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WEIGHTED NORM INEQUALITIES FOR PSEUDO-DIFFERENTIAL OPERATORS

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

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Introduction. Weighted norm inequalities appear in many places in analysis, for example, for Hardy-Littlewood maximal functions, singular integrals, pseudo-differential operators, Fourier series etc. In this note we shall investigate weighted norm inequalities for pseudo-differential operators.

Let $m(x, \xi)$ be a sufficiently regular function defined on $\boldsymbol{R}^n\times\boldsymbol{R}^n$. The pseudo-differential operator with symbol $m(x, \xi)$ is defined on the Schwartz space $\mathcal{S}(\boldsymbol{R}^n)$ of rapidly decreasing and infinitely differentiable functions by the formula:

$$m(x, D)f(x) = (2\pi)^{-n}\int_{\boldsymbol{R}^n} e^{ix\cdot\xi}m(x, \xi)\hat{f}(\xi)d\xi\,,$$

where $\hat{f}(\xi)=\int_{\boldsymbol{R}^n} e^{-ix\cdot\xi}f(x)dx$ is the Fourier transform of f.

In this note we shall consider the regularity of the following type:

$$(0.1)\qquad |\partial_\xi^\alpha m(x, \xi)|\leqq C(1+|\xi|)^{-|\alpha|}\,,\qquad |\alpha|\leqq k\,,$$

$$(0.2)\qquad |\partial_\xi^\alpha m(x+h, \xi)-\partial_\xi^\alpha m(x, \xi)|\leqq C(1+|\xi|)^{-|\alpha|}\omega(1+|\xi|, |h|)\,,\qquad |\alpha|\leqq k\,,$$

(α; multi-index), where $\omega(s, t)$ is a positive function on $(0, \infty)\times(0, \infty)$.

L^p-estimates for such operators were given by many authors, Nagase [13, 14], Coifman and Meyer [3], Journé [5], Muramatu and Nagase [12], Bourdaud [1], Yamazaki [21, 22], etc. Miller [8] showed the weighted L^p-estimates for smooth pseudo-differential operators and applied them to study the weighted Sobolev spaces. Nishigaki [15] showed the weighted L^p-estimates for the pseudo-differential operators satisfying (0.1) and (0.2), where $\omega(s, t)$ has the form $\omega(t)$ and k is sufficiently large. By a different method the author has shown the same result as Nishigaki for the case where $\omega(s, t)$ has the form $\Omega(s)\omega(t)$ and $k>n$, Yabuta [19, 20]. Recently Miyachi and the author have shown the unweighted L^p-estimates for pseudo-differential operators for the case $k>$max

$(n/p, n/p')$ and ω is general, Miyachi and Yabuta [10]. These k are in a sense sharp.

In this paper we shall show that the weighted norm inequalities for pseudo-differential operators hold for the case $k>\max(n/p, n/p')$ under a condition on ω, more restricted than in [10]. As an application we shall show that a problem rested in [20] can be solved affirmatively.

Our plan in this paper is as follows. In the next section we prepare notations and lemmas concerning weights and maximal functions and then give a sufficient condition for an operator with an integral kernel to satisfy the so-called Fefferman-Stein inequality (Proposition 1.4). In Section 2 we state the main results and their direct remarks. In Section 3 we investigate the distribution kernels of pseudo-differential operators, to check whether they satisfy the assumptions in Proposition 1.4. We prove the main results in Sections 4 and 5. In Section 6 we comment on pseudo-differential operators with double symbols $m(x, \xi, y)$. Several concluding remarks are given in the last section.

We note that the letters C, C_1, etc will always denote positive constants, which may have different values in each occasion.

1. Preliminaries and notations

$\boldsymbol{R}$ is the real line and $\boldsymbol{N}$ is the set of all nonnegative integers. For a real number s, $[s]$ denotes the integer satisfying $s-1<[s]\leqq s$. The letters α, β will denote multi-indices, i.e. $\alpha=(\alpha_1, \cdots, \alpha_n)\in \boldsymbol{N}^n$, and $|\alpha|=\alpha_1+\cdots+\alpha_n$. For x, $\xi\in\boldsymbol{R}^n$, $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$, $|x|=(x\cdot x)^{1/2}$ and $\langle x\rangle=(1+|x|^2)^{1/2}$. Differential operators are denoted by ∂^α or ∂_x^α;

$$(\partial_x^\alpha f)(x)=\left(\frac{\partial}{\partial x}\right)^\alpha f(x)=\frac{\partial^{|\alpha|}f(x)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

$C_0^\infty(\boldsymbol{R}^n)$ is the space of infinitely differentiable functions with compact support. $L^p(\boldsymbol{R}^n)$ is the space of all Lebesgue measurable functions f with $\|f\|_p=(\int_{\boldsymbol{R}^n}|f(x)|^p dx)^{1/p}<+\infty$. For a given weight function $w(x)$, i.e. a positive measurable function on $\boldsymbol{R}^n$, $L^p(w)=L^p(w(x)dx)$ is the space of all measurable functions f with $\|f\|_{L^p(w)}=(\int_{\boldsymbol{R}^n}|f(x)|^p w(x)dx)^{1/p}<+\infty$. For a Lebesgue measurable set $E\subset\boldsymbol{R}^n$, $w(E)=\int_E w(x)dx$ and $|E|$ is the Lebesgue measure of E.

For a locally integrable function f we define the sharp function $f^\sharp(x)$ of Fefferman and Stein by the formula:

$$f^\sharp(x)=\sup_{x\in Q}\inf_c |Q|^{-1}\int_Q|f(y)-c|dy,$$

where Q moves over all cubes with sides parallel to the coordinate axes, containing x, and c moves over all complex numbers.

For $1\leqq p<\infty$, $M_p f(x)$ will denote the p-th Hardy-Littlewood maximal function, i.e.

$$M_p f(x)=\sup_{x\in Q}\left(|Q|^{-1}\int_Q |f(y)|^p dy\right)^{1/p}.$$

$M_1(f(x)$ is the usual Hardy-Littlewood maximal function of f.

Weight functions we shall use are the A_p weights of Muckenhoupt, defined as follows, (Muckenhoupt [11]). For $1<p<\infty$, a positive measurable function $w(x)$ is said to belong to $A_p=A_p(\boldsymbol{R}^n)$ if and only if

$$\sup_Q\left(|Q|^{-1}\int_Q w(x)dx\right)\left(|Q|^{-1}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<+\infty .$$

For $p=1$, $w\in A_1$ if for some $C>0$

$$|Q|^{-1}\int_Q w(x)dx\leqq C \operatorname*{ess\,inf}_{y\in Q} w(y), \qquad \text{for any cube } Q.$$

We set $A_\infty=\bigcup_{p\geqq 1} A_p$. Clearly $A_p\subset A_s$ if $1\leqq p<s$.

The following two lemmas are well-known.

Lemma 1.1 [9]. *Let $w\in A_\infty$, $1\leqq p<\infty$ and $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. Then, if $w(\{M_1 f(x)>a\})<+\infty$ for each $a>0$, there exists $C>0$ such that*

$$\|f\|_{L^p(w)}\leqq\|M_1 f\|_{L^p(w)}\leqq C\|f^\#\|_{L^p(w)} .$$

Lemma 1.2 [5]. *Let $1\leqq p<\infty$. Then for any $r>p$ and any $w\in A_{r/p}$, there exists $C>0$ such that*

$$\|M_p f\|_{L^r(w)}\leqq C\|f\|_{L^r(w)} .$$

Combining the above two lemmas we have easily the following.

