



Title	Two counterexamples to Cornea's conjecture on thin sets
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Citation	Osaka Journal of Mathematics. 1995, 32(1), p. 121-124
Version Type	VoR
URL	https://doi.org/10.18910/7937
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TWO COUNTEREXAMPLES TO CORNEA'S CONJECTURE ON THIN SETS

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(Received July 5, 1993)

1. Introduction

In the paper of Cornea ([1], p. 836) is the following conjecture: A set $A \subset \mathbf{R}^d$ is thin at 0 if there exist $v_1, v_2, v_3 \in \mathbf{R}^d$ linearly independent (pairwise, if $d=2$) with $\|v_j\|=1$ and such that $T_{v_j}(A)$ is thin at 0, $j=1,2,3$, where $T_v(x) := x - \langle x, v \rangle v$. We show that this conjecture fails.

We recall that the *fine topology* on \mathbf{R}^d is the smallest topology on \mathbf{R}^d for which all superharmonic functions are continuous in the extended sense. A set $E \subset \mathbf{R}^d$ is *thin* at x if x is not a fine limit point of E . The *Wiener test* relates thinness of E to the *capacity* of certain subsets of E . We note that thinness of a set at a point is related to irregularity of boundary points relative to the Dirichlet problem. For general information see [2], [3].

2. An example in \mathbf{R}^2

We denote P_x, P_y, P_z and P_w the orthogonal projections which map \mathbf{R}^2 onto a line through the origin in such a way that the points $(0,1)$, $(1,0)$, $(1,-1)$ and $(1,1)$, respectively are mapped to the origin. We set $I_2 := \{(x,y) \in \mathbf{R}^2, 0 \leq x \leq 1, 0 \leq y \leq 1\}$, cap denotes the logarithmic capacity.

Lemma 2.1. *Given $\varepsilon > 0$ there exists a set $E \subset I_2$ such that $\text{cap}(P_x E) < \varepsilon$, $\text{cap}(P_y E) < \varepsilon$, $\text{cap}(P_z E) = 0$ and $\text{cap}(E) \geq \text{cap}(P_w E) \geq \sqrt{2}/8$.*

Proof. We set $A := \{(x,0) \in I_2, x \in \mathbf{Q}\}$, A is countable, hence $\text{cap}(A) = 0$. There exists an open set $U \supset A$ in \mathbf{R}^2 such that $\text{cap}(U) < \varepsilon$. Denote $V := \{(x,0) \in I_2\} \cap U$. We set $E := \{(x,y) \in I_2, (x,0) \in V, 0 \leq y \leq \varepsilon, x+y \in \mathbf{Q}\}$. Then

- (i) $P_x E = V \subset U$, hence $\text{cap}(P_x E) < \varepsilon$;
- (ii) $P_y E = \{(0,y) \in I_2, 0 \leq y \leq \varepsilon\}$, hence $\text{cap}(P_y E) < \varepsilon$;

(iii) $P_z E$ is countable, hence $\text{cap}(P_z E) = 0$.

Denote l the segment joining points $(0,0)$ and $(1/2, -1/2)$. Then $l \subset P_w E$, hence $\text{cap}(E) \geq \text{cap}(P_w E) \geq \text{cap}(l) = \sqrt{2}/8$. \square

REMARK 2.2. We show that the set E can be constructed to be compact: We find real numbers $0 = \alpha_0 < \beta_0 < \dots < \alpha_n < \beta_n = 1$ such that $\alpha_j - \beta_{j-1} < \varepsilon$, for $j = 1, \dots, n$, and

$$\text{cap}(\{(x,0) \in I_2, \alpha_j \leq x \leq \beta_j \text{ for some } j=0, \dots, n\}) < \varepsilon.$$

(Here we use the Wiener capacity, which is countably subadditive.)

For each $j=0, 1, \dots, n$ we construct lines $l_1^j, \dots, l_{k_j}^j$ with slopes -1 such that the point $(\beta_j, 0) \in l_1^j$, $(\alpha_j, \varepsilon) \in l_{k_j}^j$ and the distance between l_p^j and l_{p+1}^j is less than $\sqrt{2}/(\beta_j - \alpha_j)$. We set

$$\tilde{E} := \bigcup_{j=0}^n \bigcup_{p=1}^{k_j} (l_p^j \cap \{(x,y) \in I_2, \alpha_j \leq x \leq \beta_j, 0 \leq y \leq \varepsilon\}).$$

The set \tilde{E} is compact (consists only of finitely many segments) and the estimates of $\text{cap}(P_x \tilde{E})$, $\text{cap}(P_y \tilde{E})$, $\text{cap}(P_z \tilde{E})$ and $\text{cap}(P_w \tilde{E})$ can be obtained similarly as in the proof of Lemma 2.1.

