Akaike, H.
Osaka J. Math.
58 (2021), 451–475

SLOPE INEQUARITIES FOR IRREGULAR CYCLIC COVERING FIBRATIONS

Hiroto AKAIKE

(Received May 13, 2019, revised January 30, 2020)

Abstract

Let $f : S \to B$ be a finite cyclic covering fibration of a fibered surface. We study the lower bound of the slope λ_f when the relative irregularity q_f is positive.

Introduction

Let $f : S \to B$ be a surjective morphism from a smooth projective surface S to a smooth projective curve B with connected fibers. We call it a fibration of genus g when a general fiber is a curve of genus g. A fibration is called relatively minimal, when any (-1)-curve is not contained in fibers. Here we call a smooth rational curve C with $C^2 = -n$ a $(-n)$-curve. A fibration is called *smooth* when all fibers are smooth, *isotrivial* when all of the smooth fibers are isomorphic, *locally trivial* when it is smooth and isotrivial.

Assume that $f : S \to B$ is a relatively minimal fibration of genus $g \geq 2$. We denote by $K_f = K_S - f^*K_B$ a relative canonical divisor. We associate three relative invariants with f:

$$\begin{aligned} K_f^2 &= K_S^2 - 8(g-1)(b-1), \\ \chi_f &:= \chi(\mathcal{O}_S) - (g-1)(b-1), \\ e_f &:= e(S) - 4(g-1)(b-1), \end{aligned}$$

where b and $e(S)$ respectively denote the genus of the base curve B and the topological Euler-Poincaré characteristic of S. Then the following are well-known:

- (Noether) $12\chi_f = K_f^2 + e_f$.
- (Arakelov) K_f is nef.
- (Ueno) $\chi_f \geq 0$ and $\chi_f = 0$ if and only if f is locally trivial.
- (Segre) $e_f \geq 0$ and $e_f = 0$ if and only if f is smooth.

When f is not locally trivial, we put

$$\lambda_f := \frac{K_f^2}{\chi_f}$$

and call it the *slope* of f according to [5], in which Xiao succeeded in giving its effective lower bound as

$$\lambda_f \geq \frac{4(g-1)}{g}.$$

Another invariant we are interested in is the *relative irregularity* of f defined by $q_f :=$

2020 Mathematics Subject Classification. Primary 14D06, 14H10; Secondary 14D99, 14J29.

$q(S) - b$, where $q(S) := \dim H^1(S, \mathcal{O}_S)$ denotes the irregularity of S as usual. When q_f is positive, we call f an *irregular* fibration. Xiao showed in [5] that $\lambda_f \geq 4$ holds for irregular fibrations. It seems a general rule that the lower bound of the slope goes up, when the relative irregularity gets bigger.

In the present paper, we consider primitive cyclic covering fibrations of type (g, h, n) introduced in [2], where Enokizono gave the lower bound of the slope for them. Note that it is nothing more than a hyperelliptic fibration when $h = 0$ and $n = 2$. Recall that Lu and Zuo obtained the lower bound of the slope for irregular double covering fibrations in [4] and [3]. Inspired by their results, we try to generalize them to irregular primitive cyclic covering fibrations with $n \geq 3$. We give the lower bound of slope for those of type $(g, 0, n)$ in Theorems 3.4 and 4.7, and for those of type (g, h, n) with $h \geq 1$ in Theorem 3.8.

The key observation for the proof is Proposition 3.2. We apply it to the anti-invariant part of the Albanese map with respect to the action of the Galois group canonically associated to the cyclic covering fibration, and derive the "negativity" of the ramification divisor when $q_f > 0$. Recall that χ_f and (the essential part of) K_f^2 can be expressed in terms of the so-called k-th *singularity index* α_k defined for each non-negative integer k. The negativity referred above can be used to get some non-trivial restrictions on α_0 which is the most difficult one to handle with among all α_k's. Thanks to such information together with an analysis of the Albanese map, we can obtain the desired slope inequalities.

We also give a small contribution to the modified Xiao's conjecture that $q_f \leq \lceil \frac{g+1}{2} \rceil$ holds, posed by Barja, González-Alonso and Naranjo in [1]. It is known to be true for fibrations of maximal Clifford index [1] and for hyperelliptic fibrations [4] among others. We show in Theorem 4.5 that $q_f \leq (g + 1 - n)/2$ holds, when f is a primitive cyclic covering fibration of type $(g, 0, n)$ under some additional assumptions. For the history around the conjecture, see the introduction of [1].

The author express his sincere gratitude to Professor Kazuhiro Konno for suggesting this assignment, his valuable advice and support. The author also thanks Dr. Makoto Enokizono for his precious advices, allowing him to use Proposition 3.2 freely.

1. Primitive cyclic covering fibrations

We recall the basic properties of primitive cyclic covering fibrations, most of which can be found in [2].

Definition 1.1. Let $f : S \to B$ be a relatively minimal fibration of genus $g \geq 2$. We call it a primitive cyclic covering fibration of type (g, h, n), when there are (not necessarily relatively minimal) fibration $\tilde{\varphi} : \widetilde{W} \to B$ of genus $h \geq 0$ and a classical n-cyclic covering

$$\tilde{\theta} : \widetilde{S} = \mathrm{Spec}_{\widetilde{W}} \left(\bigoplus_{j=0}^{n-1} \mathcal{O}_{\widetilde{W}}(-j\tilde{\mathfrak{d}}) \right) \to \widetilde{W}$$

branched over a smooth curve $\widetilde{R} \in |n\tilde{\mathfrak{d}}|$ for some $n \geq 2$ and $\tilde{\mathfrak{d}} \in \mathrm{Pic}(\widetilde{W})$ such that f is the relatively minimal model of $\tilde{f} = \tilde{\varphi} \circ \tilde{\theta}$.

Let $f : S \to B$ be a primitive cyclic covering fibration of type (g, h, n). Let $\widetilde{F}$, F, $\widetilde{\Gamma}$ and Γ be general fibers of $\tilde{f}$, f, $\tilde{\varphi}$ and φ, respectively. Then the restriction map $\tilde{\theta}|_{\widetilde{F}} : \widetilde{F} \to \widetilde{\Gamma}$ is a classical n-cyclic covering branched over $\widetilde{R} \cap \widetilde{\Gamma}$. By the Hurwitz formula for $\tilde{\theta}|_{\widetilde{F}}$, we get

$$r := \widetilde{R}\widetilde{\Gamma} = \frac{2(g-1-n(h-1))}{n-1}. \tag{1.1}$$

From $\widetilde{R} \in |n\widetilde{\mathfrak{d}}|$, it follows that r is a multiple of n.

Let $\tilde{\psi} : \widetilde{W} \to W$ be the contraction morphism to a relative minimal model $W \to B$ of $\tilde{\varphi} : \widetilde{W} \to B$. Since $\tilde{\psi}$ is a composite of blowing-ups, we can write $\tilde{\psi} = \psi_1 \circ \cdots \psi_N$, where $\psi_i : W_i \to W_{i-1}$ denotes the blowing-up at $x_i \in W_{i-1}$ $(i = 1, \cdots, N)$, $W_0 = W$ and $W_N = \widetilde{W}$. We define a reduced curve R_i inductively as $R_{i-1} = (\psi_i)_* R_i$ starting from $R_N = \widetilde{R}$ down to $R_0 = R$. We also put $E_i = \psi_i^{-1}(x_i)$ and $m_i = \mathrm{mult}_{x_i} R_{i-1}$ $(i = 1, \cdots, N)$.

Lemma 1.2 ([2], Lemma 1.5)**.** *In the above situation, the following hold for any* $i = 1, \cdots, N$.

(1) *Either* $m_i \in n\mathbb{Z}$ *or* $n\mathbb{Z}+1$. *Furthermore,* $m_i \in n\mathbb{Z}$ *if and only if* E_i *is not contained in* R_i.

(2) $R_i = \psi_i^* R_{i-1} - n[\frac{m_i}{n}]E_i$, *where* $[t]$ *denotes the greatest integer not exceeding* t.

(3) *There exists* $\mathfrak{d}_i \in \mathrm{Pic}(W_i)$ *such that* $\mathfrak{d}_i = \psi_i^* \mathfrak{d}_{i-1} - [\frac{m_i}{n}]E_i$ *and* $R_i \sim n\mathfrak{d}_i$, $\mathfrak{d}_N = \widetilde{\mathfrak{d}}$.

REMARK 1.3. By [2], we can assume the following for any primitive cyclic covering fibrations. Let $\tilde{\sigma}$ be a generator of the covering transformation group of $\tilde{\theta}$, and σ the automorphism of S over B induced by $\tilde{\sigma}$. Then the natural morphism $\rho : \widetilde{S} \to S$ is a minimal succession of blowing-ups that resolves all isolated fixed points of σ.

We must pay a special attention when $h = 0$, since we have various relatively minimal models for $\tilde{\varphi} : \widetilde{W} \to B$. Using elementary transformations, one can show the following.

Lemma 1.4 ([2], Lemma 3.1)**.** *Let* $f : S \to B$ *be a primitive cyclic covering fibration of type* $(g, 0, n)$. *Then there is a relatively minimal model of* $\tilde{\varphi} : \widetilde{W} \to B$ *such that*

$$\mathrm{mult}_x R_h \leq \frac{r}{2} = \frac{g}{n-1} + 1$$

for all $x \in R_h$, *where* R_h *denotes the* φ*-horizontal part of* R. *Moreover if* $\mathrm{mult}_x R > \frac{r}{2}$, *then* $\mathrm{mult}_x R \in n\mathbb{Z}+1$.

When $h = 0$, we always assume that a relatively minimal model of $\tilde{\varphi} : \widetilde{W} \to B$ is as in the above lemma.

Corollary 1.5. *Let the situation be the same as in* Lemma 1.4. *If* x *is a singular point of* R *and* $m = \mathrm{mult}_x R$, *then*

$$n\left[\frac{m}{n}\right] \leq \frac{r}{2}.$$

Proof. When $m \in n\mathbb{Z}$, the inequality clearly holds by Lemma 1.4. If $m \in n\mathbb{Z}+1$, then $n[\frac{m}{n}] + 1 = m$. From Lemma 1.4, we have $m \leq \frac{r}{2} + 1$. So we get $n[\frac{m}{n}] \leq \frac{r}{2}$. □

In closing the section, we give an easy lemma that will be usuful in the sequel.

Lemma 1.6. *Let $\pi : C_1 \to C_2$ be a surjective morphism between smooth projective curves. Let R_π and Δ be the ramification divisor and the branch locus of π, respectively. Then,*

$$(\deg(\pi) - 1)\sharp\Delta \geq \deg R_\pi,$$

where $\sharp\Delta$ denotes the cardinality of Δ as a set of points.