Lemma 1.3. *Let $1\leqq p<\infty$. Let T be a linear operator from $C_0^\infty(\boldsymbol{R}^n)$ to $L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that there exists $B>0$ satisfying*

$$(1.1)\qquad (Tf)^\#(x)\leqq BM_p f(x) \qquad \textit{for } x\in\boldsymbol{R}^n .$$

Then, if $r>p$, $w\in A_{r/p}$ and $w(\{M_1(Tf)(x)>a\})<+\infty$ for each $a>0$, there exists $C>0$ such that

$$(1.2)\qquad \|Tf\|_{L^r(w)}\leqq C\|f\|_{L^r(w)} .$$

Therefore, in order to show the weighted norm inequality (1.2), it is almost sufficient to obtain the inequality (1.1). Inequalities of this type are, first, used by Fefferman and Stein [4] and are often called the *Fefferman-Stein inequalities*, (for example, Journé [5], Nishigaki [15]).

In this paper we shall show the Fefferman-Stein inequalities for pseudo-differential operators, by using the unweighted L^p-estimates and studying their distribution kernels, and then applying Lemma 1.3 we shall obtain the weighted norm inequalities.

To this end, we prepare the following proposition for operators with integral kernels.

Proposition 1.4. *Let* $1\leqq p<\infty$ *and* $1/p+1/q=1$. *Let* $\omega(t)$ *be a positive and nondecreasing function on* $(0, \infty)$ *such that* $\int_0^1 \omega(t)t^{-1}dt<+\infty$, *and let* $G(R,j)$ *be a nonnegative function on* $(0, 1)\times \boldsymbol{N}$ *such that*

$$\sup_{0<R<1} \sum_{1\leqq 2^j\leqq 1/R} G(R,j)<+\infty\,. \tag{1.3}$$

Suppose T *is a bounded linear operator on* $L^p(\boldsymbol{R}^n)$ *and has the kernel* $K(x,y)$ *satisfying the following conditions: For any* $f\in C_0^\infty(\boldsymbol{R}^n)$

$$Tf(x)=\int K(x,y)f(y)dy\,, \qquad \text{for almost all } x\in[\operatorname{supp} f]^c; \tag{1.4}$$

$$\Big(\int_{R<|x-y|<2R} |K(x,y)-K(z,y)|^q dy\Big)^{1/q}\leqq C_0R^{-n/p}\omega\Big(\frac{\max(1,|x-z|)}{R}\Big)\,, \quad \text{for } |x-z|<R/2,\ R\geqq 1; \tag{1.5}$$

$$\Big(\int_{2^jR<|x-y|<2^{j+1}R} |K(x,y)-K(z,y)|^q dy\Big)^{1/q}\leqq C_1(2^jR)^{-n/p}G(R,j)\,, \quad \text{for } |x-z|<R/2,\ 1\leqq 2^j\leqq 1/R,\ 0<R<1\,, \tag{1.6}$$

where C_0 *and* C_1 *are independent of* R, j, x *and* z. *Then there exists a positive constant* C *such that*

$$(Tf)^\sharp(x)\leqq CM_pf(x)\,, \qquad \text{for } x\in\boldsymbol{R}^n \text{ and } f\in C_0^\infty(\boldsymbol{R}^n)\,. \tag{1.7}$$

Proof. Let $f\in C_0^\infty(\boldsymbol{R}^n)$. Given $x\in\boldsymbol{R}^n$ and a cube Q centered at x with diameter d, let $B_0=\{y\in\boldsymbol{R}^n; |x-y|\leqq d\}$ and $B_j=\{y\in\boldsymbol{R}^n; 2^{j-1}d<|x-y|\leqq 2^jd\}$, $j=1, 2, \cdots$. Put $T(f\chi_{B_0{}^c})(x)=a$. Here and hereafter χ_E denotes the characteristic function of the set E. Then, for $z\in Q$

$$Tf(z)-a=T(f\chi_{B_0})(z)+\int_{B_0{}^c}[K(z,y)-K(x,y)]f(y)dy=I_1(z)+I_2(z)\,. \tag{1.8}$$

Then, since T is bounded on L^p, we have by Hölder's inequality

$$\begin{aligned}|Q|^{-1}\int_Q |I_1(z)|dz &\leqq \Big(|Q|^{-1}\int_Q |T(f\chi_{B_0})(z)|^p dz\Big)^{1/p}\\ &\leqq C\Big(|Q|^{-1}\int_{B_0} |f(y)|^p dy\Big)^{1/p}\leqq CM_pf(x)\,.\end{aligned} \tag{1.9}$$

We next treat the second term $I_2(z)$.

(i) The case $d\geqq 1$. By (1.4) and the Hölder inequality we get

$$(1.10)\qquad |I_2(z)|\leqq\sum_{j=1}^{\infty}\Big(\int_{B_j}|K(z,y)-K(x,y)|^q dy\Big)^{1/q}\Big(\int_{B_j}|f(y)|^p dy\Big)^{1/p}$$
$$\leqq\sum_{j=1}^{\infty}C(2^{j-1}d)^{-n/p}\omega(d/(2^{j-1}d))\Big(\int_{|x-y|<2^j d}|f(y)|^p dy\Big)^{1/p}$$
$$\leqq C\sum_{j=0}^{\infty}\omega(2^{-j})M_pf(x)\leqq C\Big(\int_0^1\omega(t)t^{-1}dt\Big)M_pf(x)\,.$$

(ii) The case $0<d<1$. We have

$$(1.11)\qquad I_2(z)=\int_{d<|x-y|<1}[K(z,y)-K(x,y)]f(y)dy+\int_{|x-y|\geqq 1}[\cdots]f(y)dy=J_1+J_2\,.$$

Then, setting $A_j=\{y\in\boldsymbol{R}^n;\ 2^j\leqq|x-y|<2^{j+1}\}$, we have by (1.5) and Hölder's inequality

$$(1.12)\qquad |J_2|\leqq\sum_{j=0}^{\infty}\Big(\int_{A_j}|K(z,y)-K(x,y)|^q dy\Big)^{1/q}\Big(\int_{A_j}|f(y)|^p dy\Big)^{1/p}$$
$$\leqq\sum_{j=0}^{\infty}C\,2^{-nj/p}\omega(2^{-j})\Big(\int_{|x-y|<2^{j+1}}|f(y)|^p dy\Big)^{1/p}\leqq C\,M_pf(x)\,.$$

For J_1, we obtain from (1.6) and Hölder's inequality

$$(1.13)\qquad |J_1|\leqq\sum_{1\leqq 2^j\leqq 1/d}\Big(\int_{B_j}|K(z,y)-K(x,y)|^q dy\Big)^{1/q}\Big(\int_{B_j}|f(y)|^p dy\Big)^{1/p}$$
$$\leqq\sum_{1\leqq 2^j\leqq 1/d}C(2^jd)^{-n/p}G(d,j)\Big(\int_{|x-y|<2^{j+1}d}|f(y)|^p dy\Big)^{1/p}$$
$$\leqq C\sum_{1\leqq 2^j\leqq 1/d}G(d,j)M_pf(x)\leqq C\,M_pf(x)\,.$$

Thus in both cases we get from (1.10)–(1.13)

$$(1.14)\qquad |Q|^{-1}\int_Q|I_2(z)|dz\leqq C\,M_pf(x)\,.$$

Therefore, by (1.8), (1.9) and (1.14) we obtain the desired inequality (1.7). This completes the proof of Proposition 1.4.

