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ON THE INTERPOLATION OF SOME FUNCTION ALGEBRAS

Dedicated to Professor H. Terasaka on his sixtieth birthday

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Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space X and let A be a function algebra on X , that is, a closed (by supremum norm) subalgebra in $C(X)$ containing constants and separating points of X . A closed set F_0 in X is said to be an interpolation set of A (or a closed restriction set of A) if $A|F_0 = C(F_0)$ (or $A|F_0$ is closed in $C(F_0)$), where $A|F_0 = \{f|F_0, f \in A\}$ and $f|F_0$ is the restriction of f on F_0 . In [4] I. Glicksberg characterized interpolation sets and closed restriction sets on general function algebras A , and also showed that in a Dirichlet algebra, any closed restriction set of A is an intersection of peak sets, but we see that the above fact is false in the case of a non-Dirichlet algebra. The main purpose of this paper is to consider problems of interpolation and closed restriction on a function algebra A which is not a Dirichlet algebra and which has the property that the restriction $A|\partial A$ of A by its Šilov boundary is an essential maximal algebra. Our main theorems are the following: Let A be a function algebra on a compact metric space having the above property and some additional properties (Properties (B) and (D), cf. §2, 3.). Then, (1) if F_0 is a closed restriction set of A for a closed set F_0 , F_0 contains ∂A or $F_0 \sim \partial A$ is a countable set whose cluster points are in ∂A (Theorem 2.2). (2) if F_0 is an interpolation set of A , then $F_0 \cap \partial A$ is an interpolation set of $A|\partial A$ and $F_0 \sim \partial A$ is an H^∞ -interpolating sequence, and the converse is also true (Theorem 3.2). ((2) was pointed by ([7], p. 208) in the case of the function algebra of continuous functions on the unit closed disc which is analytic on its interior, and it is a generalization).

1. Preliminaries

Let A be a linear subspace of $C(X)$. Then A is said to be a *function algebra on a compact Hausdorff space* X if it satisfies the following conditions; (i) $f \cdot g \in A$ for any $f, g \in A$, (ii) A is closed with the supremum

norm of $C(X)$, (iii) A contains the constant function 1 and (iv) for any distinct two points x, y in X , there is an $f \in A$ with $f(x) \neq f(y)$. First, we define the Šilov boundary ∂A and the essential set E of A as follows. The Šilov boundary of A is the smallest closed subset F of X such that $|f|$ takes its maximum value on F for any $f \in A$. The essential set of A is the minimal closed subset E of X such that if $f(E) = 0$ for a continuous functions f on X , then $f \in A$. A is an essential algebra if the essential set of A is X (cf. [1]). A is said to be an antisymmetric algebra (or an analytic algebra) if any real-valued function in A is always constant (or any function in A vanishing on a non-empty open set in X is always identically zero) (cf. [6]). A function algebra A is said to be a sequentially analytic algebra if any function f in A vanishing on an infinite closed set in $X \sim \partial A$ is always identically zero. If X is a metric space, we can take a sequence of points converging to a point in $X \sim \partial A$ in place of an infinite closed set in the above definition. For a sequentially analytic algebra, we can easily prove the following.

(a) Let A be a sequentially analytic algebra on X . Let X have no isolated point and let ∂A be non dense in X . Then A is analytic.

(b) If A is a sequentially analytic algebra and if $X \sim \partial A$ has a non-isolated point, then A is an integral domain, that is, $fg \equiv 0$ implies $f \equiv 0$ or $g \equiv 0$ for $f, g \in A$.

Let A be a function algebra on X and let F_0 be a closed set in X . We say " F_0 determines A " if $f \equiv 0$, whenever $f(F_0) = 0$ for an $f \in A$. For arbitrary function algebra, ∂A determines A . F_0 is said to be an interpolation set of A (or a closed restriction set of A) if $A|F_0 = C(F_0)$ (or $A|F_0$ is closed in $C(F_0)$), where $A|F_0$ denotes the set $\{f|F_0; f \in A\}$ and $f|F_0$ denotes the restriction of f on F_0 . If F_0 is an interpolation set of A , then any continuous function on F_0 can be extended to a function in A .