Proof. We put $\Delta = \{Q_1, \ldots, Q_{\sharp\Delta}\}$. For any $Q_i \in \Delta$, we put $\pi^{-1}(Q_i) = \{P^i_1, \ldots, P^i_{j_i}\}$. Note that $\deg(\pi) = r(P^i_1) + \cdots + r(P^i_{j_i})$ for any $i = 1, \ldots, \sharp\Delta$, where $r(P)$ denotes the ramification index of π around $P \in C_1$. Then, from the property of ramification divisor,

$$\begin{aligned}\deg R_\pi &= \sum_{i=1}^{\sharp\Delta}\sum_{j=1}^{j_i}(r(P^i_j) - 1)\\ &= \deg(\pi)\sharp\Delta - (j_1 + \cdots + j_{\sharp\Delta})\\ &\leq (\deg(\pi) - 1)\sharp\Delta,\end{aligned}$$

which is what we want. □

Lemma 1.7. *Let $\beta : X \to \mathbb{P}^1$ be a fibration. Take any finite sub set J of $\mathbb{P}^1$ with $\sharp J \geq 4$. For any $y \in J$, let $\Gamma_y = \sum \tilde{n}_D D$ be the fiber of β over y, and put*

$$F_{all} := \beta^*\Delta, \quad F_r := \sum_{y\in\Delta,\ D\subset\Gamma_y,\ \tilde{n}_D=1} D.$$

If there is a smooth rational curve C on X which is β-horizontal, then

$$\sharp(C \cap F_r) \geq \deg\beta|_C(\sharp J - 4) + 4 \geq \sharp J.$$

Proof. From Hurwitz formula for $\beta|_C : C \to \mathbb{P}^1$,

$$2\deg\beta|_C - 2 = \deg R_{\beta|_C},$$

where $R_{\beta|_C}$ is ramification divisor of $\beta|_C$. Let $r(P)$ be the ramification index of $\beta|_C$ at $P \in C$. Then we have

$$\deg R_{\beta|_C} = \sum_{P\in C\cap F_{all}}(r(P) - 1).$$

By definition, $r(P) \geq 2$ for any $P \in (C \cap F_{all}) \setminus (C \cap F_r)$. Hence,

$$\begin{aligned}\sum_{C\cap F_{all}}(r(P) - 1) &= \sum_{(C\cap F_{all})\setminus(C\cap F_r)}(r(P) - 1) + \sum_{C\cap F_r}(r(P) - 1)\\ &\geq \sum_{(C\cap F_{all})\setminus(C\cap F_r)}\frac{r(P)}{2} + \sum_{C\cap F_r}\frac{(r(P) - 1)}{2}\\ &= \sum_{C\cap F_{all}}\frac{r(P)}{2} - \frac{\sharp(C \cap F_r)}{2}.\end{aligned}$$

So we have

$$2\deg\beta|_C - 2 \geq \sum_{C\cap F_{all}}\frac{r(P)}{2} - \frac{\sharp(C \cap F_r)}{2}.$$

Therefore we get

$$\sharp(C \cap F_r) \geq \sum_{C \cap F_{all}} r(P) - 4 \deg \beta|_C + 4 = \deg \beta|_C (\sharp J - 4) + 4 \geq \sharp J$$

by the assumption $\sharp J \geq 4$. □

2. Singularity indices and the formulae for K_f^2 and χ_f.

We let $f : S \to B$ be a primitive cyclic covering fibration of type (g, h, n) and freely use the notation in the previous section. We obtain a classical n-cyclic covering $\theta_i : S_i \to W_i$ branched over R_i by setting

$$S_i = \operatorname{Spec}\,(\bigoplus_{j=0}^{n-1} \mathcal{O}_{W_i}(-j\mathfrak{d}_i))$$

Since R_i is reduced, S_i is a normal surface. There exists a natural birational morphism $S_i \to S_{i-1}$. Set $S' = S_0$, $\theta = \theta_0$, $\mathfrak{d} = \mathfrak{d}_0$ and $f' = \varphi \circ \theta$. Then we have a commutative diagram:

$$\begin{tikzcd}
\widetilde{S} = S_N \arrow[r] \arrow[d, "\widetilde{\theta}"] \arrow[rrrr, bend left, "\rho"] & S_{N-1} \arrow[r] \arrow[d, "\theta_{N-1}"] & \cdots \arrow[r] & S_0 = S' \arrow[d, "\theta"] & S \arrow[dl, "f"] \\
\widetilde{W} = W_N \arrow[r, "\psi_N"] \arrow[drrr, "\widetilde{\varphi}"'] & W_{N-1} \arrow[r, "\psi_{N-1}"] & \cdots \arrow[r, "\psi_1"] & W_0 = W \arrow[d, "\varphi"] & \\
& & & B &
\end{tikzcd}$$

The well-known formulae for cyclic coverings give us

$$K_{\tilde{f}}^2 = n(K_{\tilde{\varphi}}^2 + 2(n-1)\tilde{\mathfrak{d}}K_{\tilde{\varphi}} + (n-1)^2\tilde{\mathfrak{d}}^2), \tag{2.1}$$

$$\chi_{\tilde{f}} = n\chi_{\tilde{\varphi}} + \frac{1}{2}\sum_{j=0}^{n-1} j\tilde{\mathfrak{d}}(j\tilde{\mathfrak{d}} + K_{\tilde{\varphi}}), \tag{2.2}$$

(see e.g., [2]). From Lemma 1.2 and a simple calculation, we get

$$\tilde{\mathfrak{d}}^2 = \mathfrak{d}^2 - \sum_{i=1}^{N} \left[\frac{m_i}{n}\right]^2, \tag{2.3}$$

$$\tilde{\mathfrak{d}}K_{\tilde{\varphi}} = \mathfrak{d}K_{\varphi} + \sum_{i=1}^{N} \left[\frac{m_i}{n}\right] \tag{2.4}$$

and

$$K_{\tilde{\varphi}}^2 = K_{\varphi}^2 - N. \tag{2.5}$$

Definition 2.1. Let Γ_p and F_p respectively denote fibers of $\varphi : W \to B$ and $f : S \to B$ over a point $p \in B$. For any fixed $p \in B$, we consider all singular points (including infinitely near ones) of R on Γ_p. For any positive integer k, we let $\alpha_k(F_p)$ be the number of singular points of multiplicity either kn or $kn+1$ among them, and call it the k-th *singularity index* of F_p. We put $\alpha_k := \sum_{p\in B}\alpha_k(F_p)$ and call it the k-th singularity index of the fibration. We also put $\alpha_0 := (K_{\tilde{\varphi}} + \widetilde{R})\widetilde{R}$ and call it the ramification index of $\tilde{\varphi}|_{\widetilde{R}} : \widetilde{R} \to B$.

By a simple calculation, we get

$$N = \sum_{k\geq 1}\alpha_k, \tag{2.6}$$

$$\sum_{i=1}^{N}\left[\frac{m_i}{n}\right] = \sum_{k\geq 1}k\alpha_k, \tag{2.7}$$

$$\sum_{i=1}^{N}\left[\frac{m_i}{n}\right]^2 = \sum_{k\geq 1}k^2\alpha_k \tag{2.8}$$

and

$$\alpha_0 = (K_\varphi + R)R - \sum_{k\geq 1}nk(nk-1)\alpha_k. \tag{2.9}$$

Substituting (2.3), (2.4), $\cdots$,(2.8) to (2.1) and (2.2), one gets

$$K_{\tilde{f}}^2 = n(K_\varphi^2 + 2(n-1)\mathfrak{d}K_\varphi + (n-1)^2\mathfrak{d}^2) - n\sum_{k\geq 1}((n-1)k-1)^2\alpha_k$$

and

$$\chi_{\tilde{f}} = n\chi_{\tilde{\varphi}} + \frac{n(n-1)}{4}\mathfrak{d}K_\varphi + \frac{n(n-1)(2n-1)}{12}\mathfrak{d}^2 - \frac{n(n-1)}{12}\sum_{k\geq 1}((2n-1)k^2-3k)\alpha_k.$$

Since $K_f^2 \geq K_{\tilde{f}}^2$, $\chi_{\tilde{f}} = \chi_f$ and $\chi_{\tilde{\varphi}} = \chi_\varphi$, we obtain

$$K_f^2 \geq n(K_\varphi^2 + 2(n-1)\mathfrak{d}K_\varphi + (n-1)^2\mathfrak{d}^2) - n\sum_{k\geq 1}((n-1)k-1)^2\alpha_k \tag{2.10}$$

and

$$\chi_f = n\chi_\varphi + \frac{n(n-1)}{4}\mathfrak{d}K_\varphi + \frac{n(n-1)(2n-1)}{12}\mathfrak{d}^2 - \frac{n(n-1)}{12}\sum_{k\geq 1}((2n-1)k^2-3k)\alpha_k. \tag{2.11}$$

We treat the cases $h = 0$ and $h > 0$ separately.

Proposition 2.2. *Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 0, n)$ and let α_i $(i \geq 0)$ be the singularity index in* Definition 2.1. *Then,*

$$(r-1)K_f^2 \geq \frac{(r-2)n-r}{n}(n-1)\alpha_0 + \sum_{k\geq 1}^{nk\leq\frac{r}{2}}\left(\frac{n^2-1}{n}nk(r-1-(nk-1)) - (r-1)n\right)\alpha_k, \tag{2.12}$$

$$(r-1)\chi_f = \frac{(2r-3)n-r}{12n}(n-1)\alpha_0 + \sum_{k\geq 1}^{nk\leq\frac{r}{2}}\left(\frac{n^2-1}{12n}nk(r-1-(nk-1))\right)\alpha_k. \tag{2.13}$$

Proof. Note that if $\alpha_k > 0$, then $nk \leq \frac{r}{2}$ from Corollary 1.5.

We find that $R \equiv -\frac{r}{2}K_\varphi + M_0\Gamma$ for some $M_0 \in \frac{1}{2}\mathbb{Z}$, where the symbol $\equiv$ means the numerical equivalence, since $\varphi : W \to B$ is a $\mathbb{P}^1$-bundle and we have $K_W\Gamma = -2$ and $\widetilde{R}\Gamma = R\Gamma = r$. Hence we get

$$RK_\varphi = -2M_0, \tag{2.14}$$

$$R^2 = 2rM_0. \tag{2.15}$$

Therefore we have

$$(K_\varphi + R)R = 2(r-1)M_0.$$

From this equality and (2.9), we get

$$2(r-1)M_0 = \alpha_0 + \sum_{k\geq 1}^{nk\leq \frac{r}{2}} nk(nk-1)\alpha_k. \tag{2.16}$$

On the other hand, substituting (2.14) and (2.15) to (2.10), we get

$$K_f^2 \geq \frac{(r-2)n-r}{n}2(n-1)M_0 - n\sum_{k\geq 1}^{nk\leq \frac{r}{2}}((n-1)k-1)^2\alpha_k.$$

Multiplying this by $r-1$ and substituting (2.16) to it, we get (2.12). Similarly one can show (2.13). □

When $h > 0$, we have the following:

Proposition 2.3. *Let $f : S \to B$ be a primitive cyclic covering fibration of type (g, h, n) such that $h \geq 1$ and α_i $(i \geq 0)$ the singularity index in* Definition 2.1. *Put*

$$t := 2(g-1) - (h-1)(n+1),$$

$$T := \begin{cases} 2(g-1)K_\varphi R, & \text{if } h = 1, \\ -\frac{\left((g-1-n(h-1))K_\varphi-(n-1)(h-1)R\right)^2}{(n-1)(h-1)}, & \text{if } h \geq 2. \end{cases}$$

Then $t > 0$ and $T \geq 0$. Furthermore,

$$tK_f^2 \geq tx'\frac{K_\varphi^2}{(n-1)(h-1)} + ty'T + tz'\alpha_0 + \sum_{k\geq 1} a_k\alpha_k \tag{2.17}$$

and

$$t\chi_f = nt\chi_\varphi + t\bar{x}'\frac{K_\varphi^2}{(n-1)(h-1)} + t\bar{y}'T + t\bar{z}'\alpha_0 + \sum_{k\geq 1} \bar{a}_k\alpha_k \tag{2.18}$$

hold, where we define

$$x' = \frac{(g-1)(n-1)((g-1)(n+1)-2n(h-1))}{nt}, \quad \bar{x}' = \frac{(n-1)(n+1)(g-1-n(h-1))^2}{12nt},$$

$$y' = \frac{(n-1)(2-\frac{n-1}{n})}{t}, \quad \bar{y}' = \frac{(n-1)(n+1)}{12nt},$$

$$z' = \frac{2(n-1)^2(g-1)}{nt}, \quad \bar{z}' = \frac{(n-1)(2(2n-1)(g-1)-n(n+1)(h-1))}{12nt},$$

$$a_k = 12\bar{a}_k - nt, \quad \bar{a}_k = \frac{k}{12}(n^2-1)(2(g-1)+n(h-1)((n-1)k-2)).$$

In (2.17) *and* (2.18)*, we regard* $\frac{K_\varphi^2}{(n-1)(h-1)}$ *as* 0 *when* $h = 1$.