REMARK 1.5. Proposition 1.4 comprises the criterions for Fefferman-Stein's inequalities, used by Kurtz-Wheeden [6] and Rubio de Francia [16], and is a generalization of Theorem 2.4 in Yabuta [20].

2. Statement of the main results, remarks and examples

To state our results, we introduce a notion, following Yamazaki [22].

DEFINITION 2.0. We call a function $\omega(s,t)$ on $(0,\infty)\times(0,\infty)$ a *modulus of continuity* if it satisfies the following two conditions:

1) For each fixed s, $\omega(s, t)$ is positive, nondecreasing and concave with respect to t.

2) There exists a positive constant A such that the inequality

$$\omega(s', t) \leqq A\omega(s, t)$$

holds for $s/2 \leqq s' \leqq 2s$.

Then our main results are as follows.

Theorem 2.1. *Let* $1<p\leqq 2$ *and* $0<b\leqq 1$ *satisfy* $[n/p]+b>n/p$. *Let* $\omega(s,t)$ *be a modulus of continuity such that there exist* $0<\delta<1$ *and* $B>0$ *satisfying*

$$\int_0^1 \omega^2(1/t, t^\delta)t^{-1}dt<+\infty\,; \tag{2.1}$$

$$\sum_{1\leqq 2^j\leqq 1/R} \omega(2^j, R)\leqq B \qquad \text{for } 0<R\leqq 1\,. \tag{2.2}$$

Suppose $m(x, \xi)$ *satisfies the following conditions with* $k=[n/p]$:

$$|\partial_\xi^\alpha m(x, \xi)| \leqq C\langle\xi\rangle^{-|\alpha|}\,, \qquad |\alpha|\leqq k\,; \tag{2.3}$$

$$|\partial_\xi^\alpha m(x, \xi+\eta)-\partial_\xi^\alpha m(x, \xi)| \leqq C\langle\xi\rangle^{-k-b}|\eta|^b\,, \quad |\eta|<\langle\xi\rangle/2\,, \quad |\alpha|=k\,; \tag{2.4}$$

$$|\partial_\xi^\alpha m(x+h, \xi)-\partial_\xi^\alpha m(x, \xi)| \leqq C\langle\xi\rangle^{-|\alpha|}\omega(\langle\xi\rangle, |h|)\,, \quad |\alpha|\leqq k\,; \tag{2.5}$$

$$\begin{aligned}&|\partial_\xi^\alpha m(x+h, \xi+\eta)-\partial_\xi^\alpha m(x+h, \xi)-\partial_\xi^\alpha m(x, \xi+\eta)+\partial_\xi^\alpha m(x, \xi)|\\ &\leqq C\langle\xi\rangle^{-k-b}|\eta|^b\omega(\langle\xi\rangle, |h|)\,, \quad |\eta|<\langle\xi\rangle/2\,, \quad |\alpha|=k\,.\end{aligned} \tag{2.6}$$

Then there exists a positive constant C such that

$$(m(x, D)f)^\#(x) \leqq C\, M_p f(x)\,. \tag{2.7}$$

Furthermore, for any $r\geqq p$ *and any weight* $w\in A_{r/p}$ *there exists a positive constant* $C=C(p, r, w)$ *such that*

$$\|m(x, D)f\|_{L^r(w)} \leqq C(p, r, w)\|f\|_{L^r(w)}\,. \tag{2.8}$$

Theorem 2.2. *Let* $0<b\leqq 1$ *and* $\omega(s,t)$ *be a modulus of continuity satisfying* (2.1) *with* $\delta=1$ *and* (2.2). *Suppose* $m(x, \xi)$ *satisfies* (2.3)–(2.6) *with* $k=n$. *Then* $T=m(x, D)$ *satisfies:*

(P–0) $\qquad (Tf)^\#(x) \leqq C_p\, M_p f(x)\,, \qquad 1<p<\infty\,;$

(P–1) *For any* $1<p<\infty$ *and* $w\in A_p$

$$\|Tf\|_{L^p(w)} \leqq C(p, w)\|f\|_{L^p(w)}\,;$$

(P–2) *For any* $w\in A_1$

$$w(\{|Tf(x)|>a\}) \leqq \frac{C(1, w)}{a}\|f\|_{L^1(w)}\,, \qquad \text{for all } a>0\,;$$

(P–3) *For any* $w \in A_1$

$$\|Tf\|_{L^1(w)} \leqq C(1, w)\|f\|_{H^1(w)} ;$$

(P–4)
$$\|Tf\|_{BMO} = \|(Tf)^\sharp\|_{L^\infty(\boldsymbol{R}^n)} \leqq C\|f\|_{L^\infty(\boldsymbol{R}^n)} .$$

In the above $H^1(w)$ is the weighted Hardy space H^1 and BMO is the John-Nirenberg space of functions of bounded mean oscillation, i.e. $f \in$ BMO if and only if $\|f^\sharp\|_{L^\infty(\boldsymbol{R}^n)} < +\infty$, (for detailed definitions, see [19, 20], for example).

Theorem 2.3. *Let* $0 < b \leqq 1$ *and* $\omega(s, t)$ *be a modulus of continuity satisfying* $\int_0^1 \omega^2(1/t, t)t^{-1}dt < +\infty$. *Then, if* $m(x, \xi)$ *satisfies* (2.3)–(2.6) *with* $k=n$, *the conclusions* (P–1), (P–2) *and* (P–3) *in Theorem* 2.2 *hold.*

We shall prove these theorems in Sections 4 and 5.

REMARKS. 1) As the later proof will show, the condition (2.1) in the theorems is unnecessary to obtain the conclusion provided the $L^p(\boldsymbol{R}^n)$-boundedness is known. 2) Theorem 2.2 improves Theorem 3.1 in [20]. 3) Theorem 2.3 improves Theorem 5.1 in [20], and hence answer the question (c) in the section 6 there. As was shown in Proposition 3.5 in [19], the properties (P–0) and (P–4) do not hold, in general, under the assumptions in Theorem 2.3.

In the rest of this section we shall give two sufficient conditions for a modulus of continuity to satisfy (2.1) and (2.2), and then give some examples.

