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INVARIANT DIFFERENTIAL OPERATORS ON THE GRASSMANN MANIFOLD $SG_{2,n-1}(R)$

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0. Introduction. The Grassmann manifold $SG_{2,n-1}(\mathbf{R}) = SO(n+1)/SO(n-1) \times SO(2)$ with its canonical Riemannian metric is known to be a Riemannian symmetric space of rank 2. Hence the algebra $D(SG_{2,n-1}(\mathbf{R}))$ of SO(n+1)-invariant (linear) differential operators on $SG_{2,n-1}(\mathbf{R})$ is generated by two differential operators.

It is the aim of our paper to exhibit simultaneous eigenspace decomposition of appropriate generators Δ_0^{\uparrow} and Δ_1^{\uparrow} of the algebra $D(SG_{2n-1}(\mathbf{R}))$. We have obtained in [7] the followings:

(1) the eigenspace decomposition of Δ_0 restricted to $K^*(S^n, g_0)$ is given, where g_0 is the canonical metric on S^n and Δ_0 is the Lichnerowicz operator acting on the graded algebra $S^*(S^n, g_0)$ of symmetric tensor fields on the standard sphere (S^n, g_0) and $K^*(S^n, g_0)$ is the graded subalgebra of $S^*(S^n, g_0)$ generated by Killing vector fields,

(2) Radon transform Λ :

$$S^*(S^n, g_0) \rightarrow C^{\infty}(SG_{2,n-1}(R))$$

intertwines Δ_0 with the Laplace Beltrami operator Δ_0^{\wedge} on $SG_{2,n-1}(\mathbf{R})$, i.e.,

$$(\Delta_0 \xi)^{\wedge} = \Delta_0^{\wedge} \xi^{\wedge}$$

for $\xi \in S^*(S^n, g_0)$,

(3) the eigenspace decomposition obtained in (1) is transferred to that of Δ_0° , since the kernel of the Radon transform restricted to $\mathbf{K}^*(S^n, g_0)$ is the principal ideal generated by $g_0/2-1$ and the image of $\mathbf{K}^*(S^n, g_0)$ is uniformly dense in $C^{\infty}(\mathbf{SG}_{2,n-1}(\mathbf{R}))$.

In the present paper a new differential operator Δ_1 which acts on $S^*(S^n, g_0)$ with analogous properties as (1), (2), (3) above is constructed.

Especially Δ_0° together with the differential operator Δ_1° corresponding to Δ_1 by the Radon transform are found to be a set of generators of the algebra $D(SG_{2,n-1}(\mathbf{R}))$.

In 1 and 2, we recall the results obtained in [7] with some improvements.

where \langle , \rangle and $d\sigma$ are the inner product at each point of M and the volume element induced from the Riemannian metric, respectively. Now define five fundamental linear operators on the graded algebra $S^*(M)$.

DEFINITION 1.1. (1) Denote by T^* the linear map of degree 2:

$$S^{p}(M) \in \xi \mapsto \frac{1}{2} g \circ \xi \in S^{p+2}(M)$$
.

(2) Denote by T the adjoint operator of T^* :

$$(T^*\xi,\eta)=(\xi,T\eta)$$

Evidently, $S^{p}(M) \ni \xi \mapsto T\xi \in S^{p-2}(M)$, i.e., T is of degree -2.

(3) Denote by δ^* the linear map:

$${old S}^p(M) \! \ni \! \xi \mapsto \delta^* \! \xi := rac{1}{2!} [g,\xi] \! \in \! {old S}^{p+1}\!(M)$$
 ,

(4) Denote by δ the adjoint operator of $\delta^*: S^p(M) \to S^{p-1}(M)$ defined as

 $(\delta \xi, \eta) = (\xi, \delta^* \eta) \quad \hat{\xi} \in S^p(M), \ \eta \in S^{p-1}(M).$

(5) As the fifth operator let us define the degree operator d by

 $S^{p}(M) \ni \xi \mapsto d\xi := p\xi \in S^{p}(M)$.

Then δ^* and d are derivations on $S^*(M)$, i.e.,

$$\delta^*(\xi \circ \eta) = (\delta^* \xi) \circ \eta + \xi \circ \delta^* \eta$$
,

and

$$d(\xi \circ \eta) = (d\xi) \circ \eta + \xi \circ d(\eta)$$
 ,

The proofs of these two assertions are direct and easy.

Lemma 1.1. ([7] pp. 54–55)

$$[T, \delta] = 0, \quad [T^*, \delta^*] = 0, \quad [\delta^*, T] = \delta, \quad [T^*, \delta] = \delta^*,$$
$$[T, T^*] = \frac{n+2d}{2}, \quad [T^m, T^*] = \frac{mn+2dm+2m^2-2m}{2}T^{m-1}.$$

DEFINITION 1.2. $\Delta_0 := -2\sum_{a,b=0}^{m} g^{ab} \nabla_a \nabla_b - [\delta, \delta^*]$ acting on $S^*(M)$ is called the *Lichnerowicz operator* on (M, g). The restriction of Δ_0 to $C^{\infty}(M)$ coincides with the ordinary *Laplace Beltrami operator*, which we denote by the same notation Δ_0 .

Lemma 1.1. ([7] p. 55)

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$$[\Delta_0, T] = 0$$
, $[\Delta_0, T^*] = 0$.

