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INVARIANT DIFFERENTIAL OPERATORS ON THE GRASSMANN MANIFOLD $SG_{2,n-1}(\mathbf{R})$

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0. Introduction. The Grassmann manifold $SG_{2,n-1}(\mathbf{R})=SO(n+1)/SO(n-1)\times SO(2)$ with its canonical Riemannian metric is known to be a Riemannian symmetric space of rank 2. Hence the algebra $D(SG_{2,n-1}(\mathbf{R}))$ of $SO(n+1)$ -invariant (linear) differential operators on $SG_{2,n-1}(\mathbf{R})$ is generated by two differential operators.

It is the aim of our paper to exhibit simultaneous eigenspace decomposition of appropriate generators Δ_0^\wedge and Δ_1^\wedge of the algebra $D(SG_{2,n-1}(\mathbf{R}))$. We have obtained in [7] the followings:

(1) the eigenspace decomposition of Δ_0 restricted to $\mathbf{K}^*(S^n, g_0)$ is given, where g_0 is the canonical metric on S^n and Δ_0 is the Lichnerowicz operator acting on the graded algebra $\mathbf{S}^*(S^n, g_0)$ of symmetric tensor fields on the standard sphere (S^n, g_0) and $\mathbf{K}^*(S^n, g_0)$ is the graded subalgebra of $\mathbf{S}^*(S^n, g_0)$ generated by Killing vector fields,

(2) Radon transform \wedge :

$$\mathbf{S}^*(S^n, g_0) \rightarrow C^\infty(SG_{2,n-1}(\mathbf{R}))$$

intertwines Δ_0 with the Laplace Beltrami operator Δ_0^\wedge on $SG_{2,n-1}(\mathbf{R})$, i.e.,

$$(\Delta_0 \xi)^\wedge = \Delta_0^\wedge \xi^\wedge$$

for $\xi \in \mathbf{S}^*(S^n, g_0)$,

(3) the eigenspace decomposition obtained in (1) is transferred to that of Δ_0^\wedge , since the kernel of the Radon transform restricted to $\mathbf{K}^*(S^n, g_0)$ is the principal ideal generated by $g_0/2-1$ and the image of $\mathbf{K}^*(S^n, g_0)$ is uniformly dense in $C^\infty(SG_{2,n-1}(\mathbf{R}))$.

In the present paper a new differential operator Δ_1 which acts on $\mathbf{S}^*(S^n, g_0)$ with analogous properties as (1), (2), (3) above is constructed.

Especially Δ_0^\wedge together with the differential operator Δ_1^\wedge corresponding to Δ_1 by the Radon transform are found to be a set of generators of the algebra $D(SG_{2,n-1}(\mathbf{R}))$.

In 1 and 2, we recall the results obtained in [7] with some improvements.

where \langle , \rangle and $d\sigma$ are the inner product at each point of M and the volume element induced from the Riemannian metric, respectively. Now define five fundamental linear operators on the graded algebra $\mathbf{S}^*(M)$.

DEFINITION 1.1. (1) Denote by T^* the linear map of degree 2:

$$\mathbf{S}^p(M) \ni \xi \mapsto \frac{1}{2}g \circ \xi \in \mathbf{S}^{p+2}(M).$$

(2) Denote by T the adjoint operator of T^* :

$$(T^*\xi, \eta) = (\xi, T\eta).$$

Evidently, $\mathbf{S}^p(M) \ni \xi \mapsto T\xi \in \mathbf{S}^{p-2}(M)$, i.e., T is of degree -2 .

(3) Denote by δ^* the linear map:

$$\mathbf{S}^p(M) \ni \xi \mapsto \delta^*\xi := \frac{1}{2!}[g, \xi] \in \mathbf{S}^{p+1}(M),$$

(4) Denote by δ the adjoint operator of δ^* : $\mathbf{S}^p(M) \rightarrow \mathbf{S}^{p-1}(M)$ defined as

$$(\delta\xi, \eta) = (\xi, \delta^*\eta) \quad \xi \in \mathbf{S}^p(M), \eta \in \mathbf{S}^{p-1}(M).$$

(5) As the fifth operator let us define the degree operator d by

$$\mathbf{S}^p(M) \ni \xi \mapsto d\xi := p\xi \in \mathbf{S}^p(M).$$

Then δ^* and d are derivations on $\mathbf{S}^*(M)$, i.e.,

$$\delta^*(\xi \circ \eta) = (\delta^*\xi) \circ \eta + \xi \circ \delta^*\eta,$$

and

$$d(\xi \circ \eta) = (d\xi) \circ \eta + \xi \circ d(\eta),$$

The proofs of these two assertions are direct and easy.

Lemma 1.1. ([7] pp. 54–55)

$$\begin{aligned} [T, \delta] &= 0, \quad [T^*, \delta^*] = 0, \quad [\delta^*, T] = \delta, \quad [T^*, \delta] = \delta^*, \\ [T, T^*] &= \frac{n+2d}{2}, \quad [T^m, T^*] = \frac{mn+2dm+2m^2-2m}{2}T^{m-1}. \end{aligned}$$

DEFINITION 1.2. $\Delta_0 := -2\sum_{a,b=0}^m g^{ab} \nabla_a \nabla_b - [\delta, \delta^*]$ acting on $\mathbf{S}^*(M)$ is called the *Lichnerowicz operator* on (M, g) . The restriction of Δ_0 to $C^\infty(M)$ coincides with the ordinary *Laplace Beltrami operator*, which we denote by the same notation Δ_0 .