Proposition 2.4. *A modulus* $\omega(s, t)$ *of continuity satisfies* (2.1) *and* (2.2) *if one of the following conditions is satisfied:*

i) *There exist* $0 < \delta < 1$, $B_1 > 0$ *and* $B_2 > 0$ *satisfying*

(2.9)
$$\omega(s', t) \leqq B_1 \omega(s, t) \qquad \text{if } 0 < s' < s \text{ and } 0 < t \leqq 1 ,$$

(2.10)
$$\omega(1/t, t^\delta) \leqq B_2\left(\log \frac{2}{t}\right)^{-1} \qquad \text{for } 0 < t \leqq 1 ;$$

ii) *There exists* $B > 0$ *such that*

(2.11)
$$\omega(s, t) \leqq B\, \omega(s', t) \qquad \text{if } 0 < s' < s \text{ and } 0 < t \leqq 1 ,$$

and $\omega(s, t)$ *satisfies one of the following conditions,*

(2.12a)
$$\int_0^1 \omega(1/t, t)t^{-1}dt < +\infty ,$$

(2.12b)
$$\omega(1, t) \leqq B_1\left(\log \frac{2}{t}\right)^{-1} \qquad \text{for } 0 < t \leqq 1 ;$$

Proof. Suppose first i) holds. (2.10) implies (2.1). By (2.9), (2.10) and the monotonicity in t, we get for $0<R\leqq 1$ and $1\leqq 2^j\leqq 1/R$

$$\omega(2^j, R)\leqq B_1\omega(1/R, R^\delta)\leqq B_1B_2\left(\log\frac{2}{R}\right)^{-1},$$

hence we have (2.2).

Next, suppose ii) holds. By (2.11) we get for $0<R\leqq 1$, $1\leqq 2^j\leqq 1/R$, and for any fixed $0<\delta<1$,

$$\omega(1/t, t^\delta)\leqq B\,\omega(1/t^\delta, t^\delta) \text{ and } \omega(2^j, R)\leqq\omega(2^j, 2^{-j}).$$

Hence from (2.12a) we see that (2.1) and (2.2) hold. We can treat the case (2.12b) similarly. This completes the proof.

Reconsidering the above proof, we have the following consequence of Theorem 2.2.

Corollary 2.5. *Let $0<b\leqq 1$ and $\omega(s, t)$ be a modulus of continuity satisfying one of the following conditions:*

(i) *$\omega(s, t)$ satisfies (2.9) and (2.10) with $\delta=1$;*

(ii) $\int_0^1\omega(1/t, t)t^{-1}dt<+\infty$.

Then, if $m(x, \xi)$ satisfies (2.3)–(2.6) with $k=n$, $m(x, D)$ satisfies (P–0)–(P–4).

EXAMPLES of $\omega(s, t)$. 1. Let $g(t)$ be a positive, nondecreasing and concave function on $(0, \infty)$ satisfying $g(t)\leqq B(\log(2/t))^{-1}$ $(0<t<1)$ for some $B>0$. Then for any $0\leqq a<1$, $g(s^a t)$ is obviously a modulus of continuity satisfying (2.9) and (2.10). ω of this type was treated by Coifman-Meyer [3], Muramatu-Nagase [12] $(a=0)$, and Nagase [13, 14] $(0\leqq a<1)$. Note that if $g(t)$ is positive and non-decreasing then $\int_0^1 g(t)t^{-1}dt<+\infty$ implies $g(t)\leqq B(\log(2/t))^{-1}$ $(0<t<1)$ for some $B>0$. A typical example is $\omega(s, t)=s^a t$ $(0\leqq a<1)$.

2. Let $\Omega(t)$ be a positive and nondecreasing function on $(0, \infty)$ such that for some $A>0$ $\Omega(s')\leqq A\,\Omega(s)$ for $s/2<s'<2s$, and $g(t)$ be a positive, nondecreasing and concave function on $(0, \infty)$. Then, if there exists $0<\delta<1$ with $\Omega(1/t)g(t^\delta)\leqq B(\log(2/t))^{-1}$ $(0<t<1)$ for some $B>0$, $\omega(s, t)=\Omega(s)g(t)$ satisfies (i) in Proposition 2.4. ω of this type was treated by Bourdaud [1] and Yamazaki [21]. Typical one is $\omega(s, t)=\log s/(\log(2/t))^2$ (for s large and t small).

EXAMPLE 2.6. Finally in this section we state a concrete and interesting examples of pseudo-differential operators to which Theorem 2.2 can be applied. Let $1/2<a<1$ and $\varphi(\xi)\in C_0^\infty(\boldsymbol{R})$ satisfying $\varphi(\xi)=1$ on $[3/4, 5/4]$ and $\text{supp}\varphi\subset[2/3, 4/3]$. Define $m(x, \xi)$ by

$$m(x, \xi)=\sum_{j=1}^{\infty}j^{-a}\varphi(2^{-j}\xi)\exp(-i2^j x).$$

Then for this $m(x, \xi)$ the properties (P–0)–(P–4) hold. To see this, we first rewrite $m(x, \xi)$ sa follows.

$$m(x, \xi) = \sum_{j=1}^{\infty} 2^{j/2} 2^{-j/2} j^{-a} \varphi(2^{-j}\xi) \exp(-i2^j x) .$$

Take $\Omega(s)=s^{1/2}$ and $g(t)=t^{1/2} \min((\log \frac{A}{t})^{-a}, (\log A)^{-a})$, where we choose A so large that $g(t)$ is concave. g is clearly positive and increasing. As is shown in Coifman and Meyer [3, p. 39], we get $g(2^{-j})|1-\exp(i2^j h)| \leqq 2g(|h|)$. Hence using two expressions of $m(x, \xi)$ we see that it satisfies (2.3) and (2.5) with $\omega(s, t) = B\,\Omega(s)g(t)$ for $\alpha \in \boldsymbol{N}$, where B is a positive constant. One can see easily that $\omega(s, t)$ satisfies (2.1) and (2.2) with $\delta=1$. Hence by Theorem 2.2 the properties (P–0)–(P–4) hold. Note that in the case $a \geqq 1$, we can apply Theorem 2.2 more easily by taking $\omega(s, t)=C \min((\log \frac{1}{t})^{-a}, (\log 2)^{-a})$.

3. Estimates for the kernels of pseudo-differential operators

We shall investigate here the kernels of pseudo-differential operators with symbol $m(x, \xi)$. They are given by $(2\pi)^{-n}\hat{m}(x, y-x)$. Here and in the sequel $\hat{m}(x,y)$ will always denote the Fourier transform of $m(x, \xi)$ with respect to the covariables ξ. Our first estimate is the following.

Lemma 3.1. *Let* $1 \leqq p \leqq 2$, $1/p+1/q=1$ *and* $0<b\leqq 1$ *with* $[n/p]+b>n/p$. *Suppose*

$$|\partial_\xi^\alpha m(\xi)| \leqq C_\alpha |\xi|^{-|\alpha|}, \qquad |\alpha| \leqq [n/p] \; ; \tag{3.1}$$

$$|\partial_\xi^\alpha m(\xi+\eta) - \partial_\xi^\alpha m(\xi)| \leqq C_\alpha |\xi|^{-|\alpha|-b} |\eta|^b, \; |\eta|<|\xi|/2, \; |\alpha| = [n/p] . \tag{3.2}$$

Then it holds that

$$\Big(\int_{R<|y|<2R} |\hat{m}(y)|^q dy\Big)^{1/q} \leqq C\, R^{-n/p}, \; R>0 \; ; \tag{3.3}$$

$$\Big(\int_{R<|y|<2R} |\hat{m}(y+h)-\hat{m}(y)|^q dy\Big)^{1/q} \leqq C\, R^{-n/p} \Big(\frac{|h|}{R}\Big)^{[n/p]+b-n/p}, \quad |h|<R/2, \; R>0 , \tag{3.4}$$

where, if $[n/p]=n/p$ *and* $b=1$, b *in* (3.4) *should be replaced by an arbitrarily fixed positive number* $c<1$.

