The University of Osaka Institutional Knowledge Archive

Title	Compact simple Lie algebras with two involutions and submanifolds of compact symmetric spaces. I
Author(s)	Naitoh, Hiroo
Citation	Osaka Journal of Mathematics. 1993, 30(4), p. 653-690
Version Type	VoR
URL	https://doi.org/10.18910/7909
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Naitoh, H.
Osaka J. Math.
30 (1993), 653–690

COMPACT SIMPLE LIE ALGEBRAS WITH TWO INVOLUTIONS AND SUBMANIFOLDS OF COMPACT SYMMETRIC SPACES I

Dedicated to Professor Masaru Takeuchi on his sixtieth birthday

HIROO NAITOH*

(Received February 7, 1992)

Introduction. Let M be a smooth manifold of dimension m and s an integer such that $1 \leq s \leq m-1$. Let $G_s(T_p M)$ be the set of s-dimensional linear subspaces in the tangent space $T_p M$ at p and denote by $G_s(TM)$ the corresponding Grassmann bundle over M, i.e., $G_s(TM) = \cup_{p \in M} G_s(T_p M)$. Harvey-Lawson [4] introduces the notion of *Grassmann geometries*, which is described as follows. Give an arbitary subset $\mathcal{V}$ of $G_s(TM)$. An s-dimensional connected submanifold S of M is called a *$\mathcal{V}$-submanifold* if at each point p of S the tangent space $T_p S$ belongs to $\mathcal{V}$. The collection of $\mathcal{V}$-submanifolds constitutes the *$\mathcal{V}$-geometry*. Grassmann geometries are the collective name of such $\mathcal{V}$-geometries.

We consider $\mathcal{V}$-geometries of the following type. Assume that M is a compact simply connected riemannian symmetric space and denote by G the group of isometries on M. The Lie group G acts transitively on M and at the same time acts on $G_s(TM)$ via the differentials of isometries. As $\mathcal{V}$ we take a G-orbit by this action. The G-orbit $\mathcal{V}$ is a homogeneous bundle over M with homogeneous fibres, i.e., $\mathcal{V} = \cup_{p \in M} \mathcal{V}_p$, $\mathcal{V}_p = \mathcal{V} \cap G_s(T_p M)$, and G acts transitively on the family of fibres $\mathcal{V}_p$. Moreover the isotropy subgroup K_p in G at p acts transitively on the fibre $\mathcal{V}_p$. Roughly speaking, Grassmann geometries of this type correspond to classes of submanifolds with congruent tangent space. From this point of view the Grassmann geometries are important for us to study the submanifold theory of riemannian symmetric space.

In this article we especially treat the following G-orbits. Denote by R the curvature tensor on M. An s-dimensional linear subspace V in $T_p M$ is called *strongly curvature-invariant* if it satisfies that

$$(0.1) \qquad R_p(V, V)\, V \subset V \quad \text{and} \quad R_p(V^\perp, V^\perp)\, V^\perp \subset V^\perp,$$

* Partially supported by the Grant-in-aid for Scientific Research, No. 03640068

where $V^{\perp}$ denotes the orthogonal complement of V in T_pM. As $\mathcal{V}$ we take the G-orbit at a strongly curvature-invariant subspace V, and consider the set $\mathcal{S}(M)$ of such G-orbits over all s. By the first condition of (0.1) such a $\mathcal{V}$-geometry has a unique complete totally geodesic $\mathcal{V}$-submanifold, except the difference by isometries. Therefore we have the following problem: *Determine all G-orbits $\mathcal{V}$ in $\mathcal{S}(M)$ whose $\mathcal{V}$-geometries have non-totally geodesic $\mathcal{V}$-submanifolds.* The motivation to study the G-orbits of this type is due to the following facts (1), (2):

(1) The fibres $\mathcal{V}_p$ are also symmetric spaces. In fact, taking a subspace V in $\mathcal{V}_p$, by (0.1) we have a unique involutive isometry t_p on M such that $t_p(p)=p$, $(t_p)_*(x)=-x$ if $x\in V$ and x if $x\in V^{\perp}$. Then the involutive inner automorphism τ of G defined by t_p preserves the isotropy subgroup K_p and defines the isotropy subgroup in K_p at V;

(2) A sumbanifold $\boldsymbol{S}$ of M is called a *symmetric submanifold* if at each point p in $\boldsymbol{S}$ there exists a unique isometry t_p on M such that $t_p(p)=p$, $t_p(\boldsymbol{S})=\boldsymbol{S}$, $(t_p)_*(x)=-x$ if $x\in T_pM$ and x if $x\in N_pM$, where N_pM denotes the normal space. A symmetric submanifold is a submanifold with congruent tangent space. Therefore, to study the above problem is useful for the classification of symmetric submanifolds.

In the present paper we study this problem on the following situation: M is an irreducible compact simply connected symmetric space of classical and inner type, and $\mathcal{V}$ is a G-orbit such that the restriction of τ to K_p is also of inner type. Then the Lie algebra $\mathfrak{g}$ of Killing vector fields on M is compact, simple, and of classical type. To solve this problem we use a representation theoretic method of Lie algebra. In sections 1, 2 we explain how to use this. In sections 3, 4 we treat the cases that $\mathfrak{g}$ are of types A_l, B_l, respectively, where such G-orbits $\mathcal{V}$ are classified and for each $\mathcal{V}$ the above problem is solved.

In the forthcoming paper II we treat the cases that $\mathfrak{g}$ are of types C_l, D_l.

1. Main theorem and the outline of proof

Let (M, V) be a pair of compact simply connected riemannian symmetric space M and strongly curvature-invariant subspace V of T_pM. Denote by s_p the geodesic symmetry at p, and by t_p an involutive isometry of M defined in Introduction. Let $\mathfrak{g}$ be the Lie algebra of Killing vector fields on M. Then $\mathfrak{g}$ is a compact semisimple Lie algebra and the isometries s_p, t_p induce involutive automorphisms σ, τ of $\mathfrak{g}$ which commute each other and whose $(+1)$-eigenspaces, as Lie subalgebra of $\mathfrak{g}$, act faithfully on (-1)-eigenspaces. The (-1)-eigenspace of σ is naturally identified with the tangent space T_pM. Through this identification the Lie algebra $\mathfrak{g}$ uniquely admits an inner product $\langle\ ,\ \rangle$ which is preserved by σ, τ and for which the endomorphisms $\mathrm{ad}(X)$, $X\in\mathfrak{g}$, are skew symmetric. Such a quadruple $(\mathfrak{g}, \sigma, \tau, \langle\ ,\ \rangle)$ is called *orthogonal pairwise symmetric*

Lie algebra, abbreviated with OPSLA.

Conversely, given an OPSLA $(\mathfrak{g}, \sigma, \tau, \langle\ ,\ \rangle)$, we can construct a pair (M, V) of compact simply connected riemannian symmetric space M and strongly curvature-invariant subspace V as follows. Let G be the simply connected compact Lie group with Lie algebra $\mathfrak{g}$ and K the connected subgroup of G generated by the Lie subalgebra of $(+1)$-eigenspace of σ. Then M is the homogenous space G/K and the riemannian metric on M is induced from the bi-invariant metric on G defined by $\langle\ ,\ \rangle$. Put $p=K$ in M and denote by t_p an involutive isometry of M induced from an involutive automorphism of G defined by τ. Then t_p fixes the point p and the subspace V is the (-1)-eigenspace of $(t_p)_{*p}$.

Two pairs (M, V), (M', V') are equivalent to each other if there exists an isometry ϕ of M onto M' such that $\phi_*(V)=V'$, and two OPSLA's $(\mathfrak{g}, \sigma, \tau, \langle\ ,\ \rangle)$, $(\mathfrak{g}', \sigma', \tau', \langle\ ,\ \rangle')$ are equivalent to each other if there exists an automorphism ρ of $\mathfrak{g}$ onto $\mathfrak{g}'$ such that $\rho\sigma=\sigma'\rho$, $\rho\tau=\tau'\rho$, and $\langle\rho(X), \rho(Y)\rangle'=\langle X, Y\rangle$ for all $X, Y\in\mathfrak{g}$. Denote by $\mathcal{S}$, $\mathcal{O}$ the sets of equivalence classes, respectively. Then the above constructions give a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{O}$. (See Naitoh [9]). The set $\mathcal{S}$ is regarded as the collection of all $\mathcal{S}(M)$, where M moves over the isomorphism classes of compact simply connected riemannian symmetric space by isometries. The set $\mathcal{O}$ coreesponds to the isomorphism classes of semi-simple affine symmetric space by Berger [2], if we disregard the difference on the inner product $\langle\ ,\ \rangle$. (See Naitoh [9]).

We now assume that a compact simply connected riemannian symmetric space M is irreducible. Then the Lie algebra $\mathfrak{g}$ is a compact simple Lie algebra or two copies of compact simple Lie algebra. In the latter case M is a compact simply connected simple Lie group. If M is irreducible, the riemannian metric is uniquely determined except a scalar multiple. So we may consider a triple $(\mathfrak{g}, \sigma, \tau)$ instead of an OPSLA $(\mathfrak{g}, \sigma, \tau, \langle\ ,\ \rangle)$. Such a triple is called *pairwise symmetric Lie algebra*, abbreviated with PSLA. In the followings we always assume that M is irreducible.

A PSLA $(\mathfrak{g}, \sigma, \tau)$ is called a PSLA of *inner type* if σ is an inner involitive automorphism of $\mathfrak{g}$ and the restriction of τ to the subalgebra of $(+1)$-eigenspace of σ is an inner involutive automorphism of the subalgebra, and otherwise it is called a PSLA of *outer type*. If a PSLA $(\mathfrak{g}, \sigma, \tau)$ is of inner type the Lie algebra $\mathfrak{g}$ is simple since σ is inner.

We give examples of G-orbits $\mathcal{V}$ in the cases that symmetric spaces M have rank one, except the Cayley plane.

EXAMPLE 1. Let M be the m-dimensional riemannian sphere and V an r-dimensional subspace of T_pM, where $r\neq 0, m$. Then V is stongly curvature-invariant and $\mathcal{V}$-submanifolds mean r-dimensional connected submanifolds.

So the $\mathcal{V}$-geometry admits many non-totally goedesic $\mathcal{V}$-submanifolds. Moreover the PSLA corresponding to $\mathcal{V}$ is of inner type if m, r are even, and of outer type if otherwise.

EXAMPLE 2. Let M be the n-dimensional complex projective space.

(1) Let V be an r-dimensional complex subspace of T_pM, where $r \neq 0, n$. Then V is strongly curvature-invariant and $\mathcal{V}$-submanifolds mean r-dimensional complex submanifolds. So the $\mathcal{V}$-geometry admits many non-totally geodesic $\mathcal{V}$-submanifolds. Moreover the PSLA coresponding to $\mathcal{V}$ is of inner type.

(2) Let V be an n-dimensional totally real subspace of T_pM. Then V is strongly curvature-invariant and $\mathcal{V}$-submanifolds mean n-dimensional totally real submanifolds. So the $\mathcal{V}$-geometry admits many non-totally geodesic $\mathcal{V}$-submanifolds. Moreover the PSLA coresponding to $\mathcal{V}$ is of outer type.

EXAMPLE 3. Let M be the n-dimensional quaternion projective space.

(1) Let V be an r-dimensional quaternionic subspace of T_pM, where $r \neq 0, n$. Then V is strongly curvature-invariant and $\mathcal{V}$-submanifolds mean r-dimensional quaternionic submanifolds, i.e., the tangent spaces of submanifolds are preserved by the quaternion structure on M. In this case $\mathcal{V}$-submanifolds are always totally geodesic (Alekseevskii [1]). Moreover the PSLA corresponding to $\mathcal{V}$ is of inner type.

(2) Let V be an n-dimensional totally complex subspace of T_pM. Then V is strongly curvature-invariant and $\mathcal{V}$-submanifolds mean n-dimensional totally complex submanifolds. So the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds. Moreover the PSLA corresponding to $\mathcal{V}$ is of inner type.

We refer Naitoh-Takeuchi [12] for the existence of non-totally geodesic $\mathcal{V}$-submanifolds in Example 1, Example 2, and Example 3 (2). Except the above examples, there are known examples of $\mathcal{V}$-geometries which admit non-totally geodesic $\mathcal{V}$-submanifolds. (See Naitoh [9]). These examples are associated with PSLA's of outer type.

Our main theorem is now described as follows.

Main Theorem. *Let M be an irreducible compact simply connected riemannian symmetric space and $\mathcal{V}$ a G-orbit in $\mathcal{S}(M)$ associated with PSLA $(\mathfrak{g}, \sigma, \tau)$ of inner type. Then $\mathfrak{g}$ is compact simple and the following hold for $\mathfrak{g}$ of classical type*:

(1) *Let $\mathfrak{g}$ be the Lie algebra of type A_l, $l \geq 1$. In this case the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example 2,(1);*

(2) *Let $\mathfrak{g}$ be the Lie algebra of type B_l, $l \geq 2$. In this case the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example 1 (m: even and r: even);*

(3) *Let* $\mathfrak{g}$ *be the Lie algebra of type* C_l, $l \geq 3$. *In this case the* $\mathcal{V}$*-geometry admits non-totally geodesic* $\mathcal{V}$*-submanifolds if and only if it is one of the* $\mathcal{V}$*-geometries in Example* 3,(2);

(4) *Let* $\mathfrak{g}$ *be the Lie algebra of type* D_l, $l \geq 4$. *In this case no* $\mathcal{V}$*-geometry admits non-totally geodesic* $\mathcal{V}$*-submanifolds.*

Corollary. *Let* M, $\mathcal{V}$ *be the same as above. Assume moreover that* $\mathcal{V}$ *is none of G-orbits in Example* 1 (m: *even and* r: *even*), *Example* 2,(1), *Example* 3,(2). *Then a symmetric submanifold which belongs to the* $\mathcal{V}$*-geometry is always totally geodesic.*

We here remark that the symmetric submanifolds are classified for the case that M has rank one. (cf. See Naitoh-Takeuchi [12].)

In the rest of this section we explain the outline for the proof of this theorem. Let M be an irreducible compact simply connected riemannian symmetric space and $\mathcal{V}$ a G-orbit in $\mathcal{S}(M)$ with PSLA $(\mathfrak{g}, \sigma, \tau)$. Let S be a $\mathcal{V}$-submanifold. Take a point p of S and put $V = T_pS$. We may assume that σ, τ are induced from the involutive isometries s_p, t_p. Denote by $\mathfrak{k}$, $\mathfrak{p}$ the (± 1)-eigenspaces by σ, and by $\mathfrak{k}_\pm$ (resp $\mathfrak{p}_\pm$) the (± 1)-eigenspaces in $\mathfrak{k}$(resp. $\mathfrak{p}$) by τ. Then $\mathfrak{k}_+$ is a subalgebra of $\mathfrak{g}$ and vector spaces $\mathfrak{k}_-$, $\mathfrak{p}_\pm$ are $\mathfrak{k}_+$-modules. Moreover we have the following identifications:

$$\mathfrak{p} = T_pM, \quad \mathfrak{p}_- = T_pS, \quad \mathfrak{p}_+ = N_pS$$

and

$$\mathfrak{k} = \Omega_p(M), \quad \mathfrak{k}_+ = \Omega_p^+(M), \quad \mathfrak{k}_- = \Omega_p^-(M)$$

where $\Omega_p(M)$ denotes the holonomy algebra of M at p and $\Omega_p^\pm(M)$ are subspaces of $\Omega_p(M)$ defined as follows:

$$\Omega_p^+(M) = \{f \in \Omega_p(M); f(T_pS) \subset T_pS, f(N_pS) \subset N_pS\}$$

and

$$\Omega_p^-(M) = \{f \in \Omega_p(M); f(T_pS) \subset N_pS, f(N_pS) \subset T_pS\}\ .$$

Denote by α the second fundamental form of S and by A the shape operator, and for $x \in T_pS$ define an endomorphism T_x of T_pM as follows: $T_x(y) = \alpha(x, y)$, $-A_y(x)$ according as $y \in T_pS$ or $y \in N_pS$, respectively. Then, since S is a $\mathcal{V}$-submanifold, the endomorphisms T_x, $x \in T_xS$, belong to $\Omega_p^-(M)$ (Naitoh [11]). So a linear map T of $\mathfrak{p}_-$ to $\mathfrak{k}_-$ is defined by $T(x) = T_x$. It here follows by the symmetry of α that

$$[T(x), y] = T_x(y) = \alpha(x, y) = [T(y), x] \tag{1.1}$$

for $x, y \in \mathfrak{p}_- = T_pS$. Define a linear map ρ of $\mathfrak{p}_-^* \otimes \mathfrak{k}_-$ to $\wedge^2(\mathfrak{p}_-^*) \otimes \mathfrak{p}_+$ by $\rho(\lambda)(x, y) = [\lambda(x), y] - [\lambda(y), x]$ for $\lambda \in \mathfrak{p}_-^* \otimes \mathfrak{k}_-$ and $x, y \in \mathfrak{p}_-$. Here for a vector space

W, W^* denotes the dual space of W. Then ρ is a $\mathfrak{k}_+$-homomorphism and satisfies that $\rho(T)=0$. Hence, by (1.1) we have the following

Lemma 1.1 (KEY LEMMA). *If the $\mathfrak{k}_+$-homomorphism ρ is injective, a $\mathcal{V}$-submanifold is always totally geodesic.*

This homomorphism ρ is called the *homomorphism associated with* the G-orbit $\mathcal{V}$. We study on the injectivity of the complexificaiton of ρ, which is a $\mathfrak{k}_+$-homomorphism of $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$ to $\wedge^2(\mathfrak{p}_-^C)^*\otimes\mathfrak{p}_+^C$ and is denoted by the same notation ρ. Here $(*)^C$ denotes the complexification of $(*)$.

We first see a weight space decomposition of the $\mathfrak{k}_+$-module $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$. Let $(\mathfrak{g}, \sigma, \tau)$ be a PSLA of inner type. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$ and take a fundamental root system $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ in $\sqrt{-1}\mathfrak{h}$ with respect to the Cartan subalgebra $\mathfrak{h}^C$. Let $\{H_1, \cdots, H_l\}$ in $\sqrt{-1}\mathfrak{h}$ be the dual vectors of Π, i.e., $\langle\alpha_i, H_j\rangle=\delta_{ij}$ for all i, j. Then, since σ is inner, we may assume that $\sigma=\exp\operatorname{ad}(\pi\sqrt{-1}H_i)$ for some i (cf. Murakami [8]). We denote this H_i by H_σ. Let $\mathfrak{k}_s$ be the semisimple part of $\mathfrak{k}$. Either of the following cases is possible; Case (a): $\mathfrak{k}=\mathfrak{k}_s$ or Case (b): $\mathfrak{k}=\mathfrak{c}\oplus\mathfrak{k}_s$, where $\mathfrak{c}=\sqrt{-1}\boldsymbol{R}H_i$. In Case (a) put $\mathfrak{h}_s=\mathfrak{h}$. Then $\mathfrak{h}_s^C$ is a Cartan subalgebra of $\mathfrak{k}_s^C$ and the subset $\Pi_s=\{\alpha_0, \alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_l\}$ in $\sqrt{-1}\mathfrak{h}_s$ gives a fundamental root system of $\mathfrak{k}_s^C$ with respect to $\mathfrak{h}_s^C$, where α_0 is the minus multiple of the highest root of Π. In Case (b) the subalgebra $\mathfrak{h}$ is decomposed into the sum of center $\mathfrak{c}$ and maximal abelian subalgebra $\mathfrak{h}_s$ in $\mathfrak{k}_s$, where $\mathfrak{h}_s^C$ is a Cartan subalgebra of $\mathfrak{k}_s^C$. The subset $\Pi_s=\{\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_l\}$ in $\sqrt{-1}\mathfrak{h}_s$ gives a fundamental root system of $\mathfrak{k}_s^C$ with respect to this Cartan subalgebra.

