

Title	On the degrees of irreducible representations of finite groups
Author(s)	Watanabe, Atumi
Citation	Osaka Journal of Mathematics. 1981, 18(2), p. 403-406
Version Type	VoR
URL	<a href="https://doi.org/10.18910/7939">https://doi.org/10.18910/7939</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

## ON THE DEGREES OF IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS

ATUMI WATANABE

(Received November 29, 1979)

### 1. Introduction

Let  $G$  be a finite group of order  $|G|$  and  $F$  be an algebraically closed field of characteristic 0. Let  $T$  be an irreducible representation of  $G$  over  $F$  and  $d_T$  be the degree of  $T$ . As is well known,  $d_T$  divides  $|G|$ . Furthermore there exists a sharper result due to Ito [2], namely,  $d_T$  divides the index in  $G$  of every abelian normal subgroup of  $G$ . Let  $s_T$  be the order of  $\det T$ , that is,  $s_T$  is the smallest natural number such that  $|T(x)|^{s_T}=1$  for all  $x \in G$ , where  $|T(x)|$  is the determinant of  $T(x)$ . In Lemma of [4] we showed the first part of the following

**Theorem 1.** *Let  $T$  be an irreducible representation of  $G$  over  $F$ . Then we have*

- (i)  $d_T s_T \mid 2|G|$ ,
- (ii) if  $d_T$  or  $s_T$  is odd then  $d_T s_T \mid |G|$ .

The second part follows from (i) by considering the 2-part of  $d_T s_T$ , since both  $d_T$  and  $s_T$  divide  $|G|$ .

The purpose of the present paper is to prove the following theorems.

**Theorem 2.** *If  $G$  has an irreducible representation  $T$  over  $F$  with  $d_T s_T \nmid |G|$ , then the following holds.*

- (i) *A 2-Sylow subgroup  $P$  of  $G$  is cyclic and  $P \neq 1$ . Hence  $G$  has the normal 2-complement  $K$ .*
- (ii)  $C_P(K)=1$ .
- (iii)  *$T$  is induced from a representation of  $K$ .*

The converse of Theorem 2 is also true:

**Theorem 3.** *If  $G$  satisfies (i) and (ii) in Theorem 2, then  $G$  has an irreducible representation  $T$  with  $d_T s_T \nmid |G|$ .*

We also have the following

**Theorem 4.** *Let  $T$  be an irreducible representation of  $G$  over  $F$ . Then we have*

$$d_T s_T \leq |G|.$$

If  $d_T s_T = G$ , then  $G$  is cyclic.

I express my thanks to Professor H. Nagao for his valuable advice.

### 2. Proofs of the theorems

To prove our theorems we need the following Lemma.

**Lemma.** *Let  $T$  be an irreducible representation of  $G$  over  $F$ ,  $H$  a normal subgroup of  $G$  of index  $n$  and  $T_0$  be an irreducible component of  $T_H$ ,  $T_H$  the restriction of  $T$  to  $H$ . Then we have the following.*

- (i) *If  $T_H = T_0$ , then  $d_T = d_{T_0}$  and  $s_T \mid n s_{T_0}$ .*
- (ii) *If  $T = T_0^G$ , then  $d_T = n d_{T_0}$  and  $s_T \mid 2 s_{T_0}$ . Furthermore if  $2 \mid d_{T_0} s_{T_0}$  or  $s_T$  is odd, then  $s_T \mid s_{T_0}$ .*