Proof. We get $t > 0$ from $r \geq 0$, $n \geq 2$ and $g \geq 2$. We shall show that $T \geq 0$. If $h = 1$, by the canonical bundle formula, we have

$$K_\varphi \equiv \chi(\mathcal{O}_W)\Gamma + \sum_{i=1}^{l}(1 - \frac{1}{k_i})\Gamma$$

where $\{k_i \mid i = 1, \ldots, l\}$ denotes the set of multiplicities of all multiple fibers of φ, $k_i \geq 2$. Hence we get

$$K_\varphi R \geq \chi(\mathcal{O}_W)\Gamma R = \frac{2(g-1)}{n-1}\chi(\mathcal{O}_W). \tag{2.19}$$

Since W is an elliptic surface, we have $\chi(\mathcal{O}_W) \geq 0$ and hence $T \geq 0$. If $h \geq 2$, we consider the intersection matrix

$$\begin{pmatrix} K_\varphi^2 & K_\varphi \mathfrak{d} & K_\varphi \Gamma \\ K_\varphi \mathfrak{d} & \mathfrak{d}^2 & \mathfrak{d}\Gamma \\ K_\varphi \Gamma & \mathfrak{d}\Gamma & 0 \end{pmatrix}$$

for $\{K_\varphi, \ \mathfrak{d}, \ \Gamma\}$. Since we have $K_\varphi^2 \geq 0$ by Arakelov's theorem, it is not negative definite. Hence its determinant is non-negative by the Hodge index theorem, and we get

$$2(K_\varphi \mathfrak{d})(\mathfrak{d}\Gamma)(K_\varphi\Gamma) - \mathfrak{d}^2(K_\varphi\Gamma)^2 - (\mathfrak{d}\Gamma)^2 K_\varphi^2 \geq 0. \tag{2.20}$$

Since

$$\mathfrak{d}\Gamma = \frac{r}{n} = \frac{2(g-1-n(h-1))}{n(n-1)}, \quad K_\varphi\Gamma = 2(h-1),$$

the inequality (2.20) is equivalent to

$$2(g-1-n(h-1))K_\varphi\mathfrak{d} - n(n-1)(h-1)\mathfrak{d}^2 \geq \frac{1}{n(n-1)(h-1)}(g-1-n(h-1))^2 K_\varphi^2.$$

So we get

$$0 \geq ((g-1-n(h-1))K_\varphi - (n-1)(h-1)R)^2$$

and, hence, $T \geq 0$.

Now, by a direct calculation, one has

$$n(K_\varphi^2 + \frac{2(n-1)}{n}K_\varphi R + (\frac{n-1}{n})^2 R^2) = x'\frac{K_\varphi^2}{(n-1)(h-1)} + y'T + z'(K_\varphi + R)R$$

and

$$n\chi_\varphi + \frac{n-1}{4}K_\varphi R + \frac{(n-1)(2n-1)}{12n}R^2 = n\chi_\varphi + \bar{x}'\frac{K_\varphi^2}{(n-1)(h-1)} + \bar{y}'T + \bar{z}'(K_\varphi + R)R.$$

Hence we obtain from (2.10) and (2.11) that

$$K_f^2 \geq x'\frac{K_\varphi^2}{(n-1)(h-1)} + y'T + z'(K_\varphi + R)R - n\sum_{k\geq 1}((n-1)k-1)^2\alpha_k \tag{2.21}$$

and

(2.22)

$$\chi_f = n\chi_\varphi + \bar{x}'\frac{K_\varphi^2}{(n-1)(h-1)} + \bar{y}'T + \bar{z}'(K_\varphi + R)R - \frac{n(n-1)}{12}\sum_{k\geq 1}((2n-1)k^2 - 3k)\alpha_k.$$

From (2.9) and (2.21), we get

$$tK_f^2 \geq tx'\frac{K_\varphi^2}{(n-1)(h-1)} + ty'T + tz'\alpha_0 + \sum_{k\geq 1}(2(n-1)^2(g-1)k(nk-1) - nt((n-1)k-1)^2)\alpha_k.$$

Since one sees $a_k = 2(n-1)^2(g-1)k(nk-1) - nt((n-1)k-1)^2$, we obtain (2.17). Similarly, we obtain (2.18). □

3. Slope inequality for irregular cyclic covering fibrations.

The purpose of this section is to show the slope inequalities for irregular cyclic covering fibrations of type (g, h, n), $n \geq 3$. We start things in a more general setting.

Definition 3.1. Let $\tilde{\theta} : \tilde{S} \to \tilde{W}$ be a finite Galois cover (not necessarily primitive cyclic) between smooth projective varieties with Galois group G. Let $\alpha : \widetilde{S} \to \mathrm{Alb}(\widetilde{S})$ be the Albanese map. For any $\tilde{\sigma} \in G$, we denote by $\alpha(\tilde{\sigma}) : \mathrm{Alb}(\widetilde{S}) \to \mathrm{Alb}(\widetilde{S})$ the morphism induced from $\tilde{\sigma} : \widetilde{S} \to \widetilde{S}$ by the universality of the Albanese map. We put

$$\mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S}) := \mathrm{Im}\{\alpha(\tilde{\sigma}) - 1 : \mathrm{Alb}(\widetilde{S}) \to \mathrm{Alb}(\widetilde{S})\}$$

and let

$$\alpha_{\tilde{\sigma}} : \widetilde{S} \to \mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S})$$

be the morphism defined by $\alpha_{\tilde{\sigma}} := (\alpha(\tilde{\sigma}) - 1) \circ \alpha$.

The following is due to Makoto Enokizono.

Proposition 3.2. *Suppose that G is a cyclic group generated by $\tilde{\sigma}$ in the above situation. If $q_{\tilde{\theta}} := q(\widetilde{S}) - q(\widetilde{W}) > 0$, then the following hold.*

(1) $\dim \mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S}) = q_{\tilde{\theta}}$.

(2) *If* $\mathrm{Fix}(G) := \{x \in \widetilde{S} \mid \tilde{\sigma}(x) = x\} \neq \emptyset$*, then it is contracted by $\alpha_{\tilde{\sigma}}$ to the unit element* $0 \in \mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S})$.

(3) *If $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is a curve, then the geometric genus of $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is not less than $q_{\tilde{\theta}}$.*

Proof. Firstly, we show (1). By the construction of $\alpha(\tilde{\sigma}) - 1 : \mathrm{Alb}(\widetilde{S}) \to \mathrm{Alb}(\widetilde{S})$, we get the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathrm{Alb}(\widetilde{S}),\Omega^1_{\mathrm{Alb}(\widetilde{S})}) & \xrightarrow{(\alpha(\tilde{\sigma})-1)^*} & H^0(\mathrm{Alb}(\widetilde{S}),\Omega^1_{\mathrm{Alb}(\widetilde{S})}) \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\ H^0(\widetilde{S},\Omega^1_{\widetilde{S}}) & \xrightarrow[\tilde{\sigma}^*-1]{} & H^0(\widetilde{S},\Omega^1_{\widetilde{S}}). \end{array}$$

Since α^* is an isomorphism, we have $\dim \mathrm{Ker}(\alpha(\tilde{\sigma})-1)^* = \dim \mathrm{Ker}(\tilde{\sigma}^*-1)$. Since G is a cyclic group generated by $\tilde{\sigma}$, we see that $\mathrm{Ker}(\tilde{\sigma}^*-1)$ coincides with the G-invariant part $H^0(\widetilde{S},\Omega^1_{\widetilde{S}})^G$ of $H^0(\widetilde{S},\Omega^1_{\widetilde{S}})$. On the other hand, since $\tilde{\theta}^* : \mathrm{H}^0(\widetilde{W},\Omega^1_{\widetilde{W}}) \to \mathrm{H}^0(\widetilde{S},\Omega^1_{\widetilde{S}})^G$ is an isomorphism, we have $\dim(\mathrm{Ker}(\alpha(\tilde{\sigma})-1)^*) = q(\widetilde{W})$ and, hence,

$$\dim(\mathrm{Im}(\alpha(\tilde{\sigma})-1)^*) = q(\widetilde{S}) - q(\widetilde{W}).$$

It follows that $\mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S})$ is of dimension $q_{\tilde{\theta}}$.

Secondly, we show (2). We take a point x_0 in $\mathrm{Fix}(G)$ as the base point of the Albanese map $\alpha : \widetilde{S} \to \mathrm{Alb}(\widetilde{S})$. Let $x \in \mathrm{Fix}(G)$. Note that we have

$$\alpha_{\tilde{\sigma}}(x) = (\alpha(\tilde{\sigma})-1)(\alpha(x)) = \alpha(\tilde{\sigma})(\alpha(x)) - \alpha(x)$$

and that $\alpha(\tilde{\sigma})(\alpha(x)) - \alpha(x)$ is the function given by $\omega \mapsto \int_{x_0}^x \tilde{\sigma}^*\omega - \int_{x_0}^x \omega$ for $\omega \in \mathrm{H}^0(\widetilde{S},\Omega^1_{\widetilde{S}})$ modulo periods. Since x and x_0 are both in $\mathrm{Fix}(G)$, we find

$$\int_{x_0}^x \tilde{\sigma}^*\omega - \int_{x_0}^x \omega = \int_{\tilde{\sigma}(x_0)}^{\tilde{\sigma}(x)} \omega - \int_{x_0}^x \omega = 0.$$

Hence we get $\alpha_{\tilde{\sigma}}(x) = \alpha(\tilde{\sigma})(\alpha(x)) - \alpha(x) = 0$. Since x can be taken arbitrarily in $\mathrm{Fix}(G)$, we get (2).