For a function algebra which is an integral domain, we have

Theorem 1.1. Let A be a function algebra which is an integral domain and let $P \subset F_1 \cup \dots \cup F_n$, where P determines A , F_i is closed set in X and F_i is a closed restriction set of A for any i . Then $F_k \supset \partial A$ for some k .

Proof. If we assume that $f \equiv 0$ whenever $f(F_k) = 0$ for an $f \in A$, the complex homomorphism $h \rightarrow h(x)$ of $A|F_k$ is well-defined for any $x \in X$. Since $A|F_k$ is a Banach algebra, $|h(x)| \leq \|h\|_{F_k}$, so $\|h\|_X \leq \|h\|_{F_k}$ for any $h \in A$, where $\|h\|_{F_k} = \sup_{x \in F_k} |h(x)|$ and $\|h\|_X = \sup_{x \in X} |h(x)|$. Therefore $F_k \supset \partial A$. If for any i , $F_i \not\supset \partial A$, there is an $f_i \in A$ such that $f_i(F_i) = 0$ and $f_i \not\equiv 0$.

But, since $f_1 f_2 f_3 \cdots f_n \equiv 0$ on P , $f_1 f_2 \cdots f_n \equiv 0$ on X . This is a contradiction since A is an integral domain.

REMARK. We see easily that the theorem is false in the case of antisymmetric algebras.

Corollary 1.2. *Let A be a sequentially analytic algebra on a compact metric space X , and let F_0 be a closed restriction set of A for a closed set F_0 in X . Then either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a countable set whose cluster points are in ∂A (if it is an infinite set).*

Proof. Put $G_n = \{x; x \in X, d(x, \partial A) < \frac{1}{n}\}$, where $d(x, y)$ denotes the metric function on X . Then if $F_0 \sim G_n$ is an infinite set for some n , $F_0 \sim G_n$ is a set which determines A since A is a sequentially analytic algebra. Since $F_0 \sim G_n \subset F_0$ and $A|_{F_0}$ is closed in $C(F_0)$, $F_0 \supset \partial A$ by Theorem 1.1. (In the corollary, it is unnecessary that A is an integral domain).

Let A be a function algebra on a compact Hausdorff space X . By the maximal ideal space \mathfrak{M} of A we mean the set of all maximal ideal of A . \mathfrak{M} can be regarded as the set of all non-zero complex homomorphisms of A . \mathfrak{M} is a compact Hausdorff space for its weak topology and $\mathfrak{M} \supset X$. A maximal ideal M is said to be a point x in X if $M = M_x = \{f: f(x) = 0, f \in A\}$. A ideal N in A is said to be a principal ideal if $N = f_0 \cdot A = \{f_0 f; f \in A\}$ for an $f_0 \in A$.

Glicksberg [5] has proved the following theorem.

Theorem 1.3. *Let A be a function algebra on a compact Hausdorff space X . If F is a closed restriction set of A for any closed F in X , then $A = C(X)$.*

The following corollary is clear from Theorem 1.3.

Corollary 1.4. *Let A be a function algebra on a compact Hausdorff space X , and let F_0 be a closed set in X . F_0 is an interpolation set of A if and only if for any closed set $F \subset F_0$, F is a closed restriction set of A .*

For any closed set F containing ∂A , we see that $A|_F$ is closed in $C(F)$. Conversely, we have (cf. [9])

Theorem 1.5. *Let A be an essential algebra on X and let F_0 be a closed subset in X . If F is a closed restriction set of A for any closed subset F containing F_0 , then F_0 contains the Šilov boundary ∂A of A .*

Corollary 1.6. *Let A be an arbitrary function algebra on X and let F_0 be a closed subset in X which is contained in the essential set E of A . If F is a closed restriction set of A for any closed subset F containing F_0 ,*

then F_0 contains the Šilov boundary $\partial_{A|E}$ of the function algebra $A|E$ (see the next Remark).

From Corollary 1.6 we can prove Theorem 1.3.

REMARK. (1) In Corollary 1.6 we can prove the converse, that is, for any closed set F containing $\partial_{A|E}$ F is a closed restriction set. For, let $F \supset \partial_{A|E}$, then $\|f\|_E = \|f\|_{\partial_{A|E}} \leq \|f\|_F$ for any $f \in A$. Since $E \cup F \supset E$, we see that $E \cup F$ is a closed restriction set, that is, for any f in A , there is a $g \in A$ such that $\|g\|_X \leq \gamma \|f\|_{E \cup F}$ and $g = f$ on $E \cup F$. (Theorem 2.1) Therefore $\|g\|_X \leq \gamma \|f\|_{E \cup F} \leq \gamma \|f\|_F$ and $g = f$ on F , so F is a closed restriction set by Theorem 2.1.

