{if } 0 \le t \le 1, \\ (0, \dots, 0, tx_n) & \text{if } 1 < t \le 3/(2x_n). \end{cases}$$

Then γ satisifies (2.4) and (2.5) since $\delta_{G_s}(z)$ is comparable to $z_n^s - |z'|$ when $z = (z', z_n) \in G_s$ and $0 < z_n < 1$.

For the result from Theorem 2.1 in the case when $D = G_s$, see [7, Corollary 4.3].

3. Lemmas

For proofs of Theorems 2.1 and 5.1, we prepare some lemmas. Using (1.2), we have the following.

Lemma 3.1 (cf. [7, Lemma 2.2] and [23, Theorem 3]). *Let u and g be functions on D satisfying* (1.6). *Suppose* $0 < \varepsilon < 1$. *Then*

$$|u(x) - u(y)| \le C\delta_D(x)(\varphi(\delta_D(x)^{-1}))^{-1/p} \left(\fint_{\sigma B(x)} \Phi_p(g(z)) d\mu(z) \right)^{1/p} + C\delta_D(x)^{1-\varepsilon}, \tag{3.1}$$

whenever $x \in D$ *and* $y \in B(x)$, *where* $B(x) = B(x, \delta_D(x)/(2\sigma))$ *and C may depend on* ε.

Lemma 3.2 ([6, Lemma 2.2]). *Let* $\sigma > 1$ *and* μ *be a Borel measure on X satisfying the doubling condition* (1.4). *Suppose x and y can be joined by a rectifiable curve* γ *in a domain D satisfying* (2.4). *Then there exists a finite chain of balls* B_0, $B_1, \dots, B_N$ *(N may depend on* γ*) with the following properties:*

- (i) $B_j = B(z_j, \delta_D(z_j)/(2\sigma))$ *with* $z_j \in \gamma$, $z_0 = x$ *and* $y \in B_N$;
- (ii) $B_j \cap B_{j+1} \neq \emptyset$ *for all* $0 \le j \le N-1$;
- (iii) *For each* $t > 0$, *the number of* z_j *such that* $t/2 < \delta_D(z_j) \le t$ *is less than* $c_4\eta^{-1}(t)/t$, *where* c_4 *is a positive constant depending only on* σ;
- (iv) $\sum_{j=0}^{N} \chi_{\sigma B_j} \le c_5$, *where* c_5 *is a positive constant depending only on* σ *and* C_μ.

Lemma 3.3. *Let u and g be functions on D satisfying* (1.6). *Let B be an open ball with radius R. Then*

$$|u(x) - u(y)| \le C\kappa(\psi(\delta_D(x)), d_\gamma) \left(\frac{R^Q}{\mu(B)} \int_{E_\gamma} \Phi_p(g(w))\omega(\delta_D(w))\, d\mu(w) \right)^{1/p} + C \int_0^{d_\gamma} t^{-\varepsilon-1}\eta^{-1}(t)\, dt \tag{3.2}$$

whenever x and y can be joined by a rectifiable curve γ *in D satisfying* (2.4), (2.5) *and* $z \in B$, $\delta_D(z) < R$ *for all* $z \in \gamma$, *where* $E_\gamma = \cup_{z\in\gamma}\sigma B(z)$ *and* $d_\gamma = 4\max_{z\in\gamma}\delta_D(z)$.

Proof. Take a finite chain of balls $B_0, B_1, \dots, B_N$ with $B_j = B(z_j)$ as in Lemma 3.2. Pick $x_j \in B(z_{j-1}) \cap B(z_j)$ for $1 \le j \le N$; set $x_0 = x$ and $x_{N+1} = y$. By (3.1), (1.3) and (1.5), we see that

$$\begin{aligned} &|u(x_j) - u(x_{j+1})| \\ &\le C\delta_D(z_j)\omega(\delta_D(z_j))^{-1/p}(\varphi(\delta_D(z_j)^{-1}))^{-1/p}\mu(\sigma B_j)^{-1/p} \\ &\qquad \times \left(\int_{\sigma B_j} \Phi_p(g(w))\omega(\delta_D(w)) d\mu(w) \right)^{1/p} + C\delta_D(z_j)^{1-\varepsilon} \end{aligned}$$