Lemma 1.1. ([7] p. 55)

$$[\Delta_0, T] = 0, \quad [\Delta_0, T^*] = 0.$$

Lemma 1.3. ([6] Lemma 1.5) *Let (M, g) be a locally symmetric Riemannian manifold. Then*

$$[\Delta_0, \delta^*] = 0, \quad [\Delta_0, \delta] = 0.$$

$$\mathbf{Ker} \delta^*(M, g) := \sum_{p \geq 0} (\mathbf{Ker} \delta^*) \cap \mathbf{S}^p(M, g) \quad (\text{direct sum})$$

is a graded subalgebra of $\mathbf{S}^*(M)$. Each element of $\mathbf{Ker} \delta^*(M, g)$ is called a *Killing tensor field*. The graded subalgebra of $\mathbf{Ker} \delta^*(M, g)$ generated by $\mathbf{Ker} \delta^*(M, g) \cap \mathbf{S}^1(M)$, is denoted as

$$\mathbf{K}^*(M, g) = \sum_{p \geq 0} \mathbf{K}^p(M, g) (\subseteq \mathbf{Ker} \delta^*(M, g)) \quad (\text{direct sum}).$$

Theorem 1.1. ([7] p. 62) (1) $\mathbf{Ker} \delta^*(S^n, g_0)$ coincides with $\mathbf{K}^*(S^n, g_0)$.

(2) For any $\xi \in \mathbf{K}^*(S^n, g_0)$, there exists a differential operator D_ξ with ξ as its symbol tensor field such that

$$[D_\xi, \Delta_0] = 0.$$

2. Differential operators acting on $\mathbf{S}^*(S^n)$.

Lemma 2.1. ([6]) *Let M_i ($i=1, 2$) be differential manifolds. There are subalgebras $\tilde{\mathbf{E}}(M_i)$ ($i=1, 2$) of $\mathbf{E}(M_1 \times M_2)$, being canonically isomorphic to $\mathbf{E}(M_i)$ ($i=1, 2$) respectively, each one of which is the centralizer of the other in $\mathbf{E}(M_1 \times M_2)$.*

Let $\iota: S^n \rightarrow \mathbf{R}^{n+1}$ be the canonical imbedding of S^n onto the unit sphere in \mathbf{R}^{n+1} . It induces a trivialization $\tilde{\iota}: S^n \times \mathbf{R} \rightarrow \mathbf{R}^{n+1} - \{0\}$ of the real line bundle $\mathbf{R}^{n+1} - \{0\}$ defined by $(x, t) \mapsto \iota(x, t) = e^t x$. By Lemma 2.1 a vector field ξ on S^n is uniquely identified with the vector field $\tilde{\xi}$ on $S^n \times \mathbf{R}$ such that

$$(2.1) \quad [\tilde{\xi}, t] = 0 \quad \text{and} \quad [\tilde{\xi}, \partial/\partial t] = 0.$$

The condition (2.1) for $\tilde{\xi} = \sum_{a=0}^n \xi^a \frac{\partial}{\partial x^a} \in \mathbf{E}^1(\mathbf{R}^{n+1} - \{0\})$ is written as

$$(2.1)' \quad \sum_{a=0}^n \xi^a x^a = 0 \quad \text{and} \quad \sum_{b=0}^n x^b \frac{\partial \xi^a}{\partial x^b} = \xi^a \quad (0 \leq a \leq n),$$

since $r\tilde{\iota}(x, t) = e^t$ and $\tilde{\iota}_*(\partial/\partial t) = \sum_{a=0}^n x^a \frac{\partial}{\partial x^a}$, where $r^2 = \sum_{a=0}^n (x^a)^2$.

More generally, we can identify $\mathbf{E}^*(S^n)$ with the subalgebra

$$(2.2) \quad \mathbf{E}^*(S^n) := \{D \in \mathbf{E}^*(\mathbf{R}^{n+1} - \{0\}) \mid [D, r^2] = 0 \\ \text{and } [D, \sum_{a=0}^n x^a \partial/\partial x^a] = 0\}$$

of $\mathbf{E}^*(\mathbf{R}^{n+1} - \{0\})$ in virtue of Lemma 2.1. Each coefficient $\tilde{\xi}^{a_1 \dots a_k}$ of $D \in \mathbf{E}^p(S^n)$

($p \geq k \geq 0$) is a homogeneous function of degree k with respect to the variables x^0, \dots, x^n . This identification is transferred to the identification of the two algebras $S^*(S^n)$ and $\tilde{S}^*(S^n) := \tilde{\sigma}^*(\tilde{E}^*(S^n))$, where $\tilde{\sigma}$ is the symbol operator of $\tilde{E}^*(\mathbf{R}^{n+1} - \{0\})$. Let us identify $S^*(S^n)$ with $\tilde{S}^*(S^n)$ via the symbol operators σ^* of $E^*(S^n)$ and $\tilde{\sigma}^*$ of $\tilde{E}^*(S^n)$.

Namely,

$$(1/p!) \Xi^{a_1 \dots a_p} (\partial/\partial x^{a_1}) \circ \dots \circ (\partial/\partial x^{a_p}) \in S^p(\mathbf{R}^{n+1} - \{0\})$$

is in $\tilde{S}^p(S^n)$ if and only if

$$(2.3) \quad \sum_{a=0}^n \Xi^{a_1 \dots a_p-1^a} x^a = 0 \quad \text{and} \quad \sum_{a=0}^n \frac{\partial \Xi^{a_1 \dots a_p}}{\partial x^a} x^a = p \Xi^{a_1 \dots a_p}.$$

From now on, the componentwise expression of $\xi \in S^p(\mathbf{R}^{n+1} - \{0\})$ will be expressed as

$$\xi = \frac{1}{p!} \sum_{a_1, \dots, a_p=0}^n \xi^{a_1 \dots a_p} y_{a_1} \dots y_{a_p},$$

where $\xi^{a_1 \dots a_p} \in C^\infty(\mathbf{R}^{n+1} - \{0\})$. Here y_i ($0 \leq i \leq n$) are regarded as current coordinates of $T^*(\mathbf{R}^{n+1} - \{0\})_x = \{\sum y_i dx^i_x\}$ at $x = (x_0, \dots, x_n)$.