Lemma 1.3. ([6] Lemma 1.5) Let (M, g) be a locally symmetric Riemannian manifold. Then

$$\begin{split} [\Delta_0, \, \delta^*] &= 0 \,, \quad [\Delta_0, \, \delta] = 0 \,. \end{split}$$

Ker $\delta^*(M, g) := \sum_{p \geq 0} (Ker \, \delta^*) \cap S^p(M, g) \quad (direct \ sum)$

is a graded subalgebra of $S^*(M)$. Each element of Ker $\delta^*(M, g)$ is called a *Killing tensor field*. The graded subalgebra of Ker $\delta^*(M, g)$ generated by Ker $\delta^*(M, g) \cap S^1(M)$, is denoted as

$$K^*(M,g) = \sum_{p \ge 0} K^p(M,g) (\subseteq \operatorname{Ker} \delta^*(M,g)) \quad (direct \ sum).$$

Theorem 1.1. ([7] p. 62) (1) Ker $\delta^*(S^n, g_0)$ coincides with $K^*(S^n, g_0)$.

(2) For any $\xi \in \mathbf{K}^*(S^n, \xi_0)$, there exists a differential operator D_{ξ} with ξ as its symbol tensor field such that

$$[D_{\boldsymbol{\xi}}, \Delta_{\boldsymbol{0}}] = 0.$$

2. Differential operators acting on $S^*(S^n)$.

Lemma 2.1. ([6]) Let M_i (i=1, 2) be differential be manifolds. There are subalgebras $\tilde{E}(M_i)$ (i=1, 2) of $E(M_1 \times M_2)$, being canonically isomorphic to $E(M_i)$ (i=1, 2) respectively, each one of which is the centralizer of the other in $E(M_1 \times M_2)$.

Let $\iota: S^n \to \mathbb{R}^{n+1}$ be the canonical imbedding of S^n onto the unit sphere in \mathbb{R}^{n+1} . It induces a trivialization $\tilde{\iota}: S^n \times \mathbb{R} \to \mathbb{R}^{n+1} - \{0\}$ of the real line bundle $\mathbb{R}^{n+1} - \{0\}$ defined by $(x, t) \mapsto \iota(x, t) = e^t x$. By Lemma 2.1 a vector field ξ on S^n is uniquely identified with the vector field $\tilde{\xi}$ on $S^n \times \mathbb{R}$ such that

(2.1)
$$[\tilde{\xi}, t] = 0 \text{ and } [\tilde{\xi}, \partial/\partial t] = 0.$$

The condition (2.1) for $\hat{\xi} = \sum_{a=0}^{n} \hat{\xi}^{a} \frac{\partial}{\partial x^{a}} \in E^{1}(\mathbb{R}^{n+1} - \{0\})$ is written as

(2.1)'
$$\sum_{a=6}^{n} \overline{\xi}^{a} x^{a} = 0 \text{ and } \sum_{b=0}^{n} x^{b} \frac{\partial \overline{\xi}^{a}}{\partial x^{b}} = \overline{\xi}^{a} (0 \leq a \leq n),$$

since $r\tilde{\iota}(x, t) = e^t$ and $\tilde{\iota}_*(\partial/\partial t) = \sum_{a=0}^n x^a \frac{\partial}{\partial x^a}$, where $r^2 = \sum_{a=0}^n (x^a)^2$.

More generally, we can identify $E^*(S^n)$ with the subalgebra

(2.2)
$$E^{*}(S^{n}) := \{D \in E^{*}(R^{n+1} - \{0\}) | [D, r^{2}] = 0$$

and $[D, \sum_{a=0}^{n} x^{a} \partial / \partial x^{a}] = 0\}$

of $E^*(R^{n+1} - \{0\})$ in virtue of Lemma 2.1. Each coefficient $\hat{\xi}^{a_1 \cdots a_k}$ of $D \in E^p(S^n)$

 $(p \ge k \ge 0)$ is a homogeneous function of degree k with respect to the variables x^0, \dots, x^n . This identification is transferred to the identification of the two algebras $S^*(S^n)$ and $\tilde{S}^*(S^n) := \sigma^*(\tilde{E}^*(S^n))$, where σ is the symbol operator of $E^*(R^{n+1} - \{0\})$. Let us identify $S^*(S^n)$ with $\tilde{S}^*(S^n)$ via the symbol operators σ^* of $E^*(S^n)$ and σ^* of $\tilde{E}^*(S^n)$.

Namely,

$$(1/p!)\Xi^{a_1\cdots a_p}(\partial/\partial x^{a_1})\circ\cdots\circ(\partial/\partial x^{a_p})\in S^p(\mathbf{R}^{n+1}-\{0\})$$

is in $\tilde{S}^{p}(S^{n})$ if and only if

(2.3)
$$\sum_{a=0}^{n} \Xi^{a_1 \cdots a_{p-1} a} x^a = 0 \text{ and } \sum_{a=0}^{n} \frac{\partial \Xi^{a_1 \cdots a_p}}{\partial x^a} x^a = p \Xi^{a_1 \cdots a_p}.$$

From now on, the componentwise expression of $\xi \in S^{p}(\mathbb{R}^{n+1} - \{0\})$ will be expressed as

$$\xi = \frac{1}{p!} \sum_{a_1, \dots, a_p=0}^n \xi^{a_1 \cdots a_p} y_{a_1} \cdots y_{a_p},$$

where $\xi^{a_1\cdots a_p} \in C^{\infty}(\mathbb{R}^{n+1}-\{0\})$. Here $y_i \ (0 \le i \le n)$ are regarded as current coordinates of $T^*(\mathbb{R}^{n+1}-\{0\})_x = \{\sum y_i dx_{ix}^i\}$ at $x = (x_0, \cdots, x_n)$.

That is, we regard a contravariant symmetric tensor field of degree p as a homogeneous polynomial of order p with respect to y_i 's.