$$\leq C\delta_D(z_j)\omega(\delta_D(z_j))^{-1/p}(\varphi(\delta_D(z_j)^{-1}))^{-1/p}(\delta_D(z_j)/R)^{-Q/p}\mu(B)^{-1/p} \times\left(\int_{\sigma B_j}\Phi_p(g(w))\omega(\delta_D(w))d\mu(w)\right)^{1/p}+C\delta_D(z_j)^{1-\varepsilon}$$

for $0 \leq j \leq N$. Then we have by Hölder's inequality and Lemma 3.2 (iv)

$$\begin{aligned}
&|u(x)-u(y)|\\
&\leq |u(x_0)-u(x_1)|+|u(x_1)-u(x_2)|+\cdots+|u(x_N)-u(x_{N+1})|\\
&\leq C\left(\frac{R^Q}{\mu(B)}\right)^{1/p}\left(\sum_{j=0}^{N}\delta_D(z_j)^{(p-Q)/(p-1)}\omega(\delta_D(z_j))^{-1/(p-1)}(\varphi(\delta_D(z_j)^{-1}))^{-1/(p-1)}\right)^{(p-1)/p}\\
&\qquad\times\left(\int_{E_\gamma}\Phi_p(g(w))\omega(\delta_D(w))\,d\mu(w)\right)^{1/p}+C\sum_{j=0}^{N}\delta_D(z_j)^{1-\varepsilon}.
\end{aligned}$$

Hence it suffices to show that

$$\sum_{j=0}^{N}\delta_D(z_j)^{(p-Q)/(p-1)}\omega(\delta_D(z_j))^{-1/(p-1)}(\varphi(\delta_D(z_j)^{-1}))^{-1/(p-1)} \leq C\kappa(\psi(\delta_D(x)),d_\gamma)^{p/(p-1)} \tag{3.3}$$

and

$$\sum_{j=0}^{N}\delta_D(z_j)^{1-\varepsilon} \leq C\int_0^{d_\gamma}t^{-\varepsilon-1}\eta^{-1}(t)\,dt. \tag{3.4}$$

For this purpose, take natural numbers k_0 and k_1 such that $2^{-k_0+1} < d_\gamma \leq 2^{-k_0+2}$ and $2^{-k_1-1} < \psi(\delta_D(x)) \leq 2^{-k_1}$. Then we see from Lemma 3.2 (iii) that

$$\begin{aligned}
&\sum_{j=0}^{N}\delta_D(z_j)^{(p-Q)/(p-1)}\omega(\delta_D(z_j))^{-1/(p-1)}(\varphi(\delta_D(z_j)^{-1}))^{-1/(p-1)}\\
&= \sum_{k=k_0}^{k_1}\left(\sum_{2^{-k-1}<\delta_D(z_j)\leq 2^{-k}}\delta_D(z_j)^{(p-Q)/(p-1)}\omega(\delta_D(z_j))^{-1/(p-1)}(\varphi(\delta_D(z_j)^{-1}))^{-1/(p-1)}\right)\\
&\leq C\sum_{k=k_0}^{k_1}2^{-k\{(p-Q)/(p-1)-1\}}\omega(2^{-k})^{-1/(p-1)}(\varphi((2^{-k})^{-1}))^{-1/(p-1)}\eta^{-1}(2^{-k})\\
&\leq C\int_{2^{-k_1}}^{2^{-k_0+1}}t^{(1-Q)/(p-1)-1}\omega(t)^{-1/(p-1)}(\varphi(t^{-1}))^{-1/(p-1)}\eta^{-1}(t)\,dt\\
&\leq C\int_{\psi(\delta_D(x))}^{d_\gamma}t^{(1-Q)/(p-1)-1}\omega(t)^{-1/(p-1)}(\varphi(t^{-1}))^{-1/(p-1)}\eta^{-1}(t)\,dt.
\end{aligned}$$

Thus (3.3) is obtained. Further, we see that

$$\begin{aligned}
\sum_{j=0}^{N}\delta_D(z_j)^{1-\varepsilon} &= \sum_{k=k_0}^{k_1}\left(\sum_{2^{-k-1}<\delta_D(z_j)\leq 2^{-k}}\delta_D(z_j)^{1-\varepsilon}\right)\\
&\leq C\sum_{k=k_0}^{k_1}2^{-k\{(1-\varepsilon)-1\}}\eta^{-1}(2^{-k})
\end{aligned}$$

$$\leq C \int_{2^{-k_1}}^{2^{-k_0+1}} t^{-\varepsilon-1}\eta^{-1}(t)\, dt$$
$$\leq C \int_0^{d_\gamma} t^{-\varepsilon-1}\eta^{-1}(t)\, dt.$$