When $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is a curve, let B' be a normalization of $\alpha_{\tilde{\sigma}}(\widetilde{S})$. From the universality of the Abel-Jacobi map for B', we have

$$j_{B'} : \mathrm{Jac}(B') \to \mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S}),$$

where $\mathrm{Jac}(B')$ is the Jacobi variety of B'. On the other hand, since $\alpha(\widetilde{S})$ generates $\mathrm{Alb}(\widetilde{S})$ as group, $\alpha_{\tilde{\sigma}}(\widetilde{S})$ generates $\mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S})$. Since $j_{B'}$ is the group homomorphism, $\mathrm{Im}(j_{B'}) = \alpha_{\tilde{\sigma}}(\widetilde{S})$ is the sub-group of $\mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S})$. Therefore $j_{B'}$ is surjective, we get (3). □

3.1. The slope inequality in the case of $h = 0$. We consider the primitive cyclic covering fibration $f : S \to B$ of type $(g, 0, n)$ with $q_f > 0$. Since $\varphi : W \to B$ is a ruled surface, we have $q(W) = b$ and it follows $q_{\tilde{\theta}} = q_f$. We apply Proposition 3.2 to the cyclic covering $\tilde{\theta} : \widetilde{S} \to \widetilde{W}$ to find that its ramification divisor $\mathrm{Fix}(G)$ is contracted to a point by $\alpha_{\tilde{\sigma}} : \widetilde{S} \to \mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S})$, where $\tilde{\sigma}$ denotes a generator of the Galois group $G := \mathrm{Gal}(\widetilde{S}/\widetilde{W})$. So if $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is a surface (resp. a curve), from Mumford's theorem (resp. Zariski's Lemma), the intersection form is negative definite (resp. semi-definite) on $\mathrm{Fix}(G)$, and we in particular get

$$\mathrm{Fix}(G)^2 < 0 \quad (\text{resp. } \leq 0).$$

Hence, in any way, we have

$$\widetilde{R}^2 \leq 0, \tag{3.1}$$

since $\tilde{\theta}^*\widetilde{R} = n\mathrm{Fix}(G)$.

Here, we remark the following.

Lemma 3.3. *Let $f : S \to B$ be a primitive cyclic covering of type $(g, 0, n)$. If it is not locally trivial and $q_f > 0$, then $r \geq 2n$.*

Proof. We assume that $r < 2n$ and show that this leads us to a contradiction.

Recall that r is a multiple of n. If $r < 2n$, then $r \leq n$ and R has to be smooth by Lemmas 1.2 and 1.4. On the other hand, since $q_f > 0$, we already know from (3.1) that the self-intersection number of any irreducible component C of R is non-positive.

Let C_0 be the minimal section with $C_0^2 = -e$ and Γ a fiber of $\varphi : W \to B$. Note that we can choose the normalized vector bundle of rank 2 associated with W so that there are no effective divisor numerically equivalent to $C_0 - c\Gamma$ for any positive integer c. Put $C \equiv aC_0 + b\Gamma$ with two integers a, b. We have $a \geq 0$. If $a = 0$, then we have $b = 1$, that is, C is a single fiber by its irreducibility. So we may assume that a is positive.

We have $C^2 = a(2b - ae) \leq 0$. Hence $2b \leq ae$. Furthermore, since C is irreducible and $a > 0$, we have $(K_\varphi + C)C \geq 0$ by the Hurwitz formula applied for the normalization of C. Since $K_\varphi \equiv -2C_0 - e\Gamma$, we have

$$0 \leq (K_\varphi + C)C = (a - 1)(2b - ae) \leq 0,$$

by $2b \leq ae$ and $a \geq 1$. Hence we get $(K_\varphi + C)C = 0$ and, either $a = 1$ or $2b = ae$. In particular, as the first equality shows, C is smooth and $\varphi|_C : C \to B$ is unramified. Furthermore, we get $C^2 = 0$ when $a \geq 2$ by $2b = ae$. In this case, we also have $b \leq 0$, because $0 \leq CC_0 = b - ae = -b$. If $a = 1$ and $2b < e$, then $b \geq 0$ and it follows from $CC_0 = b - e < -b \leq 0$ that we have $C = C_0$ by the irreducibility of C. We remark here that we have $(a_1C_0 + b_1\Gamma)(a_2C_0 + b_2\Gamma) = 0$ when $a_i > 0$ and $2b_i = a_ie$ for $i = 1, 2$.

In summary, the only possibilities left for smooth R are (i) R consists of several fibers (including the case $R = 0$), (ii) R is the minimal section with $R^2 < 0$, and (iii) R consists of several smooth curves with self-intersection numbers 0 which are unramified over B (via φ). If (i) or (ii) is the case, then we have either $g = 0$ or $r = 1$, any of which is absurd. If (iii) is the case, then $f : S \to B$ is a locally trivial fibration, which is again inadequate. □

From $\widetilde{R}^2 = R^2 - \sum_{k\geq 1} n^2k^2\alpha_k$, (2.15) and (3.1), we get

$$2rM_0 \leq \sum_{k\geq 1}^{nk\leq \frac{r}{2}} n^2k^2\alpha_k.$$

Combining this with (2.16), we get

$$\alpha_0 \leq \sum_{k\geq 1}^{nk\leq \frac{r}{2}} \frac{nk}{r}(r - 1 - (nk - 1))\alpha_k. \tag{3.2}$$

Theorem 3.4. *Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 0, n)$ which is not locally trivial and $q_f > 0$. Then*

$$\lambda_f \geq \lambda_{g,n}^1 := 8 - \frac{8(g + n - 1)}{(n - 1)(2g - (n - 1)(n - 2))} \left(= 8 - \frac{4r}{(n - 1)(r - n)}\right).$$

Proof. For $\lambda \in \mathbb{R}$, we put

$$A(\lambda) := \frac{n-1}{n}\Big((r-2)n - r - \lambda\frac{(2r-3)n-r}{12}\Big).$$

From Proposition 2.2, we get

$$(r-1)(K_f^2 - \lambda_{g,n}^1 \chi_f) \geq A(\lambda_{g,n}^1)\alpha_0 + \sum_{k\geq 1}^{nk\leq \frac{r}{2}} a_k\alpha_k - \sum_{k\geq 1}^{nk\leq \frac{r}{2}} \lambda_{g,n}^1 \bar{a}_k \alpha_k$$

where

$$a_k := (n^2-1)k(r-1-(nk-1)) - (r-1)n,$$

$$\bar{a}_k := \frac{n^2-1}{12}k(r-1-(nk-1)).$$

We can check that $A(\lambda_{g,n}^1) \leq 0$ as follows. Since $r \geq 2n$ by Lemma 3.3, a calculation shows that the inequality

$$A(\lambda_{g,n}^1) = \frac{n-1}{n}\Big((r-2)n - r - (8 - \frac{4r}{(n-1)(r-n)})\frac{(2r-3)n-r}{12}\Big) \leq 0$$

is equivalent to

$$(r-n)(-(n-1)^2+2) + 2n^2 - 3n - r \leq 0,$$

and we can check easily its validity. Therefore $A(\lambda_{g,n}^1) \leq 0$. Hence from (3.2), we get

$$(r-1)(K_f^2 - \lambda_{g,n}^1\chi_f) \geq \sum_{k\geq 1}^{nk\leq \frac{r}{2}} \Big(\frac{(n-1)(r-1)}{4r}(8-\lambda_{g,n}^1)nk(r-nk) - (r-1)n\Big)\alpha_k. \tag{3.3}$$

For any integer k satisfying $\frac{r}{2n} \geq k \geq 1$, we have $nk(r-nk) \geq n(r-n)$. Since we have

$$\frac{(n-1)(r-1)}{4r}(8-\lambda_{g,n}^1)n(r-n) - (r-1)n = 0,$$

the coefficient of α_k in the right hand side of (3.3) is not negative. Therefore, we get $K_f^2 - \lambda_{g,n}^1\chi_f \geq 0$ as desired. □

3.2. The slope inequality in the case of $h \geq 1$. Before showing the slope inequality when $h \geq 1$ and $n \geq 3$, we study the upper bound of α_0.

Recall that we decomposed ψ into a succession of blowing-ups ψ_i as,

$$\psi : \widetilde{W} = W_N \xrightarrow{\psi_N} W_{N-1} \to \cdots \to W_1 \xrightarrow{\psi_0} W_0 = W.$$

We define the order of blowing-up ψ' appearing in ψ as follows. If the center of ψ' is a point on the branch locus of multiplicity m', we put

$$\mathrm{ord}(\psi') := \Big[\frac{m'}{n}\Big].$$

Moreover we introduce a partial order on these blowing-ups ψ' and ψ'' appearing in ψ,

$$\psi' \geq \psi'' \overset{\mathrm{def}}{\iff} \mathrm{ord}(\psi') \geq \mathrm{ord}(\psi'').$$

Lemma 3.5. *Assume that $n \geq 3$. Let x_j ($\in R_j \subset W_j$) be a singular point infinitely near to $x_i \in R_i$. Then the multiplicities satisfy $m_j \leq m_i$.*

Proof. Though this can be found in [2], Lemma 3.7, when $h = 0$, we shall give a proof for the convenience of readers.

Let x_{i+1} be the singular point of R_{i+1} infinitely near to $x_i \in R_i$. If $m_i \in n\mathbb{Z}$, then R_{i+1} coincied with $\widehat{R_i}$, the proper transform of R_i by ψ_{i+1}, by Lemma 1.2. Hence $m_{i+1} \leq m_i$ in this case. If $m_i \in n\mathbb{Z}+1$, then $R_{i+1} = \widehat{R_i} + E_{i+1}$. Hence we get $m_{i+1} \leq m_i + 1 \in n\mathbb{Z} + 2$. From Lemma 1.2 and the assumption $n \geq 3$, we get $m_{i+1} \leq m_i$. □

From Lemma 3.5, we can reorder those blowing-ups appearing in ψ so that $\psi_i \geq \psi_j$ holds whenever $i < j$. We put,

$$M := \max\{\mathrm{ord}(\psi') \mid \psi' \text{ is a blowing up in } \psi\}.$$

Then we can decompose ψ as

$$\psi : \widetilde{W} = \widehat{W}_M \xrightarrow{\hat{\psi}_M} \widehat{W}_{M-1} \quad \cdots \quad \xrightarrow{\hat{\psi}_0} \widehat{W}_0 = W$$

in such a way that $\mathrm{ord}(\psi') = M + 1 - i$ holds for any ψ' appearing in $\hat{\psi}_i$.

Lemma 3.6. *Let ψ' be a blowing-up appearing in $\hat{\psi}_i$ and $\widetilde{D}$ the proper inverse image of the exceptional curve of ψ' on $\widetilde{S}$. Then the geometric genus of $\widetilde{D}$ satisfies*

$$g(\widetilde{D}) \leq \frac{(n-1)(n(M-i)+n-2)}{2}.$$

Proof. Let m' be the multiplicity of the singular point blown up by ψ', and $\widetilde{E}$ the proper transform of the exceptional curve of ψ' on $\widetilde{W}$.

When $m' \in n\mathbb{Z} + 1$, since $\widetilde{E}$ is contained in $\widetilde{R}$, $\widetilde{D}$ is a smooth rational curve.