(2) $\partial A \cap E$ always contains $\partial_{A|E}$ (cf. [8]) and we can have an example with $\partial A \cap E \neq \partial_{A|E}$, so the conclusion ($F_0 \supset \partial A \cap E$) of Theorem 2 of [9] is false (see the above (1)).

(3) If X is a compact metric space, we have that $F_0 \supset P \cap E \neq \phi$ as the conclusion, under the hypothesis of Corollary 1.6. P here denotes the minimal boundary of A (it is the set of peak points of A and is also equal to the Choquet boundary of A) (cf. [3]). For, it is clear since $\partial_{A|E} \supset P \cap E$.

2. Closed restriction sets

Let A be a function algebra on a compact Hausdorff space X and let F_0 be a closed subset in X . A closed restriction set of A is characterized as follows.

Theorem 2.1. *Let A be a function algebra on a compact Hausdorff space X and let F_0 be a closed subset in X . Then F_0 is a closed restriction set if and only if for any $f \in A$ there is a $g \in A$ such that $\|g\|_X \leq \gamma \|f\|_{F_0}$ and $f = g$ on F_0 , where γ is a positive number which is independent of f .*

Proof. If $A|F_0$ is closed in $C(F_0)$, by Glicksberg ([4], P. 420), $A|F_0$ is isomorphic to A/kF_0 ($kF_0 = \{f \in A : f(F_0) = 0\}$), so the necessity of the theorem is clear. The sufficiency can also be proved easily.

After now, we consider function algebras satisfying the following properties:

(A). *The function algebra $A|\partial A$ is an essential maximal algebra.*

(B). *Any maximal ideal in A which is not a point of ∂A is always principal (cf. §1).*

The main theorem of this paragraph is the following

Theorem 2.2. *Let A be a function algebra on a compact metric space X satisfying the properties (A) and (B), and let F_0 be a closed set in X .*

If F_0 is a closed restriction set of A , then either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a countable set whose cluster points are in ∂A (if it is an infinite set).

Proof. The proof is clear by Corollary 1.2 (§1) and the next lemma.

Lemma 2.3. *Let A be a function algebra on a compact Hausdorff space X satisfying the properties (A) and (B). Then A is a sequentially analytic algebra.*

Proof. Let F_0 be an infinite closed set in $X \sim \partial A$ and let $f_0(F_0) = 0$ for an $f_0 \in A$. Then we have to prove that $f_0 \equiv 0$. Let x_0 be a point in F_0 which is not an isolated point. Since $M_0 = \{f : f(x_0) = 0, f \in A\}$ is a maximal ideal in A , by the hypothesis, $M_0 = g_0 A$ for some $g_0 \in A$. Since $f_0(x_0) = 0$, $f_0 = g_0 a_1$ for an $a_1 \in A$. We see here that x_0 is the sole point satisfying $g_0(x_0) = 0$, so $a_1(F_0 \sim (x_0)) = 0$, $a_1(x_0) = 0$ and $a_1 \in M_0$. Therefore, $a_1 = g_0 a_2$ for an $a_2 \in A$. By repeating the same argument, we have a sequence $\{a_n\}$ of functions in A such that

$$f_0 = g_0 a_1 = g_0^2 a_2 = \dots = g_0^k a_k = \dots \quad \dots\dots\dots (1)$$

Now, since the function g_0 does not vanish on ∂A , $g_0^{-1} | \partial A \in C(\partial A)$. But $g_0^{-1} | \partial A$ can not be extended to any function in A since $g_0(x_0) = 0$, that is, $g_0^{-1} | \partial A \notin A | \partial A$. Since $A | \partial A$ is a maximal algebra, the closed subalgebra spanned by $g_0^{-1} | \partial A$ and $A | \partial A$ is identical to $C(\partial A)$, so for any $h \in C(\partial A)$ and for any $\varepsilon > 0$,

$$\|h - (\alpha_0 + \alpha_1 g_0^{-1} + \dots + \alpha_k g_0^{-k})\|_{\partial A} < \varepsilon \quad \dots\dots\dots (2)$$

, where $\alpha_i \in A$.