That is, we regard a contravariant symmetric tensor field of degree p as a homogeneous polynomial of order p with respect to y_i 's.

Denote by $\tilde{E}(\mathbf{R}^{n+1} - \{0\})$ the set of all differential operators of $2(n+1)$ variables $x^0, \dots, x^n, y_0, \dots, y_n$ the coefficients of which being C^∞ with respect to the variables x^i 's on $\mathbf{R}^{n+1} - \{0\}$ and polynomials with respect to the variables y_j 's. Elements of $\tilde{E}(\mathbf{R}^{n+1} - \{0\})$ are differential operators acting on symmetric tensor fields on $\mathbf{R}^{n+1} - \{0\}$.

Lemma 2.2. (1) *A symmetric tensor field $\xi \in S^p(\mathbf{R}^{n+1} - \{0\})$ belongs to $\tilde{S}^p(S^n)$ if and only if*

$$\sum_{a=0}^n x^a \partial \xi / \partial y_a = 0 \quad \text{and} \quad \sum_{a=0}^n x^a \partial \xi / \partial x^a = p \xi.$$

(2) *If $\xi \in S^p(\mathbf{R}^{n+1} - \{0\})$, then $\sum_{a=0}^n y_a \partial \xi / \partial y_a = p \xi$.*

Proof. (1) is another expression of (2.3) in terms of differential operators belonging to $\tilde{E}(\mathbf{R}^{n+1} - \{0\})$. (2) is evident. Q.E.D.

DEFINITION 2.1. (1) Denote by \tilde{I} the left ideal in $\tilde{E}(\mathbf{R}^{n+1} - \{0\})$ generated by $\sum_{a=0}^n x^a \partial / \partial x^a - \sum_{a=0}^n y_a \partial / \partial y_a$ and $(1/r^2) \sum_{a=0}^n x^a \partial / \partial y_a$.

(2) Put

$$\begin{aligned} \widetilde{\mathbf{EO}}(S^n) := \\ \{D \in \widetilde{\mathbf{E}}(\mathbf{R}^{n+1} - \{0\}) \mid [D, \sum_{a=0}^n x^a \partial / \partial x^a - \sum_{a=0}^n y_a \partial / \partial y_a] \subseteq \widetilde{I} \\ \text{and } [D, (1/r^2) \sum_{a=0}^n x^a \partial / \partial y_a] \subseteq \widetilde{I}\}. \end{aligned}$$

Lemma 2.3. $D \in \widetilde{\mathbf{E}}(\mathbf{R}^{n+1} - \{0\})$ preserves $\mathbf{S}^*(S^n)$ if and only if $D \in \widetilde{\mathbf{EO}}(S^n)$.

Proof. This assertion is an immediate consequence of Lemma 2.2. Q.E.D.
Put

$$\widetilde{I}_0 := \widetilde{\mathbf{EO}}(S^n) \cap \widetilde{I}.$$

\widetilde{I}_0 is easily proved to be a two-sided ideal of $\widetilde{\mathbf{EO}}(S^n)$. So

$$\mathbf{EO}(S^n) := \widetilde{\mathbf{EO}}(S^n) / \widetilde{I}_0$$

can be regarded as an algebra of differential operators acting on symmetric tensor fields on S^n . Now we can regard the fundamental operators T^* , T , δ^* , δ , and d , as elements of $\mathbf{EO}(S^n)$. In the following, a representative in $\widetilde{\mathbf{EO}}(S^n)$ for each of these operators will be given explicitly.

Lemma 2.4. The following operators \widetilde{T} , \widetilde{T}^* , $\widetilde{\delta}$, $\widetilde{\delta}^*$ and \widetilde{d} acting on $\mathbf{S}^*(S^n)$, give representatives for the fundamental operators :

$$(1) \quad \widetilde{T}^* = (1/2) \sum_{a,b=0}^n (r^2 \delta^{ab} - x^a x^b) y_a y_b \in \widetilde{\mathbf{EO}}(S^n) \cap \mathbf{S}^2(S^n).$$

$$(2) \quad \widetilde{T} = (1/2r^2) \sum_{a=0}^n \partial^2 / \partial y_a \partial y_a \in \widetilde{\mathbf{EO}}(S^n).$$

$$(3) \quad \widetilde{\delta}^* = r^2 \sum_{a=0}^n y_a \partial / \partial x^a \in \mathbf{EO}(S^n).$$

$$(4) \quad \widetilde{\delta} = - \sum_{a=0}^n (\partial^2 / \partial x^a \partial y_a + r^{-2} \langle x, y \rangle \partial^2 / \partial y_a \partial y_a) \in \widetilde{\mathbf{EO}}(S^n),$$

where $\langle x, y \rangle := \sum x^a y_a$.

Proof. The operators \widetilde{T} , \widetilde{T}^* , $\widetilde{\delta}$, and $\widetilde{\delta}^*$ in $\mathbf{E}^*(S^n)$ are introduced in [7] and proved to correspond to T , T^* , δ , and δ^* . They are expressed as above as elements of $\widetilde{\mathbf{EO}}(S^n)$, respectively. That a representative \widetilde{d} of the degree operator d is given by the Euler operator, can be observed immediately from the second equation of Lemma 2.3. Q.E.D.

DEFINITION 2.2. Define

$$(1) \quad \kappa_{a,b} := x^a \partial / \partial x^b - x^b \partial / \partial x^a \in \mathbf{E}^1(S^n)$$

for $0 \leq a, b \leq n$ and $a \neq b$, and

$$(2) \quad \check{\kappa}_{a,b} := x^a \partial / \partial x^b - x^b \partial / \partial x^a + y_a \partial / \partial y_b - y_b \partial / \partial y_a \in \widetilde{\mathbf{EO}}(S^n)$$

for $0 \leq a, b \leq n$ and $a \neq b$.

