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REFLECTION PRINCIPLES FOR TEMPERATURE FUNCTIONS

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1. Introduction

The Schwartz reflection principle deals with a question about analytic continuation of holomorphic functions across a hyperplane. The stronger version of the Schwarz reflection principle is, so called, the edge of wedge theorem. The problems of these kinds arose in physics, in connection with quantum field theory and dispersion relations (see [2] and [7]).

A similar consideration has been given also on harmonic functions (see [1] and [9]).

We consider here the reflection principle for temperature functions. Here, a temperature function in an open set Ω means an infinitely differentiable solution of the heat equation $(\partial_t - \Delta)u(x, t) = 0$ in Ω .

In fact, temperature functions have similar properties, such as the maximum principles, the Harnack type inequality, and so on, as holomorphic functions and harmonic functions. Thus, it is interesting to consider the reflection principle for temperature functions.

The reflection principle for temperature functions was considered in [10] for the first time as far as the author knows.

It states that every temperature function in the right-hand side of the vertical line (i.e. t -axis) in the x, t -plane, which vanishes on that line and is continuous up to boundary, can be extended as a temperature function through the line to the left-hand side. So it is desirable to weaken the assumption of continuity up to boundary. In fact, it will turn out in this paper that the same conclusion can be obtained if it is only assumed that a temperature function vanishes weakly on the boundary, that is to say, vanishes in the sense of distributions. This will be done in perfectly elementary languages, which is completely different method from those in [10], [2], [7], and so on.

Finally, as an application we give a uniqueness theorem for temperature functions on a semi-infinite rod with the given initial temperature.

2. Reflection principles for holomorphic functions and harmonic functions

It is known that the reflection principle was introduced, for the first time, by H. A. Schwarz to solve some problems concerning the conformal mappings of polygonal regions. After that, many variants and improved versions have been developed so far.

The simplest version of the reflection principle for the holomorphic functions states as follows :

Theorem A (continuous version). *Let Ω be an open subset of the complex plane which is symmetric with respect to the real axis and*

$$\begin{aligned}\Omega^+ &= \{z \in \mathbb{C} | \text{Im}z > 0\}, & \Omega^- &= \{z \in \mathbb{C} | \text{Im}z < 0\}, \\ E &= \{z \in \Omega | \text{Im}z = 0\}.\end{aligned}$$

If $f(z)$ is holomorphic in Ω^+ , continuous up to E , and $f(z) = \overline{f(\bar{z})}$ on E then $f(z)$ can be holomorphically continued to a holomorphic function $F(z)$ in the whole domain Ω via the relation

$$(2.1) \quad F(z) = \overline{f(\bar{z})}, \quad z \in \Omega^-.$$

In the above theorem, the assumption for $f(z)$ requires that f has continuous boundary values on the part E of the boundary of Ω^+ . But this assumption can be weakened as follows (see [7] and [9]):

Theorem A' (distribution version). *Let Ω , Ω^+ , Ω^- and E be the same as in Theorem A. If $f(z)$ is holomorphic in Ω^+ and satisfies*

$$(2.2) \quad \lim_{y \rightarrow 0^+} \int_E f(x + iy)\phi(x)dx = \lim_{y \rightarrow 0^+} \int_E \overline{f(x - iy)}\phi(x)dx$$

for every infinitely differentiable function ϕ with compact support in E , then f has a holomorphic extension $F(z)$ in Ω with (2.1).

The condition (2.2) implies that $f(x + iy)$ has a real limit up to boundary E in the sense of distributions, so that it improves the continuous version. In fact, there are, so called, the edge of wedge theorems of which the reflection principle is only a special case.

For harmonic functions, we can find also a similar reflection principle in [1], which states as follows :

Theorem B. *Suppose Ω is an open subset of \mathbb{R}^n which is symmetric with respect to the hyperplane $t = 0$ and*

$$\begin{aligned}\Omega^+ &= \{(x, t) \in \Omega \mid x \in \mathbb{R}^{n-1}, t > 0\}, & \Omega^- &= \{(x, t) \in \Omega \mid x \in \mathbb{R}^{n-1}, t < 0\} \\ E &= \{(x, t) \in \Omega \mid x \in \mathbb{R}^{n-1}, t = 0\}.\end{aligned}$$

If u is harmonic in Ω^+ , continuous up to E , and $u(x, 0) = 0$, then u extends harmonically to v in the whole Ω via the relation

$$v(x, t) = -v(x, -t), \quad (x, t) \in \Omega^-.$$

Of course, for harmonic functions there are also a distribution version.