Thus (3.4) is obtained and the proof is completed. □

4. Proof of Theorem 2.1

Proof of Theorem 2.1. Fix $r > 0$ such that $r < \min\{\delta_D(x^*), 1/4\}$. For $x \in F(r) = \{z \in D : \delta_D(z) < r\}$, take a curve γ joining x to the John center x^* and satisfying (2.4) and (2.5). We can find $y \in \gamma$ such that $\delta_D(y) = r$ and $\delta_D(z) \leq r$ for all $z \in \gamma(x, y)$. Since $E_{\gamma(x,y)} \subset F(2r)$ and $d_{\gamma(x,y)} \leq 1$, using Lemma 3.3 with $B = B(x^*, \mathrm{diam} D)$, we have

$$|u(x) - u(y)| \leq C\kappa(\psi(\delta_D(x)))\left(\int_{F(2r)} \Phi_p(g(w))\omega(\delta_D(w))d\mu(w)\right)^{1/p} + C\int_0^1 t^{-\varepsilon-1}\eta^{-1}(t)\, dt. \tag{4.1}$$

Further, take a path γ_1 in D joining y and x^* satisfying (2.4) and (2.5). Hence it follows from Lemma 3.3 with $B = B(x^*, \mathrm{diam} D)$ that

$$|u(y) - u(x^*)| \leq C\kappa(\psi(r), d_{\gamma_1})\left(\int_D \Phi_p(g(w))\omega(\delta_D(w))d\mu(w)\right)^{1/p} + C\int_0^{d_{\gamma_1}} t^{-\varepsilon-1}\eta^{-1}(t)\, dt,$$

which implies that $\sup\{|u(y)| : \delta_D(y) = r\}$ is finite for all $r > 0$. If $\kappa(0) = \infty$, then it follows from (4.1) and (2.3) that

$$\limsup_{\delta_D(x)\to 0} \kappa(\psi(\delta_D(x)))^{-1}|u(x)| \leq C\left(\int_{F(2r)} \Phi_p(g(w))\omega(\delta_D(w))d\mu(w)\right)^{1/p}.$$

By (1.7), the left hand side is equal to zero.

On the other hand, the case $\kappa(0) < \infty$ follows readily from Lemma 3.3. Thus our theorem is proved. □

Remark 4.1. In the case of (2) in Theorem 2.1, that is, $\kappa(0) < \infty$, if we impose a strong condition on D like locally uniformity, then u will be seen to have a finite limit at each $\xi \in \partial D$. For example, we may consider a condition on D such that for each $\xi \in \partial D$, there exists a positive constant c_6 with the following property: for every $r > 0$, there exists a point $x^*(r)$ such that each $x \in D \cap B(\xi, r)$ can be joined to $x^*(r)$ by a path γ in $D \cap B(\xi, c_6 r)$ satisfying (2.4).

In fact, it follows from Lemma 3.3 that

$$|u(x) - u(y)| \leq C\kappa(0, 4c_6 r)\left(\int_D \Phi_p(g(w))\omega(\delta_D(w))d\mu(w)\right)^{1/p} + C\int_0^{4c_6 r} t^{-\varepsilon-1}\eta^{-1}(t)\, dt$$

for each $x, y \in D \cap B(\xi, r)$. Since $\kappa(0) < \infty$, this implies that u has a finite limit at ξ.

By Theorem 2.1 and Remark 4.1, we obtain the following corollary, which extends Theorem A. Note here that when D is a bounded Lipschitz domain in X, D is a bounded (η, ψ)-John domain with $\eta(t) = ct$ and $\psi(t) = c't$.

Corollary 4.2. *Let D be a bounded Lipschitz domain in X such that $\partial D \neq \emptyset$. Set*

$$\kappa(r_1, r_2) = \left(\int_{r_1}^{r_2} t^{(1-Q)/(p-1)} \omega(t)^{-1/(p-1)} (\varphi(t^{-1}))^{-1/(p-1)} dt \right)^{1-1/p}$$

for $0 \le r_1 < r_2 < \infty$; set $\kappa(r) = \kappa(r, 1)$ for $0 \le r < 1$. Assume that the pair of u and g satisfies (1.6) *and* (1.7).