By (1), $g_0^{-k} f_0 = a_k$ on ∂A ($k = 1, 2, 3, \dots$) $\dots\dots\dots (3)$

By (2) and (3),

$$\|h f_0 - (\alpha_0 f_0 + \alpha_1 a_1 + \dots + \alpha_k a_k)\|_{\partial A} \leq \varepsilon \|f_0\|_{\partial A}$$

Since $\alpha_0 f_0 + \alpha_1 a_1 + \dots + \alpha_k a_k \in A$, $h f_0 \in A | \partial A$.

$$C(\partial A) \cdot f_0 \subset A | \partial A.$$

Put $\partial A = Y$ and $B = A | Y$. Then B is an essential algebra on Y . $C(Y) \cdot f_0 \subset B$. From this we can prove that $f_0 = 0$ on Y , hence $f_0 \equiv 0$ on X . If $f_0 \not\equiv 0$ on Y , $Z = \{x : x \in Y, f_0(x) = 0\}$ is a closed subset in Y and $Z \neq Y$. We can take two open sets U, V such that

$$Z \subset V \subset \bar{V} \subset U \subset \bar{U} \subseteq Y.$$

Since $A | Y$ is essential, there is a function $p \in C(Y)$ such that $p(\bar{U}) = 0$

and that p cannot be extended to any function in A . We put

$$\begin{aligned} h_0(x) &= p(x)/f_0(x) & \text{if } x \in Y \sim V, \\ &= 0 & \text{if } x \in V, \end{aligned}$$

then $h_0 f_0 = p$. Since h_0 is continuous on Y , this is a contradiction, so $f_0 \equiv 0$ on X .

Corollary 2.4. *Let A be a function algebra on a compact metric space X which has the property (A) and is generated by function f_0 . Then if F_0 is a closed restriction set of A for a closed set F_0 in X , either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a countable set whose cluster points are in ∂A (if it is an infinite set).*

Proof. This is clear by Theorem 2.1. and next lemma.

Lemma 2.5. *If A is generated by a function f_0 , then any maximal ideal in A which is not a point in ∂A is principal, that is, A satisfies the property (B).*

Proof. Let M be a maximal ideal in A which is not a point in ∂A . Then $M = \{f : \varphi_0(f) = 0\}$ for some non-zero complex homomorphism φ_0 . Since A is generated by a function f_0 , for any $f \in M$ and for any $\varepsilon > 0$, there is a polynomial of f_0 such that

$$\|f - (a_0 + a_1 f_0 + \dots + a_n f_0^n)\| < \varepsilon \quad \dots\dots\dots (1)$$

, where a_i is a complex number.

If we put $\varphi_0(f_0) = \alpha$, $\varphi_0(f_0 - \alpha) = 0$. For the above polynomial, we set $g = a_0 + a_1 f_0 + \dots + a_n f_0^n = (f_0 - \alpha)\psi(f_0) + \beta$, where $\psi(f_0)$ is a polynomial of f_0 and β is a complex number. We easily see that $\varphi_0(g) = \beta$. By (1) we have $|\beta| < \varepsilon$. By (1) again,

$$\|f - (f_0 - \alpha)\psi(f_0)\| = \|f - g + \beta\| \leq \|f - g\| + |\beta| < 2\varepsilon \quad \dots\dots (2)$$

Now, the function f_0 cannot take the value α on ∂A . For, if $f_0(x_0) = \alpha$ for some $x_0 \in \partial A$, by (2) $f(x_0) = 0$ for any $f \in M$. Since M is not a point in ∂A , this is a contradiction. Therefore, by (2) we have

$$\left\| \frac{f}{f_0 - \alpha} - \psi_n(f_0) \right\|_{\partial A} \leq \frac{1}{n \cdot \delta}$$

, where $\delta = \min_{x \in \partial A} |f_0(x) - \alpha|$ and $\psi_n(f_0)$ is a polynomial of f_0 for any n . Since $\psi_n(f_0) \in A$, $f/f_0 - \alpha \in A \setminus \partial A$, so $f = (f_0 - \alpha)h$ on ∂A ($h \in A$). $f = (f_0 - \alpha)h$ on X . This shows that M is principal.