The proofs of the theorems stated above rely on different methods of their own. They usually involve quite amounts of rather sophisticated functional analysis.

3. Reflection principles for the temperature functions

We consider here the reflection principles for temperature functions, which will be very similar to the cases of harmonic functions and analytic functions. It states, for example, that if $u(x, t)$ is a temperature function on the right-hand side of the vertical line in the x, t -plane and vanishes in some sense on that line, then $u(x, t)$ can be extended as a temperature function through that line to the left-hand side. Of course, this can be done in such a way that the extended temperature function has opposite signs at pairs of points which are reflection of each other with respect to the vertical line.

At first, we introduce the continuous version.

Theorem 3.1 ([10]). *Let $\Omega = \{(x, t) \in \mathbb{R}^2 \mid 0 < t < T, |x| < R\}$ for $T > 0, R > 0$ and*

$$\begin{aligned}\Omega^+ &= \{(x, t) \in \Omega \mid x > 0\}, & \Omega^- &= \{(x, t) \in \Omega \mid x < 0\}, \\ E &= \{(x, t) \in \Omega \mid x = 0\}.\end{aligned}$$

If $u(x, t)$ is a temperature function in Ω^+ which is continuous up to E and $u(0, t) = 0$, then $u(x, t)$ can be extended in Ω as a temperature function by the relation

$$u(x, t) = -u(-x, t) \quad \text{on } \Omega^-.$$

As in the cases of holomorphic functions or harmonic functions we give here a distribution version of this reflection principle for temperature functions, which is the main result of this paper. But in the proof we use quite a different method, which is, so called, the heat kernel method. This method was initiated by Matsuzawa and improved by Chung and Kim (see [8] and [4]).

Theorem 3.2. *Let Ω , Ω^\pm , and E be as in Theorem 3.1. If $u(x, t)$ is a temperature function in Ω^+ and $\lim_{x \rightarrow 0^+} u(x, t) = 0$ in the sense of distributions, i.e.*

$$(3.1) \quad \lim_{x \rightarrow 0^+} \int u(x, t) \phi(t) dt = 0$$

for every infinitely differentiable function $\phi(x)$ with support in E , then $u(x, t)$ can be extended in the whole of Ω as a temperature function by the relation

$$u(x, t) = -u(-x, t) \quad \text{on } \Omega^-.$$

Proof. The proof consists of several steps.

Step I. We show first that for any compact set K of E there exists an integer k such that

$$\lim_{x \rightarrow 0^+} \int u(x, t) \varphi(t) dt = 0$$

for every k -times differentiable function $\varphi(t)$ with compact support in $(0, T)$ and that there exists a constant $C > 0$, not depending on x , such that

$$(3.2) \quad \left| \int u(x, t) \varphi(t) dt \right| \leq C \sup_K |\varphi(t)|$$

for every k -times differentiable function $\varphi(t)$ with support in K .

For simplicity, by $C_0^\infty(K)$ we denote the set of infinitely differentiable functions in \mathbb{R} with support in the set K .

Let $I = [c_1, c_2]$ be a compact interval in $(0, T)$ and for each $x \in (0, R)$ we define a linear functional Λ_x on $C_0^\infty(I)$ by

$$\Lambda_x(\phi) = \int u(x, t) \phi(t) dt, \quad \phi \in C_0^\infty(I).$$

Then there exists a constant $C(x) > 0$, depending on x , such that

$$(3.3) \quad |\Lambda_x(\phi)| \leq C(x) \sup_{t \in I} |\phi(t)|, \quad \phi \in C_0^\infty(I),$$

which implies that Λ_x is a continuous linear functional on the Fréchet space $C_0^\infty(I)$.

It follows from (3.1) that for each $\phi \in C_0^\infty(I)$ there exists $\delta > 0$ such that

$$|\Lambda_x(\phi)| \leq 1 \quad \text{for every } x \in (0, \delta).$$

Moreover, if $\delta \leq x < R$ then we get also

$$\Lambda_x(\phi) \leq C(c_1, c_2, \delta) \sup_{t \in I} |\phi(t)|$$

since we may assume in our context that $u(x, t)$ is continuous on $[\delta, R] \times (0, T)$. Therefore, it is true that for each $\phi \in C_0^\infty(I)$

$$\sup_{0 < x < R} |\Lambda_x(\phi)| < \infty.$$

Then by the uniform boundedness principle (see §6 in [9]) we can find an integer $k > 0$ and a constant $C > 0$, not depending on x , such that

$$(3.4) \quad |\Lambda_x(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_I |\partial^\alpha \phi(t)|, \quad \phi \in C_0^\infty(I), \quad 0 < x < R.$$