Denote by $E(\mathbf{R}^{n+1}{0})$ the set of all differential operators of 2(n+1) variables $x^0, \dots, x^n, y_0, \dots, y_n$ the coefficients of which being C^{∞} with respect to the variables x^i 's on $\mathbf{R}^{n+1}{0}$ and polynomials with respect to the variables y_j 's. Elements of $E(\mathbf{R}^{n+1}{0})$ are differential operators acting on symmetric tensor fields on $\mathbf{R}^{n+1} - \{0\}$.

Lemma 2.2. (1) A symmetric tensor field $\xi \in S^{p}(\mathbb{R}^{n+1} - \{0\})$ belongs to $\tilde{S}^{p}(S^{n})$ if and only if

(2) If
$$\xi \in S^{p}(\mathbb{R}^{n+1} - \{0\})$$
, then $\sum_{a=0}^{n} y_{a} \partial \xi / \partial y_{a} = p\xi$.

Proof. (1) is another expression of (2.3) in terms of differential operators belonging to $E(\mathbb{R}^{n+1} - \{0\})$. (2) is evident. Q.E.D.

DEFINITION 2.1. (1) Denote by I the left ideal in $E(\mathbf{R}^{n+1} - \{0\})$ generated by $\sum_{a=0}^{n} x^{a} \partial/\partial x^{a} - \sum_{a=0}^{n} y_{a} \partial/\partial y_{a}$ and $(1/r^{2}) \sum_{a=0}^{n} x^{a} \partial/\partial y_{a}$. (2) Put T. SUMITOMO AND K. TANDAI

$$\widetilde{EO}(S^{n}) := \{D \in \widecheck{E}(\mathbb{R}^{n+1} - \{0\}) | [D, \sum_{a=0}^{n} x^{a} \partial/\partial x^{a} - \sum_{a=0}^{n} y_{a} \partial/\partial y_{a}] \subseteq \widecheck{I} \}$$

and $[D, (1/r^{2}) \sum_{a=0}^{n} x^{a} \partial/\partial y_{a}] \subseteq \widecheck{I} \}$.

Lemma 2.3. $D \in \breve{E}(\mathbb{R}^{n+1} - \{0\})$ preserves $S^*(S^n)$ if and only if $D \in \widetilde{EO}(S^n)$.

Proof. This assertion is an immediate consequence of Lemma 2.2. Q.E.D. Put

$$\breve{I}_{0} := \widetilde{EO}(S^{n}) \cap \breve{I}.$$

 \breve{I}_0 is easily proved to be a two-sided ideal of $\widetilde{EO}(S^n)$. So

$$\boldsymbol{EO}(S^n) := \widetilde{\boldsymbol{EO}}(S^n) / \widecheck{I}_0$$

can be regarded as an algebra of differential operators acting on symmetric tensor fields on S^n . Now we can regard the fundamental operators T^* , T, δ^* , δ , and d, as elements of $EO(S^n)$. In the following, a representative in $\widetilde{EO}(S^n)$ for each of these operators will be given explicitly.

Lemma 2.4. The following operators \tilde{T} , \tilde{T}^* , $\tilde{\delta}$, $\tilde{\delta}^*$ and \tilde{d} acting on $S^*(S^n)$, give representatives for the fundamental operators:

(1)
$$\widetilde{T}^* = (1/2) \sum_{a,b=0}^n (r^2 \delta^{ab} - x^a x^b) y_a y_b \in \widetilde{EO}(S^n) \cap S^2(S^n) .$$

(2)
$$\widetilde{T} = (1/2r^2) \sum_{a=0}^{n} \partial^2 / \partial y_a \partial y_a \in \widetilde{EO}(S^n) .$$

(3)
$$\widetilde{\delta}^* = r^2 \sum_{a=0}^n y_a \partial / \partial x^a \in EO(S^n) \,.$$

(4)
$$\widetilde{\delta} = -\sum_{a=0}^{n} (\partial^2 / \partial x^a \partial y_a + r^{-2} \langle \langle x, y \rangle \rangle \partial^2 / \partial y_a \partial y_a) \in \widetilde{EO}(S^n),$$

where $\langle\!\langle x, y \rangle\!\rangle := \sum x^a y_a$.

Proof. The operators \tilde{T} , \tilde{T}^* , $\tilde{\delta}$, and $\tilde{\delta}^*$ in $E^*(S^n)$ are introduced in [7] and proved to correspond to T, T^* , δ , and δ^* . They are expressed as above as elements of $\tilde{EO}(S^n)$, respectively. That a representative \tilde{d} of the degree operator d is given by the Euler operator, can be observed immediately from the second equation of Lemma 2.3. Q.E.D.

DEFINITION 2.2. Define

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(1)
$$\kappa_{a,b} := x^a \partial/\partial x^b - x^b \partial/\partial x^a \in E^1(S^n)$$

for $0 \leq a, b \leq n$ and $a \neq b$, and

(2)
$$\widetilde{\kappa}_{a,b} := x^a \partial \partial x^b - x^b \partial \partial x^a + y_a \partial \partial y_b - y_b \partial \partial y_a \in \widetilde{EO}(S^n)$$

for $0 \leq a, b \leq n$ and $a \neq b$.

Lemma 2.5. Between $\kappa_{a,b}$ and $\check{\kappa}_{a,b}$ we have the following relation:

 $[\kappa_{a,b},\xi] = \widecheck{\kappa}_{a,b}(\xi)$

for arbitrary $\xi \in S^*(S^n, g_0)$, where the bracket product in the left-hand side is the one in $S^*(S^n, g_0)$ defined in (1.1).

Proof. This can be easily verified.

Lemma 2.6.

(1)
$$[\breve{\kappa}_{a,b}, T] = 0$$
, (2) $[\breve{\kappa}_{a,b}, T^*] = 0$,
(3) $[\breve{\kappa}_{a,b}, \delta^*] = 0$, (4) $[\breve{\kappa}_{a,b}, \delta] = 0$.