Corollary 2.6. *Let A be the algebra of all continuous functions on the closed unit disc (in the complex plane) which are analytic in its interior and let F_0 be a closed restriction set of A . Then F_0 contains ∂A (=the unit circle) or F_0 is an interpolation set of A .*

Proof. Let F_0 be a closed restriction set of A . By Corollary 2.4., F_0 contains the unit circle K in the unit disc or $F_0 \sim K$ is a countable set whose cluster points are in K . Therefore, if F_0 does not contain K , $F_0 \cap K \subseteq K$ and $F_0 \sim K$ is a countable set whose cluster points are in K , so $F_0 = (F_0 \cap K) \cup (F_0 \sim K)$ does not divide the complex plane and is also a non dense set in the complex plane. It follows that $A|F_0$ is dense in $C(F_0)$ by the Lavrent'ev approximation theorem, so $A|F_0 = C(F_0)$, that is, F_0 is an interpolation set.

Corollary 2.6. can be extended to the case which A is a more general algebra.

3. Interpolation sets

Let A_0 be the function algebra of all continuous functions on the unit disc which are analytic in its interior. Then Hoffman ([7], P. 208) has pointed that the following two statements are equivalent for a sequence of distinct points $\{z_k\}$ in the open unit disc: (a). If g is any continuous function on the closed unit disc, there exists $f \in A_0$ such that $f(z_k) = g(z_k)$, $k=1, 2, 3, \dots$. (b). $\{z_k\}$ is an interpolating sequence for H^∞ , and the set of accumulation points of $\{z_k\}$ on the unit circle has Lebesgue measure zero.

In this paragraph we consider a generalization of the above fact (Theorem 3.2.).

Let A be a function algebra on X . A is said to be a *Dirichlet algebra* if the set of all real parts of A , $Re A$ is dense in $C_R(X)$, where $C_R(X)$ is the set of all real-valued continuous functions on X . In §2 we see that if a function algebra A satisfies the property (A) and has a function f_0 as its generator and if F_0 is a closed restriction set of A (and hence, if F_0 is an interpolation set of A), then either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a sequence of points whose cluster points are in ∂A . Let A have a function f_0 as its generator. Then we can assume that A satisfies the following property: Let P be a compact set in the complex plane having a connected complement and let Γ be the boundary of P .

(*). *A is generated by a function f_0 such that $\Gamma \subset f_0(X) \subset P$, where $f_0(X) = \{f_0(x) : x \in X\}$.*¹⁾

1) Put $P = C \sim U_\infty$, where C is the complex plane and U_∞ is the connected component of $C \sim f_0(X)$ containing ∞ . Then P satisfies (*). If $A \neq C(X)$, $P \neq \emptyset$.

Let $Y = \{y_1, y_2, \dots, y_n, \dots\}$ be a sequence of points in $X \sim \partial A$. Y is said to be an H^∞ -interpolating sequence if for any bounded sequence of complex numbers $\{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$, there is an $f \in H^\infty$ such that $f(y_i) = \alpha_i$ for any i , where H^∞ denotes the set of all bounded function f on $X \sim \partial A$ such that there is a sequence of function $\{f_i\}$ in A and f_i converges uniformly to f on any compact set in $X \sim \partial A$. We easily see that if X is a compact metric space H^∞ is a Banach algebra with the norm $\|f\|_\infty = \sup_{x \in X \sim \partial A} |f(x)|$. We call H^∞ the ∞ -Hardy class relative to A .

We consider the following property for H^∞ :

(D). For any $f \in H^\infty$, there is a sequence of functions $\{f_n\}$ in A such that $\|f_n\| \leq \delta \|f\|$ and f_n converges to f on any compact set in $X \sim \partial A$, where γ is independent of f .

Let A_0 be the set of all continuous functions on the unit disc which are analytic in its interior, and let H^∞ be its ∞ -Hardy class, that is, the set of all bounded analytic functions on the open disc. For any $f \in H^\infty$, we put $f_n(z) = f\left\{\left(1 - \frac{1}{n}\right)z\right\}$ ($n = 1, 2, 3, \dots$). Then f_n can be defined as a function in A_0 . We easily see that $\|f_n\| \leq \|f\|_\infty$ and f_n converges to f on any compact set in the open unit disc.