Proof. Put $k=[n/p]$ and take a radial function $\psi(\xi) \in C_0^\infty(\boldsymbol{R}^n)$ satisfying supp $\psi \subset \{1/2<|\xi|<2\}$, $\psi(\xi)=1$ on $\{3/4<|\xi|<1\}$ and $\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi)=1$ $(\xi \neq 0)$. Define $m_j(\xi)$ $(j \in \boldsymbol{Z})$ by

$$m_j(\xi) = m(\xi)\psi(2^{-j}\xi) . \tag{3.5}$$

Then m_j satisfies (3.1) and (3.2) uniformly in j. For any α; $|\alpha|=k$ we have by integration by parts

$$\int[\partial_\xi^\alpha m_j(\xi+\eta)-\partial_\xi^\alpha m_j(\xi)]e^{-iy\cdot\xi}d\xi = (i^{|\alpha|}y^\alpha)\,(e^{iy\cdot\eta}-1)\hat{m}_j(y)\,. \tag{3.6}$$

Hence by (3.1), (3.2) we have, using Hausdorff-Young's inequality,

$$\begin{aligned}(\int_{R<|y|<2R}|y^\alpha(e^{iy\cdot\eta}-1)\hat{m}_j(y)|^q dy)^{1/q} &\leqq C(\int_{2^{j-2}<|\xi|<2^{j+2}}|\partial_\xi^\alpha m_j(\xi+\eta)-\partial_\xi^\alpha m_j(\xi)|^p d\xi)^{1/p} \\ &\leqq C(\int_{2^{j-2}<|\xi|<2^{j+2}}(|\xi|^{-k-b}|\eta|^b)^p d\xi)^{1/p} \\ &\leqq C\,2^{-(k+b-n/p)j}|\eta|^b, \qquad \text{if } |\eta|<2^{j-3}\,.\end{aligned} \tag{3.7}$$

Hence we have for any η with $|\eta|<2^{j-3}$

$$(\int_{R<|y|<2R}[|y|^k|e^{iy\cdot\eta}-1||\hat{m}_j(y)|]^q dy)^{1/q} \leqq C\,2^{-(k+b-n/p)j}|\eta|^b\,. \tag{3.8}$$

We shall treat two cases $2^jR\geqq 1$ and $2^jR<1$; it corresponds to divide the symbol $m(\xi)$ into two parts with support $\{|\xi|>(2R)^{-1}\}$ and $\{|\xi|<2/R\}$, respectively. Now suppose $2^jR>1$. Then there are a finite number of $\xi(i)$ with $|\xi(i)|=1$, independent of R, such that for any y with $R<|y|<2R$ there exists $\xi(i)$ satisfying $1/4>|y\cdot R^{-1}2^{-3}\xi(i)|>1/16$. In this case for $\eta=R^{-1}2^{-3}\xi(i)$, we have $|\eta|=2^{-3}/R<2^{j-3}$ and

$$(16|y||\eta|)^b \leqq 16|y||\eta| \leqq C|e^{iy\cdot\eta}-1|\,. \tag{3.9}$$

So, dividing the integration domain appropriately, we have from (3.8) and (3.9)

$$(\int_{R<|y|<2R}|\hat{m}_j(y)|^q dy)^{1/q} \leqq C\,R^{-k-b}2^{-(k+b-n/p)j}, \qquad \text{if } \ 2^jR\geqq 1\,. \tag{3.10}$$

Denote the left hand side of (3.10) by I_j. Then

$$\sum_{2^jR\geqq 1} I_j \leqq C\,R^{-k-b}R^{k+b-n/p} = C\,R^{-n/p}\,. \tag{3.11}$$

Next by (3.1) and (3.5)

$$\sum_{2^jR<1} I_j \leqq C\sum_{2^jR<1}(\int_{2^{j-1}<|\xi|<2^{j+1}}|m_j(\xi)|^p d\xi)^{1/p} \leqq C\sum_{2^jR<1}2^{jn/p} \leqq C\,R^{-n/p}\,. \tag{3.12}$$

This proves (3.3).

We go next to the proof of (3.4). Suppose first $2^jR\geqq 1$ as before. Using (3.6) we have the following identity for any α with $|\alpha|=k$

$$y^\alpha(e^{iy\cdot\eta}-1)\,[\hat{m}_j(y+h)-\hat{m}_j(y)] \tag{3.13}$$

$$= -[(y+h)^\alpha - y^\alpha](e^{i(y+h)\cdot\eta}-1)\hat{m}_j(y+h) - y^\alpha(e^{i(y+h)\cdot\eta}-e^{iy\cdot\eta})\hat{m}_j(y+h)$$
$$+ i^{-k}\int[\partial_\xi^\alpha m_j(\xi+\eta)-\partial_\xi^\alpha m_j(\xi)](e^{-ih\cdot\eta}-1)e^{-iy\cdot\xi}d\xi$$
$$= J_1+J_2+J_3 .$$

From (3.8) it follows that, if $|h|<R/2$ and $|\eta|<2^{j-3}$,

$$(3.14)\qquad (\int_{R<|y|<2R}|J_1|^q dy)^{1/q} \leqq C(\int_{R/2<|y|<3R}[|y|^{k-1}|h|\,|e^{iy\cdot\eta}-1|\,|\hat{m}_j(y)|]^q dy)^{1/q}$$
$$\leqq C|h|R^{-1}|\eta|^b 2^{-(k+b-n/p)j} .$$

We have by using (3.10) and $|e^{ih\cdot\eta}-1|\leqq 2(|h|\,|\eta|)^b$, $(0\leqq b\leqq 1)$,

$$(3.15)\qquad (\int_{R<|y|<2R}|J_2|^q dy)^{1/q} \leqq C\,R^k(|h|\,|\eta|)^b R^{-k-b}2^{-(k+b-n/p)j}$$
$$= C(|h|/R)^b|\eta|^b 2^{-(k+b-n/p)j} ,$$

if $|\eta|<2^{j-3}$ and $|h|<R/2$. For J_3 we get by (3.1), (3.2) and Hausdorff-Young's inequality

$$(3.16)\qquad (\int_{R<|y|<2R}|J_3|^q dy)^{1/q}$$
$$\leqq C(\int_{2^{j-2}\leqq|\xi|\leqq 2^{j+2}}|(\partial_\xi^\alpha m_j(\xi+\eta)-\partial_\xi^\alpha m_j(\xi))(e^{-ih\cdot\xi}-1)|^p d\xi)^{1/p}$$
$$\leqq C(\int_{2^{j-2}\leqq|\xi|\leqq 2^{j+2}}[|\xi|^{-k-b}|\eta|^b\min(1,\,|\xi|\,|h|)]^p d\xi)^{1/p}$$
$$\leqq C\,2^{-(k+b-n/p)j}|\eta|^b\min(1,\,2^j|h|) .$$