We now decompose $\mathfrak{k}_s$ into the sum of simple ideals $\mathfrak{k}_j$, $1\leq j\leq r$ and put $\mathfrak{h}_j=\mathfrak{h}_s\cap\mathfrak{k}_j$, and then decompose Π_s into the sum of fundamental root systems $\Pi_j=\{\alpha_{j_1}, \cdots, \alpha_{j_{r(j)}}\}$ of $\mathfrak{k}_j^C$ with respect to the Cartan subalgebras $\mathfrak{h}_j^C$. Denote by $\{K_{j_1}, \cdots, K_{j_{r(j)}}\}$ the dual vectors of Π_j in $\sqrt{-1}\mathfrak{h}_j$. Since the restriction $\bar{\tau}$ of τ to $\mathfrak{k}$ is inner, we may assume that

$$\bar{\tau}=\exp\operatorname{ad}(\pi\sqrt{-1}\textstyle\sum K_{j_{s(j)}}) \tag{1.2}$$

for some daul vectors $K_{j_{s(j)}}$, where the summation $\sum$ does not necessarily move over all j. Put $-\alpha_0=\sum_k m_k\alpha_k$, where m_k, $1\leq k\leq l$, are positive integers. Then, the following holds.

Lemma 1.2 (CF. C. YEN [13]). (1) *In Case* (a) *an involutive automorphism of $\mathfrak{k}$ given in the form* (1.2) *can be extended to an involutive automorphism of $\mathfrak{g}$ if and only if the following condition*

$$\textstyle\sum_j m_{j_{s(j)}}\equiv 0 \pmod 2 \tag{1.3}$$

is satisfied. Then every extended automorphism is involutive.

(2) *In Case* (b) *an involutive automorphism of* $\mathfrak{k}$ *given in the form* (1.2) *can be always extended to an involutive automorphism of* $\mathfrak{g}$. *Then an extended involutive automorphism is given in the following form*:

$$\exp \operatorname{ad}(\pi\sqrt{-1}(\sum K_{j_{s(j)}}+Z))$$

for some $Z\in\sqrt{-1}\,\mathfrak{c}$.

Denote by τ_0 the extended involutive automorphism of $\mathfrak{g}$ given in the above lemma: $\tau_0=\exp \operatorname{ad}(\pi\sqrt{-1}K_{\tau_0})$, where $K_{\tau_0}=\sum K_{j_{s(j)}}$ in Case (a) and $K_{\tau_0}=\sum K_{j_{s(j)}}+Z$ in Case (b). Then, since the $\mathfrak{k}$-module $\mathfrak{p}$ is irreducible, it follows by Schur's lemma that $\tau=\tau_0$ or $\tau=\tau_0\sigma$. According to each case we put $H_\tau=K_{\tau_0}$ or $H_\tau=K_{\tau_0}+H_\sigma$.

Let Δ be the set of roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$ and set

$$\begin{aligned}
\Delta_{\mathfrak{k}_+} &= \{\alpha\in\Delta;\langle\alpha, H_\sigma\rangle \text{ and } \langle\alpha, H_\tau\rangle\text{: even}\}\ ,\\
\Delta_{\mathfrak{k}_-} &= \{\alpha\in\Delta;\langle\alpha, H_\sigma\rangle\text{: even and } \langle\alpha, H_\tau\rangle\text{: odd}\}\ ,\\
\Delta_{\mathfrak{p}_+} &= \{\alpha\in\Delta;\langle\alpha, H_\sigma\rangle\text{: odd and } \langle\alpha, H_\tau\rangle\text{: even}\}\ ,\\
\Delta_{\mathfrak{p}_-} &= \{\alpha\in\Delta;\langle\alpha, H_\sigma\rangle \text{ and } \langle\alpha, H_\tau\rangle\text{: odd}\}\ .
\end{aligned}$$

Then $\Delta_{\mathfrak{k}_+}$ is the set of roots of $\mathfrak{k}_+^C$ with respect to $\mathfrak{h}^C$, and $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are the sets of weights of the $\mathfrak{k}_+$-modules $\mathfrak{k}_-^C$, $\mathfrak{p}_+^C$, $\mathfrak{p}_-^C$ with respect to $\mathfrak{h}^C$, respectively. Taking root vectors X_α, $\alpha\in\Delta$, we have the following decompositions:

$$\begin{aligned}
\mathfrak{k}_+^C &= \mathfrak{h}^C\oplus\textstyle\sum_{\alpha\in\Delta_{\mathfrak{k}_+}} CX_\alpha & \mathfrak{k}_-^C &= \textstyle\sum_{\alpha\in\Delta_{\mathfrak{k}_-}} CX_\alpha\\
\mathfrak{p}_+^C &= \textstyle\sum_{\alpha\in\Delta_{\mathfrak{p}_+}} CX_\alpha & \mathfrak{p}_-^C &= \textstyle\sum_{\alpha\in\Delta_{\mathfrak{p}_-}} CX_\alpha\ .
\end{aligned}$$

Let ω_α, $\alpha\in\Delta$, be the dual forms of X_α, $\alpha\in\Delta$. Then the set Λ of weights of the $\mathfrak{k}_+$-modlue $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$ is given by the set $\{-\alpha+\beta;\ \alpha\in\Delta_{\mathfrak{p}_-}, \beta\in\Delta_{\mathfrak{k}_-}\}$ and a basis of weight vectors is given by $\omega_\alpha\otimes X_\beta$, $\alpha\in\Delta_{\mathfrak{p}_-}$, $\beta\in\Delta_{\mathfrak{k}_-}$.

We now take a total order $<$ on $\sqrt{-1}\,\mathfrak{h}$ such that Π is the system of simple roots in Δ and denote by Δ^+ the set of positive roots in Δ. Then $\Delta_{\mathfrak{k}_+}\cap\Delta^+$, denoted by $\Delta_{\mathfrak{k}_+}^+$, is the set of positive roots in $\Delta_{\mathfrak{k}_+}$. Denote by φ the representation of $\mathfrak{k}_+^C$ on the vector space $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$. A vector u in $(\mathfrak{p}_-^C)^*\otimes\mathfrak{k}_-^C$ is called *maximal* if it holds that $\varphi(X_\gamma)\cdot u=0$ for all γ in $\Delta_{\mathfrak{k}_+}^+$. A maximal weight vector gives the highest weight vector of an irreducible submodule. By virtue of Schur's lemma, to see the injectivity of the $\mathfrak{k}_+$-homomorphism ρ assocaited with the PSLA $(\mathfrak{g}, \sigma, \tau)$, we may see whether $\rho(u)$ are zero or not for all maximal weight vectors u.

Our procedure is done as follows. Let u be a maximal vector with weight $\lambda\in\Lambda$. If u is represented as a proper linear combination of r weight vectors $\omega_{\alpha_i}\otimes X_{\beta_i}$, $1\leq i\leq r$, such that $\lambda=-\alpha_i+\beta_i$, the integer r is called the *length* of u and is denoted by $l(u)$. Dividing into the cases that $l(u)=1$, $l(u)=2$, and

$l(u)\geq 3$, we will see whether $\rho(u)$ vanishes or not. In the next section we prepare some conditions to see this.

2. Some conditions for the non-vanishing of $\rho(u)$

Let $(\mathfrak{g}, \sigma, \tau)$ be a PSLA of inner type and ρ the $\mathfrak{k}_+$-homomorphism associated with the PSLA. We retain the notations in the previous section. Let u be a maximal vector with a weight λ in Λ and represent u as follows: $u=\sum_{\alpha\in\Gamma_{\mathfrak{p}_-}} a_\alpha\, \omega_\alpha \otimes X_{\lambda+\alpha}$, where $a_\alpha(\alpha\in\Gamma_{\mathfrak{p}_-})$ are nonzero complex numbers and $\Gamma_{\mathfrak{p}_-}$ is a subset in $\Delta_{\mathfrak{p}_-}$. Put $\Gamma_{\mathfrak{k}_-}=\lambda+\Gamma_{\mathfrak{p}_-}\subset\Delta_{\mathfrak{k}_-}$. We define complex numbers $N_{\alpha,\beta}$ for $\alpha, \beta\in\Delta$ as follows: $N_{\alpha,\beta}$ is the complex number such that $[X_\alpha, X_\beta]=N_{\alpha,\beta}\, X_{\alpha+\beta}$ if $\alpha+\beta$ is a root, and $N_{\alpha,\beta}$ is zero if $\alpha+\beta$ is not a root. Then it follows that

$$\begin{aligned}\rho(u) &= \textstyle\sum_1 a_\alpha\, N_{\lambda+\alpha,\delta}\, (\omega_\alpha\wedge\omega_\delta)\otimes X_{\lambda+\alpha+\delta} \\ &= \textstyle\sum_2 a_\alpha\, N_{\lambda+\alpha,\delta}\, (\omega_\alpha\wedge\omega_\delta)\otimes X_{\lambda+\alpha+\delta} \\ &\quad + \textstyle\sum_3 (a_\alpha\, N_{\lambda+\alpha,\alpha'} - a_{\alpha'}\, N_{\lambda+\alpha',\alpha})\, (\omega_\alpha\wedge\omega_{\alpha'})\otimes X_{\lambda+\alpha+\alpha'}\,.\end{aligned}$$

Here $\sum_1$ means the summation for $\alpha\in\Gamma_{\mathfrak{p}_-}$, $\delta\in\Delta_{\mathfrak{p}_-}$ such that $\lambda+\alpha+\delta$ is a root in Δ, and $\sum_2$ means the summation for $\alpha\in\Gamma_{\mathfrak{p}_-}$, $\delta\in\Delta_{\mathfrak{p}_-}-\Gamma_{\mathfrak{p}_-}$ such that $\lambda+\alpha+\delta$ is a root, and $\sum_3$ means the summation for $\alpha, \alpha'\in\Gamma_{\mathfrak{p}_-}$ such that $\lambda+\alpha+\alpha'$ is a root and such that $\alpha<\alpha'$ for the fixed total order $<$ on $\sqrt{-1}\mathfrak{h}$. Noting that the weight vectors $(\omega_\alpha\wedge\omega_\delta)\otimes X_{\lambda+\alpha+\delta}$, $(\omega_\alpha\wedge\omega_{\alpha'})\otimes X_{\lambda+\alpha+\alpha'}$ which appear in the summations $\sum_2$, $\sum_3$ are linearly independent, we have the following

Proposition 2.1. *The vector $\rho(u)$ vanishes if and only if the following conditions* (1), (2) *are satisfied*:

(1) *$\delta+\beta$ is not a root for any $\delta\in\Delta_{\mathfrak{p}_-}-\Gamma_{\mathfrak{p}_-}$ and any $\beta\in\Gamma_{\mathfrak{k}_-}$;*

(2) *For distinct vectors $\alpha, \alpha'\in\Gamma_{\mathfrak{p}_-}$ such that $\lambda+\alpha+\alpha'$ is a root, it holds that*

$$a_\alpha\, N_{\alpha',\lambda+\alpha} = a_{\alpha'}\, N_{\alpha,\lambda+\alpha'}\,.$$

By using this proposition we show some conditions for the nonvanishing of $\rho(u)$.

Lemma 2.2. *Assume that the weight λ is a root in Δ. Then $\rho(u)$ does not vanish if it holds that $-\Gamma_{\mathfrak{p}_-}\neq\Gamma_{\mathfrak{p}_-}$. Particularly if $-\Gamma_{\mathfrak{p}_-}\cap\Gamma_{\mathfrak{p}_-}=\emptyset$, it does not vanish.*

Proof. By the above condition there exists a vector α_0 in $\Gamma_{\mathfrak{p}_-}$ such that $-\alpha_0\notin\Gamma_{\mathfrak{p}_-}$. In Proposition 2.1, (1) put $\delta=-\alpha_0$, $\beta=\lambda+\alpha_0$. Then $\delta+\beta$ is equal to λ, which is a root. Hence $\rho(u)$ does not vanish. □

Lemma 2.3. *Assume that $l(u)=1$ and put $\Gamma_{\mathfrak{p}_-}=\{\alpha\}$ and $\Gamma_{\mathfrak{k}_-}=\{\beta\}$. Then $\rho(u)$ does not vanish if and only if there exists a vector δ in $\Delta_{\mathfrak{p}_-}-\{\alpha\}$ such that $\delta+\beta$ is a root.*

Proof. In this case the condition (2) in Proposition 2.1 is always satisfied. Hence our claim is obvious. $\square$

We now consider the following condition $(*)$ on $\Gamma_{\mathfrak{k}_-}$. This consists of the conditions $(*1)$, $(*2)$:

$(*1)$ For β, β' in $\Gamma_{\mathfrak{k}_-}$ such that $\beta<\beta'$, there exists a positive root γ in $\Delta^+_{\mathfrak{k}_+}$ such that $\beta+\gamma=\beta'$;

$(*2)$ For any γ in $\Delta^+_{\mathfrak{k}_+}$, there exists at most one root β in $\Gamma_{\mathfrak{k}_-}$ such that $\beta+\gamma\in\Gamma_{\mathfrak{k}_-}$.

Lemma 2.4. *Assume that $l(u)\geq 2$ and that $\Gamma_{\mathfrak{k}_-}$ satisfies the condition $(*)$. Then $\rho(u)$ does not vanish if there exist two vectors α_0, $\alpha_1(\alpha_0<\alpha_1)$ in $\Gamma_{\mathfrak{p}_-}$ such that $\alpha_0+\beta_1=\beta_0+\alpha_1$ is a root in Δ and $\alpha_0+\beta_0$ is not a root in Δ, where $\alpha_i=\beta_i-\lambda$, $i=0, 1$.*

Proof. Represent u as above, i.e., $u=\sum_{\alpha\in\Gamma_{\mathfrak{p}_-}} a_\alpha\, \omega_\alpha\otimes X_\beta$, where $\beta=\lambda+\alpha$. By the condition $(*1)$ take a positive root γ in $\Delta^+_{\mathfrak{k}_+}$ such that $\beta_1=\beta_0+\gamma$. Then the condition $(*2)$ implies that a vector β in $\Gamma_{\mathfrak{k}_-}$ such that $\gamma+\beta\in\Gamma_{\mathfrak{k}_-}$ is only β_0. Denote by φ the representation of $\mathfrak{k}^{\mathbf{C}}_+$ on the vector space $(\mathfrak{p}^{\mathbf{C}}_-)^*\otimes\mathfrak{k}^{\mathbf{C}}_-$. Since u is a maximal vector, it follows that

$$\begin{aligned}0 &= \varphi(X_\gamma)\cdot u\\ &= -\textstyle\sum_{\alpha\in\Gamma_{\mathfrak{p}_-}} a_\alpha\, N_{\gamma,\alpha-\gamma}\, \omega_{\alpha-\gamma}\otimes X_\beta+\sum_{\alpha\in\Gamma_{\mathfrak{p}_-}} a_\alpha\, N_{\gamma,\beta}\, \omega_\alpha\otimes X_{\beta+\gamma}\\ &= (-a_{\alpha_1} N_{\gamma,\alpha_0}+a_{\alpha_0} N_{\gamma,\beta_0})\, \omega_{\alpha_0}\otimes X_{\beta_1}\end{aligned}$$

and so

$$a_{\alpha_1} N_{\gamma,\alpha_0} = a_{\alpha_0} N_{\gamma,\beta_0}\,. \tag{2.1}$$

We now show that $a_{\alpha_0} N_{\alpha_1,\beta_0}\neq a_{\alpha_1} N_{\alpha_0,\beta_1}$. If so, the vector $\rho(u)$ does not vanish by Proposition 2.1, (2). Because $\lambda+\alpha_0+\alpha_1$ is equal to $\beta_0+\alpha_1$, which is a root.

We first note that $N_{\alpha_1,\beta_0}=N_{\alpha_0+\gamma,\beta_0}$, $[X_{\alpha_1}, X_{\beta_0}]=N_{\alpha_1,\beta_0} X_{\alpha_1+\beta_0}$, and $[X_{\alpha_0}, X_\gamma]=N_{\alpha_0,\gamma} X_{\alpha_1}$. Then, since $\alpha_0+\beta_0$ is not a root, it follows that

$$\begin{aligned}N_{\alpha_1,\beta_0} X_{\alpha_1+\beta_0} &= (1/N_{\alpha_0,\gamma})\, [[X_{\alpha_0}, X_\gamma], X_{\beta_0}]\\ &= -(1/N_{\alpha_0,\gamma})\, ([[X_\gamma, X_{\beta_0}], X_{\alpha_0}]+[[X_{\beta_0}, X_{\alpha_0}], X_\gamma])\\ &= -(1/N_{\alpha_0,\gamma})\, N_{\gamma,\beta_0} N_{\beta_1,\alpha_0} X_{\beta_1+\alpha_0}\end{aligned}$$

and so $N_{\alpha_1,\beta_0} N_{\alpha_0,\gamma}=-N_{\gamma,\beta_0} N_{\beta_1,\alpha_0}$. This, together with (2.1), implies that $a_{\alpha_0} N_{\alpha_1,\beta_0}=-a_{\alpha_1} N_{\alpha_0,\beta_1}$. $\square$

We now explain how to apply these lemmas for three cases.

Case (1): $l(u)=1$. In this case a maximal weight vecotr u is represented as follows: $u=a\omega_\alpha\otimes X_\beta$, where α is the minus multiple of a dominant weight of the $\mathfrak{k}_+$-module $\mathfrak{p}_-^{\boldsymbol{C}}$ and β is a dominant weight of the $\mathfrak{k}_+$-module $\mathfrak{k}_-^{\boldsymbol{C}}$. Conversely a weight vector u constructed by such α, β is maximal. The non-vanishing of $\rho(u)$ is completely determined by Lemma 2.3.

Case (2): $l(u)=2$. In this case a maximal weight vector u is represented as follows: $u=a\,\omega_\alpha\otimes X_\beta+b\,\omega_{\alpha'}\otimes X_{\beta'}$. The maximal condition of u is described as follows: For $\gamma\in\Delta_{\mathfrak{k}_+}^+$,

$$\begin{aligned}(2.2)\qquad 0 &= \varphi(X_\gamma)\cdot u\\ &= -aN_{\gamma,\alpha-\gamma}\,\omega_{\alpha-\gamma}\otimes X_\beta+aN_{\gamma,\beta}\,\omega_\alpha\otimes X_{\beta+\gamma}\\ &\quad -bN_{\gamma,\alpha'-\gamma}\,\omega_{\alpha'-\gamma}\otimes X_{\beta'}+bN_{\gamma,\beta'}\,\omega_{\alpha'}\otimes X_{\beta'+\gamma}\end{aligned}$$

where terms which contain ω_δ, X_δ are regarded as zero if δ is not a root. Suppose that $\alpha'>\alpha$ and put $\mu=\alpha'-\alpha=\beta'-\beta>0$.

We first assume that μ is not a root in Δ. In this case, vectors $\omega_{\alpha-\gamma}\otimes X_\beta$, $\omega_\alpha\otimes X_{\beta+\gamma}$, $\omega_{\alpha'-\gamma}\otimes X_{\beta'}$, $\omega_{\alpha'}\otimes X_{\beta'+\gamma}$ are linearly independent if they appear. Hence the vectors $\alpha-\gamma, \beta+\gamma, \alpha'-\gamma, \beta'+\gamma$ are not roots for $\gamma\in\Delta_{\mathfrak{k}_+}^+$. This implies that α, α' are the minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ and that β, β' are dominant weights in $\Delta_{\mathfrak{p}_-}$. Conversely a weight vector u constructed by such $\alpha, \alpha', \beta, \beta'$ is maximal and μ is not a root. Such a u is said to be *decomposable*.