(1) *If $\kappa(0) = \infty$, then*

$$\lim_{\delta_D(x) \to 0} \kappa(\delta_D(x))^{-1} u(x) = 0.$$

(2) *If $\kappa(0) < \infty$, then u has a finite limit at each boundary point of D.*

5. Lindelöf theorem

In this section, we are concerned with Lindelöf theorems when u is a function on a uniform domain D satisfying (1.6) and (1.8), as an extension of [4, 5, 14, 21].

Let D in X with $\partial D \neq \emptyset$ be a uniform domain, that is, there exist positive constants A_1 and A_2 such that each pair of points $x, y \in D$ can be joined by a rectifiable curve γ in D for which

$$\ell(\gamma) \le A_1 d(x, y) \tag{5.1}$$

and

$$\delta_D(z) \ge A_2 \min\{\ell(\gamma(x, z)), \ell(\gamma(y, z))\} \quad \text{for all } z \in \gamma \tag{5.2}$$

(see [29]). For $\xi \in \partial D$ and $c > 1$, set

$$T(\xi; c) = \{x \in D : d(x, \xi) < c\delta_D(x)\}.$$

A function u defined on D is said to have a nontangential limit L at $\xi \in \partial D$ if

$$\lim_{T(\xi;c) \ni x \to \xi} u(x) = L$$

for every $c > 0$. For each $\tau \in \mathbf{R}$, set

$$\kappa_\tau(r_1, r_2) = \left(\int_{r_1}^{r_2} t^{(p-Q+\tau)/(p-1)} \omega(t)^{-1/(p-1)} t^{-1} dt \right)^{1-1/p}$$

and

$$\hat{\kappa}(r_1, r_2) = \left(\int_{r_1}^{r_2} t^{(1-Q)/(p-1)-1} \omega(t)^{-1/(p-1)} \eta^{-1}(t) dt \right)^{1-1/p}$$

for $0 \le r_1 < r_2 < \infty$. Here note that $\hat{\kappa}(r_1, r_2) = c^{-1+1/p} \kappa_0(r_1, r_2)$ when $\eta(t) = ct$.

Theorem 5.1. *Let u be a function on a uniform domain D with $g \ge 0$ satisfying* (1.6) *and* (1.8). *Suppose that there exists a constant $\tau \in (0, 1)$ such that $\kappa_\tau(0, 1) < \infty$. Set*

$$E = \left\{ \xi \in \partial D : \limsup_{r \to 0} \frac{r^{Q-\tau} \kappa_\tau(0, r)^p}{\mu(B(\xi, r))} \int_{B(\xi,r) \cap D} g(w)^p \omega(\delta_D(w)) \, d\mu(w) > 0 \right\}.$$

If $\xi \in \partial D \setminus E$ and there exists a rectifiable curve γ in D tending to ξ along which u has a finite limit L, then u has a nontangential limit L at ξ.

REMARK 5.2. Let $h(r;x) = r^{\tau-Q}\kappa_\tau(0,r)^{-p}\mu(B(x,r))$ for $x \in \partial D$ and $r > 0$. If $\liminf_{r\to 0} h(r;x_0) = 0$ for some $x_0 \in X$, then $\mathcal{H}_h(E) = 0$ where $\mathcal{H}_h$ is the generalized Hausdorff measure with respect to h, that is, for $F \subset X$ and $r_0 > 0$,

$$\mathcal{H}_h^{(r_0)}(F) = \inf\left\{\sum_j h(r_j;x_j); F \subset \bigcup_j B(x_j,r_j), 0 < r_j \le r_0\right\}$$

and

$$\mathcal{H}_h(F) = \lim_{r_0\to+0} \mathcal{H}_h^{(r_0)}(F).$$

For a proof of Theorem 5.1, we need the following lemmas.