Lemma 2.5. *Between $\kappa_{a,b}$ and $\check{\kappa}_{a,b}$ we have the following relation:*

$$[\kappa_{a,b}, \xi] = \check{\kappa}_{a,b}(\xi)$$

for arbitrary $\xi \in \mathbf{S}^*(S^n, g_0)$, where the bracket product in the left-hand side is the one in $\mathbf{S}^*(S^n, g_0)$ defined in (1. 1).

Proof. This can be easily verified.

Q.E.D.

Lemma 2.6.

$$\begin{aligned} (1) \quad [\check{\kappa}_{a,b}, T] &= 0, & (2) \quad [\check{\kappa}_{a,b}, T^*] &= 0, \\ (3) \quad [\check{\kappa}_{a,b}, \delta^*] &= 0, & (4) \quad [\check{\kappa}_{a,b}, \delta] &= 0. \end{aligned}$$

Proof. In virtue of the Lemma 2.5 these can be easily verified. Q.E.D.

Denote by $\kappa_{a,b}^*$ the adjoint operators of $\kappa_{a,b}$ as elements of $\mathbf{E}^1(S^n)$, and $\check{\kappa}_{a,b}^*$ the adjoint operators of $\check{\kappa}_{a,b}$ with respect to the canonical inner product defined on $\mathbf{S}^*(S^n, g_0)$.

We can see easily

$$\kappa_{a,b}^* = -\kappa_{a,b} \quad \text{and} \quad \check{\kappa}_{a,b}^* = -\check{\kappa}_{a,b}.$$

Lemma 2.7. (1) *The Laplace Beltrami operator Δ_0 on (S^n, g_0) coincides with*

$$\sum_{a < b} \kappa_{a,b}^* \kappa_{a,b}$$

as a differential operator of order 2 acting on $C^\infty(\mathbf{R}^{n+1} - \{0\})$.

(2) *The Lichnerowicz operator on (S^n, g_0) coincides with*

$$\sum_{a < b} \check{\kappa}_{a,b}^* \check{\kappa}_{a,b} \in \widetilde{\mathbf{EO}}(S^n).$$

Proof. (1) $\sum_{a < b} \kappa_{a,b}^* \kappa_{a,b}$ can be expanded as follows:

$$-r^2 \sum_{a=0}^n \frac{\partial^2}{\partial x^a \partial x^a} + \sum_{a,b=0}^n x^a x^b \frac{\partial^2}{\partial x^a \partial x^b} + n \sum_{a=0}^n x^a \frac{\partial}{\partial x^a}.$$

This operator satisfies the following three conditions: (i) its symbol tensor field coincides with the (contravariant) metric tensor g_0 ; (ii) it is a self-adjoint linear

differential operator; (iii) it annihilates the constant function 1. Such an operator must coincide with the Laplace Beltrami operator.

(2) Δ_0 on $\mathbf{S}^*(S^n, g_0)$ is known to be (cf. [7]):

$$[\delta, \delta^*] + 2d(n+d-2) - 8T^*T = - \sum_{a,b=0}^n (r^2 \delta_{ab} - x^a x^b) \frac{\partial^2}{\partial x^a \partial x^b} - 2\langle x, y \rangle \sum_{a=0}^n \frac{\partial^2}{\partial x^a \partial y_a} - 2\langle x, y \rangle^2 T + d(2n+d-3) - 4T^*T,$$

where the operator d is as in Lemma 1.1 and the notation $\langle \cdot, \cdot \rangle$ is as in Lemma 2.4. On the other hand, $\sum_{a < b} \tilde{\kappa}_{a,b}^* \tilde{\kappa}_{a,b}$ is equal to

$$-1/2 \sum_{a,b=0}^n (x^a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a} + y_a \frac{\partial}{\partial y_b} - y_b \frac{\partial}{\partial y_a})^2 = - \sum_{a,b=0}^n (r^2 \delta^{ab} - x^a x^b) \frac{\partial^2}{\partial x^a \partial x^b} + (n-1)d - 2\langle x, y \rangle \sum_{a=0}^n \frac{\partial^2}{\partial x^a \partial y_a} - 2\langle x, y \rangle^2 T + (n+d-2)d - 4T^*T.$$

This coincides with the Lichnerowicz operator reviewed above. Q.E.D.

DEFINITION 2.3. Define an endomorphism S of degree -2 on the graded algebra $\mathbf{S}^*(S^n, g_0)$:

$$\mathbf{S}^p(S^n, g_0) \ni \xi \mapsto S\xi \in \mathbf{S}^{p-2}(S^n, g_0),$$

by

$$(1) \quad S := \Delta_0 T - \lambda_{p,1} T + (16/3) T^* T^2 + (1/3) [\delta^*, T\delta] \text{ on } \mathbf{S}^p(S^n, g_0),$$

where

$$\lambda_{p,k} := 2(p-k)n + 2p^2 - 4(k+1)p + 4k^2 + 6k. \\ (\text{Eventually } \lambda_{p,1} = 2(p-1)n + 2p^2 - 8p + 10.)$$

Moreover we define

$$(2) \quad B_j^* := 2j^2 T^* + (\delta^*)^2 \quad (j \geq 1),$$

$$(3) \quad A_k^* := \left(\prod_{i=1}^k B_{2i}^* \right) T^k \quad (k \geq 1), \quad A_0^* = 1.$$

DEFINITION 2.4. (1) Denote the restriction of T^* to $\mathbf{K}^*(S^n, g_0)$ by T_0^* .