We next assume that μ is a root in Δ, and thus in $\Delta_{\mathfrak{k}_+}^+$. By the same way as above, it follows that $\alpha-\gamma, \beta+\gamma, \alpha'-\gamma, \beta'+\gamma$ are not roots for $\gamma\in\Delta_{\mathfrak{k}_+}^+$ such that $\gamma\neq\mu$. Set $\gamma=\mu$. Then, by (2.2), it follows that

$$\begin{aligned}0 = &-aN_{\mu,\alpha-\mu}\,\omega_{\alpha-\mu}\otimes X_\beta+bN_{\mu,\beta'}\,\omega_{\alpha'}\otimes X_{\beta'+\mu}\\ &+(aN_{\mu,\beta}-bN_{\mu,\alpha})\,\omega_\alpha\otimes X_{\beta'}\end{aligned}$$

where vectors $\omega_{\alpha-\mu}\otimes X_\beta$, $\omega_\alpha\otimes X_{\beta'}$, $\omega_{\alpha'}\otimes X_{\beta'+\mu}$ are linearly independent if they appear. Hence the vectors $\alpha-\mu$, $\beta'+\mu$ are not roots and it holds that $aN_{\mu,\beta}=bN_{\mu,\alpha}$. These imply the following: α is the minus multiple of a dominant weight in $\Delta_{\mathfrak{p}_-}$; β' is a dominant weight in $\Delta_{\mathfrak{k}_-}$; $\alpha'-\gamma$, $\beta+\gamma$ are not weights for $\gamma\in\Delta_{\mathfrak{k}_+}^+-\{\mu\}$. Then u is given in the following form:

$$u = a(\omega_\alpha\otimes X_\beta+(N_{\mu,\beta}/N_{\mu,\alpha})\,\omega_{\alpha'}\otimes X_{\beta'})\,.$$

Conversely a weight vector u consrtucted by such $(\alpha, \alpha', \beta, \beta'; \mu)$ is maximal. Such a u is said to be *indecomposable*.

We now consider the non-vanishing of $\rho(u)$ for a maximal weight vector u with length 2. Our procedure is done as follows: Suppose that u is indecomposable. Then $\Gamma_{\mathfrak{k}_-}$ satisfies the condition (*). We find out all objects $(\alpha, \alpha', \beta, \beta'; \mu)$ and then aplpy Lemma 2.2 or 2.4. Even there is a case that we can apply neither of these lemmas, using case by case arguments, we can check that there is a maximal weight vector v in Case (1) such that $\rho(v)$ is vanishing, or Proposition 2.1 (1) does not hold. Hence in the former case we can see that ρ is not injective, and in the latter case we can see that $\rho(u)$ is non-vanishing. Suppose next that u is decomposable. We find out all objects $(\alpha, \alpha', \beta, \beta')$ and then apply Lemma 2.2. (In this case we can not apply Lemma 2.4 because of the condition ($*1$).) Even if we can not apply Lemma 2.2, we can check the same facts as the indecomposable case. Hence our procedure is completed.

Case (3): $l(u)\geq 3$. It is not difficult to determine the weight spaces with dim≥ 3 explicitly by using case by case argument, and we can use the same arguments as in Case (2).

We last explain about the notion "family of PSLA's". Let $(\mathfrak{g}, \sigma, \tau)$ be a PSLA. From this PSLA we can construct the following new PSLA's: $(\mathfrak{g}, \sigma, \tau\sigma)$, $(\mathfrak{g}, \tau, \sigma)$, $(\mathfrak{g}, \tau, \tau\sigma)$, $(\mathfrak{g}, \sigma\tau, \sigma)$, and $(\mathfrak{g}, \sigma\tau, \tau)$. The collection of these PSLA's is said to be a *family*. For PSLA's in the same family, the subalgebras $\mathfrak{k}_+$ coincide and the collections of subspaces $\mathfrak{k}_-$, $\mathfrak{p}_+$, $\mathfrak{p}_-$ concide except order.

Let $\mathcal{F}$ be a family. By arguments in §1, if a PSLA in $\mathcal{F}$ is of inner type, all PSLA's in $\mathcal{F}$ are also of inner type. We then say that $\mathcal{F}$ is of *inner type*. Otherwise, we say that $\mathcal{F}$ is of *outer type*.

Two families $\mathcal{F}$, $\mathcal{F}'$ are said to be *equivalent* to each other if there exist a PSLA in $\mathcal{F}$ and a PSLA in $\mathcal{F}'$ which are equivalent to each other. Then the PSLA's in $\mathcal{F}$ equivalently correspond to the PSLA's in $\mathcal{F}'$.

In the following sections we will first see the equivalence of families and the equivalence of PSLA's in each family, and next see the injectivity of ρ for each PSLA.

3. The PSLA's with Lie algebra $\mathfrak{g}$ of type A_l

Let $\mathfrak{g}$ be the Lie algebra of type A_l, that is, the Lie algebra $\mathfrak{su}(l+1)$ of skew hermitian matrices of degree $l+1$ with trace 0. Then the Dynkin diagram of the fundamental root system Π is given as follows:

$$\underset{\alpha_1}{\bigcirc} - \underset{\alpha_2}{\bigcirc} - \cdots - \underset{\alpha_l}{\bigcirc} \qquad -\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_l$$

Put

$$\theta_i = \exp \operatorname{ad}(\sqrt{-1}\,\pi H_i), \quad 1\leq i\leq l.$$
$$\theta_{jk} = \exp \operatorname{ad}(\sqrt{-1}\,\pi (H_j+H_k)), \quad 1\leq j<k\leq l$$

and let $\mathcal{A}_{ij}$, $1\leq j<i\leq l$, and $\mathcal{A}_{i;jk}$, $1\leq j<i<k\leq l$, be the families which contain the PSLA's $(\mathfrak{g}, \theta_i, \theta_j)$, $(\mathfrak{g}, \theta_i, \theta_{jk})$, respectively.

Lemma 3.1. *A PSLA* $(\mathfrak{g}, \sigma, \tau)$ *of inner type is equivalent to a PSLA which belongs to one of the families* $\mathcal{A}_{ij}$ *or* $\mathcal{A}_{i\,;\,jk}$, *by an inner automorphism of* $\mathfrak{g}$.

Proof. We may assume that $\sigma=\theta_i$. Then $\mathfrak{k}=\mathfrak{c}+\mathfrak{k}_s$. We divide into the following cases: (1) $i=1$, (2) $i=l$, and (3) $1<i<l$.

Case (1): $i=1$. The Dynkin diagram of Π_s is given as follows:

$$\underset{\alpha_2}{\bigcirc} - \underset{\alpha_3}{\bigcirc} - \cdots - \underset{\alpha_l}{\bigcirc}$$

Hence we may assume that the restriction $\bar{\tau}$ of τ is given in the following form: $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\,\pi K_j)$, where $2\leq j\leq l$. Put $K_j=a_1 H_1+\cdots+a_l H_l$. Since $\langle K_j, \alpha_k\rangle=\delta_{jk}$ for $2\leq k\leq l$, it follows that $K_j=a_1 H_1+H_j$ and thus $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\,\pi H_j)$. This implies that $\tau=\tau_0=\exp \operatorname{ad}(\sqrt{-1}\,\pi H_j)$ or $\tau=\tau_0\,\sigma$. Hence the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{A}_{j1}$.

Case (2): $i=l$. By the same way as Case (1), the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{A}_{lj}$, where $1\leq j\leq l-1$.

Case (3): $1<i<l$. The Dynkin diagram of Π_s is given as follows:

$$\underset{\alpha_1}{\bigcirc} - \underset{\alpha_2}{\bigcirc} - \cdots - \underset{\alpha_{i-1}}{\bigcirc} \qquad \underset{\alpha_{i+1}}{\bigcirc} - \underset{\alpha_{i+2}}{\bigcirc} - \cdots - \underset{\alpha_l}{\bigcirc}$$

We may assume that the restriction $\bar{\tau}$ is one of the following cases (a), (b), (c).

(a) $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\,\pi K_j)$, $1\leq j\leq i-1$: It follows that $K_j=a_i H_i+H_j$ for some $a_i\in \boldsymbol{R}$ and thus $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\,\pi H_j)$. This implies that $\tau=\tau_0=\exp \operatorname{ad}(\sqrt{-1}\,\pi H_j)$ or $\tau=\tau_0\,\sigma$. Hence the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{A}_{ij}$.

(b) $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\,\pi K_k)$, $i+1\leq k\leq l$: By the same way of (a), the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{A}_{ki}$.

(c) $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\,\pi(K_j+K_k))$, $1\leq j<i<k\leq l$: It follows that $K_j+K_k=a_i H_i+H_j+H_k$ and thus $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\,\pi(H_j+H_k))$. This implies that $\tau=\tau_0=\exp \operatorname{ad}(\sqrt{-1}\,\pi(H_j+H_k))$ or $\tau=\tau_0\sigma$. Hence the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{A}_{i\,;\,jk}$. □

If $l=1$, there exists no PSLA of inner type. So we suppose that $l\geq 2$. From the above proof, we can see that the subalgebra $\mathfrak{k}_+$ for $\mathcal{A}_{ij}$ has the 2-dimensional center and that the subalgebra $\mathfrak{k}_+$ for $\mathcal{A}_{i\,;\,jk}$ has the 3-dimensional center. Hence the families $\mathcal{A}_{ij}$ are never equivalent to the families $\mathcal{A}_{i\,;\,jk}$. We first see the equivalence among the families $\mathcal{A}_{ij}$ and then the equivalence among the PSLA's which belong to each $\mathcal{A}_{ij}$.

Let V be an $(l+1)$-dimensional eucildean space which contains $\sqrt{-1}\,\mathfrak{h}$ as a subspace, and $\{e_1, \cdots, e_{l+1}\}$ an orthonormal basis which satisfies that $\alpha_i=e_i-e_{i+1}$ for all i. Then it holds that

$$H_i=e_1+\cdots+e_i-\frac{i}{l+1}\sum_{a=1}^{l+1} e_a$$

for all i. Let $W(\Delta)$ be the Weyl group of the root system Δ. Then $W(\Delta)$ equals the group of permutations of $e_1, \cdots, e_{l+1}$ and each element in $W(\Delta)$ induces an inner automorphism of $\mathfrak{g}$. For an integer k such that $1 \leq k \leq l+1$, let w_0^k be the element in $W(\Delta)$ defined by the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & l+1 \\ k & k-1 & \cdots & 2 & 1 & k+1 & \cdots & l+1 \end{pmatrix}.$$

For integers j, k such that $j, k \geq 1$ and $j+k \leq l+1$, let w_1^{jk} be the element in $W(\Delta)$ defined by the permutation

$$\begin{pmatrix} 1 & \cdots & j & j+1 & \cdots & j+k & j+k+1 & \cdots & l+1 \\ k+1 & \cdots & j+k & 1 & \cdots & k & j+k+1 & \cdots & l+1 \end{pmatrix}.$$

Then it follows that

$$w_0^k(H_i) = H_k - H_{k-i} \quad (1 \leq i < k),$$
$$w_0^k(H_i) = H_i \quad (k \leq i \leq l+1),$$

where we regard H_{l+1} as 0, and it follows that

$$w_1^{jk}(H_j) = H_{j+k} - H_k,$$
$$w_1^{jk}(H_i) = H_i \quad (j+k \leq i \leq l+1).$$

Let φ_0^k, φ_1^{jk} be inner automorphisms of $\mathfrak{g}$ induced by w_0^k, w_1^{jk}, respectively.

For a family $\mathcal{A}_{ij}$ put $i=j+k$ and $l+1=i+r$. Then $j, k, r \geq 1$ and the following holds.

Proposition 3.2. *Two families $\mathcal{A}_{ij}$, $\mathcal{A}_{i'j'}$ are equivalent to each other if and only if the triples (j, k, r), (j', k', r') coincide except order.*

Proof. Consider the PSLA $(\mathfrak{g}, \theta_i, \theta_j)$ in $\mathcal{A}_{ij}$ and the PSLA $(\mathfrak{g}, \theta_{i'}, \theta_{j'})$ in $\mathcal{A}_{i'j'}$. Then it follows that $\dim \mathfrak{k}_- = 2jk$, $\dim \mathfrak{p}_+ = 2kr$, $\dim \mathfrak{p}_- = 2jr$ and that $\dim \mathfrak{k}'_- = 2j'k'$, $\dim \mathfrak{p}'_+ = 2k'r'$, $\dim \mathfrak{p}'_- = 2j'r'$. (See (3.3) later.)

If $\mathcal{A}_{ij}$ is equivalent to $\mathcal{A}_{i'j'}$, triples (jk, kr, rj) and $(j'k', k'r', r'j')$ coincide except order. This implies that triples (j, k, r), (j', k', r') coincide except order.

To prove the converse we may prove the following equivalences:

$$(1)\ \mathcal{A}_{ij} \cong \mathcal{A}_{ik} \quad \text{and} \quad (2)\ \mathcal{A}_{ij} \cong \mathcal{A}_{r+k,r}$$

where $\mathcal{A}_{jk}$, $\mathcal{A}_{r+k,r}$ have triples (k, j, r), (r, k, j), respectively.

(1) Consider the inner automorphism φ_1^{jk}. Then it follows that $\varphi_1^{jk}\,\theta_i = \theta_i\,\varphi_1^{jk}$, $\varphi_1^{jk}\,\theta_j = \theta_i\,\theta_k\,\varphi_1^{jk}$. This implies that $(\mathfrak{g}, \theta_i, \theta_j)$ is equivalent to $(\mathfrak{g}, \theta_i, \theta_i\,\theta_k)$ and thus $\mathcal{A}_{ij}$ is equivalent to $\mathcal{A}_{ik}$.

(2) Consider the inner automorphism φ_0^{l+1}. Then it follows that $\varphi_0^{l+1}\,\theta_i = \theta_r\,\varphi_0^{l+1}$, $\varphi_0^{l+1}\,\theta_j = \theta_{k+r}\,\varphi_0^{l+1}$. This implies that $\mathcal{A}_{ij}$ is equivalent to $\mathcal{A}_{k+r,r}$. $\square$

By virtue of this proposition we may consider only the families $\mathcal{A}_{ij}$ with triple (j, k, r) such that $j \leq k \leq r$. Such a family is said to be a *proper family of type AI* and a family without the above condition is said to be simply a family of type AI.

Proposition 3.3. *Let $\mathcal{A}_{ij}$ be a proper family of type AI with triple (j, k, r) and set $(\mathfrak{g}, \sigma, \tau)=(\mathfrak{g}, \theta_i, \theta_j)$. Then the following hold:*

(1) *If $j<k<r$, all the PSLA's in $\mathcal{A}_{ij}$ are non-equivalent to each other;*

(2) *If $j=k<r$, only the following equivalences hold:*

$$(\mathfrak{g}, \sigma, \tau) \cong (\mathfrak{g}, \sigma, \sigma\tau), (\mathfrak{g}, \tau, \sigma) \cong (\mathfrak{g}, \sigma\tau, \sigma), \quad (\mathfrak{g}, \tau, \sigma\tau) \cong (\mathfrak{g}, \sigma\tau, \tau); \tag{3.1}$$

(3) *If $j<k=r$, only the following equivalences hold:*

$$(\mathfrak{g}, \sigma, \tau) \cong (\mathfrak{g}, \sigma\tau, \tau), (\mathfrak{g}, \sigma, \sigma\tau) \cong (\mathfrak{g}, \sigma\tau, \sigma), \quad (\mathfrak{g}, \tau, \sigma) \cong (\mathfrak{g}, \tau, \sigma\tau); \tag{3.2}$$

(4) *If $j=k=r$, all the PSLA's in $\mathcal{A}_{ij}$ are equivalent to each other.*

Proof. (1) Since $\dim \mathfrak{k}_- < \dim \mathfrak{p}_- < \dim \mathfrak{p}_+$, Our claim is obvious.

(2) The equivalences (3.1) follow since $\varphi_1^{jk}\theta_i=\theta_i\varphi_1^{jk}$ and $\varphi_1^{jk}\theta_j=\theta_i\theta_k\varphi_1^{jk}$ $=\theta_i\theta_j\varphi_1^{jk}$. The non-equivalences for other pairs in $\mathcal{A}_{ij}$ are easily obtained by the fact that $\dim \mathfrak{k}_- < \dim \mathfrak{p}_- = \dim \mathfrak{p}_+$.

(3) Consider the inner automorphism $(\varphi_1^{jk})^{-1}\varphi_0^{l+1}\varphi_1^{jk}$ instead of φ_1^{jk} in (2). Then our claim follows by the same way as (2).

(4) The equivalences (3.1), (3.2) hold for $k \leq r$ and $j \leq k$, respectively. Hence our claim is obvious. $\square$

In the followings the equivalences (3.1) are said to be *of first type* and the equivalences (3.2) *of second type*.

We next see the equivalence among families $\mathcal{A}_{i;jk}$ and then the equivalence among the PSLA's which belong to each $\mathcal{A}_{i;jk}$. For $\mathcal{A}_{i;jk}$ put $j=a, i=j+b$, $k=i+c, l+1=k+d$. Then $a, b, c, d \geq 1$ and the following holds.

Proposition 3.4. *Two families $\mathcal{A}_{i;jk}, \mathcal{A}_{i';j'k'}$ are equivalent to each other if and only if the quadruples $(a, b, c, d), (a', b', c', d')$ coincide except order.*

Proof. Consider the PSLA's $(\mathfrak{g}, \theta_i, \theta_{jk})$ snd $(\mathfrak{g}, \theta_{i'}, \theta_{j'k'})$. Then it follows that $\dim \mathfrak{k}_-=2(ab+cd)$, $\dim \mathfrak{p}_+=2(bc+ad)$, $\dim \mathfrak{p}_-=2(ac+bd)$ and that $\dim \mathfrak{k}'_-=2(a'b'+c'd')$, $\dim \mathfrak{p}'_+=2(b'c'+a'd')$, $\dim \mathfrak{p}'_-=2(a'c'+b'd')$,. (See (3.8) later.)

If $\mathcal{A}_{i;jk}, \mathcal{A}_{i';j'k'}$ are equivalent to each other, triples $(ab+cd, bc+ad, ac+bd)$ and $(a'b'+c'd', b'c'+a'd', a'c'+b'd')$ coincide ecxept order. This implies that quadruples $(a, b, c, d), (a', b', c', d')$ coincide except order.

To prove the converse we may see the following equivalences:

$$(1)\quad \mathcal{A}_{i\,;\,jk} \cong \mathcal{A}_{i\,;\,bk}\,, \qquad (2)\quad \mathcal{A}_{i\,;\,jk} \cong \mathcal{A}_{k-j\,;\,k-i,k}\,,$$
$$(3)\quad \mathcal{A}_{i\,;\,jk} \cong \mathcal{A}_{d+c\,;\,d,d+c+b}$$

where $\mathcal{A}_{i\,;\,bk}$, $\mathcal{A}_{k-j\,;\,k-i,k}$, $\mathcal{A}_{d+c\,;\,d,d+c+b}$ have the quadruples (b, a, c, d), (c, b, a, d), (d, c, b, a), respectively.

(1) Consider the inner automorphism φ_1^{ab}. Then it gives the equivalence of $(\mathfrak{g}, \theta_i, \theta_{jk})$ onto $(\mathfrak{g}, \theta_i, \theta_{bk}\,\theta_i)$. Hence $\mathcal{A}_{i\,;\,jk}$ is equivalent to $\mathcal{A}_{i\,;\,bk}$.