Lemma 5.3 (cf. [4, Lemma 1])**.** *Let D be a uniform domain. Then for each $\xi \in \partial D$ there exists a rectifiable curve γ_ξ in D ending at ξ such that*

$$\delta_D(z) \ge A_3\ell(\gamma_\xi(\xi,z)) \tag{5.3}$$

for all $z \in \gamma_\xi$, where A_3 is a constant depending only on A_1 and A_2.

We use the following lemma which is proved as in Lemma 3.3.

Lemma 5.4. *Let u and g be functions on D satisfying* (1.6)*. Let B be an open ball with radius R. Then*

$$|u(x)-u(y)| \le C\hat{\kappa}(\psi(\delta_D(x)),d_\gamma)\left(\frac{R^Q}{\mu(B)}\int_{E_\gamma} g(w)^p\omega(\delta_D(w))\,d\mu(w)\right)^{1/p} \tag{5.4}$$

whenever x and y can be joined by a rectifiable curve γ in D satisfying (2.4), (2.5) *and $z \in B$, $\delta_D(z) < R$ for all $z \in \gamma$, where $E_\gamma = \cup_{z\in\gamma}\sigma B(z)$ and $d_\gamma = 4\max_{z\in\gamma}\delta_D(z)$.*

Lemma 5.5 (cf. [4, Lemma 3])**.** *Let u be a function on a uniform domain D with $g \ge 0$ satisfying* (1.6) *and* (1.8)*. Set*

$$\tilde{E} = \left\{\xi \in \partial D : \limsup_{r\to 0}\frac{r^p\omega(r)^{-1}}{\mu(B(\xi,r))}\int_{B(\xi,r)\cap D} g(w)^p\omega(\delta_D(w))\,d\mu(w) > 0\right\}.$$

Supposes $\xi \in \partial D\setminus\tilde{E}$ and there exists a sequence $\{y_j\}$ such that $y_j \in \gamma_\xi$, $2^{-j-1} \le d(\xi,y_j) \le 2^{-j}$ and $u(y_j)$ has a finite limit L, where γ_ξ is as in Lemma 5.3*. Then u has a nontangential limit L at ξ.*

REMARK 5.6. Since $r^{Q-\tau}\kappa_\tau(0,r)^p \ge r^{Q-\tau}\kappa_\tau(r/2,r)^p \ge Cr^p\omega(r)^{-1}$, we see that $\tilde{E} \subset E$. When $\omega(t) = t^\alpha$ and $p > Q+\alpha-1$, we take τ such that $\max\{-p+Q+\alpha,0\} < \tau < 1$. Then we have $\kappa_\tau(0,1) < \infty$ and

$$r^{Q-\tau}\kappa_\tau(0,r)^p = r^{Q-\tau}\left(\int_0^r t^{(p-Q-\alpha+\tau)/(p-1)}t^{-1}dt\right)^{p-1} \le Cr^{p-\alpha}.$$

Hence, we see that $\tilde{E} = E$. In view of Theorem 5.1, we obtain [4, Theorem 1].

Proof of Lemma 5.5. Fix $x \in T(\xi; c)$ with $2^{-j-1} \le d(\xi, x) \le 2^{-j}$. Let γ be a rectifiable curve in D joining x and y_j satisfying (5.1) and (5.2). Take $y \in \gamma$ such that $\ell(\gamma(x,y)) = \ell(\gamma(y_j, y))$, and set $\gamma_1 = \gamma(x,y)$ and $\gamma_2 = \gamma(y_j, y)$. Then γ_i satisfies (2.4) and (2.5) with $\eta(t) = A_2 t$ and $\psi(t) = \min\{A_2, 1\}t/2$. We note that for $z \in \gamma$

$$\delta_D(z) \le d(\xi, z) \le d(\xi, x) + d(x, z) \le (A_1 + 1)d(\xi, x) + A_1 d(\xi, y_j) \tag{5.5}$$

since we have by (5.1)

$$d(x, z) \le \ell(\gamma) \le A_1 d(x, y_j) \le A_1(d(\xi, x) + d(\xi, y_j)).$$

Then we have $d_{\gamma_i} \le 4A_4 2^{-j}$ and $E_{\gamma_i} \subset B(\xi, \frac{3}{2}A_4 2^{-j}) \cap D$, where $d_{\gamma_i} = 4\max_{z\in\gamma_i} \delta_D(z)$, $E_{\gamma_i} = \cup_{z\in\gamma_i} \sigma B(z)$ as in Lemma 5.4 and $A_4 = 2A_1 + 1$. Further, we see that $\delta_D(x) \ge c^{-1} d(\xi, x) \ge c^{-1} 2^{-j-1}$ and