Assume that $m' \in n\mathbb{Z}$. From $m' = n(M+1-i)$, the intersection number of the exceptional curve of ψ' and the branch locus is $n(M + 1 - i)$. Hence the intersection number of their proper transforms on $\widetilde{W}$ is at most $n(M+1-i)$. On the other hand, we consider the composite $\pi : \widetilde{D'} \to \widetilde{D} \xrightarrow{\tilde{\theta}|_{\widetilde{D}}} \widetilde{E}$, where $\widetilde{D'} \to \widetilde{D}$ is normalization of $\widetilde{D}$, and let B_π be the branch locus π. From the Hurwitz formula for π and Lemma 1.6, we get

$$2g(\widetilde{D'}) - 2 + 2n \leq (n-1)\sharp B_\pi.$$

Since $\tilde{\theta}|_{\tilde{D}}$ is totally ramified,

$$\sharp B_\pi \leq \sharp B_{\tilde{\theta}|_{\tilde{D}}} \leq \widetilde{E}\widetilde{R} \leq n(M+1-i).$$

Therefore we get

$$g(\widetilde{D}) = g(\widetilde{D'}) \leq \frac{(n-1)(n(M-i)+n-2)}{2},$$

which is what we want. □

Proposition 3.7. *Let $f : S \to B$ be a primitive cyclic covering fibration of type (g, h, n) such that $q_{\tilde{\theta}} > 0$, $h \geq 1$, $n \geq 3$ and let α_i ($i \geq 0$) be the singularity index in* Definition 2.1. *Then,*

$$\begin{aligned}(3.4)\qquad &2(g-1-n(h-1))\alpha_0\\ &\le (g-1-n(h-1))^2\frac{K_\varphi^2}{(n-1)(h-1)}+T+\sum_{k\ge 1}\frac{12n}{(n-1)(n+1)}\bar{a}_k\alpha_k,\end{aligned}$$

where $\bar{a}_k$ *is defined in* Proposition 2.3. *If the image* $\alpha_{\tilde{\sigma}}(\widetilde{S})$ *is a curve and* $\nu(q_{\tilde{\theta}})\ge 1$*, where*

$$\nu(x):=\left\lceil\frac{2(x-1)}{n(n-1)}-\frac{n-2}{n}\right\rceil,$$

then

$$\begin{aligned}(3.5)\qquad &2(g-1-n(h-1))(\alpha_0+\sum_{k=1}^{\nu(q_{\tilde{\theta}})}nk(nk-1)\alpha_k)\\ &\le (g-1-n(h-1))^2\frac{K_\varphi^2}{(n-1)(h-1)}+T+\sum_{k\ge\nu(q_{\tilde{\theta}})+1}\frac{12n}{(n-1)(n+1)}\bar{a}_k\alpha_k.\end{aligned}$$

In (3.4) *and* (3.5)*, we regard* $\frac{K_\varphi^2}{(n-1)(h-1)}$ *as* 0 *when* $h=1$.

Proof. Firstly assume that $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is a curve of geometric genus g'. In this case, by Proposition 3.2, we have $g'\ge q_{\tilde{\theta}}$ and see that any curve of geometric genus less than g' on $\widetilde{S}$ is contracted by $\alpha_{\tilde{\sigma}}$. Hence, we know from Lemma 3.6 that for any $1\le i\le M$ satisfying

$$\frac{(n-1)(n(M-i)+n-2)}{2}\le g'-1,$$

the proper transform of the exceptional curve of $\hat{\psi}_i$ to $\widetilde{S}$ is contracted by $\alpha_{\tilde{\sigma}}$. Then, since $q_{\tilde{\theta}}\le g'$, for any $1\le i\le M$ satisfying

$$\frac{(n-1)(n(M-i)+n-2)}{2}\le q_{\tilde{\theta}}-1,$$

the same holds true. So we conclude that the total inverse image of $\widehat{R}_{M-\nu(q_{\tilde{\theta}})}$ in $\widetilde{S}$ is contracted by $\alpha_{\tilde{\sigma}}$, where $\widehat{R}_{M-\nu(q_{\tilde{\theta}})}\subset\widehat{W}_{M-\nu(q_{\tilde{\theta}})}$ is the image of $\widetilde{R}$. Therefore, the total inverse image of $\widehat{R}_{M-\nu(q_{\tilde{\theta}})}$ forms a negative semi-definite configuration. In particular, we have

$$\widehat{R}^2_{M-\nu(q_{\tilde{\theta}})}\le 0. \tag{3.6}$$

By the construction, we have

$$\widehat{R}^2_{M-\nu(q_{\tilde{\theta}})}=R^2-\sum_{k>\nu(q_{\tilde{\theta}})}n^2k^2\alpha_k=\hat{x}\frac{K_\varphi^2}{(n-1)(h-1)}+\hat{y}T+\hat{z}(K_\varphi+R)R-\sum_{k>\nu(q_{\tilde{\theta}})}n^2k^2\alpha_k,$$

where

$$\hat{x}=-\frac{(g-1-n(h-1))^2}{t},\ \hat{y}=-\frac{1}{t},\ \hat{z}=\frac{(g-1-n(h-1))}{t}.$$

Hence from (2.9), (3.6) and the above equality, we get

$$t\hat{z}(\alpha_0+\sum_{k=1}^{\nu(q_{\tilde{\theta}})}nk(nk-1)\alpha_k)\le -t\hat{x}\frac{K_\varphi^2}{(n-1)(h-1)}-t\hat{y}T+\sum_{k>\nu(q_{\tilde{\theta}})}t(n^2k^2-nk(nk-1)\hat{z})\alpha_k.$$

This is nothing more than (3.5).

Even when $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is not a curve, we have $\widetilde{R}^2 \leq 0$ by Proposition 3.2. Using this instead of (3.6), we get (3.4) by a similar argument as above. □

Theorem 3.8. *Let $f : S \to B$ be a primitive cyclic covering fibration of type (g, h, n) which is not locally trivial and such that $q_{\tilde{\theta}}$ and h are both positive, $n \geq 3$. Put*

$$F(g,h,l) = (g-1)^2 - 2n(g-1)((h+1)(n-1)(l+1)-1) - ((l+1)(n-1)^2-1)^2 n^2(h^2-1).$$

(i) *If $F(g,h,0) \geq 0$, then*

$$\lambda_f \geq \lambda_{g,h,n} := 8 - \frac{2(n+1)(g-1-n(h-1))}{3\bar{a}_1}.$$

(ii) *Assume that $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is a curve and $\nu(q_{\tilde{\theta}}) \geq 1$. If $F(g,h,\nu(q_{\tilde{\theta}})) \geq 0$, then*

$$\lambda_f \geq \lambda_{g,h,n,q_{\tilde{\theta}}} := 8 - \frac{2(n+1)(g-1-n(h-1))}{3\bar{a}_{\nu(q_{\tilde{\theta}})+1}}.$$

Proof. Here we restrict ourselves to the case that $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is a curve and show (ii) only, since (i) can be shown similarly. From (2.17) and (2.18), we obtain

$$\begin{aligned}(3.7)\qquad t(K_f^2 - \lambda_{g,h,n,q_{\tilde{\theta}}}\chi_f) \geq{}& t(x' - \lambda_{g,h,n,q_{\tilde{\theta}}}\bar{x}')\frac{K_\varphi^2}{(n-1)(h-1)} + t(y' - \lambda_{g,h,n,q_{\tilde{\theta}}}\bar{y}')T \\ &+ t(z' - \lambda_{g,h,n,q_{\tilde{\theta}}}\bar{z}')\alpha_0 + \sum_{k\geq 1}(a_k - \lambda_{g,h,n,q_{\tilde{\theta}}}\bar{a}_k)\alpha_k - n\lambda_{g,h,n,q_{\tilde{\theta}}}t\chi_\varphi.\end{aligned}$$

To apply (3.5) to the above inequality, we have to check that $z' - \lambda_{g,h,n,q_{\tilde{\theta}}}\bar{z}' \leq 0$ in advance. Since

$$\begin{aligned}(3.8)\qquad \lambda_{g,h,n,q_{\tilde{\theta}}} &\geq 8 - \frac{2(n+1)(g-1-n(h-1))}{3\bar{a}_1} \\ &= 8 - \frac{8(g-1-n(h-1))}{(n-1)(2(g-1)+n(h-1)(n-3))},\end{aligned}$$

it is sufficient to see that

$$8 - \frac{8(g-1-n(h-1))}{(n-1)(2(g-1)+n(h-1)(n-3))} \geq \frac{24(n-1)(g-1)}{2(2n-1)(g-1)-n(n+1)(h-1)},$$

which is equivalent to

$$(g-1-n(h-1))(16(g-1)n(n-2) + 8n(n+1)(h-1)((n-1)(n-3)+1)) \geq 0,$$

the validity of which can be checked directly. Therefore we get $z' - \lambda_{g,h,n,q_{\tilde{\theta}}}\bar{z}' \leq 0$. Applying (3.5) to (3.7), we obtain

$$\begin{aligned}&t(K_f^2 - \lambda_{g,h,n,q_{\tilde{\theta}}}\chi_f) \\ &\geq \frac{n-1}{12}t\Big(12(g-1) - \frac{3}{2}\lambda_{g,h,n,q_{\tilde{\theta}}}(g-1-n(h-1))\Big)\frac{K_\varphi^2}{(n-1)(h-1)} - n\lambda_{g,h,n,q_{\tilde{\theta}}}t\chi_\varphi \\ &+ \frac{(n-1)(8-\lambda_{g,h,n,q_{\tilde{\theta}}})}{8(g-1-n(h-1))}tT\end{aligned}$$

$$+\frac{nt}{12}\sum_{k=1}^{\nu(q_{\tilde\theta})}((n-1)k((2n-1)k-3)\lambda_{g,h,n,q_{\tilde\theta}}-12((n-1)k-1)^2)\alpha_k$$
$$+t\sum_{k>\nu(q_{\tilde\theta})}\left(\frac{(8-\lambda_{g,h,n,q_{\tilde\theta}})3n}{2(n+1)(g-1-n(h-1))}\bar a_k-n\right)\alpha_k.$$

We will show that $((n-1)k((2n-1)k-3)\lambda_{g,h,n,q_{\tilde\theta}}-12((n-1)k-1)^2)\geq 0$ for $1\leq k\leq \nu(q_{\tilde\theta})$. Note that

$$\begin{aligned}&(n-1)k((2n-1)k-3)\lambda_{g,h,n,q_{\tilde\theta}}-12((n-1)k-1)^2\\ =&\frac{1}{n^2}\Big(nk((n-1)(nk-1)(\lambda_{g,h,n,q_{\tilde\theta}}(2n-1)-12(n-1))+(n^2-1)(12-\lambda_{g,h,n,q_{\tilde\theta}}))-12n^2\Big).\end{aligned}$$

Firstly we will show that $\lambda_{g,h,n,q_{\tilde\theta}}\geq\frac{12(n-1)}{2n-1}$. From (3.8), it is sufficient to check that

$$8-\frac{8(g-1-n(h-1))}{(n-1)(2(g-1)+n(h-1)(n-3))}\geq\frac{12(n-1)}{2n-1}.$$

A calculation shows that it is equivalent to

$$8n(n-2)(g-1)+(4(n^2-1)(n-3)+8(2n-1))n(h-1)\geq 0,$$

which holds true clearly. So we have shown $\lambda_{g,h,n,q_{\tilde\theta}}\geq\frac{12(n-1)}{2n-1}$ and it follows that

$$(n-1)k((2n-1)k-3)\lambda_{g,h,n,q_{\tilde\theta}}-12((n-1)k-1)^2$$

is increasing in k. Evaluating at $k=1$, we get

$$\begin{aligned}&(n-1)k((2n-1)k-3)\lambda_{g,h,n,q_{\tilde\theta}}-12((n-1)k-1)^2\\ &\geq 2(n-2)((n-1)\lambda_{g,h,n,q_{\tilde\theta}}-6(n-2))\\ &\geq 0,\end{aligned}$$

by $\lambda_{g,h,n,q_{\tilde\theta}}\geq\frac{12(n-1)}{2n-1}$. Since

$$\text{(3.9)}\qquad \frac{(8-\lambda^1_{g,h,q_{\tilde\theta}})3n}{2(n+1)(g-1-n(h-1))}\bar a_k-n\geq\frac{(8-\lambda^1_{g,h,q_{\tilde\theta}})3n}{2(n+1)(g-1-n(h-1))}\bar a_{\nu(q_{\tilde\theta})+1}-n=0$$

holds for any $k\geq\nu(q_{\tilde\theta})+1$, we obtain

$$\text{(3.10)}\qquad\begin{aligned}&K_f^2-\lambda_{g,h,n,q_{\tilde\theta}}\chi_f\\ &\geq\frac{n-1}{12}\Big(12(g-1)-\frac{3}{2}\lambda_{g,h,n,q_{\tilde\theta}}(g-1-n(h-1))\Big)\frac{K_\varphi^2}{(n-1)(h-1)}\\ &\quad-n\lambda_{g,h,n,q_{\tilde\theta}}\chi_\varphi+\frac{(n-1)(8-\lambda_{g,h,n,q_{\tilde\theta}})}{8(g-1-n(h-1))}T.\end{aligned}$$