Proof. In virtue of the Lemma 2.5 these can be easily verified. Q.E.D.

Denote by $\kappa_{a,b}^*$ the adjoint operators of $\kappa_{a,b}$ as elements of $E^1(S^n)$, and $\check{\kappa}_{a,b}^*$ the adjoint operators of $\check{\kappa}_{a,b}$ with respect to the canonical linner product defined on $S^*(S^n, g_0)$.

We can see easily

$$\kappa_{a,b}^* = -\kappa_{a,b}$$
 and $\breve{\kappa}_{a,b}^* = -\breve{\kappa}_{a,b}$.

Lemma 2.7. (1) The Laplace Beltrami operator Δ_0 on (S^*, g_0) coincides with

$$\sum_{a < b} \kappa^*_{a,b} \kappa_{a,b}$$

as a differential operator of order 2 acting on C[∞](Rⁿ⁺¹- {0}).
(2) The Lichnerowicz operator on (Sⁿ, g₀) coincides with

$$\sum_{a < b} \breve{\kappa}^*_{a,b} \breve{\kappa}_{a,b} \in \widetilde{EO}(S^n).$$

Proof. (1) $\sum_{a < b} \kappa_{a,b}^* \kappa_{a,b}$ can be expanded as follows:

$$-r^{2}\sum_{a=0}^{n}\frac{\partial^{2}}{\partial x^{a}\partial x^{a}}+\sum_{a,b=0}^{n}x^{a}x^{b}\frac{\partial^{2}}{\partial x^{a}\partial x^{b}}+n\sum_{a=0}^{n}x^{a}\frac{\partial}{\partial x^{a}}$$

This operator satisfies the following three conditions: (i) its symbol tensor field coincides with the (contravariant) metric tensor g_0 ; (ii) it is a self-adjoint linear

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Q.E.D.

differential operator; (iii) it annihilates the constant function 1. Such an operator must coincide with the Laplace Beltrami operator.

(2) Δ_0 on $S^*(S^n, g_0)$ is known to be (cf. [7]):

$$\begin{split} [\delta, \delta^*] + 2d(n+d-2) - 8T^*T &= -\sum_{a,b=0}^n (r^2 \delta_{ab} - x^a x^b) \frac{\partial^2}{\partial x^a \partial x^b} \\ -2 \langle \langle x, y \rangle \rangle \sum_{a=0}^n \frac{\partial^2}{\partial x^a \partial y_a} - 2 \langle \langle x, y \rangle \rangle^2 T + d(2n+d-3) - 4T^*T \,, \end{split}$$

where the operator d is as in Lemma 1.1 and the notation $\langle \langle , \rangle \rangle$ is as in Lemma 2.4. On the other hand, $\sum_{a < b} \breve{\kappa}^*_{a,b} \breve{\kappa}_{a,b}$ is equal to

$$-\frac{1}{2}\sum_{a,b=0}^{n}\left(x^{a}\frac{\partial}{\partial x^{b}}-x^{b}\frac{\partial}{\partial x^{a}}+y_{a}\frac{\partial}{\partial y_{b}}-y_{b}\frac{\partial}{\partial y_{a}}\right)^{2}=$$
$$-\sum_{a,b=0}^{n}\left(r^{2}\delta^{ab}-x^{a}x^{b}\right)\frac{\partial^{2}}{\partial x^{a}\partial x^{b}}+(n-1)d-2\langle\langle x,y\rangle\rangle\sum_{a=0}^{n}\frac{\partial^{2}}{\partial x^{a}\partial y_{a}}$$
$$-2\langle\langle x,y\rangle\rangle^{2}T+(n+d-2)d-4T^{*}T.$$

This coincides with the Lichnerowicz operator reviewed above. Q.E.D.

DEFINITION 2.3. Define an endomorphism S of degree -2 on the graded algebra $S^*(S^n, g_0)$:

$$\mathbf{S}^{p}(S^{n},g_{0}) \ni \xi \mapsto S\xi \in \mathbf{S}^{p-2}(S^{n},g_{0}),$$

by

(1)
$$S := \Delta_0 T - \lambda_{p,1} T + (16/3) T^* T^2 + (1/3) [\delta^*, T\delta] \text{ on } S^p(S^n, g_0),$$

where

$$\begin{split} \lambda_{p,k} &:= 2(p-k)n + 2p^2 - 4(k+1)p + 4k^2 + 6k \,. \\ (\text{Eventually } \lambda_{p,1} &= 2(p-1)n + 2p^2 - 8p + 10.) \end{split}$$

Moreover we define

(2)
$$B_j^* := 2j^2T^* + (\delta^*)^2 \quad (j \ge 1),$$

(3)
$$A_k^* := (\prod_{i=1}^k B_{2i}^*) T^k \ (k \ge 1), \ A_0^* = 1.$$

DEFINITION 2.4. (1) Denote the restriction of T^* to $K^*(S^n, g_0)$ by T_0^* .

(2) Denote the image of T_0^* by $Im \ T_0^* (\subseteq K^*(S^n, g_0))$ and denote the orthogonal complement of $Im \ T_0^*$ in $K^*(S^n, g_0)$ by

$$P^*(S^n, g_0)$$

We have

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$$\boldsymbol{K}^*(S^n, g_0) = Im \ T^*_0 \oplus \boldsymbol{P}^*(S^n, g_0)$$
$$\boldsymbol{P}^*(S^n, g_0) = \sum_{p=0}^{\infty} \boldsymbol{P}^p(S^n, g_0) \quad (direct \ sum)$$

with $P^{p}(S^{n}, g_{0}) = P^{*}(S^{n}, g_{0}) \cap K^{p}(S^{n}, g_{0}).$

Lemma 2.8. (1) As an endomorphism of degree -2 on the graded algebra $S^*(S^n)$, S preserves $K^*(S^n, g_0)$ invariant.