COUNTEREXAMPLE 2.3. *There is a set E in \mathbf{R}^2 such that E is not thin at the origin and the projections $P_x E$, $P_y E$ and $P_z E$ are thin at the origin.*

Proof. Let E_n be the set E in Lemma 2.1 constructed with $\varepsilon = 1/2^{n^3}$. Set

$$E := \bigcup_{n=3}^{\infty} \frac{1}{2^{n+1}} \cdot ((1,1) + \frac{1}{2} \cdot E_n).$$

Denote $A_n := \{a \in \mathbf{R}^2, 1/2^{n+1} \leq \|a\| \leq 1/2^n\}$. Then

$$\text{cap}(P_x E \cap A_n) = \text{cap}(P_x(E \cap A_n)) < 1/2^{n^3}.$$

$$\text{cap}(P_y E \cap A_n) = \text{cap}(P_y(E \cap A_n)) < 1/2^{n^3}.$$

$$\text{cap}(P_z E \cap A_n) = \text{cap}(P_z(E \cap A_n)) = 0, \text{ and}$$

$$\text{cap}(E \cap A_n) \geq \text{cap}(P_w(E \cap A_n)) \geq \sqrt{2}/(16 \cdot 2^{n+1}).$$

Hence $P_x E$, $P_y E$ and $P_z E$ are thin, and E is not thin at the origin due to the Wiener test. \square

3. An example in R^3

We denote P_{xy} , P_{xz} , P_{yz} and P_{wy} the orthogonal projections, which map R^3 onto a plane through the origin in such a way, that $(0,0,1)$, $(0,1,0)$, $(1,0,0)$ and $(1,0,1)$, respectively are mapped to the origin. We set $I_3 := \{(x,y,z) \in R^3, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$, c denotes the Newton capacity.

Lemma 3.1. *Given $\varepsilon > 0$ there exists a set $E \subset I_3$ such that $c(P_{xy}E) < \varepsilon$, $c(P_{xz}E) < \varepsilon$, $c(P_{yz}E) < \varepsilon$ and $c(E) \geq c(P_{wy}E) \geq c(P_{wy}\{(x,y,0) \in I_3\})$, ($=: b > 0$).*

Proof. We set $A := \{(x,y,0) \in I_3, x \in Q\}$, hence $c(A) = 0$. There exists an open set $U \supset A$ in R^3 such that $c(U) < \varepsilon$. Denote $V := \{(x,y,0) \in I_3\} \cap U$.

We set $L := \{(0,y,0) \in I_3\}$, hence $c(L) = 0$. We find $\delta < \varepsilon$ such that $c(\{(0,y,z) \in I_3, 0 \leq z \leq \delta\}) < \varepsilon$.

We set $E := \{(x,y,z) \in I_3, (x,y,0) \in V, 0 \leq z \leq \delta\}$. Then

- (i) $P_{xy}E \subset V$, hence $c(P_{xy}E) \leq c(U) < \varepsilon$;
- (ii) $P_{xz}E \subset \{(x,0,z) \in I_3, 0 \leq z \leq \delta\}$, hence $c(P_{xz}E) < \varepsilon$ due to construction of δ ;
- (iii) $P_{yz}E \subset \{(0,y,z) \in I_3, 0 \leq z \leq \delta\}$, hence $c(P_{yz}E) < \varepsilon$ due to construction of δ .

Nevertheless $P_{wy}E$ contains the set $P_{wy}(\{(x,y,0) \in I_3\})$, hence $c(E) \geq c(P_{wy}E) \geq b$. \square

REMARK 3.2. We show that the set E can be constructed in such a way that it is a compact set consisting of finitely many rectangles (like in Remark 2.2).

COUNTEREXAMPLE 3.3. *There is a set E in R^3 such that E is not thin at the origin and the projections $P_{xy}E$, $P_{xz}E$ and $P_{yz}E$ are thin at the origin.*

Proof. Let E_n be the set E in Lemma 3.1 constructed with $\varepsilon = 1/2^n$. Set

$$E := \bigcup_{n=10}^{\infty} \frac{1}{2^{n+1}} \cdot ((1,1,1) + \frac{1}{2} \cdot E_n).$$

The rest of the proof runs like in the proof of Counterexample 2.3. \square

REMARK 3.4. A counterexample to Cornea's conjecture in dimension $d > 3$ can be derived from the set E in Counterexample 3.3. It suffices to consider the set $E \times R^{d-3} \subset R^d$ because, for any set $F \subset R^3$, $F \times R^{d-3}$ is thin at 0 in R^d if and only if F itself is thin at 0 in R^3 (in the sense of potential theory in R^3).

REMARK 3.5. Cornea states in [1], Remark on the page 836, that the conjecture is true for a set A contained in a set of the form $\bigcup_{j=0}^{\infty} G_j$, where G_j is a Lipschitz manifold (graph of a Lipschitz function). It should be compared with

Counterexample 2.3, where the set obtained is contained in countably many lines.

References

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