First, we shall prove the following theorem.

Theorem 3.1. Let A be a function algebra on a compact metric space X which has the property (A) and is generated by a function f_0 . Then if F_0 is an interpolation set of A for a closed set F_0 in X , the following conditions are satisfied:

- (i) $F_0 \cap \partial A$ is an interpolation set of $A|_{\partial A}$.
- (ii) For any finite set $\{y_1, y_2, \dots, y_n\}$ in $F_0 \sim \partial A$ and for any finite set $\{c_1, c_2, \dots, c_n\}$ of complex numbers, there is an $f \in A$ such that $f(y_i) = c_i$ ($i = 1, 2, \dots, n$) and $\|f\| \leq \gamma \sup_{i \leq n} |c_i|$, where γ is a positive number which is independent of $\{y_1, y_2, \dots, y_n\}$ and of $\{c_1, c_2, \dots, c_n\}$.

Conversely, the conditions (i) and (ii) imply that F_0 is an interpolation set of A .

The main theorem of this paragraph is the following

Theorem 3.2. Let A be a function algebra on a compact metric space X satisfying the property (A) and having a generator f_0 . Then if F_0 is an interpolation set of A , the following conditions are satisfied:

- (i) $F_0 \cap \partial A$ is an interpolation set of $A|_{\partial A}$.
- (ii') $F_0 \sim \partial A$ is an H^∞ -interpolating sequence.

Conversely, if the ∞ -Hardy class H^∞ relative to A has the property

(D), then the conditions (i) and (ii') imply that F_0 is an interpolation set of A .

Proof of Theorem 3.1. (i) Let f be any continuous function on $M=F_0 \cap \partial A$, and let f^* be a continuous extension of f on F_0 . Since $A|F_0=C(F_0)$, there is a function $g \in A$ such that $g=f^*$ on F_0 , so $g=f$ on M .

(ii) Let $\{c_1, c_2, \dots, c_n\}$ be a sequence of complex numbers and let h be a continuous function on F_0 such that

$$\begin{aligned} h(y_i) &= c_i & (i = 1, 2, \dots, n) \\ h(x) &= 0 & (x \in F_0 \sim \{y_1, y_2, \dots, y_n\}). \end{aligned}$$

Since $A|F_0=C(F_0)$, there is an $f \in A$ such that $f(x)=h(x)$ on F_0 . We here can assume that $\|f\|_X \leq \gamma \|h\|_{F_0}$ (γ is independent of h) by Theorem 2.1, so $f(y_i)=c_i$ ($i=1, 2, \dots, n$) and $\|f\|_X \leq \gamma \sup_{i \leq n} |c_i|$.

Conversely, let A satisfy the conditions (i) and (ii). We will show that $A|F_0=C(F_0)$. Put $M=\partial A \cap F_0$. For any continuous function f on F_0 , $f|M \in C(M)$. By (i) there is an $f' \in A$ such that $f'=f$ on M . If we put $f_1=f'-f$, then $f_1(M)=0$. If we prove that $f_1=g$ on F_0 for a function $g \in A$, then $f=h$ on F_0 for some $h \in A$, so the theorem will be proved. Therefore, for any $f_1 \in C(F_0)$, $f_1(M)=0$, we are only to prove that $f_1=g$ on F_0 for some $g \in A$. We can assume that $\|f_1\|_{F_0}=1$. By Theorem 2.2 we put $F_0 \sim \partial A = \{y_1, y_2, y_3, \dots\}$. Since $f_1(M)=0$, there is a positive integer n_1 such that