(2) A_k^* also preserves $\mathbf{K}^*(S^n, g_0)$ invariant.

For the proof c.f. [7] Lemma 4.3.

Denote the orthogonal projection:

$$\boldsymbol{K^*}(S^n, g_0) \to \boldsymbol{P^*}(S^n, g_0)$$

by Π_0 . Π_0 can be proved to be commutative with Δ_0 (cf. [7]). Let

$$C_k^* := \Pi_0 A_k^*.$$

 C_k^* 's satisfy

$$\Delta_0 C_k^* - \lambda_{p,k} C_k^* + \frac{1}{(k+1)(2k+1)} C_{k+1}^* = 0 \text{ on } K^p(S^n, g_0).$$

(cf. [7] p. 69, Lemma 4.3 and (4. 10).)

Define

$$P_{p,k} := \frac{n+2p-4k-3}{k!(n+2p-2k-3)!!} \sum_{i=k}^{\lfloor p/2 \rfloor} \left(\frac{(-1)^{i-k}(n+2p-2k-2i-5)!!}{(2i)!(i-k)!} \right) C_i^*,$$

where $p \ge 2k \ge 0$. Denote the image of $P_{p,k}$: $K^{p}(S^{n}, g_{0}) \rightarrow P^{p}(S^{n}, g_{0})$ by $E_{p,k}$.

Theorem 2.1. (1) For $p \ge 2k \ge 0$ we have

$$\Delta_0 P_{p,k} = \lambda_{p,k} P_{p,k} \text{ on } K^p(S^n, g_0),$$

where $\lambda_{p,k}$ is as in Definition 2.3 (1).

(2) We have the two direct sums:

$$\boldsymbol{K}^{p}(S^{n},g_{0}) = \sum_{k=0}^{\lfloor p/2 \rfloor} (T^{*})^{k} (\boldsymbol{P}^{p-2k}(S^{n},g_{0})) ,$$
$$\boldsymbol{P}^{p}(S^{n},g_{0}) = \sum_{k=0}^{\lfloor p/2 \rfloor} E_{p,k} ,$$

which thus together with (1) give the eigenspace decomposition of Δ_0 on $K^*(S^n, g_0)$. (3) Every $E_{p,k}$ is nonzero for $n \ge 3$.

For the proof of (1) refer to [7] lemma 4.4. For the proof of (2) and (3) cf. [7] p. 69 and pp. 75-76.

In the remainder of this section we assume $n+1 \ge 4$.

DEFINITION 2.5. Define

(1)
$$D_{abcd} := \frac{1}{2^3} \sum_{e,f,\mathcal{E},h=0}^n \delta^{efgh}_{abcd} \breve{\kappa}_{ef} \breve{\kappa}_{gh} \in \widetilde{EO}(S^n),$$

(2)
$$\Delta_1 := \frac{1}{4!} \sum_{a,b,c,d=0}^n D^*_{abcd} D_{abcd} \in \widetilde{EO}(S^n).$$

Notice that D_{abcd} is a self-adjoint operator.

Theorem 2.2.

(1)
$$[\breve{\kappa}_{ab}, \Delta_0] = 0$$

(2)
$$[\breve{\kappa}_{ab}, \Delta_1] = 0$$

Proof. These are obtained by direct calculations. Q.E.D. Theorem 2.3.

$$[T^*, \Delta_1] = 0$$

$$[T, \Delta_1] = 0$$

$$[\delta^*, \Delta_1] = 0$$

$$[\delta, \Delta_1] = 0$$

Proof. From Lemma 2.7 we obtain easily. Q.E.D.

Note that thus Δ_1 preserves $P^p(S^n, g_0)$ invariant.

3. The eigenspace decomposition of $K^*(S^n, g_0)$.

In this section we assume $n+1 \ge 4$.

Theorem 3.1. As a differential operator acting on $S^*(S^n)$

$$\Delta_{1} = -4T^{*}T\Delta_{0} + d(n+d-3)\Delta_{0} - 16(T^{*})^{2}T^{2} - 2T^{*}\delta^{2} - 2(\delta^{*})^{2}T - (n+2d)\delta^{*}\delta + 4(2d-3)n + 2d^{2} - 10d + 11)T^{*}T - d(d-1)(n+d-2)(n+d-3).$$

Proof. From the definition of Δ_1 in 2 and Lemma 2.4 we can obtain the result by direct calculations. Q.E.D.

Lemma 3.1.

$$\Delta_1 = (d+1)(n+d-2) \{ \Delta_0 - d(n+d-1) \} - 2\delta^* T \delta^* + (n+d-2)\delta\delta^* - 6T^* S \text{ on } S^*(S^n, g_0) .$$

Proof. Apply T^* to the operator S in its definition reviewed in 2. Then

we can express T^*S in terms of fundamental operators, from which we can eliminate $-4T^*\Delta_0 T$ in virtue of Theorem 3.1. The resulting relation is the required one. Q.E.D.

Theorem 3.2. We have

$$\Delta_1 = \sum_{k=0}^{\lfloor p/2 \rfloor} \mu_{p,k} P_{p,k} \text{ on } \mathbf{P}^p(S^n, g_0) ,$$

where

$$u_{p,k} = (p-2k)(p+1)(n+p-2)(n+p-3-2k)$$

Thus $\mathbf{P}^{b}(S^{n}, g_{0}) = \sum_{k=0}^{\lfloor b/2 \rfloor} E_{b,k}$ gives the eigenspace decomposition of Δ_{1} restricted to $\mathbf{P}^{p}(S^{n},g_{0}).$

Proof. Restricting Δ_1 on $E_{p,k}$ we have from Lemma 2.8 (1) and Lemma 3.1,

$$\Delta_{1}P_{p,k} = (p+1)(n+p-2)\{\Delta_{0}P_{p,k}-p(n+p-1)P_{p,k}\}$$

= (p+1)(n+p-2)\{2(p-k)n+2p^{2}-4(k+1)p + 4k^{2}+6k-p(n+p-1)\}P_{p,k}

which coincides with the desired eigenvalue.