(2) Consider the inner automorphism φ_0^k. Then it gives the equivalence of $(\mathfrak{g}, \theta_i, \theta_{jk})$ onto $(\mathfrak{g}, \theta_{k-i,k}, \theta_{k-j})$. Hence $\mathcal{A}_{i\,;\,jk}$ is equivalent to $\mathcal{A}_{k-j\,;\,k-i,k}$.

(3) In the same way as Proposition 3.2, (2), the equivalence (3) is obtained by the automorphism φ_0^{l+1}. □

By virtue of this proposition we may consider only the families $\mathcal{A}_{i\,;\,jk}$ with quadruple (a, b, c, d) such that $a \leq b \leq c \leq d$. Such a family is said to be a *proper family of type AII* and a family without the above condition is said to be simply a family of type *AII*.

Proposition 3.5. *Let $\mathcal{A}_{i\,;\,jk}$ be a proper family of type AII with quadruple (a, b, c, d) and set $(\mathfrak{g}, \sigma, \tau) = (\mathfrak{g}, \theta_i, \theta_{jk})$. Then the following hold:*

(1) *If $a<b<c<d$, all the PSLA's in $\mathcal{A}_{i\,;\,jk}$ are non-equivalent to each other;*

(2) *If $a=b<c\leq d$ or $a\leq b<c=d$, only the equivalences of first type hold;*

(3) *If $a<b=c<d$, only the equivalences of second type hold;*

(4) *If $a=b=c<d$, $a<b=c=d$, or $a=b=c=d$, all the PSLA's in $\mathcal{A}_{i\,;\,jk}$ are equivalent to each other.*

Proof. (1) It follows that $\dim \mathfrak{k}_- > \dim \mathfrak{p}_- > \dim \mathfrak{p}_+$. Hence our claim is obvious.

(2) Assume first that $a=b<c\leq d$ and consider the inner automorphism φ_1^{ab}. Then it follows that $\varphi_1^{ab}\,\theta_i = \theta_i\,\varphi_1^{ab}$ and $\varphi_1^{ab}\,\theta_{jk} = \theta_i\,\theta_{bk}\,\varphi_1^{ab} = \theta_i\,\theta_{jk}\,\varphi_1^{ab}$. Hence the equivalences of first type are obtained.

Assume next that $a\leq b<c=d$ and consider the inner automorphism $\varphi = (\varphi_0^{l+1})^{-1}\,\varphi_1^{dc}\,\varphi_0^{l+1}$. The equivalences of first type are similarly obtained.

For both cases it holds that $\dim \mathfrak{k}_- > \dim \mathfrak{p}_- = \dim \mathfrak{p}_+$. This shows the non-equivalences for other pairs in $\mathcal{A}_{i\,;\,jk}$.

(3) Consider the inner automorphism $\varphi = (\varphi_1^{ab})^{-1}\,\varphi_0^k\,\varphi_1^{ab}$. Then it gives the equivalences of second type. The non-equivalences for other pairs in $\mathcal{A}_{i\,;\,jk}$ follow by the fact that $\dim \mathfrak{k}_- = \dim \mathfrak{p}_- > \dim \mathfrak{p}_+$.

(4) The equivalences of (2) hold for $b\leq c$ and those of (3) hold for $a\leq b=c\leq d$. Hence our claim is obvious. □

We now see the injectivity of the $\mathfrak{k}_+$-homomorphism ρ associated with each PSLA.

Fix a positive integer r and denote by $\boldsymbol{Z}^r$ the set of ordered r-tuples of integers. Set

$$R = R_1 = \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\ \overbrace{0\cdots0}^{c})\in\boldsymbol{Z}^r;\ a\geq0, b\geq0, c\geq0\}\ ,$$
$$R^2 = \{\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right);\ \alpha, \beta\in R\}\ .$$

The set R is considered as follows: Represent the roots of type A_l by linear combinations of the fundamental root system and identify them with the l-tuples of coefficients. Suppose that $r\leq l$. Then R (resp. $R-\{(0\cdots0)\}$) is the set of r-tuples of coefficients which are taken out from positions fixed consecutively, provided that $r\neq l$ (resp. $r=l$). Moreover denote by $R^2[\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right)_i]$ (resp. $R^2[\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right)_i, \left(\begin{smallmatrix}c\\ d\end{smallmatrix}\right)_j]$) the subset in R^2 of pairs $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ such that the i-components of α, β are a, b (resp. such that the i-components of α, β are a, b and the j-components are c, d), and for $\lambda\in\boldsymbol{Z}^r$ denote by $R^2_\lambda[*]$ the subset in $R^2[*]$ of pairs $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$ satisfying that $\lambda=-\alpha+\beta$. Then we can check the following lemma by a usual argument.

Lemma 3.6. *Let λ be an r-tuples in $\boldsymbol{Z}^r$. Then the following hold:*

(1) *The following each set has at most* 2 *elements:* $R^2_\lambda[\left(\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right)_r]$, $R^2_\lambda[\left(\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right)_1]$, $R^2_\lambda[\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)_1, \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)_r]$;

(2) *The set* $R^2_\lambda[\left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)_r]$ (*resp.* $R^2_\lambda[\left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)_1]$) *has at most* 1 *element if* $\lambda\neq(0\cdots0)$, *and has just r elements with form*

$$\begin{pmatrix}0 & \cdots & 0 & 1 & \cdots & 1\\ 0 & \cdots & 0 & 1 & \cdots & 1\end{pmatrix},\quad \left(\textit{resp.}\ \begin{pmatrix}1 & \cdots & 1 & 0 & \cdots & 0\\ 1 & \cdots & 1 & 0 & \cdots & 0\end{pmatrix}\right)$$

if $\lambda=(0\cdots0)$;

(3) *The set* $R^2_\lambda[\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)_1, \left(\begin{smallmatrix}-1\\ 0\end{smallmatrix}\right)_r]$ *has at most* 1 *element if* $\lambda\neq(1\cdots1)$, *and has just* $r-1$ *elements with form*

$$\begin{pmatrix}0 & \cdots & 0 & -1 & \cdots & -1\\ 1 & \cdots & 1 & 0 & \cdots & 0\end{pmatrix}$$

if $\lambda=(1\cdots1)$.

In the following we represent a root of type A_l by a linear combination of the fundamental root system Π and identify it with an l-tuple of coefficients.

Case AI: The families $\mathcal{A}_{ij}$ with triple (j, k, r)

Put $\sigma=\theta_i$ and $\tau=\theta_j$. Then, for each PSLA in $\mathcal{A}_{ij}$, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given as follows: (N is locally described.)

(a) $\mathcal{V} = (\mathfrak{g}, \sigma, \tau)$: $M = SU(l+1)/S(U(j+k)\times U(r))$.
In this case $N = \mathfrak{su}(j+r)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(r))$;
(b) $\mathcal{V} = (\mathfrak{g}, \sigma, \sigma\tau)$: $M = SU(l+1)/S(U(j+k)\times U(r))$.

In this case $N = \mathfrak{su}(k+r)/\mathfrak{s}(\mathfrak{u}(k)\oplus\mathfrak{u}(r))$;
(c) $\mathcal{V} = (\mathfrak{g}, \tau, \sigma)$: $M = SU(l+1)/S(U(j)\times U(k+r))$.
In this case $N = \mathfrak{su}(j+r)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(r))$;
(d) $\mathcal{V} = (\mathfrak{g}, \tau, \sigma\tau)$: $M = SU(l+1)/S(U(j)\times U(k+r))$.
In this case $N = \mathfrak{su}(j+k)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(k))$;
(e) $\mathcal{V} = (\mathfrak{g}, \sigma\tau, \sigma)$: $M = SU(l+1)/S(U(k)\times U(j+r))$.
In this case $N = \mathfrak{su}(k+r)/\mathfrak{s}(\mathfrak{u}(k)\oplus\mathfrak{u}(r))$;
(f) $\mathcal{V} = (\mathfrak{g}, \sigma\tau, \tau)$: $M = SU(l+1)/S(U(k)\times U(j+r))$.
In this case $N = \mathfrak{su}(j+k)/\mathfrak{s}(\mathfrak{u}(j)\oplus\mathfrak{u}(k))$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}_+}, \Delta^+_{\mathfrak{k}_-}, \Delta^+_{\mathfrak{p}_+}, \Delta^+_{\mathfrak{p}_-}$ of Δ^+ are given as follows: (Here $\Delta^+_{(*)}=\Delta_{(*)}\cap\Delta^+$.)

$$\begin{aligned}
(3.3)\qquad \Delta^+_{\mathfrak{k}_+} &= \{\delta\in\Delta^+;\ \delta_i = \delta_j = 0\} \\
&= \left\{\delta\in\Delta^+;\ \delta = \begin{matrix}(0\cdots01\cdots10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots0) \\ (0\cdots\overset{j}{0}\cdots01\cdots10\cdots\overset{i}{0}\cdots0) \\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots10\cdots0)\end{matrix}\right\}, \\
\Delta^+_{\mathfrak{k}_-} &= \{\delta\in\Delta^+;\ \delta_i = 0,\ \delta_j = 1\} \\
&= \{\delta\in\Delta^+;\ \delta = (0\cdots01\cdots\overset{j}{1}\cdots10\cdots\overset{i}{0}\cdots0)\}, \\
\Delta^+_{\mathfrak{p}_+} &= \{\delta\in\Delta^+;\ \delta_i = 1,\ \delta_j = 0\} \\
&= \{\delta\in\Delta^+;\ \delta = (0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots10\cdots0)\}, \\
\Delta^+_{\mathfrak{p}_-} &= \{\delta\in\Delta^+;\ \delta_i = \delta_j = 1\} \\
&= \{\delta\in\Delta^+;\ \delta = (0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots10\cdots0)\}.
\end{aligned}$$

If $l\geq 3$, the dominant weights in $\Delta_{\mathfrak{k}_-}, \Delta_{\mathfrak{p}_+}, \Delta_{\mathfrak{p}_-}$ are given by (3.4), (3.5), (3.6), respecively:

$$(3.4)\qquad (1\cdots\overset{j}{1}\cdots1\overset{i}{0}\cdots0),\quad -(0\cdots0\overset{j}{1}0\cdots\overset{i}{0}\cdots0).$$

$$(3.5)\qquad (0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots1),\quad -(0\cdots\overset{j}{0}\cdots0\overset{i}{1}0\cdots0).$$

$$(3.6)\qquad (1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots1),\quad -(0\cdots0\overset{j}{1}\cdots\overset{i}{1}0\cdots0).$$

If $l=2$, the subset $\Delta_{\mathfrak{k}_+}$ is empty and so the weights in $\Delta_{\mathfrak{k}_-}, \Delta_{\mathfrak{p}_+}, \Delta_{\mathfrak{p}_-}$ are all dominant.

We now see the injectivity of ρ for Case (c): $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$. Note that in this case ρ is a homomorphism of $(\mathfrak{p}^{\mathbf{C}}_-)^*\otimes\mathfrak{p}^{\mathbf{C}}_+$ to $\wedge^2(\mathfrak{p}^{\mathbf{C}}_-)^*\otimes\mathfrak{k}^{\mathbf{C}}_-$.

We first suppose that $l\geq 3$. Then the minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by (α1), (α2) and the dominant weights in $\Delta_{\mathfrak{p}_+}$ are given by (β1), (β2):

$$(\alpha 1)\quad -(1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots1),\qquad (\alpha 2)\quad (0\cdots0\overset{j}{1}\cdots\overset{i}{1}0\cdots0).$$

$$(\beta 1)\quad (0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots1),\qquad (\beta 2)\quad -(0\cdots\overset{j}{0}\cdots0\overset{i}{1}0\cdots0).$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α,β) is one of pairs $((\alpha\, s),(\beta\, t))$, where $s,t=1,2$. Applying Lemma 2.3 to each pair, we obtain that $\rho(u)=0$ for pairs $((\alpha 1),(\beta 1))$ $(j=1)$, $((\alpha 2),(\beta 2))$ $(j=1)$ and $\rho(u)\neq 0$ for the other pairs.

Case (2): $l(u)=2$. Note that in this case there exists no decomposable maximal weight vector. Hence we suppose that u is indecomposable. As in §2 consider the object $(\alpha,\alpha',\beta,\beta';\mu)$ associated with u. Since this object is determined by α,β',μ, we consider the triple $(\alpha,\beta';\mu)$ instead of it. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$$(\mu 1)\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots 01)\,,\qquad (\mu 2)\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}10\cdots 0)\,.$$

Then the triples $(\alpha,\beta';\mu)$ are given in the following:

(1) $((\alpha\,1),(\beta\,1);(\mu\,1)),\ i\neq l$; (2) $((\alpha\,1),(\beta\,2);(\mu\,1)),\ i=l-1$;

(3) $((\alpha\,2),(\beta\,1);(\mu\,2)),\ i=l-1$; (4) $((\alpha\,2),(\beta\,2);(\mu\,2)),\ i\neq l$.

Lemma 2.4 is available for cases (1), (4) and Lemma 2.2 is available for cases (2), (3). So it follows that $\rho(u)\neq 0$.

Case (3): $l(u)\geq 3$ We see the weight spaces with $\dim\geq 3$. Let λ be a weight in Λ and let α,β be roots such that $\lambda=-\alpha+\beta$, where $\alpha\in\Delta_{\mathfrak{p}_-}$ and $\beta\in\Delta_{\mathfrak{p}_+}$. Denote by a_k, b_k, $\lambda_k (1\leq k\leq l)$ the k-th components of α,β,λ, respectively. Since $a_j=\pm 1$ and $b_j=0$, it follows that $\lambda_j=\mp 1$. Suppose that $\lambda_j=1$. (For the case that $\lambda_j=-1$ we can similarly do the argument mentioned below.) Then it follows by (3.3) that $\lambda_i=0, 2$.

If $\lambda_i=0$, the pair $\binom{\alpha}{\beta}$ has the form $\begin{pmatrix} & \overset{j}{-1} & \cdots & \overset{i}{-1} \\ 0\cdots & 0 & & -1\end{pmatrix}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 3.6 that $\lambda=(0\cdots 01\cdots\overset{j}{1}\cdots 10\cdots\overset{i}{0}\cdots 0)$, and the pair $\binom{\alpha}{\beta}$ has the form

$$\begin{pmatrix} 0 & \cdots 0-1\cdots & \overset{j}{-1} & \cdots & \overset{i}{-1} & \cdots & -10 & \cdots 0 \\ 0 & \cdots & 0 & \cdots 0-1\cdots & -1 & \cdots & -10 & \cdots 0\end{pmatrix}. \tag{3.7}$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.2 that $\rho(u)\neq 0$.

If $\lambda_i=2$, the pair $\binom{\alpha}{\beta}$ has the form $\begin{pmatrix} & \overset{j}{-1} & \cdots & \overset{i}{-1} \\ 0\cdots & 0 & & 1\end{pmatrix}$. By Lemma 3.6, the weight space for this λ has at most dimension 2.

We next suppose that $l=2$. Then it holds that

$$\Delta^+_{\mathfrak{k}_-}=\{(10)\}\,,\quad \Delta^+_{\mathfrak{p}_+}=\{(01)\}\,,\quad \Delta^+_{\mathfrak{p}_-}=\{(11)\}\,.$$

So a weight λ in Λ is one of $\pm(10)$, $\pm(12)$ and each weight space has dimension 1. Hence we may consider Case (1): $l(u)=1$. By Lemma 2.3 it holds that

$\rho(u)=0$ for $\lambda=\pm(12)$ and $\rho(u)\neq 0$ fro the other λ.

Summing up the above arguments, we have the following result for the PSLA of Case (c); the homomorphism ρ is not injective if and only if $j=1$. For the other cases we have similar results; ρ is not injective only for the following cases: Cases (a), (b), $i=l$; Case (d), $j=1$; Cases (e), (f), $j=i-1$. These cases imply the cases of Example 2, (1) in §1.

Theroem 3.7. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type AI. Then the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example 2, (1).*

Case AII: The families $\mathcal{A}_{i\,;\,jk}$ with quadruple (a, b, c, d)

Put $\sigma=\theta_i$ and $\tau=\theta_{jk}$. Then, for each PSLA in $\mathcal{A}_{i\,;\,jk}$, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given in the following: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=SU(l+1)/S(U(a+b)\times U(c+d))$.
In this case $N=\mathfrak{su}(a+c)/\mathfrak{s}(\mathfrak{u}(a)\oplus\mathfrak{u}(c))\oplus\mathfrak{su}(b+d)/\mathfrak{s}(\mathfrak{u}(b)\oplus\mathfrak{u}(d))$;
(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=SU(l+1)/S(U(a+b)\times U(c+d))$.
In this case $N=\mathfrak{su}(b+c)/\mathfrak{s}(\mathfrak{u}(b)\oplus\mathfrak{u}(c))\oplus\mathfrak{su}(a+d)/\mathfrak{s}(\mathfrak{u}(a)\oplus\mathfrak{u}(d))$;
(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=SU(l+1)/S(U(b+c)\times U(a+d))$.
In this case $N=\mathfrak{su}(a+c)/\mathfrak{s}(\mathfrak{u}(a)\oplus\mathfrak{u}(c))\oplus\mathfrak{su}(b+d)/\mathfrak{s}(\mathfrak{u}(b)\oplus\mathfrak{u}(d))$;
(d) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma\tau)$: $M=SU(l+1)/S(U(b+c)\times U(a+d))$.
In this case $N=\mathfrak{su}(a+b)/\mathfrak{s}(\mathfrak{u}(a)\oplus\mathfrak{u}(b))\oplus\mathfrak{su}(c+d)/\mathfrak{s}(\mathfrak{u}(c)\oplus\mathfrak{u}(d))$;
(e) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \sigma)$: $M=SU(l+1)/S(U(a+c)\times U(b+d))$.
In this case $N=\mathfrak{su}(b+c)/\mathfrak{s}(\mathfrak{u}(b)\oplus\mathfrak{u}(c))\oplus\mathfrak{su}(a+d)/\mathfrak{s}(\mathfrak{u}(a)\oplus\mathfrak{u}(d))$;
(f) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \tau)$: $M=SU(l+1)/S(U(a+c)\times U(b+d))$.
In this case $N=\mathfrak{su}(a+b)/\mathfrak{s}(\mathfrak{u}(a)\oplus\mathfrak{u}(b))\oplus\mathfrak{su}(c+d)/\mathfrak{s}(\mathfrak{u}(c)\oplus\mathfrak{u}(d))$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}\pm}$, $\Delta^+_{\mathfrak{p}\pm}$ of Δ^+ are given as follows:

$$
\begin{aligned}
(3.8)\qquad \Delta^+_{\mathfrak{k}+} &= \{\delta\in\Delta^+;\ \delta_i=0,\ (\delta_j, \delta_k)=(0,0), (1,1)\} \\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots01\cdots10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0) \\
(0\cdots\overset{j}{0}\cdots01\cdots10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0) \\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots10\cdots\overset{k}{0}\cdots0) \\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots10\cdots0)
\end{matrix}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\Delta^+_{\mathfrak{k}-} &= \{\delta\in\Delta^+;\ \delta_i=0,\ (\delta_j, \delta_k)=(1,0), (0,1)\} \\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots01\cdots\overset{j}{1}\cdots10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0) \\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots10\cdots0)
\end{matrix}\right\},
\end{aligned}
$$

$$
\Delta^+_{\mathfrak{p}+} = \{\delta\in\Delta^+;\ \delta_i=1,\ (\delta_j, \delta_k)=(0,0), (1,1)\}
$$

$$
= \left\{ \delta \in \Delta^+;\ \delta = \begin{matrix} (0\cdots\overset{j}{0}\cdots 01\cdots\overset{i}{1}\cdots 10\cdots\overset{k}{0}\cdots 0) \\ (0\cdots 01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots 10\cdots 0) \end{matrix} \right\},
$$

$$
\Delta^+_{\mathfrak{p}_-} = \{\delta \in \Delta^+;\ \delta_i = 1,\ (\delta_j, \delta_k) = (1, 0), (0, 1)\}
$$

$$
= \left\{ \delta \in \Delta^+;\ \delta = \begin{matrix} (0\cdots 01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots 10\cdots\overset{k}{0}\cdots 0) \\ (0\cdots\overset{j}{0}\cdots 01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots 10\cdots 0) \end{matrix} \right\}.
$$

If $l \geq 4$, the dominant weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are given by (3.9), (3.10), (3.11), respectively:

$$
\left\{ \begin{matrix} (1\cdots\overset{j}{1}\cdots 1\overset{i}{0}\cdots\overset{k}{0}\cdots 0), & -(0\cdots 0\overset{j}{1}0\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots 0), \\ (0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}\cdots 1), & -(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots 0\overset{k}{1}0\cdots 0). \end{matrix} \right. \tag{3.9}
$$

$$
\left\{ \begin{matrix} (0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots 1\overset{k}{0}\cdots 0), & -(0\cdots\overset{j}{0}\cdots 0\overset{i}{1}0\cdots\overset{k}{0}\cdots 0), \\ (1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots 1), & -(0\cdots 0\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}0\cdots 0). \end{matrix} \right. \tag{3.10}
$$

$$
\left\{ \begin{matrix} (1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots 1\overset{k}{0}\cdots 0), & -(0\cdots 0\overset{j}{1}\cdots\overset{i}{1}0\cdots\overset{k}{0}\cdots 0), \\ (0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots 1), & -(0\cdots\overset{j}{0}\cdots 0\overset{i}{1}\cdots\overset{k}{1}0\cdots 0). \end{matrix} \right. \tag{3.11}
$$

If $l=3$, the subset $\Delta_{\mathfrak{k}_+}$ is empty and so the weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are all dominant.