Proof. (i) is clear. We prove (ii). Clearly  $d_T = n d_{T_0}$ . We set  $s_T = s$ ,  $d_{T_0} = d_0$  and  $s_{T_0} = s_0$ . Let  $x_1, \dots, x_n$  be a complete set of coset representatives of  $H$  in  $G$ . We extend  $T_0$  to all elements of  $G$  by setting  $T_0(x) = 0$  for all  $x \notin H$ . We may assume that  $T(x)$  is a  $n \times n$  matrix of blocks whose  $(i, j)$ -th entry is the  $d_0 \times d_0$  matrix  $T_0(x_i^{-1} x x_j)$ :

$$T(x) = \begin{pmatrix} T_0(x_1^{-1} x x_1) & \dots & T_0(x_1^{-1} x x_n) \\ \dots & \dots & \dots \\ T_0(x_n^{-1} x x_1) & \dots & T_0(x_n^{-1} x x_n) \end{pmatrix} \quad (x \in G).$$

Hence for each  $x \in G$ , we have  $|T(x)| = (-1)^{d_0 m} |T_0(y_1)| \cdots |T_0(y_n)|$ , where  $y_i \in H$ ,  $i = 1, \dots, n$ , and  $m$  is an integer. Therefore, for each  $x \in G$ ,  $|T(x)|^{2s_0} = 1$  and hence  $s \mid 2s_0$ . If  $s$  is odd then  $s \mid s_0$ , and if  $d_0$  or  $s_0$  is even then  $|T(x)|^{s_0} = 1$  for each  $x \in G$  and hence  $s \mid s_0$ .

Proof of Theorem 2. We prove (i) by induction on  $|G|$ . Put  $d_T = d$  and  $s_T = s$ . Since  $ds \nmid |G|$ , by Theorem 1, (ii)  $2 \mid d$  and  $2 \mid s$ . In particular  $P \neq 1$  and  $2 \mid |G: G'|$ , where  $G'$  is the commutator subgroup of  $G$ . Let  $H$  be a normal subgroup of  $G$  of index 2 and  $T_0$  be an irreducible component of  $T_H$ . By Clifford's theorem,  $T_H = T_0$  or  $T = T_0^G$ . Put  $d_{T_0} = d_0$  and  $s_{T_0} = s_0$ .

(a) Suppose  $T_H = T_0$ . Since  $ds \nmid |G|$ , by Lemma, (i)  $d_0 s_0 \nmid |H|$  and hence both  $d_0$  and  $s_0$  are even. Therefore by the induction hypothesis,  $P \cap H$  is cyclic. Suppose  $P$  is not cyclic. For each  $x \in P$ ,  $\langle x^2 \rangle \neq P \cap H$  and  $|T(x)|^2 = |T_0(x^2)|$ . Hence  $|T_0(x^2)|^{s_0/2} = 1$  and  $|T(x)|^{s_0} = 1$ . On the other hand, for each 2-regular element  $x \in G$ ,  $|T(x)|^{s_0} = 1$ , because  $s \mid 2s_0$ . Therefore for each  $x \in G$ ,  $|T(x)|^{s_0} = 1$  and hence  $s \mid s_0$ . Thus  $ds \mid d_0 s_0$  and  $d_0 s_0 \mid 2|H| = |G|$ , which is a contradiction. Therefore  $P$  is cyclic.

(b) Suppose  $T = T_0^G$ . We may assume  $4 \mid |G|$ . Suppose  $d_0 s_0$  is odd.