Now let $J = [a, b]$ be a compact interval in $(0, T)$ and $\psi \in C_0^k(J)$, where $C_0^k(J)$ denote the set of k -times differentiable functions in \mathbb{R}^n with compact support in J . We choose $\chi \in C_0^\infty(-1, 1)$ satisfying that

$$0 \leq \chi(t) \leq 1 \text{ and } \int \chi(t) dt = 1.$$

If we put $\chi_j(t) = j\chi(jt)$ and

$$\varphi_j(t) = \psi(t) * \chi_j(t) = \int \psi(t-s)\chi_j(s) ds.$$

then φ_j is an infinitely differentiable function whose support is contained in the interval $[a - \frac{1}{j}, b + \frac{1}{j}]$. Moreover it is easy to see that

$$(3.5) \quad \sum_{|\alpha| \leq k} \sup_{0 < t < T} |\partial^\alpha \varphi_j(t) - \partial^\alpha \psi(t)| \rightarrow 0$$

as $j \rightarrow \infty$. If $0 < c_1 < a < b < c_2 < T$, then we can see that $\varphi_j(t)$ belongs to $C_0^\infty(I)$ for sufficiently large j , where $I = [c_1, c_2]$. Thus we may assume that $\varphi_j \in C_0^\infty(I)$ for all j .

On the other hand, it is clear from (3.4) and (3.5) that for each $x \in (0, R)$

$$(3.6) \quad \lim_{j \rightarrow \infty} \int u(x, t) \varphi_j(t) dt = \int u(x, t) \psi(t) dt.$$

In view of (3.4) we have

$$|\int u(x, t) \varphi_j(t) dt| \leq C \sum_{t \in I} \sup |\partial^\alpha \varphi_j(t)|, \quad 0 < x < T, \quad j = 1, 2, \dots,$$

and it follows from (3.5) and (3.6) that as $j \rightarrow \infty$

$$(3.7) \quad |\int u(x, t) \psi(t) dt| \leq C \sum_{|\alpha| \leq k} \sup_J |\partial^\alpha \psi(t)|, \quad \psi \in C_0^k(J), \quad 0 < x < R.$$

Here, C is a constant which is independent of x and ψ .

Now let (x_j) be any sequence in $(0, R)$ which converges to zero. Then it follows from (3.1) and (3.5) that for every $\varepsilon > 0$ there exists $N > 0$ such that $j \geq N$ implies

$$\sum_{|\alpha| \leq k} \sup_{t \in I} |\partial^\alpha \psi(t) - \partial^\alpha \varphi_j(t)| \leq \frac{\varepsilon}{2}$$

and $|\Lambda_{x_j}(\varphi_N(t))| \leq \varepsilon/2C$, where C is the constant in (3.7). Therefore, applying (3.7) we obtain that for given $\varepsilon > 0$

$$\begin{aligned} |\Lambda_{x_j}(\psi)| &\leq |\Lambda_{x_j}(\psi - \varphi_N)| + |\Lambda_{x_j}(\varphi_N)| \\ &\leq C \sum_{|\alpha| \leq k} \sup_{t \in J} |\partial^\alpha \psi(t) - \partial^\alpha \varphi_N(t)| + \frac{\varepsilon}{2} \\ &< \varepsilon, \end{aligned}$$

which implies that

$$(3.8) \quad \lim_{x \rightarrow 0+} \int u(x, t)\psi(t)dt = 0, \quad \psi \in C_0^k(0, T).$$

Step II. In this step for any compact interval $I = [a, b]$ in $(0, T)$ and a compact subset K in $[0, T)$ with $I + K = \{x + y | x \in I \text{ and } y \in K\} \subset (0, T)$, we show that

$$(3.9) \quad \max_{t \in I} \left| \int u(x, t+s)\phi(s)ds \right| \rightarrow 0$$

as $x \rightarrow 0+$ for every $\phi \in C_0^k(K)$.

To do this, let τ_t be the translation given by

$$(\tau_t \phi)(s) = \phi(s - t).$$

For a fixed $\phi \in C_0^k(K)$ we define the mapping $P : I \rightarrow C_0^k(I + K)$ by

$$P(t) = \tau_t \phi \quad (= \phi(\cdot - t)).$$

Then it is easy to see that P is continuous, since $\partial^\alpha \phi$ is uniformly continuous for each $|\alpha| \leq k$.