(2) Denote the image of T_0^* by $Im T_0^* (\subseteq \mathbf{K}^*(S^n, g_0))$ and denote the orthogonal complement of $Im T_0^*$ in $\mathbf{K}^*(S^n, g_0)$ by

$$\mathbf{P}^*(S^n, g_0).$$

We have

$$K^*(S^n, g_0) = \text{Im } T_0^* \oplus P^*(S^n, g_0)$$

$$P^*(S^n, g_0) = \sum_{p=0}^{\infty} P^p(S^n, g_0) \quad (\text{direct sum})$$

with $P^p(S^n, g_0) = P^*(S^n, g_0) \cap K^p(S^n, g_0)$.

Lemma 2.8. (1) *As an endomorphism of degree -2 on the graded algebra $S^*(S^n)$, S preserves $K^*(S^n, g_0)$ invariant.*

(2) *A_k^* also preserves $K^*(S^n, g_0)$ invariant.*

For the proof c.f. [7] Lemma 4.3.

Denote the orthogonal projection:

$$K^*(S^n, g_0) \rightarrow P^*(S^n, g_0)$$

by Π_0 . Π_0 can be proved to be commutative with Δ_0 (cf. [7]). Let

$$C_k^* := \Pi_0 A_k^*.$$

C_k^* 's satisfy

$$\Delta_0 C_k^* - \lambda_{p,k} C_k^* + \frac{1}{(k+1)(2k+1)} C_{k+1}^* = 0 \text{ on } K^p(S^n, g_0).$$

(cf. [7] p. 69, Lemma 4.3 and (4. 10).)

Define

$$P_{p,k} := \frac{n+2p-4k-3}{k!(n+2p-2k-3)!!} \sum_{i=k}^{\lfloor p/2 \rfloor} \left(\frac{(-1)^{i-k} (n+2p-2k-2i-5)!!}{(2i)! (i-k)!} \right) C_i^*,$$

where $p \geq 2k \geq 0$. Denote the image of $P_{p,k}: K^p(S^n, g_0) \rightarrow P^p(S^n, g_0)$ by $E_{p,k}$.

Theorem 2.1. (1) *For $p \geq 2k \geq 0$ we have*

$$\Delta_0 P_{p,k} = \lambda_{p,k} P_{p,k} \text{ on } K^p(S^n, g_0),$$

where $\lambda_{p,k}$ is as in Definition 2.3 (1).

(2) *We have the two direct sums:*

$$K^p(S^n, g_0) = \sum_{k=0}^{\lfloor p/2 \rfloor} (T^*)^k (P^{p-2k}(S^n, g_0)),$$

$$P^p(S^n, g_0) = \sum_{k=0}^{\lfloor p/2 \rfloor} E_{p,k},$$

which thus together with (1) give the eigenspace decomposition of Δ_0 on $K^*(S^n, g_0)$.

(3) *Every $E_{p,k}$ is nonzero for $n \geq 3$.*

For the proof of (1) refer to [7] lemma 4.4. For the proof of (2) and (3) cf. [7] p. 69 and pp. 75–76.

In the remainder of this section we assume $n+1 \geq 4$.

DEFINITION 2.5. Define

$$(1) \quad D_{abcd} := \frac{1}{2^3} \sum_{e,f,g,h=0}^n \delta_{abcd}^{efgh} \tilde{\kappa}_{ef} \tilde{\kappa}_{gh} \in \tilde{\mathbf{EO}}(S^n),$$

$$(2) \quad \Delta_1 := \frac{1}{4!} \sum_{a,b,c,d=0}^n D_{abcd}^* D_{abcd} \in \tilde{\mathbf{EO}}(S^n).$$

Notice that D_{abcd} is a self-adjoint operator.

Theorem 2.2.

$$(1) \quad [\tilde{\kappa}_{ab}, \Delta_0] = 0.$$

$$(2) \quad [\tilde{\kappa}_{ab}, \Delta_1] = 0.$$

Proof. These are obtained by direct calculations. Q.E.D.

Theorem 2.3.

$$(1) \quad [T^*, \Delta_1] = 0.$$

$$(2) \quad [T, \Delta_1] = 0.$$

$$(3) \quad [\delta^*, \Delta_1] = 0.$$

$$(4) \quad [\delta, \Delta_1] = 0.$$

Proof. From Lemma 2.7 we obtain easily. Q.E.D.

Note that thus Δ_1 preserves $\mathbf{P}^p(S^n, g_0)$ invariant.

3. The eigenspace decomposition of $K^*(S^n, g_0)$.

In this section we assume $n+1 \geq 4$.

Theorem 3.1. *As a differential operator acting on $S^*(S^n)$*

$$\Delta_1 = -4T^*T\Delta_0 + d(n+d-3)\Delta_0 - 16(T^*)^2T^2 - 2T^*\delta^2 - 2(\delta^*)^2T - (n+2d-4)\delta^*\delta + 4(2d-3)n + 2d^2 - 10d + 11)T^*T - d(d-1)(n+d-2)(n+d-3).$$

Proof. From the definition of Δ_1 in 2 and Lemma 2.4 we can obtain the result by direct calculations. Q.E.D.

Lemma 3.1.

$$\Delta_1 = (d+1)(n+d-2) \{ \Delta_0 - d(n+d-1) \} - 2\delta^*T\delta^* + (n+d-2)\delta\delta^* - 6T^*S \text{ on } S^*(S^n, g_0).$$

Proof. Apply T^* to the operator S in its definition reviewed in 2. Then

we can express T^*S in terms of fundamental operators, from which we can eliminate $-4T^*\Delta_0T$ in virtue of Theorem 3.1. The resulting relation is the required one. Q.E.D.