Lemma 3.2.

) Ker
$$T^k \cap P^p(S^n, g_0) \subset \sum_{l=0}^{k-1} E_{p,l}$$
,

where $p \geq 2k \geq 0$.

(2) Let $\xi \in \mathbf{P}^{\mathfrak{p}}(S^n, g_0)$ be an eigen tensor field of Δ_0 . Then $\xi \in E_{\mathfrak{p}, k}$ if and only if $T^k \xi \neq 0$ and $T^{k+1} \xi = 0$.

Proof. (1) From the definition of the projection operator $P_{p,k}$ in 2 the assertion follows immediately. (2) follows from (1) directly. Q.E.D.

Theorem 3.3. Let $\xi \in E_{p,k}$ and let ξ be a simultaneous eigen tensor field of Δ_i (i=0, 1). Then ξ has the eigenvalues $\lambda_{p,k}$ and $\mu_{p,k}$ for Δ_0 and Δ_1 , respectively.

Proof. From the commutativity of T with Δ_0 and from that $T^k \xi \neq 0$, $T^k \xi$ is proved to be an eigen tensor field of Δ_0 with eigenvalue $\lambda_{p,k}$. On the other hand, as $(k+1)\delta T^k\xi = [\delta^*, T^{k+1}]\xi = \delta^*T^{k+1}\xi = 0$, we obtain

$$\Delta_{1}(T^{k}\xi) = (p-2k)(n+p-2k-3)\Delta_{0}(T^{k}\xi) - (p-2k)(n+p-2k-3)(p-2k-1)(n+p-2k-2)T^{k}\xi$$

Thus $T^{k}\xi$ is a simultaneous eigen tensor field of Δ_{0} and Δ_{1} . $\mu_{p,k}$ is regained as an eigenvalue if we substitute the $\lambda_{p,k}$ in place of Δ_0 in the right-hand side of

Q.E.D.

the expression of $\Delta_1(T^k\xi)$. The commutativity of Δ_1 with T obtained in Theorem 2.3 implies that $\Delta_1\xi = \mu_{p,k}\xi$. Q.E.D.

4. Main theorems.

Denote by $W_2(\mathbb{R}^{n+1})$ the manifold of all 2-frames in the Euclidean space (\mathbb{R}^{n+1}, g_0) of dimension n+1 and call it a *Stiefel manifold*. The submanifold of $W_2(\mathbb{R}^{n+1})$ defined as the set of orthonormal 2-frames is denoted by $V_2(\mathbb{R}^{n+1})$ and we call it an *orthogonal Stiefel manifold*. $V_2(\mathbb{R}^{n+1})$ is identified with the homogeneous space

$$SO(n+1)/SO(n-1)$$
.

Denote by $SG_{2,n-1}(\mathbf{R})$ the Grassmann manifold of all oriented 2-planes in \mathbf{R}^{n+1} passing through the origin. As is well known $SG_{2,n-1}(\mathbf{R})$ is identified with the homogeneous space

$$SO(n+1)/SO(n-1) \times SO(2)$$
.

The orthogonal Stiefel manifold $V_2(\mathbf{R}^{n+1})$ can be regarded as a principal bundle with the base space $SG_{2,n-1}(\mathbf{R})$ and the structural group SO(2), where the projection π_v is defined canonically.

Theorem 4.1. (cf. [7]) Let P(M,G) be a principal bundle with a Lie group G as its fibre. Let $\mathbf{F}^{c}(P)$ be the subalgebra of $\mathbf{E}(P)$ which consists of G-invariant differential operators on P. Then there is an isomorphism : $\mathbf{E}^{c}(P)/\mathbf{J} \cong \mathbf{E}(M)$, where \mathbf{J} is the two-sided ideal of $\mathbf{E}^{c}(P)$ generated over \mathbf{R} by G-invariant vertical vector fields.

Applying Theorem 4.1 to the principal bundle

$$V_2(\mathbf{R}^{n+1}) \rightarrow SG_{2,n-1}(\mathbf{R})$$

with SO(2) as fibre, we obtain

(4.1)
$$E(SG_{2,n-1}(R)) \simeq E^{SO(2)}(V_2(R^{n+1}))/J$$
,

where J is the two-sided ideal in $E^{SO(2)}(V_2(\mathbb{R}^{n+1}))$ generated by SO(2)-invariant vertical vector fields. On the other hand, there is a polar decomposition of the Stiefel manifold $W_2(\mathbb{R}^{n+1})$:

$$(4.2) W_2(\mathbf{R}^{n+1}) \cong P_2 \times V_2(\mathbf{R}^{n+1}),$$

where P_2 is the space of real positive definite 2×2 symmetric matrices. (cf. ([7]) Applying Lemma 2.1, the polar decomposition assures the existence of two subalgebras each one of which is the centralizer of the other in $E(W_2(\mathbb{R}^{n+1}))$ and the second one is canonically isomorphic to $E(V_2(\mathbb{R}^{n+1}))$. Thus a differential

operator $D \in E(V_2(\mathbb{R}^{n+1}))$ can be identified with a differential operator $D^{\dagger} \in E(W_2(\mathbb{R}^{n+1}))$ satisfying

$$egin{aligned} & [D^{\dagger}, \,
ho_{m{lpha}m{eta}}^2] = 0 \, , \ & [D^{\dagger}, \, rac{\partial}{\partial
ho_{m{lpha}m{eta}}^2}] = 0 \, , \end{aligned}$$

where each of $\rho_{\alpha\beta}^2(1 \le \alpha, \beta \le 2)$ denotes the (α, β) -component of the P_2 -part ρ^2 in the polar decomposition (4.2). The totality of such operators is designated as $E^{\dagger}(V_2(\mathbb{R}^{n+1}))$.