We now see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$. In this case ρ is a homomorphism of $(\mathfrak{p}_-^C)^* \otimes \mathfrak{k}_-^C$ to $\wedge^2(\mathfrak{p}_-^C)^* \otimes \mathfrak{p}_+^C$.

We first suppose that $l \geq 4$. Then the minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha 1)\sim(\alpha 4)$ and the dominant weights in $\Delta_{\mathfrak{k}_-}$ are given by $(\beta 1)\sim(\beta 4)$:

$$
\begin{array}{llll}
(\alpha 1) & -(1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots 1\overset{k}{0}\cdots 0), & (\alpha 2) & (0\cdots 0\overset{j}{1}\cdots\overset{i}{1}0\cdots\overset{k}{0}\cdots 0), \\
(\alpha 3) & -(0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots 1), & (\alpha 4) & (0\cdots\overset{j}{0}\cdots 0\overset{i}{1}\cdots\overset{k}{1}0\cdots 0), \\
(\beta 1) & (1\cdots\overset{j}{1}\cdots 1\overset{i}{0}\cdots\overset{k}{0}\cdots 0), & (\beta 2) & -(0\cdots 0\overset{j}{1}0\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots 0), \\
(\beta 3) & (0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}\cdots 1), & (\beta 4) & -(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots 0\overset{k}{1}0\cdots 0).
\end{array}
$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha \otimes X_\beta$. Then the pair (α, β) is one of pairs $((\alpha\, r), (\beta\, s))$, where $r, s=1, 2, 3, 4$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u) \neq 0$ for all the pairs.

Case (2): $l(u)=2$. We first suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$$
\begin{array}{llll}
(\mu 1) & (10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots 0), & (\mu 2) & (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots 01\overset{k}{0}\cdots 0), \\
(\mu 3) & (0\cdots 01\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots 0), & (\mu 4) & (0\cdots\overset{j}{0}\cdots\overset{i}{0}10\cdots\overset{k}{0}\cdots 0), \\
(\mu 5) & (0\cdots\overset{j}{0}10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots 0), & (\mu 6) & (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots 01), \\
(\mu 7) & (0\cdots\overset{j}{0}\cdots 01\overset{i}{0}\cdots\overset{k}{0}\cdots 0), & (\mu 8) & (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}10\cdots 0).
\end{array}
$$

Then such the triples $(\alpha, \beta'; \mu)$ as Case (2) of Case AI are given in the following:

(1) $((\alpha 1), (\beta 1); (\mu 1)), j \geq 2$;	(2) $((\alpha 1), (\beta 2); (\mu 1)), j = 2$;
(3) $((\alpha 1), (\beta 3); (\mu 2)), k-i = 2$;	(4) $((\alpha 1), (\beta 4); (\mu 2)), k-i \geq 2$;
(5) $((\alpha 2), (\beta 1); (\mu 3)), j = 2$;	(6) $((\alpha 2), (\beta 2); (\mu 3)), j \geq 2$;
(7) $((\alpha 2), (\beta 3); (\mu 4)), k-i \geq 2$;	(8) $((\alpha 2), (\beta 4); (\mu 4), k-i = 2$;
(9) $((\alpha 3), (\beta 1); (\mu 5)), i-j = 2$;	(10) $((\alpha 3), (\beta 2); (\mu 5)), i-j \geq 2$;
(11) $((\alpha 3), (\beta 3); (\mu 6)), k \leq l-1$;	(12) $((\alpha 3), (\beta 4); (\mu 6)), k = l-1$;
(13) $((\alpha 4), (\beta 1); (\mu 7)), i-j \geq 2$;	(14) $((\alpha 4), (\beta 2); (\mu 7)), i-j = 2$;
(15) $((\alpha 4), (\beta 3); (\mu 8)), k = l-1$;	(16) $((\alpha 4), (\beta 4); (\mu 8)), k \leq l-1$.

Lemma 2.4 is available for cases (1), (4), (6), (7), (10), (11), (13), (16) and Lemma 2.2 is available for the other cases. So it follows that $\rho(u) \neq 0$.

We next suppose that u is decomposable. Put $u = a\, \omega_{\alpha_1} \otimes X_{\beta_1} + b\, \omega_{\alpha_2} \otimes X_{\beta_2}$. Then the following two cases are considerable:

(1) $\lambda = (1\cdots\overset{j}{1}\overset{i}{1}\overset{k}{1}\cdots1)$, where $i = j+1, k = i+1$ and the pairs (α_i, β_i) are $((\alpha 1), (\beta 3)), (\alpha 3), (\beta 1))$;

(2) $\lambda = -(\overset{j}{0}\cdots0\overset{i}{1}0\cdots\overset{k}{0})$, where $j = 1, k = l$ and the pairs (α_i, β_i) are $((\alpha 2), (\beta 1)), ((\alpha 4), (\beta 3))$.

In these cases the weights λ are roots and Lemma 2.2 is available. So it follows that $\rho(u) \neq 0$.

Case (3): $l(u) \geq 3$. We see the weight spaces with $\dim \geq 3$. Let λ be a weight in Λ and let α, β be weights such that $\lambda = -\alpha + \beta$, where $\alpha \in \Delta_{\mathfrak{p}_-}$ and $\beta \in \Delta_{\mathfrak{k}_-}$. Denote by a_k, b_k, λ_k $(1 \leq k \leq l)$ the k-th components of α, β, λ, respectively. Since $a_i = \pm 1$ and $b_i = 0$, it follows that $\lambda_i = \mp 1$. Suppose that $\lambda_i = 1$. (For the case that $\lambda_i = -1$ we can similarly do the arguments mentioned below.) Then it follows by (3.8) that $\lambda_j = 0, \pm 1, 2$.

Case (i): $\lambda_j = 0$. Then it moreover follows by (3.8) that $\lambda_k = 0, 2$.

If $\lambda_k = 0$, the pair $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ has either of the forms

$$\begin{array}{c} \qquad\ {\scriptstyle j} \qquad\quad {\scriptstyle i} \qquad\quad\ \ {\scriptstyle k} \\ \begin{pmatrix} 0\cdots0 & -1\cdots & -1 \\ 0\cdots0\cdots & 0 & -1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \quad\ {\scriptstyle j} \qquad\quad\ {\scriptstyle j} \qquad\ \ {\scriptstyle k} \\ \begin{pmatrix} -1\cdots-1 & 0\cdots0 \\ -1 & 0\cdots0\cdots0 \end{pmatrix} \end{array}.$$

If the weight space for λ has the dimension more than 3, it follows by Lemma 3.6 that $\lambda = (0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots10\cdots\overset{k}{0}\cdots0)$, and the pair $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ has either of the forms

$$(3.12)\qquad \left\{ \begin{array}{l} \begin{array}{c} \qquad\quad {\scriptstyle j} \qquad\qquad\qquad\quad {\scriptstyle i} \qquad\qquad\qquad\qquad\quad {\scriptstyle k} \\ \begin{pmatrix} 0\cdots0\cdots0-1\cdots-1 & \cdots\cdots\cdots & -1\cdots-10\cdots0 \\ 0\cdots0 \quad \cdots\cdots\cdots & 0\cdots0-1\cdots & -1\cdots-10\cdots0 \end{pmatrix} \end{array}, \\ \begin{array}{c} \qquad\qquad\quad {\scriptstyle j} \qquad\qquad\qquad\quad {\scriptstyle i} \qquad\qquad\qquad\quad {\scriptstyle k} \\ \begin{pmatrix} 0\cdots0-1\cdots-1 & \cdots\cdots\cdots & -1\cdots-10\cdots0\cdots0 \\ 0\cdots0-1\cdots-1\cdots-10\cdots & 0 & \cdots\cdots\cdots\ 0\cdots0 \end{pmatrix} \end{array}. \end{array} \right.$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.2 that $\rho(u)\neq 0$.

If $\lambda_k=2$, the pair $\binom{\alpha}{\beta}$ has the form $\begin{array}{c} \begin{matrix} \quad j & \quad i & \quad k \end{matrix} \\ \begin{pmatrix} 0\cdots 0 & -1\cdots & -1 \\ 0\cdots 0\cdots & 0 & 1 \end{pmatrix} \end{array}$. By Lemma 3.6, the weight space with this λ has at most dimension 2.

Case (ii): $\lambda_j=1$. Then it moreover fololws by (3.8) that $\lambda_k=\pm 1$.

If $\lambda_k=1$, the pair $\binom{\alpha}{\beta}$ has either of the forms

$$\begin{array}{c} \begin{matrix} & \quad j & \quad i & \quad k \end{matrix} \\ \begin{pmatrix} & -1\cdots & -1 & 0\cdots 0 \\ 0\cdots & 0\cdots & 0 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{c} \begin{matrix} \quad j & \quad i & \quad k \end{matrix} \\ \begin{pmatrix} 0\cdots 0 & -1\cdots & -1 & \\ 1 & 0\cdots & 0\cdots 0 \end{pmatrix} \end{array}.$$

If the weight space for λ has the dimension more than 3, it follows by Lemma 3.6 that $\lambda=(0\cdots 01\cdots \overset{j}{1}\cdots \overset{i}{1}\cdots \overset{k}{1}\cdots 10\cdots 0)$, and the pair $\binom{\alpha}{\beta}$ has either of the forms

$$(3.13)\qquad \left\{\begin{array}{l} \begin{array}{c} \begin{matrix} & & j & i & & & k & \end{matrix} \\ \begin{pmatrix} 0\cdots 0-1\cdots & -1\cdots & -1\cdots & -1 & 0\cdots & 0\cdots\cdots\cdots 0 \\ 0\cdots\cdots\cdots & 0\cdots & 0\cdots & 0 & 1\cdots & 1\cdots 10\cdots 0 \end{pmatrix} \end{array}, \\ \begin{array}{c} \begin{matrix} & j & & & i & k & \end{matrix} \\ \begin{pmatrix} 0\cdots\cdots\cdots & 0\cdots & 0 & -1\cdots & -1\cdots & -1 & \cdots -10\cdots 0 \\ 0\cdots 01\cdots & 1\cdots & 1 & 0\cdots & 0\cdots & 0 & \cdots\cdots\cdots 0 \end{pmatrix} \end{array}. \end{array}\right.$$

Hence for a maximal vector u in this weight space, it follows by Lemma 2.2 that $\rho(u)\neq 0$.

If $\lambda_k=-1$, the pair $\binom{\alpha}{\beta}$ has the form $\begin{array}{c} \begin{matrix} & \quad j & \quad i & \quad k \end{matrix} \\ \begin{pmatrix} & -1\cdots & -1 & 0\cdots 0 \\ 0\cdots & 0\cdots & 0 & -1 \end{pmatrix} \end{array}$. By Lemma 3.6 the weight space for this λ has at most dimension 2.

Case (iii): $\lambda_j=-1$ (resp. $\lambda_j=2$). The pair $\binom{\alpha}{\beta}$ has the form

$$\begin{array}{c} \begin{matrix} & \quad j & \quad i & \quad k \end{matrix} \\ \begin{pmatrix} 0\cdots & 0 & -1\cdots & -1 & \\ & -1 & 0\cdots & 0\cdots 0 \end{pmatrix} \end{array} \quad \left(\text{resp. } \begin{array}{c} \begin{matrix} \quad j & \quad i & \quad k \end{matrix} \\ \begin{pmatrix} -1\cdots & -1 & 0\cdots 0 \\ 1 & 0\cdots & 0\cdots 0 \end{pmatrix} \end{array}\right).$$

By Lemma 3.6, the weight spaces with these λ have at most dimension 2.

We next suppose that $l=3$. It holds that

$$\Delta^+_{\mathfrak{k}-} = \{(100), (001)\}, \quad \Delta^+_{\mathfrak{p}+} = \{(010), (111)\},$$
$$\Delta^+_{\mathfrak{p}-} = \{(110), (011)\}.$$

So a weight λ in Λ is one of $\pm(010)$, $\pm(11-1)$, $\pm(-111)$, $\pm(210)$, $\pm(111)$, $\pm(012)$, and the weight spaces for $\lambda=\pm(010)$, $\pm(111)$ have dimension 2 and the weight spaces for the other λ have dimension 1. Lemma 2.3 is available for the cases with dimension 1 and Lemma 2.2 is available for the cases with dimension 2. Hence it follows that $\rho(u)\neq 0$ for all maximal weight vectors u.

Summing up the above arguments, we have the following result for the PSLA of Case (a); the homomorphism ρ is always injective. Similarly for the other cases ρ is injective.

Theorem 3.8. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type AII. Then the $\mathcal{V}$-geometry does not admit a non-totally geodesic $\mathcal{V}$-submanifold.*

4. The PSLA's with Lie algebra $\mathfrak{g}$ of type B_l

Let $\mathfrak{g}$ be the Lie algebra of type B_l, $l \geq 2$, that is, the Lie algebra $\mathfrak{so}(2l+1)$ of real skew symmetric matrices of degree $2l+1$. Then the Dynkin diagram of the fundamental root system Π is given as follows:

$$\underset{\alpha_1}{\bigcirc} - \underset{\alpha_2}{\bigcirc} - \cdots - \underset{\alpha_{l-1}}{\bigcirc} \Rightarrow \underset{\alpha_l}{\bigcirc} \qquad -\alpha_0 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l$$

Put θ_i, θ_{jk} as in §3 and let $\mathcal{B}_{ij}$, $1 \leq j < i \leq l$, and $\mathcal{B}_{i\,;\,jk}$, $1 \leq j < i < k \leq l$, be the families which contain the PSLA's $(\mathfrak{g}, \theta_i, \theta_j)$, $(\mathfrak{g}, \theta_i, \theta_{jk})$, respectively.

Lemma 4.1. *A PSLA $(\mathfrak{g}, \sigma, \tau)$ of inner type is equivalent to a PSLA which belongs to one of the families $\mathcal{B}_{ij}$ or $\mathcal{B}_{i\,;\,jk}$, by an inner automorphism of $\mathfrak{g}$.*

Proof. We may assume that $\sigma = \theta_i$. We divide into the following cases: (1) $i=1$, (2) $i=2$, and (3) $2 < i \leq l$.

Case (1): $i=1$. In this case $\mathfrak{k} = \mathfrak{c} + \mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:

$$\underset{\alpha_2}{\bigcirc} - \underset{\alpha_3}{\bigcirc} - \cdots - \underset{\alpha_{l-1}}{\bigcirc} \Rightarrow \underset{\alpha_l}{\bigcirc}$$

Hence we may assume that the restriction $\bar{\tau}$ of τ is given as follows: $\bar{\tau} = \exp \operatorname{ad}(\sqrt{-1}\pi K_j)$, where $2 \leq j \leq l$. Put $K_j = a_1 H_1 + \cdots + a_l H_l$. Since $\langle K_j, \alpha_k \rangle = \delta_{jk}$ for $2 \leq k \leq l$, it follows that $K_j = a_1 H_1 + H_j$ and thus $\bar{\tau} = \exp \operatorname{ad}(\sqrt{-1}\pi H_j)$. This implies that $\tau = \tau_0 = \exp \operatorname{ad}(\sqrt{-1}\pi H_j)$ or $\tau = \tau_0 \sigma$. So the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{B}_{j1}$.

Case (2): $i=2$. In this case $\mathfrak{k} = \mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:

$$\underset{\alpha_0}{\bigcirc} \quad \underset{\alpha_1}{\bigcirc} \quad \underset{\alpha_3}{\bigcirc} - \underset{\alpha_4}{\bigcirc} - \cdots - \underset{\alpha_{l-1}}{\bigcirc} \Rightarrow \underset{\alpha_l}{\bigcirc}$$

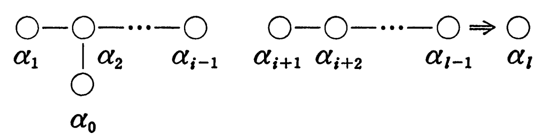
If we put $\bar{\tau} = \exp \operatorname{ad}(\sqrt{-1}\pi K)$, the following cases are considerable: (i) $K = K_0$; (ii) $K = K_1$; (iii) $K = K_j$, $3 \leq j \leq l$; (iv) $K = K_0 + K_1$; (v) $K = K_0 + K_j$, $3 \leq j \leq l$; (vi) $K = K_1 + K_j$, $3 \leq j \leq l$; (vii) $K = K_0 + K_1 + K_j$, $3 \leq j \leq l$. By Lemma 1.2 (1), only the cases (iii), (iv), (vii) have involutive extensions of $\bar{\tau}$. Using the fact that $\langle K_r, \alpha_s\rangle=\delta_{rs}$ for $r, s=0, 1, 3, \cdots, l$, we represent the vectors K_r by the vectors $H_1, \cdots, H_l$.

For Case (iii) it follows that $K_j=-H_2+H_j$ and thus $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\pi\, H_j)$. This implies that $\tau=\tau_0=\exp \operatorname{ad}(\sqrt{-1}\pi\, H_j)$, or $\tau=\tau_0\,\sigma$. So the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{B}_{j2}$.

For Case (iv) it follows that $K_0+K_1=H_1-H_2$ and thus $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\pi\, H_1)$. This implies that $\tau=\tau_0=\exp \operatorname{ad}(\sqrt{-1}\pi\, H_1)$, or $\tau=\tau_0\,\sigma$. So the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{B}_{21}$.

For Case (vii) it follows that $K_0+K_1+K_j=H_1-2H_2+H_j$ and thus $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\pi\,(H_1+H_j))$. This implies that $\tau=\tau_0=\exp \operatorname{ad}(\sqrt{-1}\pi\,(H_1+H_j))$, or $\tau=\tau_0\,\sigma$. So the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{B}_{2\,;\,1j}$.