Hence, as in the proof of (3.3) we obtain from (3.13)–(3.16)

$$(3.17)\qquad L_j = (\int_{R<|y|<2R}|\hat{m}_j(y+h)-\hat{m}_j(y)|^q dy)^{1/q}$$
$$\leqq C\,R^{-k-b}\left(\frac{|h|}{R}\right)^b 2^{-(k+b-n/p)j} + C\,R^{-k-b}2^{-(k+b-n/p)j}\min(1,\,2^j|h|) ,$$

if $|h|<R/2$ and $2^jR\geqq 1$. So, if $|h|<R/2$ and $k+b-n/p<1$, we get

$$(3.18)\qquad \sum_{2^jR\geqq 1} L_j \leqq C\,R^{-k-b}(|h|/R)^b R^{k+b-n/p}$$
$$+C\,R^{-k-b}\sum_{|h|\leqq 2^{-j}\leqq R}2^{-(k+b-n/p)j}2^j|h| + C\,R^{-k-b}\sum_{2^{-j}\leqq|h|}2^{-(k+b-n/p)j}$$
$$\leqq C\,R^{-n/p}(|h|/R)^b + C\,R^{-k-b}|h|^{k+b-n/p}$$
$$\leqq C\,R^{-n/p}(|h|/R)^b + C\,R^{-n/p}(|h|/R)^{k+b-n/p}$$
$$\leqq C\,R^{-n/p}(|h|/R)^{k+b-n/p} ,$$

since $0<k+b-n/p\leqq b\leqq 1$. If $k+b-n/p=1$, i.e., if $[n/p]=n/p$ and $b=1$, for any $0<c<1$ there is $C_c>0$ such that

$$R^{-k-b}\sum_{|h|\leqq 2^{-j}\leqq R}2^{-(k+b-n/p)j}2^j|h|\sim C\,R^{-n/p}(|h|/R)\log(R/|h|) \tag{3.19}$$
$$\leqq C_c\,R^{-n/p}(|h|/R)^c\,.$$

As for the case $2^jR<1$, we have by Hausdorff-Young's inequality for $R>0$ and $|h|<R/2$,

$$\sum_{2^jR<1}L_j\leqq C\sum_{2^jR<1}\left(\int|m_j(\xi)\,(e^{ih\cdot\xi}-1)|^p d\xi\right)^{1/p}\leqq C\sum_{2^jR<1}\left(\int_{|\xi|<2^{j+1}}(|\xi||h|)^p d\xi\right)^{1/p} \tag{3.20}$$
$$\leqq C|h|\sum_{2^jR<1}2^{(1+n/p)j}\leqq C|h|R^{-1-n/p}\leqq C\,R^{-n/p}(|h|/R)^b\,.$$

Hence we obtain the desired estimate (3.4). This completes the proof.

Lemma 3.2. *Let p, q, b be as in Lemma 3.1. Suppose $m(\xi)$ satisfies the following:*

$$|\partial_\xi^\alpha m(\xi)|\leqq C\langle\xi\rangle^{-|\alpha|}\,,\qquad |\alpha|\leqq[n/p]\,; \tag{3.21}$$

$$|\partial_\xi^\alpha m(\xi+\eta)-\partial_\xi^\alpha m(\xi)|\leqq C\langle\xi\rangle^{-|\alpha|-b}|\eta|^b,\ |\eta|<\langle\xi\rangle/2,\ |\alpha|=[n/p]\,. \tag{3.22}$$

Then it follows

$$\left(\int_{R<|y|<2R}|\hat{m}(y)|^q dy\right)^{1/q}\leqq C\,R^{-[n/p]-b},\ R\geqq 1/10\,. \tag{3.23}$$

Proof. We use the notations in the proof of Lemma 3.1. By Lemma 3.1 we may assume $R\geqq 1$. Then from (3.10) it follows that

$$\sum_{j\geqq 0}I_j\leqq C\,R^{-k-b}\,. \tag{3.24}$$

Set $\varphi(\xi)=1-\sum_{j\geqq 0}\psi(2^{-j}\xi)$ and $m_I(\xi)=m(\xi)\varphi(\xi)$. Then (3.21) and (3.22) also hold for this m_I. Hence as before we have

$$\left(\int_{R<|y|<2R}[|y|^k|e^{iy\cdot\eta}-1||\hat{m}_I(y)|]^q dy\right)^{1/q}\leqq C|\eta|^b,\qquad \text{if } |\eta|<1/2\,. \tag{3.25}$$

Considering η with $|\eta|=1/(4R)$, we have as in (3.10)

$$\left(\int_{R<|y|<2R}|\hat{m}_I(y)|^q dy\right)^{1/p}\leqq C\,R^{-k-b}\,. \tag{3.26}$$

Combining (3.24) and (3.26) we have (3.23) for $R\geqq 1$. This completes the proof.

Lemma 3.3. *Let p, q, b be as in Lemma 3.1. Suppose $m(\xi)$ satisfies (3.1), (3.2) and* supp $m(\xi)\subset\{|\xi|>1/R\}$ $(0<R\leqq 1)$. *Then it holds that*

$$(3.27)\qquad \Big(\int_{2^lR<|y|<2^{l+1}R} |\hat{m}(y)|^q dy\Big)^{1/q} \leqq C\,R^{-n/p}2^{-([n/p]+b)l},\ l\in \boldsymbol{N},$$

where C is independent of R, l and m.

Proof. We use the notations in the proof of Lemma 3.1. Taking account of the support of m, we have only to consider j with $2^{j+1}>1/R$. In this case $2^{j+l}R\geqq 1$ for $l=1,2,\cdots$. Hence from (3.10) we get (3.27) for $l=1,2,\cdots$. (3.27) for $l=0$ follows from (3.3). This completes the proof.

Lemma 3.4. *Let p, q and b be the same as in Lemma 3.1, and $\omega(s,t)$ be a modulus of continuity. Suppose $m(x,\xi)$ satisfies (2.3)–(2.6) with $k=[n/p]$ and* $\operatorname{supp}_\xi m(x,\xi)\subset\{1\leqq|\xi|\leqq 1/R\}$, $(0<R\leqq 1)$. *Then we have*

$$(3.28)\qquad \Big(\int_{2^lR<|y|<2^{l+1}R} |\hat{m}(x+h,y)-\hat{m}(x,y)|^q dy\Big)^{1/q}$$
$$\leqq C(2^lR)^{-n/p}H(R,l),\qquad \text{for } x,h\in\boldsymbol{R}^n,\ |h|<R/2,\ l\in\boldsymbol{Z}$$

where $H(R,l)$ satisfies for some positive constant C_1

$$(3.29)\qquad \sum_{l=-\infty}^{\infty} H(R,l)\leqq C_1\sum_{1\leqq 2^j\leqq 1/R}\omega(2^j,R),$$

and C is independent of R and l.

Proof. Let ψ be the same as in the proof of Lemma 3.1 and $m_j(x,\xi)=m(x,\xi)\psi(2^{-j}\xi)$. Then m_j satisfies (2.3)–(2.6) uniformly in j.