Then since  $d=2d_0$ ,  $s$  is even and  $s \mid 2s_0$  we have  $ds=4r$  with  $r$  odd. By Theorem 1, (i)  $r \mid |G|$  and hence  $ds \mid |G|$ , which is a contradiction. Thus  $2 \mid d_0s_0$  and by Lemma, (ii)  $s \mid s_0$ . Then  $ds \mid 2d_0s_0$  and hence  $d_0s_0 \nmid |H|$ . By the induction hypothesis,  $P \cap H$  is cyclic. Suppose  $P$  is not cyclic. For each  $x \in P \cap H$ ,  $|T(x)| = |T_0(x)| |T_0(y^{-1}xy)| = |T_0(xy^{-1}xy)|$ , where  $y$  is an element of  $P$  which does not belong to  $P \cap H$ . Since  $P \cap H$  is a cyclic 2-group and  $x$  and  $y^{-1}xy$  are of the same order,  $xy^{-1}xy$  does not generate  $P \cap H$ . Hence  $|T_0(xy^{-1}xy)|^{s_0/2} = 1$  and  $|T(x)|^{s_0/2} = 1$ . For each  $x \in P$  which does not belong to  $P \cap H$ ,  $|T(x)| = |T_0(x^2)|$ , because  $d_0$  is even. Since  $P$  is not cyclic,  $|T_0(x^2)|^{s_0/2} = 1$ , hence  $|T(x)|^{s_0/2} = 1$ . Therefore for each  $x \in P$ ,  $|T_0(x)|^{s_0/2} = 1$ . On the other hand, for each 2-regular element  $x \in G$ ,  $|T_0(x)|^{s_0/2} = 1$  because  $s \mid s_0$ . Hence  $s \mid s_0/2$  and  $ds \mid 2d_0 \cdot s_0/2 = d_0s_0$ . By Theorem 1, (i) we have  $ds \mid |G|$ . This is a contradiction. Therefore  $P$  is cyclic. By Burnside's theorem  $G$  has the normal 2-complement  $K$ . Thus (i) is proved.

Now we show (iii). Let  $T_1$  be an irreducible component of  $T_K$ ,  $\tilde{K}$  the inertial group of  $T_1$  in  $G$  and let  $\tilde{T}$  be an irreducible representation of  $\tilde{K}$  such that  $T = \tilde{T}^G$  and that  $T_1$  is an irreducible component of  $\tilde{T}_K$ . Put  $d_{T_1} = d_1$ ,  $s_{T_1} = s_1$ ,  $d_{\tilde{T}} = \tilde{d}$  and  $s_{\tilde{T}} = \tilde{s}$ . Since  $\tilde{K}/K$  is cyclic,  $\tilde{T}_K = T_1$  and  $\tilde{d} = d_1$  (see the proof of [1, (9.12)]). As  $\tilde{d}$  is odd,  $\tilde{d}\tilde{s} \mid |\tilde{K}|$  by Theorem 1, (ii). If  $2 \mid \tilde{d}\tilde{s}$ , then  $s \mid \tilde{s}$  by Lemma, (ii). Hence  $ds \mid |G: \tilde{K}| \tilde{d}\tilde{s} \mid |G: \tilde{K}| |\tilde{K}| = |G|$ . This yields a contradiction. Hence  $2 \nmid \tilde{d}\tilde{s}$ . Since  $|\tilde{K}: K|$  is a power of 2, by Theorem 1, (i)  $\tilde{d}\tilde{s} \mid |K|$ . Therefore  $ds \mid 2|G: \tilde{K}| \tilde{d}\tilde{s} \mid 2|G: \tilde{K}| |K|$ . Thus we see  $\tilde{K} = K$ . This completes the proof of (iii).

Finally we prove (ii). From (iii),  $|P| \mid d$ . From (i),  $C_P(K)$  is a central subgroup of  $G$ . Hence  $d \mid |G: C_P(K)|$ . Therefore  $C_P(K) = 1$ . This completes the proof of the theorem.

Proof of Theorem 3. We set  $|P| = 2^a$ ,  $P = \langle x \rangle$  and  $y = x^{2^a-1}$ . Since  $C_P(K) = 1$ ,  $y$  induces a non-identity automorphism of  $K$ . By [3, Satz 108], there is a conjugate class of  $K$  which is not fixed by  $y$ . Hence  $y$  does not fix some irreducible representation of  $K$  over  $F$ , say  $T_0$ . Since  $\langle yK \rangle$  is the unique minimal subgroup of  $G/K$ ,  $K$  is the inertial group of  $T_0$  in  $G$ . Hence  $T_0^G$  is an irreducible representation of  $G$ . We set  $T = T_0^G$ . Then  $2^a \mid d_T$  and we see  $|T(x)| = -1$ . Hence  $d_T s_T \nmid |G|$ .