Connecting the isomorphism (4.1) with the identification "†" we obtain representatives in $E(W_2(\mathbb{R}^{n+1}))$ of elements in $E(SG_{2,n-1}(\mathbb{R}))$.

Let $\operatorname{Geod}(S^n, g_0)$ be the space of oriented geodesics on (S^n, g_0) with respect to the canonical metric. We have a natural identification

$$\iota: \mathbf{SG}_{2,n-1}(\mathbf{R}) \to \mathbf{Geod}(S^n, g_0).$$

Let $\xi \in S^{p}(S^{n})$. Define $\xi^{\wedge} \in C^{\infty}(SG_{2,n-1}(R))$ by

$$\xi^{\wedge}(\Gamma) = \frac{1}{2\pi p!} \int_{\gamma=\iota(\Gamma)} \langle \xi, \dot{\gamma}^{p} \rangle ds$$

where $\dot{\gamma}^{p}$ is the *p*-th symmetric power in $S^{*}(\gamma)$ of the unit tangent vector field $\dot{\gamma}$ along $\gamma = \iota(\Gamma)$. The mapping defined by

$$\boldsymbol{S}^{\ast}(\boldsymbol{S}^{n},\boldsymbol{g}_{0}) \ni \boldsymbol{\xi} \mapsto \boldsymbol{\xi}^{\wedge} \in C^{\infty}(\boldsymbol{S}\boldsymbol{G}_{2,n-1}(\boldsymbol{R}))$$

is called the Radon transform.

DEFINITION 4.1. (1) Denote by (P^{ab}) the system of normalized Plucker coordinates P^{ab} of the Grassmann manifold $SG_{2,n-1}(\mathbf{R})$, where a system of Plucker coordinates P^{ab} is said to be *normalized* if and only if

$$\sum_{a < b} (P^{ab})^2 = 1.$$

(2) Denoted by

$$\boldsymbol{R}(P^{ab}:n\geq b>a\geq 0)$$

the subalgebra of $C^{\infty}(\mathbf{SG}_{2,n-1}(\mathbf{R}))$ generated by the normalized Plücker coordinates.

Theorem 4.2. (cf. [7])

(1)
$$(\kappa_{a,b})^{\wedge} = P^{ab},$$

where (P^{ab}) are the system of normalized Plücker coordinates of the Grassmann manifold $SG_{2,n-1}(R)$).

(2) The image of the Radon transform restricted to $\mathbf{K}^*(S^n, g_0)$ is the uniformly

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dense subalgebra $\mathbf{R}(P^{ab}: n \geq b > a \geq 0)$ of $C^{\infty}(\mathbf{SG}_{2,n-1}(\mathbf{R}))$.

(3) The kernel of the Radon transform retricted to $K^*(S^n, g_0)$ is the ideal generated by $g_0/2-1$.

Corollary.

$$(\xi \circ \eta)^{\wedge} = \xi^{\wedge} \eta^{\wedge}$$

for $\xi \in \mathbf{K}^*(S^n, g_0)$ and $\eta \in \mathbf{S}^*(S^n)$.

Proof. The assertion follows from the fact that $\langle \xi, \dot{\gamma}^p \rangle$ is constant along $\gamma = \iota(\Gamma)$. Q.E.D.

From now on we often confuse an element of $E(SG_{2,n-1}(\mathbf{R}))$ with its representative in $E(\mathbf{W}_2(\mathbf{R}^{n+1}))$ as well as an element of $C^{\infty}(SG_{2,n-1}(\mathbf{R}))$ with its representative in $C^{\infty}(\mathbf{W}_2(\mathbf{R}^{n+1}))$. For an element $q=(q_1, q_2)$ of $\mathbf{W}_2(\mathbf{R}^{n+1})$ the components of q_1, q_2 will be denoted by $q_1^a, q_2^a(0 \le a \le n)$, respectively.

DEFINITION 4.2. Define

$$\hat{\kappa}_{a,b}:=q_1^a\partial/\partial q_1^b{-}q_1^b\partial/\partial q_1^a{+}q_2^a\partial/\partial q_2^b{-}q_2^b\partial/\partial q_2^a$$
 ,

where $0 \leq a < b \leq n$, and

$$\Delta_0^{\wedge} := \sum_{a < b} \hat{\kappa}^*_{a,b} \hat{\kappa}_{a,b}$$
,

where $\hat{\kappa}_{a,b}^*$ is a representative via the connecting isomorphism of (4.1) with \dagger , of the adjoint operator of Killing vector field on $(\mathbf{SG}_{2,n-1}(\mathbf{R}), g_1)$ corresponding to $\hat{\kappa}_{a,b}$, where g_1 is the canonically normalized SO(n+1)-invariant Riemannian metric on $\mathbf{SG}_{2,n-1}(\mathbf{R})$. Moreover we define

$$D^{\wedge}_{abcd} := \frac{2^3}{1} \sum_{e,f,g,h=0}^n \delta^{efgh}_{abcd} \hat{k}_{e,f} \hat{k}_{g,h},$$

$$\Delta^{\wedge}_1 := \frac{1}{4!} \sum_{a,b,e,d=0}^n (D^{\wedge}_{abcd})^* D^{\wedge}_{abcd}.$$

Notice that Δ_0^{\uparrow} is a representative of the Laplace Beltrami operator on $(\mathbf{SG}_{2,n-1}(\mathbf{R}), g_1)$ and expressed explicitly as

$$-(\delta^{ab}-q^a_{a}q^b_{eta}(
ho^2)^{lphaeta})
ho^2_{\gamma\delta}\partial^2/\partial q^a_{\gamma}\partial q^b_{\delta}+(n-1)q^a_{a}\partial/\partial q^a_{a}$$
 ,

where the convention of dummy indices is adopted. (cf. [7] p. 64.)