Case (3): $2<i\leq l$. In this case $\mathfrak{k}=\mathfrak{k}_s$ and the Dynkin diagram of Π_s is given as follows:



If we put $\bar{\tau}=\exp \operatorname{ad}(\sqrt{-1}\pi\, K)$, the following cases are considerable: (i) $K=K_j$, $0\leq j\leq i-1$; (ii) $K=K_k$, $i+1\leq k\leq l$; (iii) $K=K_j+K_k$, $0\leq j\leq i-1$, $i+1\leq k\leq l$. Here the cases that $j=0, 1$ don't occur by Lemma 1.2 (1). Similarly to Case (2), the PSLA $(\mathfrak{g}, \sigma, \tau)$ belongs to $\mathcal{B}_{ij}$, $\mathcal{B}_{ki}$, $\mathcal{B}_{i\,;\,jk}$ according to Cases (i), (ii), (iii). $\square$

From the above proof, we can see that the subalgebras $\mathfrak{k}_+$ for $\mathcal{B}_{ij}$ are different from those for $\mathcal{B}_{i\,;\,jk}$. Hence the families $\mathcal{B}_{ij}$ are never equivalent to the familiies $\mathcal{B}_{i\,;\,jk}$.

We first see the equivalences among the families $\mathcal{B}_{ij}$ and the equivalences among the PSLA's wihch belong to each $\mathcal{B}_{ij}$.

Put $V=\sqrt{-1}\mathfrak{h}$ and take an orthonomal basis $\{e_1, \cdots, e_l\}$ such that $\alpha_i=e_i-e_{i+1}$ for $1\leq i\leq l-1$, and $\alpha_l=e_l$. Then it holds that $H_i=e_1+\cdots+e_i$ for all i, and the Weyl group $W(\Delta)$ is generated by the permutations of $e_1, \cdots, e_l$ and the mappings w_i^-, $1\leq i\leq l$: $w_i^-(e_i)=-e_i$ and $w_i^-(e_j)=e_j$ for $j\neq i$. Define elements $w_0^k(1\leq k\leq l)$ and $w_1^{jk}(j, k\geq 1, j+k\leq l)$ in $W(\Delta)$ in the same way as in §3. Then it follows that

$$w_0^k(H_i)=H_k-H_{k-i}\quad (1\leq i<k)\,,$$
$$w_0^k(H_i)=H_i\quad (k\leq i\leq l)\,,$$

$$w_1^{jk}(H_j)=H_{j+k}-H_k\,,$$
$$w_1^{jk}(H_i)=H_i\quad (j+k\leq i\leq l)\,.$$

Let φ_0^k, φ_1^{jk} be inner automorphisms of $\mathfrak{g}$ induced by w_0^k, w_1^{jk}, respectively.

For a family $\mathscr{B}_{ij}$ put $i=j+k$ and $l+1=i+r$. Then $j, k, r\geq 1$ and the following holds.

Proposition 4.2. *Two families $\mathscr{B}_{ij}$, $\mathscr{B}_{i'j'}$ are equivalent to each other if and only if $r=r'$ and the pairs (j, k), (j', k') concide except order.*

Proof. Consider the PSLA $(\mathfrak{g}, \theta_i, \theta_j)$ in $\mathscr{B}_{ij}$ and the PSLA $(\mathfrak{g}, \theta_{i'}, \theta_{j'})$ in $\mathscr{B}_{i'j'}$. Then it follows that $\mathfrak{k}_+=\mathfrak{so}(2j)\oplus\mathfrak{so}(2k)\otimes\mathfrak{so}(2r-1)$ and $\mathfrak{k}'_+=\mathfrak{so}(2j')\oplus\mathfrak{so}(2k')\otimes\mathfrak{so}(2r'-1)$.

Suppose that $\mathscr{B}_{ij}$ is equivalent to $\mathscr{B}_{i'j'}$. Since $\mathfrak{k}_+$ is isomorphic to $\mathfrak{k}'_+$, it follows that $r=r'$ and pairs (j, k), (j', k') coincide except order.

To prove the converse we may prove the following equivalence: $\mathscr{B}_{ij}\cong\mathscr{B}_{i,k}$, where $\mathscr{B}_{i,k}$ has the triple (k, j, r). This is similarly given by φ_1^{jk} as Proposition 3.2. □

By virtue of this proposition we may consider only the families $\mathscr{B}_{ij}$ with triple (j, k, r) such that $j\leq k$. Such a family is said to be a *proper family of type BI* and a family without the above condition is said to be simply a family of type *BI*.

Proposition 4.3. *Let $\mathscr{B}_{ij}$ be a proper family of type BI with triple (j, k, r) and set $(\mathfrak{g}, \sigma, \tau)=(\mathfrak{g}, \theta_i, \theta_j)$. Then the following hold*:

(1) *If $j<k$, all the PSLA's in $\mathscr{B}_{ij}$ are non-equivalent to each other*;

(2) *If $j=k$, only the equivalences of first type hold.*

Proof. We note that $\dim\mathfrak{k}_-=4jk$, $\dim\mathfrak{p}_+=2k(2r-1)$, $\dim\mathfrak{p}_-=2j(2r-1)$. (See (4.1) later.) Thus $\mathfrak{p}_\pm$ are not isomorphic to $\mathfrak{k}_-$.

(1) In this case it moreover follows that $\dim\mathfrak{p}_-<\dim\mathfrak{p}_+$. Hence our claim is obvious.

(2) Consider the inner automorphism φ_1^{jk}. By the same way as Proposition 3.3 (2), the equivalences of first type are obtained. The non-equivalences for the other pairs are obtained by the above note. □

We next see the equivalences among families $\mathscr{B}_{i;jk}$ and the equivalences among the PSLA's which belong to each $\mathscr{B}_{i;jk}$.

For a family $\mathscr{B}_{i;jk}$ put $j=a$, $i=j+b$, $k=i+c$, $l+1=k+d$. Then $a, b, c, d\geq 1$ and the following holds.

Proposition 4.4. *Two families $\mathscr{B}_{i;jk}$, $\mathscr{B}_{i';j'k'}$ are equivalent to each other if and only if $d=d'$ and the triples (a, b, c), (a', b', c') coincide except order.*

Proof. Consider the PSLA's $(\mathfrak{g}, \theta_i, \theta_{jk})$, $(\mathfrak{g}, \theta_{i'}, \theta_{j'k'})$. Then it follows that $\mathfrak{k}_+=\mathfrak{so}(2a)\oplus\mathfrak{so}(2b)\oplus\mathfrak{so}(2c)\oplus\mathfrak{so}(2d-1)$ and $\mathfrak{l}'_+=\mathfrak{so}(2a')\oplus\mathfrak{so}(2b')\oplus\mathfrak{so}(2c')\oplus\mathfrak{so}(2d'-1)$.

Suppose that $\mathscr{B}_{i;jk}$ are equivalent to $\mathscr{B}_{i';j'k'}$. Since $\mathfrak{k}_+$ is isomorphic to $\mathfrak{k}'_+$, it follows that $d=d'$ and triples $((a, b, c), (a', b', c')$ coincide except order.

To prove the converse we may show the following equivalences:

$$(1)\ \mathscr{B}_{i;jk} \cong \mathscr{B}_{i;bk} \quad \text{and} \quad (2)\ \mathscr{B}_{i;jk} \cong \mathscr{B}_{k-j;k-i,k}\,.$$

Similarly to Proposition 3.4, these equivalences are obtained by the inner automorphisms φ_1^{ab}, φ_0^k, respectively. $\square$

By virtue of this proposition we may consider only the families $\mathscr{B}_{i;jk}$ with quadruple (a, b, c, d) such that $a \leq b \leq c$. Such a family is said to be a *proper family of type BII* and a family without the above condition is said to be simply a family of type *BII*.

Proposition 4.5. *Let $\mathscr{B}_{i;jk}$ be a proper family of type BII with quadruple (a, b, c, d) and set $(\mathfrak{g}, \sigma, \tau)=(\mathfrak{g}, \theta_i, \theta_{jk})$. Then the following hold:*

(1) *If $a<b<c$, all the PSLA's in $\mathscr{B}_{i;jk}$ are non-equivalent to each other;*

(2) *If $a=b<c$, only the equivalences of first type hold;*

(3) *If $a<b=c$, only the equivalences of second type hold;*

(4) *If $a=b=c=d$, all the PSLA's in $\mathscr{B}_{i;jk}$ are equivalent to each other.*

Proof. We note that

$$\mathfrak{k}_- = (\mathfrak{so}(2a+2b)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2b))\oplus(\mathfrak{so}(2c+2d-1)/\mathfrak{so}(2c)\oplus\mathfrak{so}(2d-1))\,,$$
$$\mathfrak{p}_+ = (\mathfrak{so}(2b+2c)/\mathfrak{so}(2b)\oplus\mathfrak{so}(2c))\oplus(\mathfrak{so}(2a+2d-1)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2d-1))\,,$$
$$\mathfrak{p}_- = (\mathfrak{so}(2a+2c)/\mathfrak{so}(2a)\oplus\mathfrak{so}(2c))\oplus(\mathfrak{so}(2b+2d-1)/\mathfrak{so}(2b)\oplus\mathfrak{so}(2d-1))\,.$$

(1) In this case $\mathfrak{k}_-, \mathfrak{p}_+, \mathfrak{p}_-$ are not isomorphic to each other. Hence our claim is obvious.

(2) Similarly to Proposition 3.5 (2), the equivalences of first type are obtained by the inner automorphism φ_1^{ab}. Also by the above note, $\mathfrak{p}_\pm$ are not isomorphic to $\mathfrak{k}_-$. This implies the non-equivalences of the other pairs.

(3) Similarly to Proposition 3.5 (3), the equivalences of second type are obtained by the inner automorphism $(\varphi_1^{ab})^{-1}\varphi_0^k\varphi_1^{ab}$. Also by the above note, $\mathfrak{k}_-, \mathfrak{p}_-$ are not isomorphic to $\mathfrak{p}_+$. This implies the non-equivalences of the other pairs.

(4) Similarly to Proposition 3.4 (4), the equivalences of first and second types hold. Hence our claim is obvious. $\square$

We now see the injectivity of the $\mathfrak{k}_+$-homomorphism ρ for each PSLA.

Similarly to §3, fix a positive integer r and set

$$\boldsymbol{R}_1 = \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\ \overbrace{0\cdots0}^{c})\in\boldsymbol{Z}^r;\ a\geq0,\ b\geq0,\ c\geq0\}\ ,$$
$$\boldsymbol{R}_2 = \{\pm(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b}\ \overbrace{2\cdots2}^{c})\in\boldsymbol{Z}^r;\ a\geq0,\ b\geq0,\ c>0\}\ ,$$

$$R = R_1 \cup R_2\,, \quad R^2 = \{\begin{pmatrix}\alpha\\ \beta\end{pmatrix};\ \alpha, \beta \in R\}\ .$$

Moreover let $R^2[\binom{a}{b}_i]$, $R^2[\binom{a}{b}_i, \binom{c}{d}_j]$, $R^2_\lambda[*]$ be subsets of R^2 defined as in §3. Then we can check the following lemma by a usual argument.

Lemma 4.6. *Let λ be an r-tuples in $\boldsymbol{Z}^r$. Then the following hold:*

(1) *The following each set has at most* 2 *elements:*

$R^2_\lambda[\binom{-1}{1}_r]$, $R^2_\lambda[\binom{-1}{1}_1, \binom{0}{0}_r]$, $R^2_\lambda[\binom{-1}{1}_1, \binom{-2}{2}_r]$, $R^2_\lambda[\binom{0}{1}_1, \binom{-1}{2}_r]$, $R^2_\lambda[\binom{0}{1}_1, \binom{1}{0}_r]$, $R^2_\lambda[\binom{1}{1}_1, \binom{0}{2}_r]$;

(2) *For the sets $R^2_\lambda[\binom{1}{1}_r]$ and $R^2_\lambda[\binom{0}{1}_1, \binom{-1}{0}_r]$ Lemma* 3.6 (2) *and* (3) *hold respectively;*

(3) *The set $R^2_\lambda[\binom{1}{1}_1, \binom{0}{0}_r]$ (resp. $R^2_\lambda[\binom{1}{1}_1, \binom{2}{2}_r]$) has at most* 1 *element if $\lambda \neq (0\cdots0)$, and has just $r-1$ elements with form*

$$\begin{pmatrix}1\cdots1 & 0\cdots0\\ 1\cdots1 & 0\cdots0\end{pmatrix}, \quad \left(\textit{resp.}\ \begin{pmatrix}1\cdots1 & 2\cdots2\\ 1\cdots1 & 2\cdots2\end{pmatrix}\right)$$

if $\lambda=(0\cdots0)$;

(4) *The set $R^2_\lambda[\binom{-1}{1}_1, \binom{-2}{0}_r]$ has at most* 1 *element if $\lambda \neq (2\cdots2)$, and has just $r-1$ elements with form*

$$\begin{pmatrix}-1\cdots-1 & -2\cdots-2\\ 1\ \cdots\ 1 & 0\ \cdots\ 0\end{pmatrix}$$

if $\lambda=(2\cdots2)$;

(5) *The set $R^2_\lambda[\binom{1}{2}_r]$ has at most* 1 *element if $\lambda \neq (\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b})$ $(a \geq 0, b > 0)$, and has just r elements with forms*

$$\begin{pmatrix}\overbrace{0\cdots0}^{a} & \overbrace{0\cdots0\ 1\cdots1}^{b}\\ 0\cdots0 & 1\cdots1\ 2\cdots2\end{pmatrix}, \begin{pmatrix}\overbrace{0\cdots0\ 1\cdots1}^{a} & \overbrace{1\cdots1}^{b}\\ 0\cdots0\ 1\cdots1 & 2\cdots2\end{pmatrix}$$

if $\lambda=(\overbrace{0\cdots0}^{a}\ \overbrace{1\cdots1}^{b})$;

(6) *The set $R^2_\lambda[\binom{1}{1}_1]$ has at most* 2 *elements if $\lambda \neq (0\cdots0)$, and has just $2r-1$ elements with forms*

$$\begin{pmatrix}1\cdots1 & 0\cdots0\\ 1\cdots1 & 0\cdots0\end{pmatrix}, \begin{pmatrix}1\cdots1 & 2\cdots2\\ 1\cdots1 & 2\cdots2\end{pmatrix}$$

if $\lambda=(0\cdots0)$;

(7) *The set $R^2_\lambda[\binom{-1}{1}_1]$ has at most* 2 *elements if $\lambda \neq (2\cdots2)$, and has just $2r-1$ elements with forms*

$$\begin{pmatrix}-1\cdots-1 & 0\cdots0\\ 1\ \cdots\ 1 & 2\cdots2\end{pmatrix}, \begin{pmatrix}-1\cdots-1 & -2\cdots-2\\ 1\ \cdots\ 1 & 0\ \cdots\ 0\end{pmatrix}$$

if $\lambda=(2\cdots 2)$.

In the following we represent a root of type B_l by a linear combination of the fundamental root system Π and identify it with an l-tuple of coefficients.

Case BI: The families $\mathscr{B}_{ij}$ with triple (j, k, r)

Put $\sigma=\theta_i$ and $\tau=\theta_j$. Then, for each PSLA in $\mathscr{B}_{ij}$, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given as follows: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=SO(2l+1)/S(O(2j+2k)\times O(2r-1))$.
In this case $N=\mathfrak{so}(2j+2r-1)/\mathfrak{s}(\mathfrak{o}(2j)\oplus\mathfrak{o}(2r-1))$;
(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=SU(2l+1)/S(O(2j+2k)\times O(2r-1))$.
In this case $N=\mathfrak{so}(2k+2r-1)/\mathfrak{s}(\mathfrak{o}(2k)\oplus\mathfrak{o}(2r-1))$;
(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=SO(2l+1)/S(O(2j)\times O(2k+2r-1))$.
In this case $N=\mathfrak{so}(2j+2r-1)/\mathfrak{s}(\mathfrak{o}(2j)\oplus\mathfrak{o}(2r-1))$:
(d) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma\tau)$: $M=SO(2l+1)/S(O(2j)\times O(2k+2r-1))$.
In this case $N=\mathfrak{so}(2j+2k)/\mathfrak{s}(\mathfrak{o}(2j)\oplus\mathfrak{o}(2k))$;
(e) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \sigma)$: $M=SO(2l+1)/S(O(2k)\times O(2j+2r-1))$.
In this case $N=\mathfrak{so}(2k+2r-1)/\mathfrak{s}(\mathfrak{o}(2k)\oplus\mathfrak{o}(2r-1))$;
(f) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \tau)$: $M=SO(2l+1)/S(O(2k)\times O(2j+2r-1))$.
In this case $N=\mathfrak{so}(2j+2k)/\mathfrak{s}(\mathfrak{o}(2j)\oplus\mathfrak{o}(2k))$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta^+_{\mathfrak{k}_+}$, $\Delta^+_{\mathfrak{k}_-}$, $\Delta^+_{\mathfrak{p}_+}$, $\Delta^+_{\mathfrak{p}_-}$ of Δ^+ are given as follows:

(4.1)
$$\begin{aligned}
\Delta^+_{\mathfrak{k}_+} &= \{\delta\in\Delta^+;\ \delta_i=\delta_j=0, 2\}\\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots 01\cdots 10\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 0)\\
(0\cdots \overset{j}{0}\cdots 01\cdots 10\cdots \overset{i}{0}\cdots 0)\\
(0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots 10\cdots 0)\\
(0\cdots 01\cdots 12\cdots \overset{j}{2}\cdots \overset{i}{2}\cdots 2)\\
(0\cdots \overset{j}{0}\cdots 01\cdots 12\cdots \overset{i}{2}\cdots 2)\\
(0\cdots \overset{j}{0}\cdots \overset{i}{0}\cdots 01\cdots 12\cdots 2)
\end{matrix}\right\},
\end{aligned}$$

$$\begin{aligned}
\Delta^+_{\mathfrak{k}_-} &= \{\delta\in\Delta^+;\ \delta_i=0, 2,\ \delta_j=1\}\\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots 01\cdots \overset{j}{1}\cdots 10\cdots \overset{i}{0}\cdots 0)\\
(0\cdots 01\cdots \overset{j}{1}\cdots 12\cdots \overset{i}{2}\cdots 2)
\end{matrix}\right\},
\end{aligned}$$

$$\begin{aligned}
\Delta^+_{\mathfrak{p}_+} &= \{\delta\in\Delta^+;\ \delta_i=1,\ \delta_j=0, 2\}\\
&= \left\{\delta\in\Delta^+;\ \delta=\begin{matrix}
(0\cdots \overset{j}{0}\cdots 01\cdots \overset{i}{1}\cdots 10\cdots 0)\\
(0\cdots \overset{j}{0}\cdots 01\cdots \overset{i}{1}\cdots 12\cdots 2)
\end{matrix}\right\},
\end{aligned}$$

$$\Delta^+_{\mathfrak{p}_-} = \{\delta\in\Delta^+;\ \delta_i=\delta_j=1\}$$

$$=\left\{\delta\in\Delta^+;\ \delta=\begin{matrix}(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots10\cdots0)\\(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots12\cdots2)\end{matrix}\right\}.$$

If $l\geq3$, the dominant weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are given by (4.2), (4.3), (4.4), respectively:

$$\left\{\begin{matrix}(1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots2)\,, & (1\cdots\overset{j}{1}\overset{i}{0}\cdots0)\quad(i=j+1)\,,\\ -(\overset{j}{1}0\cdots\overset{i}{0}\cdots0)\quad(j=1)\,, & -(\overset{j}{1}\overset{i}{2}\cdots2)\quad(j=1,i=2)\,.\end{matrix}\right.\tag{4.2}$$

$$(0\cdots\overset{j}{0}1\cdots\overset{i}{1}2\cdots\cdots2)\,,\qquad-(0\cdots\overset{j}{0}\overset{i}{1}0\cdots0)\quad(i=j+1)\,.\tag{4.3}$$

$$(1\cdots1\cdots1\overset{j}{2}\cdots\overset{i}{2})\,,\qquad-(\overset{j}{1}\cdots\overset{i}{1}0\cdots0)\quad(j=1)\,.\tag{4.4}$$

If $l=2$, the subset $\Delta_{\mathfrak{k}_+}$ is empty and so the weights in $\Delta_{\mathfrak{k}_-}$, $\Delta_{\mathfrak{p}_+}$, $\Delta_{\mathfrak{p}_-}$ are all dominant.