Theorem 3.2. *We have*

$$\Delta_1 = \sum_{k=0}^{\lfloor p/2 \rfloor} \mu_{p,k} P_{p,k} \text{ on } P^p(S^n, g_0),$$

where

$$\mu_{p,k} = (p-2k)(p+1)(n+p-2)(n+p-3-2k).$$

Thus $P^p(S^n, g_0) = \sum_{k=0}^{\lfloor p/2 \rfloor} E_{p,k}$ gives the eigenspace decomposition of Δ_1 restricted to $P^p(S^n, g_0)$.

Proof. Restricting Δ_1 on $E_{p,k}$ we have from Lemma 2.8 (1) and Lemma 3.1,

$$\begin{aligned} \Delta_1 P_{p,k} &= (p+1)(n+p-2) \{ \Delta_0 P_{p,k} - p(n+p-1) P_{p,k} \} \\ &= (p+1)(n+p-2) \{ 2(p-k)n + 2p^2 - 4(k+1)p \\ &\quad + 4k^2 + 6k - p(n+p-1) \} P_{p,k} \end{aligned}$$

which coincides with the desired eigenvalue. Q.E.D.

Lemma 3.2.

$$(1) \quad \text{Ker } T^k \cap P^p(S^n, g_0) \subset \sum_{i=0}^{k-1} E_{p,i},$$

where $p \geq 2k \geq 0$.

(2) *Let $\xi \in P^p(S^n, g_0)$ be an eigen tensor field of Δ_0 . Then $\xi \in E_{p,k}$ if and only if $T^k \xi \neq 0$ and $T^{k+1} \xi = 0$.*

Proof. (1) From the definition of the projection operator $P_{p,k}$ in 2 the assertion follows immediately. (2) follows from (1) directly. Q.E.D.

Theorem 3.3. *Let $\xi \in E_{p,k}$ and let ξ be a simultaneous eigen tensor field of Δ_i ($i=0, 1$). Then ξ has the eigenvalues $\lambda_{p,k}$ and $\mu_{p,k}$ for Δ_0 and Δ_1 , respectively.*

Proof. From the commutativity of T with Δ_0 and from that $T^k \xi \neq 0$, $T^k \xi$ is proved to be an eigen tensor field of Δ_0 with eigenvalue $\lambda_{p,k}$. On the other hand, as $(k+1)\delta T^k \xi = [\delta^*, T^{k+1}] \xi = \delta^* T^{k+1} \xi = 0$, we obtain

$$\begin{aligned} \Delta_1(T^k \xi) &= (p-2k)(n+p-2k-3)\Delta_0(T^k \xi) - \\ &= (p-2k)(n+p-2k-3)(p-2k-1)(n+p-2k-2)T^k \xi. \end{aligned}$$

Thus $T^k \xi$ is a simultaneous eigen tensor field of Δ_0 and Δ_1 . $\mu_{p,k}$ is regained as an eigenvalue if we substitute the $\lambda_{p,k}$ in place of Δ_0 in the right-hand side of

the expression of $\Delta_1(T^k\xi)$. The commutativity of Δ_1 with T obtained in Theorem 2.3 implies that $\Delta_1\xi = \mu_{p,k}\xi$. Q.E.D.

4. Main theorems.

Denote by $W_2(\mathbf{R}^{n+1})$ the manifold of all 2-frames in the Euclidean space (\mathbf{R}^{n+1}, g_0) of dimension $n+1$ and call it a *Stiefel manifold*. The submanifold of $W_2(\mathbf{R}^{n+1})$ defined as the set of orthonormal 2-frames is denoted by $V_2(\mathbf{R}^{n+1})$ and we call it an *orthogonal Stiefel manifold*. $V_2(\mathbf{R}^{n+1})$ is identified with the homogeneous space

$$SO(n+1)/SO(n-1).$$

Denote by $SG_{2,n-1}(\mathbf{R})$ the Grassmann manifold of all oriented 2-planes in \mathbf{R}^{n+1} passing through the origin. As is well known $SG_{2,n-1}(\mathbf{R})$ is identified with the homogeneous space

$$SO(n+1)/SO(n-1) \times SO(2).$$

The orthogonal Stiefel manifold $V_2(\mathbf{R}^{n+1})$ can be regarded as a principal bundle with the base space $SG_{2,n-1}(\mathbf{R})$ and the structural group $SO(2)$, where the projection π_* is defined canonically.

Theorem 4.1. (cf. [7]) *Let $P(M,G)$ be a principal bundle with a Lie group G as its fibre. Let $\mathbf{E}^G(P)$ be the subalgebra of $\mathbf{E}(P)$ which consists of G -invariant differential operators on P . Then there is an isomorphism: $\mathbf{E}^G(P)/\mathbf{J} \cong \mathbf{E}(M)$, where \mathbf{J} is the two-sided ideal of $\mathbf{E}^G(P)$ generated over \mathbf{R} by G -invariant vertical vector fields.*

Applying Theorem 4.1 to the principal bundle

$$V_2(\mathbf{R}^{n+1}) \rightarrow SG_{2,n-1}(\mathbf{R})$$

with $SO(2)$ as fibre, we obtain

$$(4.1) \quad \mathbf{E}(SG_{2,n-1}(\mathbf{R})) \cong \mathbf{E}^{SO(2)}(V_2(\mathbf{R}^{n+1}))/\mathbf{J},$$

where \mathbf{J} is the two-sided ideal in $\mathbf{E}^{SO(2)}(V_2(\mathbf{R}^{n+1}))$ generated by $SO(2)$ -invariant vertical vector fields. On the other hand, there is a polar decomposition of the Stiefel manifold $W_2(\mathbf{R}^{n+1})$:

$$(4.2) \quad W_2(\mathbf{R}^{n+1}) \cong P_2 \times V_2(\mathbf{R}^{n+1}),$$

where P_2 is the space of real positive definite 2×2 symmetric matrices. (cf. ([7]) Applying Lemma 2.1, the polar decomposition assures the existence of two subalgebras each one of which is the centralizer of the other in $\mathbf{E}(W_2(\mathbf{R}^{n+1}))$ and the second one is canonically isomorphic to $\mathbf{E}(V_2(\mathbf{R}^{n+1}))$. Thus a differential