$$\delta_D(y_j) \ge A_3 \ell(\gamma_\xi(\xi, y_j)) \ge A_3 d(\xi, y_j) \ge A_3 2^{-j-1}$$

by Lemma 5.3. Hence, we obtain by Lemma 5.4

$$\begin{aligned}
&|u(x) - u(y_j)| \\
&\le |u(x) - u(y)| + |u(y_j) - u(y)| \\
&\le C\kappa_0(\psi(\delta_D(x)), d_{\gamma_1}) \left(\frac{2^{-jQ}}{\mu(B(\xi, \frac{3}{2}A_4 2^{-j}))} \int_{E_{\gamma_1}} g(w)^p \omega(\delta_D(w))\, d\mu(w) \right)^{1/p} \\
&\quad + C\kappa_0(\psi(\delta_D(y_j)), d_{\gamma_2}) \left(\frac{2^{-jQ}}{\mu(B(\xi, \frac{3}{2}A_4 2^{-j}))} \int_{E_{\gamma_2}} g(w)^p \omega(\delta_D(w))\, d\mu(w) \right)^{1/p} \\
&\le C\kappa_0(A_5 2^{-j-1}, 4A_4 2^{-j}) \left(\frac{2^{-jQ}}{\mu(B(\xi, \frac{3}{2}A_4 2^{-j}))} \int_{B(\xi, \frac{3}{2}A_4 2^{-j}) \cap D} g(w)^p \omega(\delta_D(w))\, d\mu(w) \right)^{1/p} \\
&\le C 2^{-j} \omega(2^{-j})^{-1/p} \left(\frac{1}{\mu(B(\xi, \frac{3}{2}A_4 2^{-j}))} \int_{B(\xi, \frac{3}{2}A_4 2^{-j}) \cap D} g(w)^p \omega(\delta_D(w))\, d\mu(w) \right)^{1/p},
\end{aligned}$$

where $A_5 = \psi(\min\{c^{-1}, A_3\})$. Since $\xi \notin \tilde{E}$ and $\lim_{j\to\infty} u(y_j) = L$, u has a nontangential limit L at ξ. □

Proof of Theorem 5.1. For $r > 0$ sufficiently small, take $x_1(r) \in \gamma \cap \partial B(\xi, r)$ and $x_2(r) \in \gamma_\xi \cap \partial B(\xi, r)$. Then $x_1(r)$ and $x_2(r)$ can be connected by a rectifiable curve γ_0 in D with (5.1) and (5.2). Take $y(r) \in \gamma_0$ such that $\ell(\gamma_0(x_1(r), y(r))) = \ell(\gamma_0(x_2(r), y(r)))$, and set $\gamma_1 = \gamma_0(x_1(r), y(r))$ and $\gamma_2 = \gamma_0(x_2(r), y(r))$. Then γ_1 and γ_2 satisfy (2.4) and (2.5) with $\eta(t) = A_2 t$ and $\psi(t) = \min\{A_2, 1\}t/2$. We see from (5.5) that $d_{\gamma_i} \le 4A_4 r$ and $E_{\gamma_i} \subset B(\xi, \frac{3}{2}A_4 r) \cap D$ for $i = 1, 2$. By Lemma 5.4 replacing $\omega(t)$ with $\omega(t)t^{-\tau}$, we have

$$\begin{aligned}
&|u(x_1(r)) - u(x_2(r))| \\
&\le |u(x_1(r)) - u(y(r))| + |u(x_2(r)) - u(y(r))| \\
&\le C\kappa_\tau(0, d_{\gamma_1}) \left(\frac{r^Q}{\mu(B(\xi, \frac{3}{2}A_4 r))} \int_{E_{\gamma_1}} g(w)^p \omega(\delta_D(w)) \delta_D(w)^{-\tau}\, d\mu(w) \right)^{\frac{1}{p}} \\
&\quad + C\kappa_\tau(0, d_{\gamma_2}) \left(\frac{r^Q}{\mu(B(\xi, \frac{3}{2}A_4 r))} \int_{E_{\gamma_2}} g(w)^p \omega(\delta_D(w)) \delta_D(w)^{-\tau}\, d\mu(w) \right)^{\frac{1}{p}}.
\end{aligned} \tag{5.6}$$