If $F(g,h,\nu(q_{\tilde\theta}))\geq 0$ and $h=1$, then by (2.19) we have $T=2(g-1)K_\varphi.R\geq\frac{4(g-1)^2}{n-1}\chi_\varphi$. Hence it follows from (3.10) that

$$K_f^2-\lambda_{g,1,n,q_{\tilde\theta}}\chi_f\geq\frac{n+1}{3\bar a_{\nu(q_{\tilde\theta})+1}}F(g,1,\nu(q_{\tilde\theta}))\chi_\varphi\geq 0$$

which gives us (ii) for $h=1$. If $F(g,h,\nu(q_{\tilde\theta}))\geq 0$ and $h\geq 2$, then we can use Xiao's slope inequality $K_\varphi^2 \geq \frac{4(h-1)}{h}\chi_\varphi$ and $T \geq 0$, to get

$$K_f^2 - \lambda_{g,h,n,q_{\tilde{\theta}}}\chi_f \geq \frac{n+1}{3h\bar{a}_{\nu(q_{\tilde{\theta}})+1}} F(g,h,\nu(q_{\tilde{\theta}}))\chi_\varphi \geq 0$$

from (3.10). Hence we have shown (ii) also for $h \geq 2$. □

4. Special irregular cyclic covering fibrations of ruled surfaces.

Let $f : S \to B$ be a primitive cyclic covering fibration of type $(g, 0, n)$ with $q_f > 0$ and suppose that it is not locally trivial. Let $\alpha_{\tilde{\sigma}} : \widetilde{S} \to \mathrm{Alb}_{\tilde{\sigma}}(\widetilde{S})$ be the morphism defined as in Definition 3.1 for the generator $\tilde{\sigma}$ of the covering transformation group G of $\tilde{\theta} : \widetilde{S} \to \widetilde{W}$. Moreover we assume that there is a component C of $\mathrm{Fix}(G)$ such that $C^2 = 0$. Note then that $\alpha_{\tilde{\sigma}}(\widetilde{S})$ is a curve by Proposition 3.2.

Proposition 4.1. *In the above situation, there are fibrations $\tilde{f}' : \widetilde{S} \to B'$, $\tilde{\varphi}' : \widetilde{W} \to \mathbb{P}^1$ and a morphism $\theta' : B' \to \mathbb{P}^1$ with B' a smooth curve such that $\widetilde{R}$ is $\tilde{\varphi}'$-vertical, $q_f \leq g(B')$, and they fit into the commutative diagram:*

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\tilde{\theta}"] \arrow[d, "\tilde{f}'"'] & \widetilde{W} \arrow[d, "\tilde{\varphi}'"] \\ B' \arrow[r, "\theta'"'] & \mathbb{P}^1. \end{tikzcd}$$

Proof. We can obtain $\tilde{f}' : \widetilde{S} \to B'$ from the Stein factorization of $\alpha_{\tilde{\sigma}} : \widetilde{S} \to \alpha_{\tilde{\sigma}}(\widetilde{S})$. Hence we have $g(B') \geq q_f$ by Proposition 3.2, (3). We will show that the automorphism $\tilde{\sigma} : \widetilde{S} \to \widetilde{S}$ induces an automorphism of B'. We assume that there is a fiber F' of $\tilde{f}'$ such that $\tilde{\sigma}^*F'$ has a $\tilde{f}'$-horizontal component. Let F'_C be the fiber of $\tilde{f}'$ which contains the curve C with $C^2 = 0$. Then, from Zariski's lemma, we see that $F'_C = aC$ for some positive integer a and it follows $F'_C = \tilde{\sigma}^*F'_C$, since C is a component of $\mathrm{Fix}(G)$. Hence

$$0 < \tilde{\sigma}^*F'F'_C = (\tilde{\sigma}^*F')(\tilde{\sigma}^*F'_C) = F'F'_C = 0,$$

a contradiction. Therefore $\tilde{\sigma}$ maps fibers to fibers, and descends down to give an automorphism $\tilde{\sigma}_{B'} : B' \to B'$. Furthermore we have the commutative diagram

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\tilde{\theta}"] \arrow[d, "\tilde{f}'"'] & \widetilde{W} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{\varphi}'"] & B \\ B' \arrow[r, "\theta'"'] & D', & \end{tikzcd}$$

where $\theta' : B' \to D' := B'/\langle\tilde{\sigma}_{B'}\rangle$ denotes the quotient map. In order to complete the proof, it suffices to see that $D' = \mathbb{P}^1$. This can be shown as follows. Any general fiber of $\tilde{\varphi}$ is $\tilde{\varphi}'$-horizontal by $\widetilde{R}.\Gamma > 0$. Since $\tilde{\varphi}$ is ruled, we see that $\mathbb{P}^1$ dominates D' and it follows $D' = \mathbb{P}^1$. □

The contraction $\varphi : \widetilde{W} \to W$ is composed of several blowing-ups. We decompose it as $\psi = \check{\psi} \circ \bar{\psi}$ as follows. Let $\bar{\psi} : \widetilde{W} \to \overline{W}$ be the longest succession of blowing-downs such that we still have the morphism $\bar{\varphi}'$ satisfying $\tilde{\varphi}' = \bar{\varphi}' \circ \bar{\psi}$. Then we have the following commutative diagram.

$$\begin{tikzcd} & \mathbb{P}^1 & \\ \widetilde{W} \arrow[ur, "\tilde{\varphi}'"] \arrow[r, "\bar{\psi}"] \arrow[dr, "\tilde{\varphi}"'] & \overline{W} \arrow[u, "\bar{\varphi}'"] \arrow[r, "\check{\psi}"] \arrow[d, "\bar{\varphi}"] & W \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

Let $\overline{R} := \bar{\psi}_*\widetilde{R}$ be the image of $\widetilde{R}$ by $\bar{\psi}$.

Lemma 4.2. *The morphism $\check{\psi} : \overline{W} \to W$ is not the identity map.*

Proof. We will prove this by contradiction. Suppose that $\check{\psi}$ is the identity map. As one sees from the proof of Lemma 3.3, any irreducible curve D on W with $D^2 \leq 0$ is smooth, and $\varphi|_D : D \to B$ is an unramified covering when $D^2 = 0$ and D is not a fiber of φ. Hence, any irreducible fiber of $\bar{\varphi}' : W \to \mathbb{P}^1$ has to be smooth. Suppose that there is a fiber of $\bar{\varphi}'$ whose reduced scheme is reducible, and take an irreducible component D_0. Then $D_0^2 < 0$ and, from the proof of Lemma 3.3, we conclude that D_0 coincides with the minimal section. The unicity implies that we cannot have such reducible singular fibers. Therefore, a singular fiber of $\bar{\varphi}'$, if any, is a multiple fiber whose support is a smooth irreducible curve. Since R is a reduced divisor with support in fibers of $\bar{\varphi}'$ by Proposition 4.1, we see that R is smooth and $\varphi|_R : R \to B$ is unramified. Then $f : S \to B$ is a locally trivial fibration, which is inadequate. □

Assume that $\theta' : B' \to \mathbb{P}^1$ is branched over $\Delta \subset \mathbb{P}^1$. For any $y \in \Delta$, let $\widetilde{\Gamma}'_y = \sum \tilde{n}_C C$ be the fiber of $\tilde{\varphi}'$ over y, and put

$$\widetilde{R}_{all} := \tilde{\varphi}'^{*}\Delta, \quad \widetilde{R}_r := \sum_{y\in\Delta,\ C\subset\widetilde{\Gamma}'_y,\ \tilde{n}_C=1} C.$$

Lemma 4.3. *In the above situation, $\widetilde{R}_r \leq \widetilde{R}$.*

Proof. We put

$$G_{\tilde{f}'} := \{\tau \in G \mid \tau(\widetilde{F}') = \widetilde{F}' \text{ for any fiber } \widetilde{F}' \text{ of } \tilde{f}'\}.$$

Since $\tilde{f}' \circ \tau = \tau$ for any $\tau \in G_{\tilde{f}'}$, the morphism $\tilde{f}'$ induces the morphism $\pi : \widetilde{S}/G_{\tilde{f}'} \to \widetilde{W}$ and we have the following commutative diagram.

$$\begin{tikzcd} \widetilde{S} \arrow[dr] \arrow[drr, bend left, "\tilde{\theta}"] \arrow[ddr, bend right, "\tilde{f}'"'] & & \\ & \widetilde{S}/G_{\tilde{f}'} \arrow[r, "\pi"] \arrow[d] & \widetilde{W} \arrow[d, "\tilde{\varphi}'"] \\ & B' \arrow[r, "\theta'"'] & \mathbb{P}^1 \end{tikzcd}$$

Note that the degree of π is equal to that of θ'. We claim that $\mathrm{Fix}(G_{\tilde{f}'}) = \mathrm{Fix}(G)$. This can be seen as follows. It is clear that $\mathrm{Fix}(G_{\tilde{f}'}) \supset \mathrm{Fix}(G)$. If there is a point $x \in \mathrm{Fix}(G_{\tilde{f}'}) \setminus \mathrm{Fix}(G)$, then we have $\tilde{\sigma}(x) \neq x$ for the generator $\tilde{\sigma}$ of G. On the other hand, since $G_{\tilde{f}'}$ is a subgroup of G of order $n/\deg\theta'$, we have $G_{\tilde{f}'} = \langle \tilde{\sigma}^{\deg\theta'} \rangle$. Hence the number of G-orbits of x is at most $\deg\theta'$. This contradicts that $\tilde{\theta} : \widetilde{S} \to \widetilde{W}$ is totally ramified. Therefore $\mathrm{Fix}(G_{\tilde{f}'}) = \mathrm{Fix}(G)$. Hence $\widetilde{S}/G_{\tilde{f}'}$ is smooth. Let R_π be the branch locus of π. Since $\tilde{\theta}$ is totally ramified, one can check easily that $R_\pi = \widetilde{R}$. Hence it is sufficient to prove that $\widetilde{R}_r \leq R_\pi$. Let C be any component of $\widetilde{R}_r$. We can take analytic local coordinates $(U_{\mathbb{P}^1}, x)$ on $\mathbb{P}^1$, $(U_{\widetilde{W}}, y, z)$ on $\widetilde{W}$ and $(U_{B'}, w)$ on B' such that $\tilde{\varphi}'(C)$ is defined by $x = 0$, C is defined by $y = 0$, $\theta'^* x = w^{\deg\theta'}$ and $\tilde{\varphi}'^* x = y$. $U_{B'} \times_{\mathbb{P}^1} U_{\widetilde{W}}$ is defined by $y = w^{\deg\theta'}$ in $U_{B'} \times U_{\widetilde{W}}$. So $U_{B'} \times_{\mathbb{P}^1} U_{\widetilde{W}} \to U_{\widetilde{W}}$ is ramified over $C \cap U_{\widetilde{W}}$ and $U_{B'} \times_{\mathbb{P}^1} U_{\widetilde{W}}$ is smooth. Hence the natural morphism $\widetilde{S}/G_{\tilde{f}'} \to B' \times_{\mathbb{P}^1} \widetilde{W}$ is an isomorphism around $U_{B'} \times_{\mathbb{P}^1} U_{\widetilde{W}}$. Therefore we get $C \leq R_\pi = \widetilde{R}$. □

We suppose that $\check{\psi} = \check{\psi}_1 \circ \cdots \circ \check{\psi}_u$, where $\check{\psi}_i : \check{W}_i \to \check{W}_{i-1}$ is a blowing-up at $\check{x}_{i-1} \in \check{W}_{i-1}$, $\check{W}_0 = W$ and $\check{W}_u = \overline{W}$. Let $\check{R}_i$ be the image of $\overline{R}$ in $\check{W}_i$, and let $\check{x}_i$ be a singular point of $\check{R}_i$ of multiplicity $\check{m}_i$.