Integrating by parts with respect to ξ, we have for α with $|\alpha|=k$

$$(3.30)\qquad \begin{aligned} Y_{j,\alpha} &= i^{|\alpha|}y^\alpha(e^{iy\cdot\eta}-1)\,[\hat{m}(x+h,y)-\hat{m}(x,y)]\\ &=\int[\partial_\xi^\alpha m_j(x+h,\xi+\eta)-\partial_\xi^\alpha m_j(x+h,\xi)-\partial_\xi^\alpha m_j(x,\xi+\eta)\\ &\quad+\partial_\xi^\alpha m_j(x,\xi)]e^{-iy\cdot\xi}d\xi\,. \end{aligned}$$

Hence by Hausdorff-Young's inequality and (2.6) we have for any $s>0$

$$(3.31)\qquad \begin{aligned} \Big(\int_{s<|y|<2s}|Y_{j,\alpha}|^q dy\Big)^{1/q} &\leqq C\Big(\int_{2^{j-2}<|\xi|<2^{j+2}}[\langle\xi\rangle^{-k-b}|\eta|^b\omega(\langle\xi\rangle,|h|)]^p d\xi\Big)^{1/p}\\ &\leqq C\,2^{-(k+b-n/p)j}|\eta|^b\omega(2^j,|h|), \end{aligned}$$

if $|\eta|<2^{j-3}$ and $2^js\geqq 1$. Hence as in Lemma 3.1 we have

$$(3.32)\qquad \Big(\int_{s<|y|<2s}|\hat{m}_j(x+h,y)-\hat{m}_j(x,y)|^q dy\Big)^{1/q}\leqq C\,2^{-(k+b-n/p)j}s^{-k-b}\omega(2^j,|h|)$$

if $2^js\geqq 1$. On the other hand, if $2^js<1$, by (2.3) and Hausdorff-Young's inequality we get

$$(3.33)\qquad \Big(\int_{s<|y|<2s} |\hat{m}_j(x+h, y)-\hat{m}_j(x, y)|^q dy\Big)^{1/q}$$

$$\leqq C\Big(\int_{2^{j-1}<|\xi|<2^{j+1}} |m_j(x+h, \xi)-m_j(x, \xi)|^p d\xi\Big)^{1/p} \leqq C\, 2^{jn/p}\omega(2^j, |h|)\,.$$

So using (3.32) and (3.33) we have

$$(3.34)\qquad \Big(\int_{2^l R<|y|<2^{l+1}R} |\hat{m}_j(x+h, y)-\hat{m}_j(x, y)|^q dy\Big)^{1/q} \leqq C(2^l R)^{-n/p} F(R; j, l, |h|)\,,$$

where $F(R; j, l, t)=(2^{j+l}R)^{-([n/p]+b-n/p)}\omega(2^j, t)$ $(2^{j+l}R\geqq 1)$, $=(2^{j+l}R)^{n/p}\omega(2^j, t)$ $(2^{j+l}R<1)$. By elementary calculation we obtain

$$\sum_{l=-\infty}^{\infty} F(R; j, l, R) = \sum_{2^{j-l}R\geqq 1} \cdots + \sum_{2^{j+l}R<1} \cdots \leqq C\,\omega(2^j, R)\,.$$

Since $\mathrm{supp}_\xi\, m(x, \xi)\subset \{1\leqq |\xi|\leqq 1/R\}$, we have only to consider j with $1\leqq 2^j\leqq 1/R$. Hence setting $H(R, l)=\sum_{1\leqq 2^j\leqq 1/R} F(R; j, l, R)$, we get the desired estimate (3.28), by summing (3.34) with respect to j: $1\leqq 2^j\leqq 1/R$. This completes the proof.

4. Proof of Theorem 2.1

We shall show that the conditions in Proposition 1.4 are satisfied under the hypotheses in Theorem 2.1. Hereafter q will denote the conjugate exponent of p, i.e. $1/p+1/q=1$. First, we note that by Theorem 3.1 in Miyachi and Yabuta [10] $m(x, D)$ is bounded on $L^p(\boldsymbol{R}^n)$, in this case. Next we study the kernel $K(x, y)=(2\pi)^{-n}\hat{m}(x, y-x)$ of $m(x, D)$.

For $R\geqq 1$, we have, using Lemma 3.2

$$(4.1)\qquad \Big(\int_{R<|x-y|<2R} |K(x,y)-K(z,y)|^q dy\Big)^{1/q} \leqq (2\pi)^{-n}\Big[\Big(\int_{R<|x-y|<2R} |\hat{m}(x,y-x)|^q dy\Big)^{1/q}$$

$$+\Big(\int_{R/2<|y-z|<3R} |\hat{m}(z, y-z)|^q dy\Big)^{1/q}\Big] \leqq C\, R^{-n/p-b}\,, \qquad \text{for } |x-z|<R/2\,.$$

Assume then $0<R\leqq 1$. By Lemma 3.1 we have

$$(4.2)\qquad \Big(\int_{R<|x-y|<2R} |\hat{m}(z,y-x)-\hat{m}(z,y-z)|^q dy\Big)^{1/q}\leqq C\, R^{-n/p}(|x-z|/R)^{[n/p]+b-n/p}\,,$$

$$|x-z|<R/2\,.$$

Let φ and ψ be the same as in the proof of Lemma 3.2, and put

$$m_I(x, \xi) = m(x, \xi)\varphi(\xi) \quad\text{and}\quad m_j(x, \xi) = m(x, \xi)\psi(2^{-j}\xi) \quad (j = 0, 1, \cdots)\,.$$

Then

$$(4.3)\qquad m(x, \xi) = m_I(x, \xi)+\sum_{1\leqq 2^j\leqq 1/R} m_j(x, \xi)+\sum_{2^j>1/R} m_j(x, \xi)$$

$$= m_I(x, \xi)+m_{II}(x, \xi)+m_{III}(x, \xi)\,.$$

Then $m_I(x, \xi)$ satisfies (2.3). Since $\mathrm{supp}_\xi\, m_I(x, \xi)\subset\{|\xi|<1]$, by (2.3) we have clearly $|\hat{m}_I(x, y)|\leqq C_0$, and hence

$$(4.4)\qquad \Big(\int_{2^jR<|x-y|<2^{j+1}R} |\hat{m}_I(x, y-x)-\hat{m}_I(z, y-x)|^q dy\Big)^{1/q} \leqq C(2^jR)^{n/q} = C\, 2^j R(2^jR)^{-n/p}\,.$$

Next, since $\mathrm{supp}_\xi\, m_{II}(x, \xi)\subset\{1/2<|\xi|<2/R\}$, we have by Lemma 3.4

$$(4.5)\qquad \Big(\int_{2^jR<|y-y|<2^{j+1}R} |\hat{m}_{II}(x, y-x)-\hat{m}_{II}(z, y-x)|^q dy\Big)^{1/q} \leqq C(2^jR)^{-n/p} H(R, j)\,,\quad |x-z|<R/2\,,$$

where $\sum_{1\leqq 2^j\leqq 1/R} H(R, j)\leqq C_1 \sum_{1\leqq 2^j\leqq 1/R} \omega(2^j, R)$. Since the support of $m_{III}(x, \xi)$ is contained in $\{|\xi|>1/R\}$, we have by Lemma 3.3

$$(4.6)\qquad \Big(\int_{2^jR<|x-y|<2^{j+1}R} |\hat{m}_{III}(x, y-x)-\hat{m}_{III}(z, y-x)|^q dy\Big)^{1/q}$$
$$\leqq \Big(\int_{2^jR<|y|<2^{j+1}R} |\hat{m}_{III}(x, y)|^q dy\Big)^{1/q}+\Big(\int_{2^jR<|y|<2^{j+1}R} |\hat{m}_{III}(z, y)|^q dy\Big)^{1/q}$$
$$\leqq C(2^jR)^{-n/p} 2^{-([n/p]+b-n/p)j}\,,\quad |x-z|<R/2\,.$$

Therefore from (4.1)–(4.6) we see that the hypotheses in Proposition 1.4 are satisfied by $m(x,D)$ if we take $\omega(t)=t^b$ and $G(R,j)=2^jR+H(R, j)+2^{-([n/p]+b-n/p)j}$ in Proposition 1.4. Hence we have the desired estimate (2.7).