Theorem 4.3. The natural SO(n+1)-action on $S^*(S^n)$ and $C^{\infty}(SG_{2,n-1}(\mathbf{R}))$ commutes with the Radon transform.

Proof. This follows from the definition of the Radon transform. Q.E.D.

Corollary. Let $\eta^{\wedge} \in C^{\infty}(W_2(\mathbb{R}^{n+1}))$ be the image of $\eta \in S^*(S^n)$ by the Radon transform. Then

$$\hat{\kappa}_{a,b}\eta^{\wedge} = (\kappa_{a,b}\eta)^{\wedge}.$$

Proof. The assertion is an infinitesimal version of Theorem 4.3. Q.E.D.

Theorem 4.4. Let $\eta \in S^*(S^n)$ and let η^{\wedge} be its Radon transform.

(1)
$$(\Delta_0 \eta)^{\wedge} = \Delta_0^{\wedge} \eta^{\wedge},$$

(2)
$$(\Delta_1 \eta)^{\wedge} = \Delta_1^{\wedge} \eta^{\wedge}$$

Proof. By the definitions of Δ_0 , Δ_1 , $(\Delta_0)^{\wedge}$, and $(\Delta_1)^{\wedge}$, the assertion follows from the previous corollary. Q.E.D.

DEFINITION 4.3. Denote by $E_{p,k}^{\wedge}$ the image of $E_{p,k}$ by the Radon transform.

Theorem 4.5. Assume n+1>4.

(1) Δ_0° and Δ_1° are generators of the algebra $D(SG_{2,n-1}(\mathbf{R}))$ of SO(n+1)invariant differential operators on $SG_{2,n-1}(\mathbf{R})$.

(2)
$$\Delta^{\wedge}_{0|E^{\wedge}_{p,k}} = \lambda_{p,k} \mathbf{1}_{p,k} \quad and \quad \Delta^{\wedge}_{1|E^{\wedge}_{p,k}} = \mu_{p,k} \mathbf{1}_{p,k}$$

where $\mathbf{1}_{p,k}$ is the identity operator of $E_{p,k}^{\wedge}$. The totality of

 $E_{p,k}^{\wedge}, p \geq 2k \geq 0$,

gives all of the simultaneous eigenspaces of Δ_0^{\wedge} and Δ_1^{\wedge} .

(3) Each $E_{p,k}^{\wedge}$ is an SO(n+1)-irreducible subspace.

Proof. Notice that $SG_{2,2}(\mathbf{R})$ is known to be globally homothetic to $S^2 \times S^2$ with the canonical metric. So we omit to detail of $D(SG_{2,2}(\mathbf{R}))$ as the reduced case.

(1) That Δ_0° and Δ_1° are invariant differential operators is a direct consequence of Theorem 4.4. It is known that the algebra $D(SG_{2,n-1}(\mathbf{R}))$ is generated by two operators of order 2 and 4, respectively. (cf. [3] Ch.II.) It remains to show that Δ_0° and Δ_1° are algebraically independent. Suppose that they are not so, then we can write

$$\Delta_1^{\wedge} = a (\Delta_0^{\wedge})^2 + b \Delta_0^{\wedge} + c$$

for some constants a, b and $c((a, b, c) \neq (0, 0, 0))$. On the other hand, in virtue of Theorem 3.2 we can easily verify that Δ_1^{\uparrow} acts trivially on $\sum_{k=0}^{\infty} E_{2k,k}^{\circ}$ (direct sum). If we restict the action of Δ_1^{\uparrow} to each $E_{2k,k}^{\circ}$, we would obtain an polynomial equation of one variable of order at most four with an infinite number of solutions $k=1, 2, \cdots$, from which we can conclude a=b=c=0. This is a contradiction. Thus our assertion (1) is proved.

(2) follows from Theorem 4.4 and Theorem 4.2 (2). In order to prove (3) we need

Lemma 4.2. If $\lambda_{p,k} = \lambda_{p',k'}$ and $\mu_{p,k} = \mu_{p',k'}$ then p = p' and k = k'.

Proof. We can easily verify

- (1) $\lambda_{p,k} > \lambda_{p,k+1}$,
- (2) $\lambda_{p,k} < \lambda_{p+1,k} ,$

 $\lambda_{p,k} < \lambda_{p+1,k+1},$

On the other hand, we can see

- $(4) \qquad \qquad \mu_{p,k} > \mu_{p,k+1},$
- (5) $\mu_{p,k} < \mu_{p+1,k}$,
- but
- $\mu_{p,k} > \mu_{p+1,k+1}$

in contrast with (3). From these the required property follows immediately. Q.E.D.

Proof of Theorem 4.5. (3) In virtue of Lemma 4.2, $E_{p,k}^{\circ}$ is known to be maximal in $\mathbf{R}(P^{ab}: n \ge b > a \ge 0)$ as the subspace of simultaneous eigenfunctions of the eigenvalues $\lambda_{p,k}$ and $\mu_{p,k}$ for Δ_0° and Δ_1° , respectively. For the irreducibility of $E_{p,k}^{\circ}$, we refer to [2] p. 401, Corollary 3.3. Q.E.D.

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