$$\{y_i | |f_1(y_i)| \geq 1/4\} \subset \{y_1, y_2, y_3, \dots, y_{n_1}\}.$$

Since $A|\partial A$ is a maximal essential algebra, $M \neq \partial A$. And since M is an interpolation set of $A|\partial A$, there is a function $\psi \in A|\partial A$ such that $\psi(M)=1$, $\psi \not\equiv 1$ and $\|\psi\|_{\partial A}=1$.²⁾ (cf. [4]). Since $y_i \notin \partial A$, by ([2] or [6], §5), $|\psi(y_i)| < 1$ ($i=1, 2, 3, \dots$). By taking a sufficiently large integer m , the value of $1-\psi^m$ on y_i ($i=1, 2, \dots, n_1$) can be arbitrarily near 1. Also, since $\psi(M)=1$, there is a positive integer n_2 ($n_2 \geq n_1$) such that $\{y_i : |(1-\psi^m)(y_i)| < 1/4\gamma\} \subset \{y_{n_2+1}, y_{n_2+2}, \dots\}$, where γ is that in the condition (ii). By (ii) there is a function $p \in A$, such that $p(y_i)=f_1(y_i)$ ($i=1, 2, \dots, n_1$), $p(y_i)=0$ ($i=n_1+1, \dots, n_2$) and $\|p\| \leq \gamma \|f_1\|_{F_0} = \gamma$. For a sufficiently large integer m ,

$$\begin{aligned} |(1-\psi^m)(y_i) \cdot p(y_i) - f_1(y_i)| &< 1/2 & (i = 1, 2, \dots, n_1) \\ |(1-\psi^m)(y_i) \cdot p(y_i)| &< 1/4 & (i = n_1+1, n_1+2, \dots), \\ \text{so } |(1-\psi^m)(y_i) \cdot p(y_i) - f_1(y_i)| &< 1/2 & (i = n_1+1, n_1+2, \dots). \end{aligned}$$

2) Since $f_0(M) \subset \Gamma$ (Lemma 3.3. Footnote) we can find the function ψ by the similar method as the proof of Lemma 3.3.

Put $(1-\psi^m)p=g_1$, then $g_1 \in A$, $\|f_1-g_1\|_{F_0} < 1/2$ and $\|g_1\|_X \leq 2\gamma$. If we $g_1^* = f_1 - g_1$, $g_1^*(M) = 0$ and $\|g_1^*\|_{F_0} < 1/2$. By repeating the same argument, we have a sequence $\{g_n\} \subset A$ such that $\|g_n\|_X < 2^{-(n-2)} \cdot \gamma$ and $g_n^* = f_1 - g_1 - g_2 \cdots - g_n$ satisfies that $g_n^*(M) = 0$ and $\|g_n^*\|_{F_0} < 2^{-n}$ for any n . Put $h_n = g_1 + g_2 + \cdots + g_n$, then $h_n \in A$. If $m > n$, $\|h_m - h_n\|_X = \|g_{n+1} + \cdots + g_m\|_X \leq \|g_{n+1}\|_X + \cdots + \|g_m\|_X \leq 2^{-(n-2)}\gamma$, so h_n converges to some $h \in A$, and $\|f_1 - h_n\|_{F_0} = \|g_n^*\|_{F_0} < 2^{-n}$. This shows that $f_1 = h$ on F_0 for some $h \in A$.

Before the proof of Theorem 3.2 we need the following lemmas.

Lemma. 3.3. *Let A be a function algebra having a generator f_0 and let a sequence of points $\{y_n\} \subset X \sim \partial A$ converges to point y in $X \sim \partial A$. Then*

$$\rho(y_n, y) = \sup_{\substack{f \in A \\ f \neq 0}} |f(y_n) - f(y)| / \|f\| \text{ converges to } 0.$$

Proof. Since A is generated by f_0 , the set of all polynomials $a_0 + a_1 f_0 + \cdots + a_m f_0^m$ (a_i is a complex number) is dense in A . For any polynomial g of f_0 , $g = a_0 + a_1 f_0 + \cdots + a_m f_0^m$, we put $g'(z) = a_0 + a_1 z + \cdots + a_m z^m$. We consider the polynomial $g'(z)$ as function on P . Put $\|g'\| = \sup_{z \in P} |g'(z)|$. Then there is a complex number z_0 ($z_0 \in \Gamma$) such that $\|g'\| = |g'(z_0)|$. By the property (*), there is a point $x_0 \in X$ such that $z_0 = f_0(x_0)$, so $\|g'\| = |a_0 + a_1 f_0(x_0) + a_2 f_0^2(x_0) + \cdots + a_m f_0^m(x_0)| \leq \|g\|$. Therefore,

$$|g(y_n) - g(y)| / \|g\| \leq |g'[f_0(y_n)] - g'[f_0(y)]| / \|g'\| \dots\dots\dots (1)$$