Since I is compact, it is clear that

$$\begin{aligned} P(I) &= \{\tau_t \phi \in C_0^k(I + K) | t \in I\} \\ &= \{\phi(\cdot - t) | t \in I\} \end{aligned}$$

is a compact subset of $C_0^k(I + K)$. Then for every $\varepsilon > 0$ there exist real numbers $t_1, t_2, \dots, t_m \in I$ such that for every $t \in I$,

$$(3.10) \quad \sum_{|\alpha| \leq k} \sup_{I+K} |\partial^\alpha \phi(\cdot - t) - \partial^\alpha \phi(\cdot - t_j)| \leq \frac{\varepsilon}{2C}$$

for at least one j , where C is the constant in (3.2).

In view of the previous step we see that

$$\begin{aligned} \Lambda_x(\tau_t \phi) &= \int u(x, s) \phi(s - t) ds \\ &= \int u(x, t + s) \phi(s) ds \end{aligned}$$

converges to zero as $x \rightarrow 0+$. Hence given $\varepsilon > 0$ we can find $\delta > 0$ such that $0 < x < \delta$ implies

$$(3.11) \quad |\Lambda_x(\tau_{t_j} \phi)| < \frac{\varepsilon}{2}, \quad j = 1, 2, \dots, m.$$

If we apply (3.2) in the previous step to the compact set $I + K$ instead of I , it follows from (3.10) and (3.11) that for all x in $(0, \delta)$,

$$\begin{aligned} |\Lambda_x(\phi(\cdot - t))| &\leq |\Lambda_x(\phi(\cdot - t) - \phi(\cdot - t_j))| + |\Lambda_x \phi(\cdot - t_j)| \\ &\leq C \sum_{|\alpha| \leq k} \sup_{I+K} |\partial^\alpha \phi(\cdot - t) - \partial^\alpha \phi(\cdot - t_j)| + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

for all $t \in I$ and t_j as in (3.10). This implies

$$\max_{t \in I} \left| \int u(x, t + s) \phi(s) ds \right| = \max_{t \in I} |\Lambda_x(\tau_t \phi)| \rightarrow 0$$

as $x \rightarrow 0+$ for each $\phi \in C_0^k(K)$, which is the required.

Step III. Now we complete the proof in this step. To do this we choose real numbers a and b with $0 < a < b < T$.

Consider the continuous function f on the real line defined by

$$f(t) = \begin{cases} \frac{t^{k+1}}{(k+1)!}, & t \geq 0, \\ 0, & t < 0 \end{cases}$$

and an infinitely differentiable function $\theta(t)$ on the real line satisfying

$$\theta(t) = \begin{cases} 1, & |t| \leq a, \\ 0, & |t| > b. \end{cases}$$

If we put $v(t) = \theta(t)f(t)$ and $K = [0, b]$, then the function $v(t)$ belongs to $C_0^k(K)$ and

$$(3.12) \quad \left(\frac{d}{dt}\right)^{k+1} v(t) = \delta(t) + w(t)$$

for some infinitely differentiable function $w(t)$ with support in $I = [a, b]$ where δ is the Dirac measure on the real line and $\left(\frac{d}{dt}\right)^{k+1} v(t)$ means the weak derivative, i.e., the derivative in the sense of distributions.

Now we define a couple of functions $g(x, t)$ and $h(x, t)$ on $\Omega_b^+ = \{(x, t) \in \mathbb{R}^2 | 0 < x < R, 0 < t < T - b\}$ by

$$(3.13) \quad g(x, t) = \int u(x, t + s)v(s)ds$$

and

$$(3.14) \quad h(x, t) = \int u(x, t + s)w(s)ds$$

Then it is easy to see that $g(x, t)$ and $h(x, t)$ are temperature functions in Ω_b^+ . Moreover, applying the result in the previous steps, we can see that for each compact interval I in $(0, T)$,

$$\max_{t \in I} |g(x, t)| \rightarrow 0$$

and

$$\max_{t \in I} |h(x, t)| \rightarrow 0$$

as $x \rightarrow 0+$, which implies that $g(x, t)$ and $h(x, t)$ can be defined to be $g(0, t) = 0$ and $h(0, t) = 0$ for $0 < t < T - b$ so that they are continuous up to $E_b = \{(0, t) \in \Omega | 0 < t < T - b\}$.