operator $D \in \mathbf{E}(V_2(\mathbf{R}^{n+1}))$ can be identified with a differential operator $D^\dagger \in \mathbf{E}(W_2(\mathbf{R}^{n+1}))$ satisfying

$$[D^\dagger, \rho_{\alpha\beta}^2] = 0, \\ [D^\dagger, \frac{\partial}{\partial \rho_{\alpha\beta}^2}] = 0,$$

where each of $\rho_{\alpha\beta}^2 (1 \leq \alpha, \beta \leq 2)$ denotes the (α, β) -component of the P_2 -part ρ^2 in the polar decomposition (4. 2). The totality of such operators is designated as $\mathbf{E}^\dagger(V_2(\mathbf{R}^{n+1}))$.

Connecting the isomorphism (4. 1) with the identification “†” we obtain representatives in $\mathbf{E}(W_2(\mathbf{R}^{n+1}))$ of elements in $\mathbf{E}(SG_{2,n-1}(\mathbf{R}))$.

Let $\mathbf{Geod}(S^n, g_0)$ be the space of oriented geodesics on (S^n, g_0) with respect to the canonical metric. We have a natural identification

$$\iota: SG_{2,n-1}(\mathbf{R}) \rightarrow \mathbf{Geod}(S^n, g_0).$$

Let $\xi \in \mathbf{S}^p(S^n)$. Define $\xi^\wedge \in C^\infty(SG_{2,n-1}(\mathbf{R}))$ by

$$\xi^\wedge(\Gamma) = \frac{1}{2\pi p!} \int_{\gamma=\iota(\Gamma)} \langle \xi, \dot{\gamma}^p \rangle ds,$$

where $\dot{\gamma}^p$ is the p -th symmetric power in $\mathbf{S}^*(\gamma)$ of the unit tangent vector field $\dot{\gamma}$ along $\gamma=\iota(\Gamma)$. The mapping defined by

$$\mathbf{S}^*(S^n, g_0) \ni \xi \mapsto \xi^\wedge \in C^\infty(SG_{2,n-1}(\mathbf{R}))$$

is called the *Radon transform*.

DEFINITION 4.1. (1) Denote by (P^{ab}) the system of normalized Plücker coordinates P^{ab} of the Grassmann manifold $SG_{2,n-1}(\mathbf{R})$, where a system of Plücker coordinates P^{ab} is said to be *normalized* if and only if

$$\sum_{a < b} (P^{ab})^2 = 1.$$

(2) Denoted by

$$\mathbf{R}(P^{ab}: n \geq b > a \geq 0)$$

the subalgebra of $C^\infty(SG_{2,n-1}(\mathbf{R}))$ generated by the normalized Plücker coordinates.

Theorem 4.2. (cf. [7])

$$(1) \quad (\kappa_{a,b})^\wedge = P^{ab},$$

where (P^{ab}) are the system of normalized Plücker coordinates of the Grassmann manifold $SG_{2,n-1}(\mathbf{R})$.

(2) The image of the Radon transform restricted to $\mathbf{K}^*(S^n, g_0)$ is the uniformly

dense subalgebra $\mathbf{R}(P^{ab}: n \geq b > a \geq 0)$ of $C^\infty(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}))$.

(3) The kernel of the Radon transform restricted to $\mathbf{K}^*(S^n, g_0)$ is the ideal generated by $g_0/2-1$.

Corollary.

$$(\xi \circ \eta)^\wedge = \xi^\wedge \eta^\wedge$$

for $\xi \in \mathbf{K}^*(S^n, g_0)$ and $\eta \in \mathbf{S}^*(S^n)$.

Proof. The assertion follows from the fact that $\langle \xi, \dot{\gamma}^p \rangle$ is constant along $\gamma = \iota(\Gamma)$. Q.E.D.

From now on we often confuse an element of $\mathbf{E}(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}))$ with its representative in $\mathbf{E}(\mathbf{W}_2(\mathbf{R}^{n+1}))$ as well as an element of $C^\infty(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}))$ with its representative in $C^\infty(\mathbf{W}_2(\mathbf{R}^{n+1}))$. For an element $q=(q_1, q_2)$ of $\mathbf{W}_2(\mathbf{R}^{n+1})$ the components of q_1, q_2 will be denoted by $q_1^a, q_2^a (0 \leq a \leq n)$, respectively.

DEFINITION 4.2. Define

$$\hat{\kappa}_{a,b} := q_1^a \partial / \partial q_1^b - q_1^b \partial / \partial q_1^a + q_2^a \partial / \partial q_2^b - q_2^b \partial / \partial q_2^a,$$

where $0 \leq a < b \leq n$, and

$$\Delta_0^\wedge := \sum_{a < b} \hat{\kappa}_{a,b}^* \hat{\kappa}_{a,b},$$

where $\hat{\kappa}_{a,b}^*$ is a representative via the connecting isomorphism of (4.1) with \dagger , of the adjoint operator of Killing vector field on $(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}), g_1)$ corresponding to $\hat{\kappa}_{a,b}$, where g_1 is the canonically normalized $SO(n+1)$ -invariant Riemannian metric on $\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R})$. Moreover we define

$$D_{abcd}^\wedge := \frac{2^3}{1} \sum_{e,f,g,h=0}^n \delta_{abcd}^{efgh} \hat{\kappa}_{e,f} \hat{\kappa}_{g,h},$$

$$\Delta_1^\wedge := \frac{1}{4!} \sum_{a,b,c,d=0}^n (D_{abcd}^\wedge)^* D_{abcd}^\wedge.$$

Notice that Δ_0^\wedge is a representative of the Laplace Beltrami operator on $(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}), g_1)$ and expressed explicitly as

$$-(\delta^{ab} - q_\alpha^a q_\beta^b (\rho^2)^{\alpha\beta}) \rho_{\gamma\delta}^2 \partial^2 / \partial q_\gamma^a \partial q_\delta^b + (n-1) q_\alpha^a \partial / \partial q_\alpha^a,$$

where the convention of dummy indices is adopted. (cf. [7] p. 64.)