Now, we easily see that $f_0(x) \notin \Gamma$ if $x \notin \partial A$. For, let $f_0(x) \in \Gamma$ for a point $x \notin \partial A$. If $f_0(x) = f_0(x')$ for another point x' , then $f(x) = f(x')$ for any $f \in A$, since A is generated by f_0 . This contradiction shows that there exists no point x' (different from x) such that $f_0(x) = f_0(x')$. If we put $u_0 = f_0(x)$, then $u_0 \in \Gamma$. The function algebra of continuous functions of Γ which admit a continuous extension to P that is analytic on the interior of P is a Dirichlet algebra ([10]) and the one point set u_0 is a closed restriction set. Therefore $\{u_0\}$ is peak set ([4]), so there is a continuous function ψ on P which is analytic on the interior of P such that $\psi(u_0) = 1$ and $|\psi(u)| < 1$ for any $u \in P$ ($u_0 \neq u$). Put $h = \psi \circ f_0$. Then $h \in A$, since ψ is approximated uniformly on P by polynomials of z . We see that $\|h\| = 1$ and x is the sole point satisfying $|h(x)| = 1$. Since $x \notin \partial A$, this is a contradiction, so $f_0(x) \notin \Gamma$ if $x \notin \partial A$.³⁾ Coming back argument, let $y_n, y \notin \partial A$ and $y_n \rightarrow y$. Then $f_0(y_n) \notin \Gamma$ for any n , $f_0(y) \notin \Gamma$ and $f_0(y_n) \rightarrow f_0(y)$. Since $g'(z)$ is analytic in the interior of P ,

3) We can prove that $f_0(\partial A) = \Gamma$. For, f_0 is a homeomorphism of X onto $f_0(X)$ and $\Gamma \subset f_0(X) \subset P$.

$$|g'[f_0(y_n)] - g'[f_0(y)]| / \|g'\| \leq M |f_0(y_n) - f_0(y)| \dots\dots\dots (2)$$

, where M is a constant number which is independent of n . By (1), (2),

$$|g(y_n) - g(y)| / \|g\| \leq M\eta_n, \quad \text{where } \eta_n \rightarrow 0 \text{ for } n \rightarrow \infty.$$

Since the set of all such functions g is dense in A , $\rho(y_n, y) \rightarrow 0$.

This lemma implies the following

Lemma 3.4. *Let A be a function algebra on a compact metric space X having a generator f_0 . Then any equibounded sequence of functions in A is equicontinuous on any compact subset K in $X \sim \partial A$.*

Proof of Theorem 3.2. In order to prove (ii'), let $\{c_1, c_2, \dots, c_n, \dots\}$ be a bounded sequence of complex numbers. If $F_0 \sim \partial A = \{y_1, y_2, \dots, y_n, \dots\}$, by Theorem 3.1 (ii), there is a $g_n \in A$ for any n such that $g_n(y_i) = c_i$ ($i = 1, 2, \dots, n$) and $\|g_n\|_X \leq \gamma \sup_i |c_i|$, so $\{g_n\}$ is equibounded. By Lemma 3.4 $\{g_n\}$ is equicontinuous on any compact subset in $X \sim \partial A$. Therefore, by the diagonal argument, there is a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that g_{n_i} converges uniformly to some h on any compact subset in $X \sim \partial A$. By definition, $h \in H^\infty$ and $h(y_i) = c_i$ ($1, 2, 3, \dots$). Conversely, let H^∞ have the property (D). Then since H^∞ is a Banach algebra and since the sequence of points $\{y_1, y_2, \dots, y_n, \dots\}$ ($= F_0 \sim \partial A$) is an H^∞ -interpolating sequence, for any bounded function f on $\{y_1, y_2, \dots, y_n, \dots\}$ there is an $h \in H^\infty$ such that $f(y_i) = h(y_i)$ ($i = 1, 2, 3, \dots$) and $\|h\|_\infty \leq \gamma \sup |f(y_i)|$ (γ is independent of f) by the same argument as [7] (P. 196). Therefore, we can prove that F_0 is an interpolation set of A by the similar method as Theorem 3.1, since H^∞ has the property (D).

TOKYO JOSHI DAIGAKU

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Added in proof. M. Hasumi also proved Theorem 3.2. without the property (A) by use of the maximal ideal space of A . Some theorems of this paper can be extended to the case which A is a more general algebra.