Lemma 4.4. *Assume that $n \geq 3$. For $1 \leq i \leq u-1$, we have $\check{m}_i \geq \frac{2q_f}{n-1} + 2$. Moreover if there is $\check{m}_i$ such that equality sign holds, then $\frac{2q_f}{n-1} + 2 \in n\mathbb{Z}$ and $\deg\theta' = n$.*

Proof. Let $\mathcal{E} \subset \overline{W}$ be any (-1)-curve contracted by $\check{\psi}$. Note that $\bar{\varphi}'|_{\mathcal{E}} : \mathcal{E} \to \mathbb{P}^1$ is surjective and $\mathcal{E}\overline{R} \in n\mathbb{Z}$. We will show that $\mathcal{E}\overline{R} \geq \frac{2q_f}{n-1} + 2$. If $n \geq \frac{2q_f}{n-1} + 2$, then the assertion is clear from Lemma 1.2, (1). Hence we may assume that $n < \frac{2q_f}{n-1} + 2$. In particular, we have $q_f \geq n-1$. By Lemma 1.6 and $q_f \geq n-1$, we get

$$\sharp\Delta \geq \frac{2}{\deg\theta' - 1} g(B') + 2 \geq 4.$$

So we apply Lemma 1.7 to $\bar{\varphi}' : \overline{W} \to \mathbb{P}^1$ as $J = \Delta$, $F_{all} = \overline{R}_{all} := \bar{\varphi}'^* \Delta$ and $F_r = \overline{R}_r$, we have $\sharp(\mathcal{E} \cap \overline{R}_r) \geq (\deg \bar{\varphi}'|_{\mathcal{E}})(\sharp\Delta - 4) + 4$. Therefore we get

$$\begin{aligned}(\mathcal{E}\overline{R}) \geq (\mathcal{E}\overline{R}_r) \geq \sharp(\mathcal{E} \cap \overline{R}_r) &\geq (\deg \bar{\varphi}'|_{\mathcal{E}}) \left(\frac{2}{\deg\theta' - 1} g(B') - 2 \right) + 4 \\ &\geq \frac{2}{\deg\theta' - 1} g(B') + 2 \\ &\geq \frac{2}{\deg\theta' - 1} q_f + 2.\end{aligned}$$

For any $\check{x}_i$, let $\check{x}_{i+j_i}$ be the singular point infinitely near to $\check{x}_i$ with $i + j_i$ maximal blown up by $\check{\psi}$, $\mathcal{E}_{i+j_i}$ the exceptional curve. Then, from the above argument, we get

$$\frac{2}{\deg\theta' - 1} q_f + 2 \leq (\mathcal{E}_{i+j_i}\overline{R}) = \check{m}_{i+j_i} \leq \check{m}_i.$$

Moreover if the equality signs hold everywhere, then we get

$$\check{m}_i = \frac{2}{\deg\theta' - 1} q_f + 2 = (\mathcal{E}_{i+j_i}\overline{R}) \in n\mathbb{Z}.$$

Since $\deg\theta' \leq n$, we are done. □

Theorem 4.5. *Let $f : S \to B$ be a locally non-trivial primitive cyclic covering fibration of type $(g, 0, n)$ with $q_f > 0$ and $n \geq 3$. Assume that there is a component C of* $\mathrm{Fix}(\tilde{\sigma})$ *such that $C^2 = 0$. Then,*

$$q_f \leq \frac{g - n + 1}{2}.$$

Proof. There is a singular point x of R which is blown up by $\check{\psi} : \overline{W} \to W$ from Lemma 4.2. If $m := \mathrm{mult}_x R \leq \frac{r}{2}$, then

$$\frac{2q_f}{n-1} + 2 \leq m \leq \frac{r}{2} = \frac{g}{n-1} + 1$$

from Lemma 4.4. So we get $q_f \leq (g - n + 1)/2$. If $m > \frac{r}{2}$, then we have $m \in n\mathbb{Z} + 1$ from Lemma 1.4. Let $\check{x}$ be the last singular point, infinitely near to x, blown up by $\check{\psi}$ and $\check{m}$ its multiplicity. Then it holds that $\check{m} \in n\mathbb{Z}$. Indeed, if $\check{m} \in n\mathbb{Z} + 1$, the exceptional curve arizing from $\check{x}$ is contained in branch locus. It contradicts the definition of $\check{\psi}$. So we get $\check{m} + 1 \leq m$. Therefore we get

$$\frac{2q_f}{n-1} + 3 \leq \check{m} + 1 \leq m \leq \frac{r}{2} + 1 = \frac{g}{n-1} + 2$$

from Lemma 1.4 and Lemma 4.4. It follows $q_f \leq (g - n + 1)/2$. □

Therefore, the Modified Xiao's Conjecture is true in this particular case.

Now, we turn our attention to the slope. Let $\check{\alpha}_k$ be the number of the singular points of R with multiplicity nk or $nk + 1$ appearing in $\check{\psi}$. Then $\check{\alpha}_k \geq 0$ and by Lemma 4.4 one has

$$\check{\alpha}_k = 0 \tag{4.1}$$

for any k satisfying $nk + 1 \leq \frac{2q_f}{n-1} + 2$. We put $\bar{\alpha}_k := \alpha_k - \check{\alpha}_k$ then by (4.1),

$$\bar{\alpha}_k = \alpha_k \tag{4.2}$$

for any k satisfying $nk + 1 \leq \frac{2q_f}{n-1} + 2$. By the construction of $\bar{\varphi}'$, $\overline{R}$ is contained in fibers of $\bar{\varphi}'$, hence we get

$$\overline{R}^2 \leq 0.$$

On the other hand, we have

$$\overline{R}^2 = R^2 - \sum_{\frac{2q_f}{n-1}+2 \leq nk}^{nk \leq \frac{r}{2}} n^2 k^2 \check{\alpha}_k.$$

As $R^2 = 2rM_0$ by (2.15), we get

$$2rM_0 \leq \sum_{\frac{2q_f}{n-1}+2 \leq nk}^{nk \leq \frac{r}{2}} n^2 k^2 \check{\alpha}_k. \tag{4.3}$$

Hence

$$
\begin{aligned}
\alpha_0 + \sum_{k\geq 1}^{nk+1\leq \frac{2q_f}{n-1}+2} nk(nk-1)\alpha_k &\leq \alpha_0 + \sum_{k\geq 1}^{nk\leq \frac{r}{2}} nk(nk-1)\bar{\alpha}_k \\
&\leq \sum_{\frac{2q_f}{n-1}+2\leq nk}^{nk\leq \frac{r}{2}} \frac{nk}{r}(r-1-(nk-1))\check{\alpha}_k \\
&\leq \sum_{\frac{2q_f}{n-1}+2\leq nk}^{nk\leq \frac{r}{2}} \frac{nk}{r}(r-1-(nk-1))\alpha_k,
\end{aligned}
$$

where the first and the last inequalities above follow immediately from $\bar{\alpha}_k \geq 0$ and (4.2), and the second one follows from (2.16) and (4.3). Hence we have shown:

Proposition 4.6. *Under the same assumptions as in* Proposition 4.1*,*

$$
\alpha_0 + \sum_{k\geq 1}^{nk+1\leq \frac{2q_f}{n-1}+2} nk(nk-1)\alpha_k \leq \sum_{\frac{2q_f}{n-1}+2\leq nk}^{nk\leq \frac{r}{2}} \frac{nk}{r}(r-1-(nk-1))\alpha_k.
$$

Using this, we will prove the following:

Theorem 4.7. *Let $f : S \to B$ be a locally non-trivial primitive cyclic covering fibration of type $(g, 0, n)$ such that there is a component $C \subset \mathrm{Fix}(G)$ with $C^2 = 0$. If $q_f > 0$ and $n \geq 3$, then we have*

$$
\lambda_f \geq \lambda^2_{g,n,q_f} := 8 - \frac{2n(g+n-1)}{(g-q_f)(q_f+n-1)} \left(= 8 - \frac{4rn}{(n-1)(2+\frac{2q_f}{n-1})(r-(2+\frac{2q_f}{n-1}))}\right). \tag{4.4}
$$

Proof. We first remark that, for two real numbers x, y with $x + y \leq r$, we have $x(r - x) \geq y(r - y)$ if and only if $x \geq y$. Since we have $n + (2 + 2q_f/(n - 1)) \leq r$ by the proof of Theorem 4.5 and $r \geq 2n$, this observation works for $x = n$, $y = 2 + 2q_f/(n - 1)$.

(i) The case of $n \geq 2 + \frac{2q_f}{n-1}$, i.e., $\frac{(n-2)(n-1)}{2} \geq q_f$.

Since $n \geq 2 + \frac{2q_f}{n-1}$, we get

$$
n(r-n) \geq \left(2 + \frac{2q_f}{n-1}\right)\left(r - \left(2 + \frac{2q_f}{n-1}\right)\right).
$$

Therefore, $\lambda^1_{g,n} \geq \lambda^2_{g,n,q_f}$, and (4.4) follows from Theorem 3.4.

(ii) The case of $n < 2 + \frac{2q_f}{n-1}$.

In this case, we have

$$
\left(2 + \frac{2q_f}{n-1}\right)\left(r - \left(2 + \frac{2q_f}{n-1}\right)\right) \geq n(r-n)
$$

and, hence, $\lambda^1_{g,n} \leq \lambda^2_{g,n,q_f}$. Then we have $A(\lambda^2_{g,n,q_f}) \leq 0$, since the function $A(\lambda)$ defined in the proof of Proposition 3.4 is decreasing in λ and we have already proved $A(\lambda^1_{g,n}) \leq 0$ there.