Take $z \in \gamma_i$ and $w \in \sigma B(z)$. Then note that

$$|r - d(\xi, w)| \le d(x_i(r), w) \le d(x_i(r), z) + d(z, w) \le \frac{2 + A_2}{A_2}\delta_D(w), \tag{5.7}$$

since we have $d(x_i(r), z) \le \ell(\gamma_i(x_i(r), z)) \le A_2^{-1}\delta_D(z)$ by (5.2) and $\delta_D(z) \le 2\delta_D(w)$. Hence we have

$$\begin{aligned} &|u(x_1(r)) - u(x_2(r))|^p \\ &\quad \le \; C\kappa_\tau(0, 4A_4 r)^p r^Q \mu(B(\xi, \tfrac{3}{2}A_4 r))^{-1} \int_{B(\xi, \frac{3}{2}A_4 r)\cap D} g(w)^p \omega(\delta_D(w))|r - d(\xi, w)|^{-\tau}\, d\mu(w). \end{aligned}$$

Moreover, since $0 < \tau < 1$, we see that

$$\int_{2^{-j-1}}^{2^{-j}} |r - d(\xi, w)|^{-\tau} dr \le C2^{-j(1-\tau)}.$$

Hence it follows that

$$\begin{aligned} &\inf_{2^{-j-1}\le r\le 2^{-j}} |u(x_1(r)) - u(x_2(r))|^p \\ &\quad \le \; C\int_{2^{-j-1}}^{2^{-j}} \Big(\kappa_\tau(0, 4A_4 r)^p r^Q \mu(B(\xi, \tfrac{3}{2}A_4 r))^{-1} \\ &\qquad\qquad \times \int_{B(\xi, \frac{3}{2}A_4 r)\cap D} g(w)^p \omega(\delta_D(w))|r - d(\xi, w)|^{-\tau} d\mu(w)\Big)\frac{dr}{r} \\ &\quad \le \; C\kappa_\tau(0, 4A_4 2^{-j})^p 2^{-j(Q-1)} \mu(B(\xi, \tfrac{3}{2}A_4 2^{-j}))^{-1} \\ &\qquad\qquad \times \int_{B(\xi, \frac{3}{2}A_4 2^{-j})\cap D} g(w)^p \omega(\delta_D(w)) \left(\int_{2^{-j-1}}^{2^{-j}} |r - d(\xi, w)|^{-\tau} dr\right) d\mu(w) \\ &\quad \le \; C2^{-j(Q-\tau)}\kappa_\tau(0, 4A_4 2^{-j})^p \mu(B(\xi, \tfrac{3}{2}A_4 2^{-j}))^{-1} \int_{B(\xi, \frac{3}{2}A_4 2^{-j})\cap D} g(w)^p \omega(\delta_D(w)) d\mu(w). \end{aligned}$$

Since $\xi \notin E$, we can find a sequence $\{r_j\}$ such that $2^{-j-1} \le r_j \le 2^{-j}$ and

$$\lim_{j\to\infty} |u(x_1(r_j)) - u(x_2(r_j))|^p = 0.$$

Since u has a finite limit L at ξ along γ, we have

$$\lim_{j\to\infty} u(x_2(r_j)) = \lim_{j\to\infty} u(x_1(r_j)) = L.$$

Thus u has a nontangential limit L at ξ by Lemma 5.5. □

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References

[1] A. Björn and J. Björn: Nonlinear Potential Theory on Metric Spaces, EMS Tracts in Mathematics, **17**, European Mathematical Society (EMS), Zürich, 2011.

[2] L. Carleson: Selected problems on exceptional sets. Van Nostrand Mathematical Studies, No. 13 D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[3] N. DeJarnette: *Is an Orlicz-Poincaré inequality an open ended condition, and what does that mean?*, J. Math. Anal. Appl **423** (2015), 358–376.

[4] T. Futamura: *Lindelöf theorems for monotone Sobolev functions on uniform domains*, Hiroshima Math. J. **34** (2004), 413–422.

[5] T. Futamura and Y. Mizuta: *Lindelöf theorems for monotone Sobolev functions*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 271–277.