We now see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g},\sigma,\tau)$. In this case ρ is a homomorphism of $(\mathfrak{p}_-^{\boldsymbol{C}})^*\otimes\mathfrak{k}_-^{\boldsymbol{C}}$ to $\wedge^2(\mathfrak{p}_-^{\boldsymbol{C}})^*\otimes\mathfrak{p}_+^{\boldsymbol{C}}$.

We first suppose that $l\geq3$. Then the minus multiple of dominant weights in $\Delta_{\mathfrak{p}_-}$ are given by $(\alpha1)$, $(\alpha2)$ and the dominant weights in $\Delta_{\mathfrak{k}_-}$ are given by $(\beta1)\sim(\beta4)$:

$$\begin{array}{llll}(\alpha1) & -(1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots2)\,, & (\alpha2) & (\overset{j}{1}\cdots\overset{i}{1}0\cdots0)\quad(j=1)\,,\\ (\beta1) & (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots2)\,, & (\beta2) & (1\cdots\overset{j}{1}\overset{i}{0}\cdots0)\quad(i=j+1)\,,\\ (\beta3) & -(\overset{j}{1}0\cdots\overset{i}{0}\cdots0)\quad(j=1)\,, & (\beta4) & -(\overset{j}{1}\overset{i}{2}\cdots2)\quad(j=1,i=2)\,.\end{array}$$

Case (1): $l(u)=1$, Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α,β) is one of the pairs $((\alpha\,s),(\beta\,t))$, where $s=1,2$ and $r=1,2,3,4$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)=0$ for pairs $((\alpha1),(\beta1))$ $(i=l)$, $((\alpha1),(\beta2))$ $(i=l,\ i=j+1)$, $((\alpha2),(\beta3))$ $(i=l,j=1)$ and $\rho(u)\neq0$ for the other cases.

Case (2): $l(u)=2$. We first suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}_+}$:

$$(\mu1)\quad(10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots0)\quad(j\geq2)\,,\qquad(\mu2)\quad(1\overset{j}{2}\cdots\overset{i}{2}\cdots2)\quad(j=2)\,.$$

Then such the triples $(\alpha,\beta';\mu)$ as in §3 (Case (2) of type AI) are given in the following;

(1) $((\alpha1),(\beta1);(\mu1)),\ j\geq2$, (2) $((\alpha1),(\beta1);(\mu2)),\ j=2$,
(3) $((\alpha1),(\beta2);(\mu1)),\ i=j+1,\ j\geq2$, (4) $((\alpha1),(\beta2);(\mu2)),\ i=j+1,\ j=2$.

Lemma 2.4 is available for all cases and so it follows that $\rho(u)\neq0$.

We next suppose that u is decomposable. Put $u=a\,\omega_{\alpha_1}\otimes X_{\beta_1}+b\,\omega_{\alpha_2}\otimes X_{\beta_2}$.

Then the following two cases are considerable:

(1) $\lambda=(\overset{j}{0}1\cdots\overset{i}{1}2\cdots 2)$, where $j=1$ and the pairs (α_i, β_i) are $((\alpha 1), (\beta 3))$, $((\alpha 2), (\beta 1))$;

(2) $\lambda=-(\overset{j}{0}\overset{i}{1}0\cdots 0)$, where $j=1$, $i=2$ and the pairs (α_i, β_i) are $((\alpha 1), (\beta 4))$, $((\alpha 2), (\beta 2))$.

In these cases the weights λ are roots and Lemma 2.2 is available except the above case (1) $(i=l)$. So it follows that $\rho(u)\neq 0$ except the excepitional case. By virtue of Case (1) we do not need to consider the exceptional case.

Case (3): $l(u)\geq 3$. We see the weight spaces with $\dim\geq 3$. Let λ be a weight in Λ and let α, β be weights such that $\lambda=-\alpha+\beta$, where $\alpha\in\Delta_{\mathfrak{p}_-}$ and $\beta\in\Delta_{\mathfrak{k}_+}$. Denote by a_k, b_k, $\lambda_k (1\leq k\leq l)$ the k-th components of α, β, λ, respectively. Since $a_j=\pm 1$ and $b_j=\pm 1$, it follows that $\lambda_j=0, \pm 2$.

If $\lambda_j=0$, it moreover follows by (4.1) that $\lambda_i=\pm 1$. We see only the case that $\lambda_i=1$. Because we can similarly see the case that $\lambda_i=-1$. By (4.1), the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has either of the forms $\begin{pmatrix} \overset{j}{-1}\cdots\overset{i}{-1} & \\ -1 \quad\; 0 & \cdots 0\end{pmatrix}$, $\begin{pmatrix} \overset{j}{1}\cdots\overset{i}{1} & \\ 1 \quad 2 & \cdots 2\end{pmatrix}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 4.6 that $\lambda=(0\cdots\overset{j}{0}\cdots 01\cdots\overset{i}{1}\cdots 10\cdots 0)$ or $(0\cdots\overset{j}{0}\cdots 01\cdots\overset{i}{1}\cdots 12\cdots 2)$, and for the former (resp. the latter) the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has either of the forms

$$(4.5)\qquad \begin{pmatrix} 0\cdots 0-1\cdots\overset{j}{-1} & \cdots\cdots & \overset{i}{-1}\cdots -10\cdots 0\\ 0\cdots 0-1\cdots -1\cdots -10\cdots & 0 & \cdots\cdots 0\end{pmatrix},$$

$$\begin{pmatrix} 0\cdots 01\cdots\overset{j}{1} & \cdots\cdots & \overset{i}{1}\cdots 12\cdots 2\\ 0\cdots 01\cdots 1\cdots 12\cdots & 2 & \cdots\cdots 2\end{pmatrix}$$

$$\left(\text{resp.}\ \begin{pmatrix} 0\cdots 0-1\cdots\overset{j}{-1} & \cdots\cdots & \overset{i}{-1}\cdots -1-2\cdots -2\\ 0\cdots 0-1\cdots -1\cdots -10\cdots & 0 & \cdots\cdots\cdots 0\end{pmatrix},\ \begin{pmatrix} 0\cdots 01\cdots\overset{j}{1} & \cdots\cdots & \overset{i}{1}\cdots 10\cdots 0\\ 0\cdots 01\cdots 1\cdots 12\cdots & 2 & \cdots\cdots 2\end{pmatrix}\right).$$

Suppose that u belongs to this weight space. If $i\neq l$, it follows by Lemma 2.2 that $\rho(u)\neq 0$. If $i=l$, we can not apply Lemma 2.2 and Lemma 2.4. But, by virtue of Case (1), we do not need to consider this case.

Suppose that $\lambda_j=2$. (For the case that $\lambda_j=-2$ we can similarly do the argument mentioned below.) Then it moreover follows by (4.1) that $\lambda_i=1, 3$. For each case the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has the following form, respectively: $\begin{pmatrix} \overset{j}{-1}\cdots\overset{i}{-1} & \\ 1 \quad\; 2 & \cdots 2\end{pmatrix}$, $\begin{pmatrix} \overset{j}{-1}\cdots\overset{i}{-1} & \\ 1 \quad\; 0 & \cdots 0\end{pmatrix}$. By Lemma 4.6, the weight spaces with these λ have at most dimension 2.

We next suppose that $l=2$. In this case it holds that

$$\Delta_{\mathfrak{k}_-}^+ = \{(10), (12)\}\ ,\quad \Delta_{\mathfrak{p}_-}^+ = \{(11)\}\ .$$

So a weight λ in Λ is one of $\pm(01)$, $\pm(21)$, $\pm(23)$, and the weight spaces for $\lambda=\pm(01)$ have dimension 2 and the weight spaces for the other λ have dimension 1. Lemma 2.3 is available for the cases with dimension 1 and it consequently follows that $\rho(u)=0$ for these cases. By virtue of the cases with dimension 1, we do not need to consider other cases.

Summing up the above arguments, we have the following result for the PSLA of Case (a); the homomorphism ρ is not injective if and only if $i=l$, i.e., $r=1$. For the other cases we have similar results; ρ is not injective only for Cases (b), $i=l$. These cases imply the cases of Example 1 (m: even and r: even) in §1.

Theorem 4.7. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type BI. Then the $\mathcal{V}$-geometry admits non-totally geodesic $\mathcal{V}$-submanifolds if and only if it is one of the $\mathcal{V}$-geometries in Example* 1 (*m: even and r: even*).

Case BII: The families $\mathscr{B}_{i;jk}$ with quadruple (a, b, c, d)

Put $\sigma=\theta_i$ and $\tau=\theta_{jk}$. Then, for each PSLA in $\mathscr{B}_{i;jk}$, the corresponding symmetric space M and the totally geodesic $\mathcal{V}$-submanifold N are given in the following: (N is locally described.)

(a) $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$: $M=SO(2l+1)/S(O(2a+2b)\times O(2c+2d-1))$.
In this case $N=(\mathfrak{so}(2a+2c)/\mathfrak{s}(\mathfrak{o}(2a)\oplus\mathfrak{o}(2c)))\oplus(\mathfrak{so}(2b+2d-1)/\mathfrak{s}(\mathfrak{o}(2b)\oplus\mathfrak{o}(2d-1)))$;

(b) $\mathcal{V}=(\mathfrak{g}, \sigma, \sigma\tau)$: $M=SO(2l+1)/S(O(2a+2b)\times O(2c+2d-1))$.
In this case $N=(\mathfrak{so}(2b+2c)/\mathfrak{s}(\mathfrak{o}(2b)\oplus\mathfrak{o}(2c)))\oplus(\mathfrak{so}((2a+2d-1)/\mathfrak{s}(\mathfrak{o}(2a)\oplus\mathfrak{o}(2d-1)))$;

(c) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma)$: $M=SO(2l+1)/S(O(2b+2c)\times O(2a+2d-1))$.
In this case $N=(\mathfrak{so}(2a+2c)/\mathfrak{s}(\mathfrak{o}(2a)\oplus\mathfrak{o}(2c)))\oplus(\mathfrak{so}(2b+2d-1)/\mathfrak{s}(\mathfrak{o}(2b)\oplus\mathfrak{o}(2d-1)))$;

(d) $\mathcal{V}=(\mathfrak{g}, \tau, \sigma\tau)$: $M=SO(2l+1)/S(O(2b+2c)\times O(2a+2d-1))$.
In this case $N=(\mathfrak{so}(2a+2b)/\mathfrak{s}(\mathfrak{o}(2a)\oplus\mathfrak{o}(2b)))\oplus(\mathfrak{so}(2c+2d-1)/\mathfrak{s}(\mathfrak{o}(2c)\oplus\mathfrak{o}(2d-1)))$;

(e) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \sigma)$: $M=SO(2l+1)/S(O(2a+2c)\times O(2b+2d-1))$.
In this case $N=(\mathfrak{so}(2b+2c)/\mathfrak{s}(\mathfrak{o}(2b)\oplus\mathfrak{o}(2c)))\oplus(\mathfrak{so}(2a+2d-1)/\mathfrak{s}(\mathfrak{o}(2a)\oplus\mathfrak{o}(2d-1)))$;

(f) $\mathcal{V}=(\mathfrak{g}, \sigma\tau, \tau)$: $M=SO(2l+1)/S(O(2a+2c)\times O(2b+2d-1))$.
In this case $N=(\mathfrak{so}(2a+2b)/\mathfrak{s}(\mathfrak{o}(2a)\oplus\mathfrak{o}(2b)))\oplus(\mathfrak{so}(2c+2d-1)/\mathfrak{s}(\mathfrak{o}(2c)\oplus\mathfrak{o}(2d-1)))$.

For the PSLA $(\mathfrak{g}, \sigma, \tau)$, the subsets $\Delta_{\mathfrak{k}_\pm}^+$, $\Delta_{\mathfrak{p}_\pm}^+$ of Δ^+ are given as follows:

$$
(4.6)\quad \Delta^+_{\mathfrak{k}_+} = \{\delta\in\Delta^+;\ \delta_i = 0, 2,\ (\delta_j, \delta_k) = (0, 0)\,(0, 2), (2, 0), (1, 1), (2, 2)\}
$$

$$
= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix}
(0\cdots01\cdots10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)\\
(0\cdots\overset{j}{0}\cdots01\cdots10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots10\cdots\overset{k}{0}\cdots0)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots10\cdots0)\\
(0\cdots01\cdots12\cdots\overset{j}{2}\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2)\\
(0\cdots\overset{j}{0}\cdots01\cdots12\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots12\cdots\overset{k}{2}\cdots2)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots01\cdots12\cdots2)
\end{matrix} \right\}
$$

$$
\Delta^+_{\mathfrak{k}_-} = \{\delta\in\Delta^+;\ \delta_i = 0, 2,\ (\delta_j, \delta_k) = (0, 1), (1, 0), (1, 2), (2, 1)\}
$$

$$
= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix}
(0\cdots01\cdots\overset{j}{1}\cdots10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots10\cdots0)\\
(0\cdots01\cdots\overset{j}{1}\cdots12\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2)\\
(0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots01\cdots\overset{k}{1}\cdots12\cdots2)
\end{matrix} \right\},
$$

$$
\Delta^+_{\mathfrak{p}_+} = \{\delta\in\Delta^+;\ \delta_i = 1,\ (\delta_j, \delta_k) = (0, 0), (0, 2), (2, 0), (1, 1), (2,2)\}
$$

$$
= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix}
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots10\cdots\overset{k}{0}\cdots0)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots10\cdots0)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots12\cdots\overset{k}{2}\cdots2)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots12\cdots2)
\end{matrix} \right\},
$$

$$
\Delta^+_{\mathfrak{p}_-} = \{\delta\in\Delta^+;\ \delta_i = 1,\ (\delta_j, \delta_k) = (0, 1), (1, 0), (1, 2), (2, 1)\}
$$

$$
= \left\{ \delta\in\Delta^+;\ \delta = \begin{matrix}
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots10\cdots\overset{k}{0}\cdots0)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots10\cdots0)\\
(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots12\cdots\overset{k}{2}\cdots2)\\
(0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots12\cdots2)
\end{matrix} \right\}.
$$

If $l\geq 4$, the dominant weights in $\Delta_{\mathfrak{k}_-}, \Delta_{\mathfrak{p}_+}, \Delta_{\mathfrak{p}_-}$ are given by (4.7), (4.8), (4.9), respectively:

$$
(4.7)\quad \begin{array}{ll}
(0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}2\cdots2), & (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2)\,,\\
(1\cdots\overset{j}{1}\overset{i}{0}\cdots\overset{k}{0}\cdots0)\quad (i=j+1), & -(0\cdots\overset{j}{0}\cdots\overset{i}{0}\overset{k}{1}0\cdots0)\quad (k=i+1)\,,\\
-(\overset{j}{1}0\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)\quad (j=1), & -(\overset{j}{1}\overset{i}{2}\cdots\overset{k}{2}\cdots2)\quad (j=1, i=2)\,.
\end{array}
$$

$$
(4.8)\quad \begin{array}{ll}
(0\cdots\overset{j}{0}1\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots2), & (1\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots2)\,,\\
-(0\cdots\overset{j}{0}\overset{i}{1}0\cdots\overset{k}{0}\cdots0)\ (i=j+1), & (0\cdots\overset{j}{0}1\cdots\overset{i}{1}\overset{k}{0}\cdots0)\quad (k=i+1)\,,\\
-(\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}0\cdots0)\quad (j=1), & -(0\cdots\overset{j}{0}\overset{i}{1}\overset{k}{2}\cdots2)\quad (i=j+1, k=i+1)\,.
\end{array}
$$

$$(4.9)\quad \begin{aligned} &(0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots2), && (1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots2)\,, \\ &-(0\cdots\overset{j}{0}\overset{i}{1}\cdots\overset{k}{1}0\cdots0)\ \ (i=j+1), && (1\cdots\overset{j}{1}\cdots\overset{i}{1}\overset{k}{0}\cdots0)\ \ (k=i+1)\,, \\ &-(\overset{j}{1}\cdots\overset{i}{1}0\cdots\overset{k}{0}\cdots0)\ \ (j=1), && -(\overset{j}{1}\cdots\overset{i}{1}\overset{k}{2}\cdots\cdots2)\ \ (j=1, k=i+1)\,. \end{aligned}$$

If $l=3$, the subset $\Delta_{\mathfrak{k}+}$ is empty and so the weights in $\Delta_{\mathfrak{k}-}, \Delta_{\mathfrak{p}+}, \Delta_{\mathfrak{p}-}$ are all dominant.

We now see the injectivity of ρ for Case (a): $\mathcal{V}=(\mathfrak{g}, \sigma, \tau)$. In this case ρ is a homomorphism of $(\mathfrak{p}_-^{\mathbf{C}})^*\otimes\mathfrak{k}_-^{\mathbf{C}}$ to $\wedge^2(\mathfrak{p}_-^{\mathbf{C}})^*\otimes\mathfrak{p}_+^{\mathbf{C}}$.

We first suppose that $l\geq4$. Then the minus multiple of dominant weights in $\Delta_{\mathfrak{p}-}$ are given by $(\alpha1)\sim(\alpha6)$ and the dominant weights in $\Delta_{\mathfrak{k}-}$ are given by $(\beta1)\sim(\beta6)$:

$$\begin{aligned} &(\alpha1)\ \ -(0\cdots\overset{j}{0}1\cdots\overset{i}{1}\cdots\overset{k}{1}2\cdots2), && (\alpha2)\ \ -(1\cdots\overset{j}{1}\cdots\overset{i}{1}2\cdots\overset{k}{2}\cdots2)\,, \\ &(\alpha3)\ \ (0\cdots\overset{j}{0}\overset{i}{1}\cdots\overset{k}{1}0\cdots0)\ (i=j+1), && (\alpha4)\ \ -(1\cdots\overset{j}{1}\cdots\overset{i}{1}\overset{k}{0}\cdots0)\ (k=i+1)\,, \\ &(\alpha5)\ \ (\overset{j}{1}\cdots\overset{i}{1}0\cdots\overset{k}{0}\cdots0)\ (j=1), && (\alpha6)\ \ (\overset{j}{1}\cdots\overset{i}{1}\overset{k}{2}\cdots2)\ (j=1, k=i+1)\,, \\ &(\beta1)\ \ (0\cdots\overset{j}{0}\cdots\overset{i}{0}1\cdots\overset{k}{1}2\cdots2), && (\beta2)\ \ (1\cdots\overset{j}{1}2\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2)\,, \\ &(\beta3)\ \ (1\cdots\overset{j}{1}\overset{i}{0}\cdots\overset{k}{0}\cdots0)\ (i=j+1)\,, && (\beta4)\ \ -(0\cdots\overset{j}{0}\cdots\overset{i}{0}\overset{k}{1}0\cdots0)\ (k=i+1)\,, \\ &(\beta5)\ \ -(\overset{j}{1}0\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)\ (j=1)\,, && (\beta6)\ \ -(\overset{j}{1}\overset{i}{2}\cdots\overset{k}{2}\cdots2)\ (j=1, i=2)\,. \end{aligned}$$

Case (1): $l(u)=1$. Represent u as follows: $u=a\,\omega_\alpha\otimes X_\beta$. Then the pair (α, β) is one of the pairs $((\alpha\, r), (\beta\, s))$, where $r, s=1, 2, 3, 4, 5, 6$. Applying Lemma 2.3 for each pair, we obtain that $\rho(u)\neq0$ for all the pairs.