From Proposition 2.2, we have

$$
(r-1)(K_f^2 - \lambda^2_{g,n,q_f}\chi_f) \geq A(\lambda^2_{g,n,q_f})\alpha_0 + \sum_{k\geq 1}^{nk\leq \frac{r}{2}} (a_k - \lambda^2_{g,n,q_f}\bar{a}_k)\alpha_k \tag{4.5}
$$

where a_k and $\bar{a}_k$ are the same as in the proof of Proposition 3.4. Applying Proposition 4.6 to (4.5), we get

$$(r-1)(K_f^2 - \lambda_{g,n,q_f}^2 \chi_f) \geq \sum_{k\geq 1}^{nk<\frac{2q_f}{n-1}+2} (-A(\lambda_{g,n,q_f}^2) nk(nk-1) + a_k - \lambda_{g,n,q_f}^2 \bar{a}_k)\alpha_k$$
$$+ \sum_{2+\frac{2q_f}{n-1}\leq nk}^{nk\leq \frac{r}{2}} (A(\lambda_{g,n,q_f}^2)\frac{nk}{r}(r-nk) + a_k - \lambda_{g,n,q_f}^2 \bar{a}_k)\alpha_k.$$

First, we will show

$$-A(\lambda_{g,n,q_f}^2) nk(nk-1) + a_k - \lambda_{g,n,q_f}^2 \bar{a}_k \geq 0 \tag{4.6}$$

for any positive integer k with $nk+1 \leq 2 + \frac{2q_f}{n-1}$. By a simple calculation, we get

$$\begin{aligned} &- A(\lambda_{g,n,q_f}^2) nk(nk-1)\alpha_0 + a_k - \lambda_{g,n,q_f}^2 \bar{a}_k \\ &= nk\Big(\frac{(n-1)(r-1)}{12n}(nk-1)(\lambda_{g,n,q_f}^2(2n-1) - 12(n-1)) \\ &\qquad + \frac{(n^2-1)(r-1)}{12n}(12 - \lambda_{g,n,q_f}^2)\Big) - (r-1)n. \end{aligned} \tag{4.7}$$

We claim that $\lambda_{g,n,q_f}^2 \geq \frac{12(n-1)}{2n-1}$. It is equivalent to

$$4(n+1)q_f(g-n+1-q_f) + (2n-4)g - 2n(n-1)(2n-1) \geq 0.$$

From $g-n+1 \geq 2q_f$ and $q_f \geq \frac{(n-2)(n-1)}{2} + 1$, we easily see that it holds true. Since $\lambda_{g,n,q_f}^2 \geq \frac{12(n-1)}{2n-1}$, the right hand side of (4.7), which is incleasing in k, is not less than

$$\begin{aligned} &n\Big(\frac{(n-1)(r-1)}{12n}(n-1)(\lambda_{g,n,q_f}^2(2n-1) - 12(n-1)) + \frac{(n^2-1)(r-1)}{12n}(12 - \lambda_{g,n,q_f}^2)\Big) - (r-1)n \\ &= \frac{n(n-1)(r-1)}{6}\Big(\lambda_{g,n,q_f}^2(n-2) - 6(n-3)\Big) - (r-1)n \\ &= n(r-1)\frac{n(n-2)}{2n-1} \\ &\geq 0. \end{aligned}$$

Therefore we get (4.6).

Secondly, we will show

$$A(\lambda_{g,n,q_f}^2)\frac{nk}{r}(r-nk) + a_k - \lambda_{g,n,q_f}^2 \bar{a}_k \geq 0 \tag{4.8}$$

for any positive integer k satisfying $\frac{r}{2} \geq nk \geq 2 + \frac{2q_f}{n-1}$. By a simple calculation, we get

$$\begin{aligned} &A(\lambda_{g,n,q_f}^2)\frac{nk}{r}(r-nk) + a_k - \lambda_{g,n,q_f}^2 \bar{a}_k \\ &= \frac{(n-1)(r-1)}{4r} nk(r-nk)(8 - \lambda_{g,n,q_f}^2) - (r-1)n. \end{aligned} \tag{4.9}$$

Since we have

$$nk(r-nk) \geq \left(2+\frac{2q_f}{n-1}\right)\left(r-\left(2+\frac{2q_f}{n-1}\right)\right)$$

for any positive integer k satisfying $2+\frac{2q_f}{n-1} \leq nk \leq \frac{r}{2}$, the right hand side of (4.9) is not less than

$$\frac{(n-1)(r-1)}{4r}\left(2+\frac{2q_f}{n-1}\right)\left(r-\left(2+\frac{2q_f}{n-1}\right)\right)(8-\lambda^2_{g,n,q_f})-(r-1)n=0.$$

Thus, we obtain $K_f^2-\lambda^2_{g,n,q_f}\chi_f \geq 0$. □

5. An example.

We construct primitive cyclic covering fibrations of type $(g,0,n)$ with relative irregularity q_f satisfying $g+n-1=m(q_f+n-1)$ for any integer $m \geq 2$. Hence, when $m=2$, this implies that the bound of q_f in Theorem 4.5 is sharp. Also, our examples show that the slope bound (4.4) in Theorem 4.7 is sharp.

Let $\varphi : W := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \bigoplus \mathcal{O}_{\mathbb{P}^1}(e)) \to B := \mathbb{P}^1$ be the Hirzebruch surface of degree $e \geq 0$. Denote by Γ and C_0 a fiber of φ and the section with $C_0^2=-e$, respectively. We know that $mC_0+b\Gamma$ is very ample if and only if $b > me$. So we take b_0 with $b_0 > me$.

We take two general members D, D' of $|mC_0+b_0\Gamma|$ which intersect each other transversely. Let Λ be the pencil generated by D and D'. Then Λ define the rational map $\varphi_\Lambda : W \cdots \to \mathbb{P}^1$. Let ψ be a minimal succession blowing-ups which eliminates the base points of Λ. We get a relatively minimal fibration $\tilde{\varphi}' : \widetilde{W} \to \mathbb{P}^1$ by putting $\tilde{\varphi}'=\varphi_\Lambda \circ \psi$. Denote by $\widetilde{\Gamma}'$ a general fiber of $\tilde{\varphi}'$ and $K_{\widetilde{W}}$ a canonical divisor of $\widetilde{W}$. By a simple calculation, we get

$$K_{\widetilde{W}}^2=8-x,\ K_{\widetilde{W}}\widetilde{\Gamma}'=\frac{m-1}{m}x-2m,\ \widetilde{\Gamma}'^2=0, \tag{5.1}$$

where x is the number of blowing-ups in ψ. Note that $x=(mC_0+b_0\Gamma)^2$.

Let $\Delta \subset \mathbb{P}^1$ be a set of $\frac{2q}{n-1}+2$ general points, where q is an integer satisfying $\frac{2q}{n-1}+2 \in n\mathbb{Z}$. Then there is a divisor $\mathfrak{d}'$ on $\mathbb{P}^1$ such that $n\mathfrak{d}'=\Delta$. Let $\widetilde{R}=(\tilde{\varphi}')^*\Delta$ be the fiber of $\tilde{\varphi}'$ over Δ. Since Δ is general, we can assume that $\widetilde{R}$ is both reduced and smooth.

We consider a classical cyclic n-covering

$$\theta' : B' = \mathrm{Spec}_{\mathbb{P}^1}\left(\bigoplus_{j=0}^{n-1}\mathcal{O}_{\mathbb{P}^1}(-j\mathfrak{d}')\right) \to \mathbb{P}^1.$$

Since Δ is general, we can assume that the fiber product $\widetilde{S} := B \times_{\mathbb{P}^1} \widetilde{W}$ is smooth. Noting that the morphism $\tilde{\theta} : \widetilde{S} \to \widetilde{W}$ induced by θ' is nothing but the natural one

$$\widetilde{S} = \mathrm{Spec}_{\mathbb{P}^1}\left(\bigoplus_{j=0}^{n-1}\mathcal{O}_{\widetilde{W}}(-j(\varphi')^*\mathfrak{d}')\right) \to \widetilde{W},$$

one gets a commutative diagram

$$\begin{tikzcd}
B & \\
\widetilde{S} \arrow[u, "\tilde{f}"] \arrow[r, "\tilde{\theta}"] \arrow[d, "\tilde{f}'"'] & \widetilde{W} \arrow[ul, "\tilde{\varphi}"'] \arrow[d, "\tilde{\varphi}'"] \\
B' \arrow[r, "\theta'"'] & \mathbb{P}^1
\end{tikzcd}$$

where $\tilde{f} := \tilde{\varphi} \circ \tilde{\theta}$.

By the construction, we get $q_f = q = g(B')$. From the formulae

$$K_{\widetilde{S}}^2 = n(K_{\widetilde{W}} + \frac{n-1}{n}\widetilde{R})^2, \quad \chi(\mathcal{O}_{\widetilde{S}}) = n\chi(\mathcal{O}_{\widetilde{W}}) + \frac{1}{2}\sum_{j=1}^{n-1} \frac{1}{n} j\widetilde{R}\Big(\frac{1}{n} j\widetilde{R} + K_{\widetilde{W}}\Big),$$

and (5.1), we get

$$K_{\widetilde{S}}^2 = \left(4\frac{m-1}{m}\left(\frac{q}{n-1}+1\right)(n-1) - n\right)x + 8\left(n - m(n-1)\left(\frac{q}{n-1}+1\right)\right), \tag{5.2}$$

$$\chi(\mathcal{O}_{\widetilde{S}}) = \frac{m-1}{2m}\left(\frac{q}{n-1}+1\right)(n-1)x + n - m(n-1)\left(\frac{q}{n-1}+1\right). \tag{5.3}$$

Let g be the genus of fibration $f : \widetilde{S} \to B$. Then it is easy to see that

$$\frac{2g}{n-1} + 2 = m\left(\frac{2q}{n-1} + 2\right).$$

Hence we get

$$K_{\tilde{f}}^2 = K_{\widetilde{S}}^2 - 8(g-1)(g(B)-1) = \left(4\frac{m-1}{m}\left(\frac{q}{n-1}+1\right)(n-1) - n\right)x,$$

$$\chi_{\tilde{f}} = \chi(\mathcal{O}_{\widetilde{S}}) - (g-1)(g(B)-1) = \frac{m-1}{2m}\left(\frac{q}{n-1}+1\right)(n-1)x.$$

Therefore we get

$$\lambda_{\tilde{f}} = \frac{K_{\tilde{f}}^2}{\chi_{\tilde{f}}} = 8 - \frac{2n(g+n-1)}{(g-q_f)(q_f+n-1)}$$

by $q = q_f$.

We remark that $\tilde{f}$ is relatively minimal. In fact the singular points of R, the image of $\widetilde{R}$ in W, are all of multiplicity $\frac{2q_f}{n-1} + 2 \in n\mathbb{Z}$ and can be resolved by a single blowing-up. So there is no $\tilde{\varphi}$-vertical $(-n)$-curve in $\widetilde{W}$. Therefore there is no $\tilde{f}$-vertical (-1)-curve in $\widetilde{S}$.

References

[1] M.A. Barja, V.G. Alonso and J.C. Naranjo: *Xiao's Conjecture for general fibred surfaces*, J. Reine Angew Math. **739** (2018), 297–308.

[2] M. Enokizono: *Slopes of fibered surfaces with a finite cyclic automorphism*, Michigan Math. J. **66** (2017), 125–154.
[3] X. Lu and K. Zuo: *On the slope conjecture of Barja and Stoppino for fibred surfaces*, arXiv:1504.06276v1[math.A.G].
[4] X. Lu and K. Zuo: *On the slope of hyperelliptic fibrations with positive relative irregularity*, Trans. Amer. Math. Soc. **369** (2017), 909–934.
[5] G. Xiao: *Fibred algebraic surfaces with low slope*, Math. Ann. **276** (1987), 449–466.

Department of Mathematics
Graduate School of Science
Osaka University
1–1 Machikaneyama, Toyonaka
Osaka 560–0043
Japan
e-mail: u802629d@ecs.osaka-u.ac.jp