Theorem 4.3. The natural $SO(n+1)$ -action on $\mathbf{S}^*(S^n)$ and $C^\infty(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}))$ commutes with the Radon transform.

Proof. This follows from the definition of the Radon transform. Q.E.D.

Corollary. Let $\eta^\wedge \in C^\infty(W_2(\mathbf{R}^{n+1}))$ be the image of $\eta \in \mathbf{S}^*(S^n)$ by the Radon transform. Then

$$\hat{\kappa}_{a,b}\eta^\wedge = (\kappa_{a,b}\eta)^\wedge.$$

Proof. The assertion is an infinitesimal version of Theorem 4.3. Q.E.D.

Theorem 4.4. Let $\eta \in \mathbf{S}^*(S^n)$ and let η^\wedge be its Radon transform.

- (1) $(\Delta_0\eta)^\wedge = \Delta_0^\wedge\eta^\wedge,$
- (2) $(\Delta_1\eta)^\wedge = \Delta_1^\wedge\eta^\wedge.$

Proof. By the definitions of $\Delta_0, \Delta_1, (\Delta_0)^\wedge,$ and $(\Delta_1)^\wedge,$ the assertion follows from the previous corollary. Q.E.D.

DEFINITION 4.3. Denote by $E_{p,k}^\wedge$ the image of $E_{p,k}$ by the Radon transform.

Theorem 4.5. Assume $n+1 > 4.$

(1) Δ_0^\wedge and Δ_1^\wedge are generators of the algebra $\mathbf{D}(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}))$ of $SO(n+1)$ -invariant differential operators on $\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}).$

- (2) $\Delta_0^\wedge|_{E_{p,k}^\wedge} = \lambda_{p,k} \mathbf{1}_{p,k}$ and $\Delta_1^\wedge|_{E_{p,k}^\wedge} = \mu_{p,k} \mathbf{1}_{p,k}$
 where $\mathbf{1}_{p,k}$ is the identity operator of $E_{p,k}^\wedge.$ The totality of

$$E_{p,k}^\wedge, p \geq 2k \geq 0,$$

gives all of the simultaneous eigenspaces of Δ_0^\wedge and $\Delta_1^\wedge.$

- (3) Each $E_{p,k}^\wedge$ is an $SO(n+1)$ -irreducible subspace.

Proof. Notice that $\mathbf{S}\mathbf{G}_{2,2}(\mathbf{R})$ is known to be globally homothetic to $S^2 \times S^2$ with the canonical metric. So we omit to detail of $\mathbf{D}(\mathbf{S}\mathbf{G}_{2,2}(\mathbf{R}))$ as the reduced case.

(1) That Δ_0^\wedge and Δ_1^\wedge are invariant differential operators is a direct consequence of Theorem 4.4. It is known that the algebra $\mathbf{D}(\mathbf{S}\mathbf{G}_{2,n-1}(\mathbf{R}))$ is generated by two operators of order 2 and 4, respectively. (cf. [3] Ch.II.) It remains to show that Δ_0^\wedge and Δ_1^\wedge are algebraically independent. Suppose that they are not so, then we can write

$$\Delta_1^\wedge = a(\Delta_0^\wedge)^2 + b\Delta_0^\wedge + c$$

for some constants a, b and c ($(a, b, c) \neq (0, 0, 0)$). On the other hand, in virtue of Theorem 3.2 we can easily verify that Δ_1^\wedge acts trivially on $\sum_{k=0}^\infty E_{2k,k}^\wedge$ (direct sum). If we restrict the action of Δ_1^\wedge to each $E_{2k,k}^\wedge,$ we would obtain a polynomial equation of one variable of order at most four with an infinite number of solutions $k=1, 2, \dots,$ from which we can conclude $a=b=c=0.$ This is a contradiction. Thus our assertion (1) is proved.

(2) follows from Theorem 4.4 and Theorem 4.2 (2). In order to prove (3) we need

Lemma 4.2. *If $\lambda_{p,k} = \lambda_{p',k'}$ and $\mu_{p,k} = \mu_{p',k'}$ then $p = p'$ and $k = k'$.*

Proof. We can easily verify

$$(1) \quad \lambda_{p,k} > \lambda_{p,k+1},$$

$$(2) \quad \lambda_{p,k} < \lambda_{p+1,k},$$

$$(3) \quad \lambda_{p,k} < \lambda_{p+1,k+1},$$

On the other hand, we can see

$$(4) \quad \mu_{p,k} > \mu_{p,k+1},$$

$$(5) \quad \mu_{p,k} < \mu_{p+1,k},$$

but

$$(6) \quad \mu_{p,k} > \mu_{p+1,k+1}$$

in contrast with (3). From these the required property follows immediately. Q.E.D.

Proof of Theorem 4.5. (3) In virtue of Lemma 4.2, $E_{p,k}^{\wedge}$ is known to be maximal in $\mathbf{R}(P^{ab}: n \geq b > a \geq 0)$ as the subspace of simultaneous eigenfunctions of the eigenvalues $\lambda_{p,k}$ and $\mu_{p,k}$ for Δ_0^{\wedge} and Δ_1^{\wedge} , respectively. For the irreducibility of $E_{p,k}^{\wedge}$, we refer to [2] p. 401, Corollary 3.3. Q.E.D.

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