Case (2): $l(u)=2$. We first suppose that u is indecomposable. Consider the following elements in $\Delta_{\mathfrak{k}+}$:

$$\begin{aligned} &(\mu1)\ \ (0\cdots\overset{j}{0}10\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)\,, && (\mu2)\ \ (0\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}10\cdots0)\,, \\ &(\mu3)\ \ (0\cdots\overset{j}{0}1\overset{i}{2}\cdots\overset{k}{2}\cdots2)\ (i=j+2)\,, && (\mu4)\ \ (10\cdots\overset{j}{0}\cdots\overset{i}{0}\cdots\overset{k}{0}\cdots0)\,, \\ &(\mu5)\ \ (0\cdots\overset{j}{0}\cdots\overset{i}{0}10\cdots\overset{k}{0}\cdots0)\,, && (\mu6)\ \ (1\overset{j}{2}\cdots\overset{i}{2}\cdots\overset{k}{2}\cdots2)\ (j=2)\,, \\ &(\mu7)\ \ (0\cdots\overset{j}{0}\cdots\overset{i}{0}1\overset{k}{2}\cdots2)\ (k=i+2)\,. \end{aligned}$$

Then such the triples $(\alpha, \beta'; \mu)$ as in §3 (Case (2) of type AI) are given in the following:

(1) $((\alpha1), (\beta2); (\mu1)), i-j\geq2$, (2) $((\alpha1), (\beta5); (\mu1)), j=1, i-j\geq2$,
(3) $((\alpha1), (\beta1); (\mu2)), k\neq l$, (4) $((\alpha1), (\beta4); (\mu2)), k\neq l, k=i+1$,
(5) $((\alpha1), (\beta2); (\mu3)).\ i-j=2$, (6) $((\alpha1), (\beta5); (\mu3)), j=1, i-j=2$,
(7) $((\alpha2), (\beta2); (\mu4)), j\geq2$, (8) $((\alpha2), (\beta3); (\mu4)), j\geq2, i=j+1$,
(9) $((\alpha2), (\beta1); (\mu5)), k-i\geq2$, (10) $((\alpha2), (\beta2); (\mu6)), j=2$,
(11) $((\alpha2), (\beta3); (\mu6)), j=2, i=j+1$,
(12) $((\alpha2), (\beta1); (\mu7)), k-i=2$,
(13) $((\alpha3), (\beta1); (\mu2)), i=j+1, k\neq l$,
(14) $((\alpha3), (\beta4); (\mu2)), i=j+1, k=i+1, k\neq l$,

(15) $((\alpha 4), (\beta 2); (\mu 4)), k=i+1, j\geq 2$,
(16) $((\alpha 4), (\beta 3); (\mu 4)), j\geq 2, i=j+1, k=i+1$,
(17) $((\alpha 4), (\beta 2); (\mu 6)), j=2, k=i+1$,
(18) $((\alpha 4), (\beta 3); (\mu 6)), j=2, i=j+1, k=i+1$,
(19) $((\alpha 5), (\beta 1); (\mu 5)), k-i\geq 2, j=1$,
(20) $((\alpha 5), (\beta 1); (\mu 7)), k-i=2, j=1$.

Lemma 2.4 is available for all cases and thus it follows that $\rho(u)\neq 0$.

We next suppose that u is decomposable. Put $u=a\,\omega_{\alpha_1}\otimes X_{\beta_1}+b\,\omega_{\alpha_2}\otimes X_{\beta_2}$. Then the weight λ is a root and Lemma 2.2 is available except the following cases (1), (2):

(1) $\lambda=(1\cdots\overset{j}{1}\overset{i}{1}\cdots\overset{k}{1})$, where $i=j+1, k=l$ and the pairs (α_i, β_i) are $((\alpha 1), (\beta 3)), (\alpha 3), (\beta 2))$;

(2) $\lambda=-(\overset{j}{1}\overset{i}{1}\cdots\overset{k}{1})$, where $j=1, i=2, k=l$ and the pairs (α_i, β_i) are $((\alpha 1), (\beta 6)), ((\alpha 3), (\beta 5))$.

For the exceptional cases the condition (1) of Proposition 2.1 does not hold. Hence it follows that $\rho(u)\neq 0$.

Case (3): $l(u)\geq 3$. We see the weight spaces with $\dim\geq 3$. Let λ be a weight in Λ and let α, β be roots such that $\lambda=-\alpha+\beta$, where $\alpha\in\Delta_{\mathfrak{p}_-}$ and $\beta\in\Delta_{\mathfrak{k}_-}$. Denote by $a_k, b_k, \lambda_k (1\leq k\leq l)$ the k-th components of α, β, λ, respectively. Since $a_i=\pm 1$ and $b_i=0, \pm 2$, it follows by (4.1) that $\lambda_i=\pm 1, \pm 3$. Consider only the cases that $\lambda_i=1, 3$. (For the cases that $\lambda_i=-1, -3$, we can similarly do the argument mentioned below.)

Suppose that $\lambda_i=3$. Then it moreover follows by (4.1) that $\lambda_j=1, 2$. For each case the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has the following form, respectively:

$$\begin{array}{c}\begin{array}{ccccccc} & j & & i & & k & \end{array}\\ \begin{pmatrix} 0\cdots & 0 & & -1 & \cdots & -1 & \\ & 1 & & 2 & \cdots & 2 & \cdots 2\end{pmatrix}\end{array},\quad \begin{array}{c}\begin{array}{cccccc} j & & i & & k & \end{array}\\ \begin{pmatrix} -1 & \cdots & -1 & & * & \\ 1 & & 2 & \cdots & 2 & \cdots 2\end{pmatrix}\end{array}.$$

By Lemma 4.6, the weight space with this λ has at most dimension 2.

Suppose that $\lambda_i=1$. Then it moreover follows by (4.1) that $\lambda_j=0, \pm 1, 2$. If $\lambda_j=-1$ (resp. $\lambda_j=2$), the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has the following form:

$$\begin{array}{c}\begin{array}{ccccccc} & j & & i & & k & \end{array}\\ \begin{pmatrix} 0\cdots & 0 & & -1 & \cdots & -1 & \\ & -1 & & 0 & \cdots & 0 & \cdots 0\end{pmatrix}\end{array}\ \left(\text{resp. } \begin{array}{c}\begin{array}{cccccc} j & & i & & k & \end{array}\\ \begin{pmatrix} -1 & \cdots & -1 & & * & \\ 1 & & 0 & \cdots & 0 & \cdots 0\end{pmatrix}\end{array}\right).$$

By Lemma 4.6, the weight space with this λ has at most dimension 2.

Consider the case that $\lambda_j=1$. Then it moreover follows by (4.1) that $\lambda_k=\pm 1,3$. If $\lambda_k=-1$ (resp. $\lambda_k=3$), the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has the form

$$\begin{array}{c}\begin{array}{ccccccc} & j & & i & k & & \end{array}\\ \begin{pmatrix} & -1 & \cdots & -1 & 0 & \cdots 0 \\ 0\cdots & 0 & \cdots & 0 & -1 & \end{pmatrix}\end{array}\ \left(\text{resp. } \begin{array}{c}\begin{array}{cccccc} & j & & i & k & \end{array}\\ \begin{pmatrix} & -1 & \cdots & -1 & -2 & \cdots -2 \\ 0\cdots & 0 & \cdots & 0 & 1 & \end{pmatrix}\end{array}\right).$$

By Lemma 4.6, the weight space with this λ has at most dimension 2. If $\lambda_k=1$, the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

$\begin{pmatrix} 0\cdots\overset{j}{0} & \overset{i}{-1}\cdots\overset{k}{-1} & \\ 1 & 0\cdots & 0\cdots0 \end{pmatrix}$, $\begin{pmatrix} 0\cdots\overset{j}{0} & \overset{i}{1}\cdots\overset{k}{1} & \\ 1 & 2\cdots2\cdots2 & \end{pmatrix}$, $\begin{pmatrix} & \overset{j}{-1}\cdots\overset{i}{-1} & \overset{k}{0}\cdots0 \\ 0\cdots & 0\cdots\ 0 & 1 \end{pmatrix}$,

$\begin{pmatrix} & \overset{j}{-1}\cdots\overset{i}{-1} & \overset{k}{-2}\cdots-2 \\ 0\cdots & 0\cdots\ 0 & -1 \end{pmatrix}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 4.6 that

$$\lambda = (0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots10\cdots0)\,,$$
$$(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots12\cdots2)\,.$$

For the former λ the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

(4.10)
$$\begin{pmatrix} 0\cdots\cdots\cdots\overset{j}{0}\cdots0 & -1\cdots\overset{i}{-1}\cdots\overset{k}{-1}\cdots-10\cdots0 \\ 0\cdots01\cdots1\cdots1 & 0\cdots\ \ 0\cdots\ \ 0\cdots\cdots\cdots0 \end{pmatrix},$$
$$\begin{pmatrix} 0\cdots\cdots\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots12\cdots2 \\ 0\cdots01\cdots1\cdots12\cdots2\cdots2\cdots\cdots\cdots2 \end{pmatrix},$$
$$\begin{pmatrix} 0\cdots0-1\cdots\overset{j}{-1}\cdots\overset{i}{-1}\cdots-1 & 0\cdots\overset{k}{0}\ \cdots\ 0 \\ 0\ \cdots\ 0\cdots\ 0\cdots\ 0 & 1\cdots1\cdots10\cdots0 \end{pmatrix},$$
$$\begin{pmatrix} 0\cdots0-1\cdots\overset{j}{-1}\cdots\overset{i}{-1}\cdots-1 & -2\cdots\overset{k}{-2}\ \cdots\ -2 \\ 0\ \cdots\ 0\cdots\ 0\cdots\ 0 & -1\cdots-1\cdots-1-2\cdots-2 \end{pmatrix}.$$

(For the latter λ we can similarly see the form of $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$.) Hence it follows by Lemma 2.2 and Proposition 2.1 (1) that $\rho(u)\neq0$ for a maximal vector u in the weight space with weight λ. Proposition 2.1 (1) is applied to the case that $\lambda=(0\cdots01\cdots\overset{j}{1}\cdots\overset{i}{1}\cdots\overset{k}{1}\cdots1)$.

Consider the case that $\lambda_j=0$. Then it moreover follows by (4.1) that $\lambda_k=0, 2$. If $\lambda_k=0$, the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

$\begin{pmatrix} 0\cdots\overset{j}{0} & \overset{i}{-1}\cdots\overset{k}{-1} \\ 0\cdots0\cdots & 0\quad -1 \end{pmatrix}$, $\begin{pmatrix} \overset{j}{-1}\cdots\overset{i}{-1} & \overset{k}{0}\cdots0 \\ -1 & 0\cdots0\cdots0 \end{pmatrix}$, $\begin{pmatrix} \overset{j}{1}\cdots\overset{i}{1} & \overset{k}{2}\cdots2 \\ 1 & 2\cdots2\cdots2 \end{pmatrix}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 4.6 that

$$\lambda = (0\cdots\overset{j}{0}\cdots01\cdots\overset{i}{1}\cdots10\cdots\overset{k}{0}\cdots0)$$

and the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

(4.11)
$$\begin{pmatrix} 0\cdots\overset{j}{0}\cdots0-1\cdots\overset{i}{-1}\ \cdots\cdots\cdots\ \overset{k}{-1}\cdots-1\ 0\cdots0 \\ 0\cdots0\ \cdots\cdots\cdots\ 0\cdots0-1\cdots-1\cdots-1\ 0\cdots0 \end{pmatrix},$$

$$\begin{array}{c} \qquad\qquad j \qquad\qquad\quad i \qquad\qquad\qquad k \qquad\qquad\qquad\qquad\qquad \\ \begin{pmatrix} 0\cdots 0\cdots 0-1\cdots -1 \ \cdots\cdots\ -1\cdots -1\ -2\cdots -2 \\ 0\cdots 0\ \cdots\cdots\ 0\cdots 0-1\cdots -1\cdots -1\ -2\cdots -2 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \qquad\qquad\qquad\quad j \qquad\qquad\qquad i \qquad\qquad\quad k \qquad\quad \\ \begin{pmatrix} 0\cdots 0\ -1\cdots -1\ \cdots\cdots\ -1\cdots -10\cdots 0\cdots 0 \\ 0\cdots 0\ -1\cdots -1\cdots -10\cdots\ 0\ \cdots\cdots\ 0\cdots 0 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \qquad\qquad\quad j \qquad\qquad i \qquad\quad k \qquad \\ \begin{pmatrix} 0\cdots 0\ 1\cdots 1\cdots\cdots 1\cdots 12\cdots 2\cdots 2 \\ 0\ldots 0\ 1\cdots 1\cdots 12\cdots 2\cdots\cdots 2\cdots 2 \end{pmatrix} \end{array}.$$

Hence it follows by Lemma 2.2 that $\rho(u)\neq 0$ for a maximal vector u in this weight space. If $\lambda_k=2$, the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the forms:

$\begin{array}{c} \quad j \quad\ \ i \quad\ \ k \\ \begin{pmatrix} 0\cdots 0 & -1\cdots -1 \\ 0\cdots 0\cdots\ 0 & 1 \end{pmatrix} \end{array}$, $\begin{array}{c} \quad j \quad\ \ i \quad\ \ k \\ \begin{pmatrix} -1\cdots -1 & -2\cdots -2 \\ -1\quad 0\cdots & 0\cdots\ 0 \end{pmatrix} \end{array}$, $\begin{array}{c} j\ \ i\ \ k \\ \begin{pmatrix} 1\cdots 1 & 0\cdots 0 \\ 1 & 2\cdots 2\cdots 2 \end{pmatrix} \end{array}$. If the weight space for λ has the dimension more than 3, it follows by Lemma 4.6 that

$$\lambda=(0\cdots\overset{j}{0}\cdots 01\cdots\overset{i}{1}\cdots 12\cdots\overset{k}{2}\cdots 2)$$

and the pair $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ has one of the following forms:

$$(4.12)\qquad \begin{array}{c} \qquad\qquad j \qquad\qquad\quad i \qquad\qquad\quad k \qquad\qquad\qquad\qquad\qquad \\ \begin{pmatrix} 0\cdots 0\cdots 0-1\cdots -1\cdots\cdots -1\cdots -1\ -2\cdots -2 \\ 0\cdots 0\ \cdots\cdots\ 0\cdots 01\cdots\ 1\cdots\ 1\ \ 0\cdots\ 0 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \qquad\qquad j \qquad\qquad\quad i \qquad\qquad\quad k \qquad\qquad\qquad\qquad \\ \begin{pmatrix} 0\cdots 0\cdots 0-1\cdots -1\cdots\cdots -1\cdots -1\ 0\cdots 0 \\ 0\cdots 0\ \cdots\cdots\ 0\cdots 01\cdots\ 1\cdots\ 1\ 2\cdots 2 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \qquad\qquad\qquad\quad j \qquad\qquad\qquad i \qquad\qquad\qquad k \qquad\qquad \\ \begin{pmatrix} 0\cdots 0\ -1\cdots -1\ \cdots\cdots\ -1\cdots -1-2\cdots -2\cdots -2 \\ 0\cdots 0\ -1\cdots -1\cdots -10\cdots\ 0\ \cdots\cdots\ 0\cdots\ 0 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \qquad\qquad\quad j \qquad\qquad i \qquad\quad k \qquad \\ \begin{pmatrix} 0\cdots 0\ 1\cdots 1\cdots\cdots 1\cdots 10\cdots 0\cdots 0 \\ 0\cdots 0\ 1\cdots 1\cdots 12\cdots 2\cdots\cdots 2\cdots 2 \end{pmatrix} \end{array}.$$

Hence it follows by Lemma 2.2 that $\rho(u)\neq 0$ for a maximal vector u in this weight space.

We next suppose that $l=3$. Then it holds that

$$\Delta_{\mathfrak{k}_-}^+=\{(100),(001),(122)\}\ ,\quad \Delta_{\mathfrak{p}_-}^+=\{(110),(011),(112)\}\ .$$

So a weight λ in Λ is one of the following: $\pm(010)$, $\pm(11-1)$, $\pm(-111)$, $\pm(210)$, $\pm(111)$, $\pm(012)$, $\pm(212)$, $\pm(113)$, $\pm(133)$, $\pm(232)$, $\pm(234)$. The weight spaces for $\lambda=\pm(111)$ have dimension 4 and the weight spaces for $\lambda=\pm(010)$, $\pm(012)$ have dimension 3 and the weight spaces for the other λ have dimension 1. Lemma 2.3 is available for the cases with dimension 1 and Lemma 2.2 is available for the cases with dimension 3 and the cases with dimension 4 except the following two; The exceptions are the cases that a maximal vector u has length 2 and is associated with two pairs

$$\pm\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

in R^2 according to $\lambda = \pm(111)$. For these cases Proposition 2.1 (1) does not hold. Hence it follows that $\rho(u) \neq 0$ for all cases.

Summing up the above arguments, we have the following result for the PSLA of Case (a); the homomorphism ρ is always injective. Similarly for the other cases ρ is always injective.

Theorme 4.8. *Let $\mathcal{V}$ be the G-orbit which corresponds to a PSLA in a family of type BII. Then the $\mathcal{V}$-geometry does not admit non-totally geodesic $\mathcal{V}$-submanifolds.*

References

[1] D.V. Alekseevskii: *Compact quaternion spaces*, Functional Anal, Appl. **2** (1968), 106–114.

[2] M. Berger: *Les espaces symétriques non compacts*, Ann. Sci. Ecole Norm. Sup. **74** (1957), 85–177.

[3] N. Bourbaki: Groupes et algèbres de Lie, Hermann, Paris, 1968.

[4] R. Harvey-H.B. Laswon: *Calibrated qeometries*, Acta Math. **148** (1982), 47–157.

[5] S. Helgason: Differental Geometry, Lie groups and Symmetric spaces, Academic Press, New York, 1978.

[6] J.E. Humphreys: Introduction to Lie Algebras and Representation Theory, Springer-Verlag, New York, 1972.

[7] S. Kobayashi-K. Nomizu: Foundations of differential geometry I, II, Wiley, New York, 1963, 1969.

[8] S. Murakami: *Sur la classification des algèbres de Lie réelles et simples*, Osaka J. Math. **2** (1965), 291–307.

[9] H. Naitoh: *Symmetric submanifolds of compact symmetric spaces*, Tsukuba J. Math. **10** (1986), 215–242.

[10] ————: *Symmetric submanifolds and generalized Gauss maps*, Tsukuba J. Math. **14** (1990), 113–132.

[11] ————: *Submanifolds of symmetric spaces and Gauss maps*, in "Progress in Differential Geometry, ed. by K. Shiohama", Adv. Stud. in Pure Math. **22** (1993), 197–211.

[12] H. Naitoh-M. Takeuchi: *Symmetric submanifolds of symmetric spaces*, Sugaku Exp. **2** (1989), 157–188.

[13] C. Yen: *Sur les espaces symétriques non compacts*, Scientia Sinica **14** (1965), 31–38.

Department of Mathematics
Yamaguchi University
Yamaguchi 753, Japan