[6] T. Futamura and Y. Mizuta: *Boundary behavior of monotone Sobolev functions on John domains in a metric space*, Complex Var. Theory Appl. **50** (2005), 441–451.

[7] T. Futamura and T. Shimomura: *Boundary behavior of monotone Sobolev functions in Orlicz spaces on John domains in a metric space*, J. Geom. Anal. **28** (2018), 1233–1244.

[8] P. Hajłasz and P. Koskela: Sobolev met Poincaré, Mem. Amer. Math. Soc. **145** (2000), no. 688.

[9] J. Heinonen, T. Kilpeläinen and O. Martio: Nonlinear potential theory of degenerate elliptic equations, Oxford Univ. Press, New York, 1993.

[10] P. Koskela, J.J. Manfredi and E. Villamor: *Regularity theory and traces of $\mathcal{A}$-harmonic functions*, Trans. Amer. Math. Soc. **348** (1996), 755–766.

[11] H. Lebesgue: *Sur le probléme de Dirichlet*, Rend. Circ. Mat. Palermo **24** (1907), 371–402.

[12] S. Lisini: *Absolutely continuous curves in extended Wasserstein-Orlicz spaces*, ESAIM: COCV. **22** (2016), 670–687.

[13] J.J. Manfredi: *Weakly monotone functions*, J. Geom. Anal. **4** (1994), 393–402.

[14] J.J. Manfredi and E. Villamor: *Traces of monotone Sobolev functions*, J. Geom. Anal. **6** (1996), 433–444.

[15] J.J. Manfredi and E. Villamor: *Traces of monotone Sobolev functions in weighted Sobolev spaces*, Illinois J. Math. **45** (2001), 403–422.

[16] V.G. Maz'ya and S.V. Poborichi: Differentiable functions on bad domains, World Scientific, River Edge, NJ, 1997.

[17] Y. Mizuta: *On the boundary limits of harmonic functions with gradient in L^p*, Ann. Inst. Fourier (Grenoble) **34** (1984), 99–109.

[18] Y. Mizuta: *On the boundary limits of harmonic functions*, Hiroshima Math. J. **18** (1988), 207–217.

[19] Y. Mizuta: *On the existence of weighted boundary limits of harmonic functions*, Ann. Inst. Fourier (Grenoble) **40** (1990), 811–833.

[20] Y. Mizuta: *Boundary limits of polyharmonic functions in Sobolev-Orlicz spaces*, Complex Variables **27** (1995), 117–131.

[21] Y. Mizuta: *Tangential limits of monotone Sobolev functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **20** (1995), 315–326.

[22] Y. Mizuta: Potential theory in Euclidean spaces, Gakkōtosho, Tokyo, 1996.

[23] Y. Mizuta and T. Shimomura: *Boundary limits of spherical means for BLD and monotone BLD functions in the unit ball*, Ann. Acad. Sci. Fenn. Math. **24** (1999), 45–60.

[24] Y. Mizuta and T. Shimomura: *Differentiability and Hölder continuity of Riesz potentials of Orlicz functions*, Analysis (Munich) **20** (2000), 201–223.

[25] W. Orlicz: *Über konjugierte Exponentenfolgen*, Studia Math. **3** (1931), 200–211.

[26] M.A. Ragusa and A. Tachikawa: *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), 710–728.

[27] M.A. Ragusa, A. Tachikawa and H. Takabayashi: *Partial regularity of $p(x)$-harmonic maps*, Trans. Amer. Math. Soc. **365** (2013), 3329–3353.

[28] Yu.G. Reshetnyak: Space mappings with bounded distortion, Translations of Mathematical Monographs **73**, American Mathematical Society, Providence, RI, 1989.

[29] J. Väisälä: *Uniform domains*, Tohoku Math. J. **40** (1988), 101–118.

[30] E. Villamor and B.Q. Li: *Boundary limits for bounded quasiregular mappings*, J. Geom. Anal. **19** (2009), 708–718.

[31] M. Vuorinen: Conformal geometry and quasiregular mappings, Lectures Notes in Math. **1319**, Springer, Berlin-Heidelberg-New York, 1988.

[32] H. Wallin: *On the existence of boundary values of a class of Beppo Levi functions*, Trans. Amer. Math. Soc. **120** (1965), 510–525.

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