Proof of Theorem 4. We prove by induction on  $|G|$ . If  $G$  is abelian, then the theorem is trivial. We assume that the theorem is true for any proper subgroup of  $G$ . First we prove  $ds \leq |G|$ . Suppose  $ds > |G|$ . By Theorem 1, (i)  $ds = 2|G|$ . Since  $d \mid |G|$ ,  $s$  is even and  $2 \mid |G: G'|$ . Let  $H$  be a normal subgroup of  $G$  of index 2 and  $T_0$  be an irreducible component of  $T_H$ . By the induction hypothesis and Lemma,  $T = T_0^G$  and  $d_{T_0} s_{T_0} = |H|$ . By the induction hypothesis,  $H$  is cyclic,  $d = 2$  and  $s = |G|$ . Hence  $G$  is abelian, which con-

tradicts  $d=2$ . Thus we have proved  $ds \leq |G|$ . Next we prove the remaining part of the theorem. Suppose  $ds = |G|$ . We may assume  $G \neq 1$ . Since  $d < |G|$ ,  $s \neq 1$  and hence  $G \neq G'$ . Let  $L$  be a normal subgroup of  $G$  of prime index  $p$  and  $T_1$  be an irreducible component of  $T_L$ . We prove  $L$  is cyclic. By the induction hypothesis and Lemma, if  $T_L = T_1$  or if  $T = T_1^G$  and  $d_{T_1} s_{T_1}$  is even, we see easily  $L$  is cyclic. Put  $d_{T_1} = d_1$  and  $s_{T_1} = s_1$ . If  $T = T_1^G$  and  $d_1 s_1$  is odd, then we see  $|L| = d_1 s_1$  or  $|L| = 2d_1 s_1$ . By the induction hypothesis,  $|L| = d_1 s_1$  implies  $L$  is cyclic. In the case  $|L| = 2d_1 s_1$ , let  $U$  be the normal 2-complement of  $L$ . As  $d_1 s_1$  is odd, by Clifford's theorem and Lemma, (ii) we see that  $(T_1)_U$  is irreducible and  $s_1$  is the order of  $\det (T_1)_U$  and that  $d_1 s_1 = |U|$ . Hence by the induction hypothesis,  $U$  is cyclic. Hence  $d_1 = 1$  and  $s_1 = |U|$  and hence  $|L'| \mid 2$ . On the other hand  $L' \subset U$ . Therefore  $L' = 1$  and  $L$  is cyclic. Thus we have proved that  $L$  is cyclic. If  $T_L = T_1$ , then  $d = 1$  and  $s = |G|$ , hence  $G$  is cyclic. Suppose  $T = T_1^G$ , then  $d = p$ ,  $|G'| = p$  and  $s = |G : G'|$ . Let  $M$  be any normal subgroup of  $G$  of prime index. By the argument applied to  $L$  and by  $d = p$ ,  $|G : M| = p$ . Hence  $G/G'$  is a  $p$ -group and hence  $G$  is a  $p$ -group. Since  $|G'| = p$ ,  $G'$  is a central subgroup of  $G$ . By  $s = |G : G'|$ ,  $G/G'$  is cyclic. Therefore  $G$  is abelian. This contradicts  $d = p$ , and this completes the proof.

---

### References

- [1] W. Feit: *Characters of finite groups*, Benjamin, New York, 1967.
- [2] N. Ito: *On the degrees of irreducible representations of a finite group*, Nagoya Math. J. **3** (1951), 5-6.
- [3] A. Speiser: *Die Theorie der Gruppen von endlicher Ordnung*, 3rd., Berlin.
- [4] A. Watanabe: *On Fong's reductions*, Kumamoto J. Sci. (Math.) **13** (1979), 48-54.

Department of Mathematics  
Faculty of Science  
Kumamoto University  
(860) Kurokami 2-39-1,  
Kumamoto, Japan