Therefore, applying the continuous version of the reflection principle (Theorem 3.1) we can extend $g(x, t)$ and $h(x, t)$ to $\Omega_b = \{(x, t) | -R < x < R, 0 < t < T - b\}$ as temperature functions by the relation

$$(3.15) \quad g(x, t) = -g(-x, t), \quad h(x, t) = -h(-x, t)$$

respectively on $\Omega_b^- = \{(x, t) \in \Omega_b | x < 0\}$.

On the other hand, the relation (3.12) gives

$$(3.16) \quad \begin{aligned} \left(-\frac{d}{dt}\right)^{k+1} g(x, t) &= \int u(x, t + s) \left(\frac{d}{ds}\right)^{k+1} v(s)ds \\ &= u(x, t) + h(x, t). \end{aligned}$$

Thus we can get an extension of $u(x, t)$ in Ω_b^- by

$$u(x, t) = \left(-\frac{d}{dt}\right)^{k+1} g(x, t) - h(x, t).$$

Moreover, (3.15) gives

$$u(x, t) = -u(-x, t) \quad \text{on } \Omega_b^-.$$

But since the real number b can be chosen arbitrarily in $(0, T)$ we obtain an extension of $u(x, t)$ to the whole of Ω , which completes the proof. \square

For every $\phi \in C_0^\infty(I)$, $I = [a, b] \subset (0, T)$ it is true that

$$|\phi(t)| \leq C, \quad 0 < t < T$$

for some $C > 0$. Hence, modifying the above proof we actually can prove the following, which looks easier in application :

Corollary 3.3. *Let Ω , Ω^\pm , and E be as in Theorem 3.1. If $u(x, t)$ is a temperature function in Ω^+ and*

$$\lim_{x \rightarrow 0^+} \int_{[0, T]} |u(x, t)| dt = 0,$$

then $u(x, t)$ can be extended in the whole of Ω as a temperature function by the relations

$$u(x, t) = -u(-x, t) \quad \text{on } \Omega^-.$$

4. An application

Recently, Chung and Kim([3], [5]) gave somewhat improved results for the uniqueness for the solution of the Cauchy problem of the heat equation in infinite rod. Actually they relaxed the growth condition in direction of time, which had originally required to be uniformly bounded in that direction. They can be stated in a simple form as follows:

Theorem 4.1 ([3]). *Let $u(x, t)$ be a continuous function on $\mathbb{R}^n \times [0, T]$ satisfying the heat equation in $\mathbb{R}^n \times (0, T)$ and the followings:*

(i) *There exist constants $a > 0$, $0 < \alpha < 1$, and $C > 0$ such that*

$$|u(x, t)| \leq C \exp \left[\left(\frac{a}{t}\right)^\alpha + a|x|^2 \right], \quad (x, t) \in \mathbb{R}^n \times (0, T),$$

(ii) $u(x, 0) = 0, x \in \mathbb{R}^n$.

Then $u(x, t)$ is identically zero in $\mathbb{R}^n \times [0, T]$.

Now applying the main result of this paper and the above uniqueness theorem we give the uniqueness theorem for temperature functions in a semi-infinite rod.

Theorem 4.2. Let $u(x, t)$ be a continuous function on $(0, \infty) \times [0, T]$ satisfying the heat equation in $(0, \infty) \times (0, T)$ and the followings:

(i) There exist constants $a > 0, 0 < \alpha < 1$, and $C > 0$ such that

$$|u(x, t)| \leq C \exp \left[\left(\frac{a}{t} \right)^\alpha + ax^2 \right], \quad (x, t) \in (0, \infty) \times (0, T),$$

(ii) $u(x, 0) = 0$ on $(0, \infty)$,

(iii) $\lim_{x \rightarrow 0^+} \int u(x, t)\phi(t)dt = 0$ for every $\phi \in C_0^\infty(0, T)$.

Then $u(x, t)$ is identically zero on $[0, \infty) \times [0, T]$.

Proof. At first, in view of (iii) we can apply Theorem 3.2 to get a temperature function $\tilde{u}(x, t)$ on $\mathbb{R} \times (0, T)$ as an extension of $u(x, t)$. Then it is easy to see that the relation $\tilde{u}(x, t) = -\tilde{u}(-x, t)$ for $x < 0$ and $0 < t < T$ makes $\tilde{u}(x, t)$ satisfy the conditions in Theorem 4.1. Therefore, $\tilde{u}(x, t)$ is identically zero, which gives the conclusion. \square

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