We next go to the proof of the second assertion. It is clear that there exists a real number s with $1<s<p$ such that all the assumptions in Theorem 2.1 are satisfied for p replaced by s. Hence by the first step, we get

$$(4.7)\qquad (m(x, D)f)^\sharp(x)\leqq C\, M_s f(x)\,.$$

Now let $f(x)\in C_0^\infty(\boldsymbol{R}^n)$ with $\mathrm{supp}\, f\subset\{|x|<A\}$ and $w(x)\in A_{r/p}$. Then from the theory of A_p weights we have

$$(4.8)\qquad \int w(x)\,(1+|x|)^{-nr/p}dx<+\infty\,.$$

From Lemma 3.1 we get for x with $|x|>2A$

$$(4.9)\qquad |m(x, D)f(x)| = \Big|(2\pi)^{-n}\int \hat{m}(x, y-x)f(y)dy\Big|$$
$$\leqq C\Big(\int_{|x|/2<|x-y|<2|x|} |\hat{m}(x, y-x)|^q dy\Big)^{1/q}\Big(\int |f(y)|^p dy\Big)^{1/p} \leqq C|x|^{-n/p}\,.$$

Since we have already $m(x, D)f\in L^p(\boldsymbol{R}^n)$, we get by (4.9)

$$(4.10)\qquad M_1(m(x, D)f)(x) \leqq C|x|^{-n/p}, \qquad \text{for } |x|>2A.$$

Combining (4.8) with (4.10) we get

$$(4.11)\qquad w(\{M_1(m(x, D)f)(x)>a\})<+\infty, \qquad \text{for any } a>0.$$

Since $w\in A_{r/p}\subset A_{r/s}$, we see by Lemma 1.3 that (4.7) and (4.11) imply the desired estimate (2.8). This completes the proof.

5. Proofs of Theorems 2.2 and 2.3

We shall prove Theorem 2.2 by showing that to $m(x,D)$ one can apply the arguments in Journé [5, Chapter 4], i.e. $m(x,D)$ is a Calderón-Zygmund operator in a generalized sense. We begin now with the following lemma.

Lemma 5.1. *Let* $0<b\leqq 1$ *and* $\omega(s, t)$ *be a modulus of continuity satisfying* (2.2) i.e. *for some* $B>0$ $\sum_{1\leqq 2^j\leqq 1/R} \omega(2^j, R)\leqq B$, $(0<R\leqq 1)$. *Suppose* $m(x, \xi)$ *satisfies* (2.3)–(2.6) *in Section* 2 *with* $k=n$. *Then we have for any cube Q with diameter d and center a, and any* $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$

$$(5.1)\qquad \int_{|y-a|>d} |K(x, y)-K(z, y)|\,|f(y)|\,dy \leqq C \sup_{I\supset Q} |I|^{-1}\int_I |f(y)|\,dy,\ x, z\in Q;$$

$$(5.2)\qquad \int_{|y-a|>d} |K(y, x)-K(y, z)|\,|f(y)|\,dy \leqq C \sup_{I\supset Q} |I|^{-1}\int_I |f(y)|\,dy,\ x, z\in Q,$$

where $K(x, y) = (2\pi)^{-n}\,\hat{m}(x, y-x)$ *and I's are cubes.*

Proof. As in the proof of Theorem 2.1 in Section 4, we see that $K(x,y)$ satisfies the assumptions in Proposition 1.4 ($p=1$), except the $L^1(\boldsymbol{R}^n)$-boundedness. Hence as the proof of Proposition 1.4 shows, we see that (5.1) holds. (5.2) is more easy. Indeed, by (2.3), (2.4) and Lemma 3.1 ($p=1$) we get

$$(5.3)\qquad |\hat{m}(y, x-y)-\hat{m}(y, z-y)| \leqq C|x-y|^{-n}(|x-z|/|x-y|)^b, \qquad |x-z|<|x-y|/2.$$

Hence $K^*(x,y)=K(y,x)$ satisfies the assumptions in Proposition 1.4 ($p=1$). So, as above we obtain (5.2), which completes the proof.

From (5.1) we have easily for any cube Q with diameter d and center a, and any $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$

$$(5.4)\qquad \left|\int_{|y-a|>d} [K(x, y)-K(z, y)]f(y)dy\right| \leqq C\,M_1f(z), \qquad \text{for } x, z\in Q.$$

From (5.2) we have for any $w\in A_1$ and any cube Q with diameter d and center a, and any $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$

$$|\int_{|x-a|>d}\int_Q [K(x,y)-K(x,z)]f(y)w(x)dydx| \leqq Cw(Q)|Q|^{-1}\int_Q |f(y)|dy\,, \tag{5.5}$$
for all $z\in Q$,

since one gets $|I|^{-1}\int_I w(x)dx \leqq C \operatorname{ess\,inf}_{z\in I} w(z) \leqq |Q|^{-1}\int_Q w(x)dx$ for $Q\subset I$, by the definition of A_1. By (2.3), (2.4) and Lemma 3.1, we have $|\hat{m}(x,y-x)|\leqq C|x-y|^{-n}$ and hence for any cube Q with center a and diameter d, and any $f\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$

$$|\int_{d/2<|y-a|<d} K(a,y)f(y)dy| \leqq C\,M_1 f(a)\,. \tag{5.6}$$

Therefore, in this case, $K(x,y)$ satisfies the kernel conditions, discussed in Journé [5, p. 47]. Thus, if one can show the $L^r(\boldsymbol{R}^n)$-boundedness for some $1<r<\infty$, then the standard conlcusions for the usual Calderón-Zygmund operators hold, (for the details, see [5, Chapter 4] or Yabuta [19, 20]).

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. From the assumption on ω, we get easily $\int_0^1 \omega^2(1/t,t)t^{-1}dt<+\infty$. Hence, by the arguments in Bourdaud [1, p. 1032] or Coifman-Meyer [3, pp. 43–45] we have the $L^2(\boldsymbol{R}^n)$-boundedness for $m(x,D)$. This, combined with the above consideration, gives the proof.

Proof of Theorem 2.3. We shall show (P–1) in the following. Then (P–2) and (P–3) follow from (5.2) as in the proof of 5.1 in [20]. Now as in Theorem 2.2, $m(x,D)$ is bounded on $L^2(\boldsymbol{R}^n)$, i.e.

$$\|m(x,D)f\|_2 \leqq C\|f\|_2\,. \tag{5.7}$$

Next we decompose the symbol $m(x,\xi)$ as follows: Take a nonnegative function $u(x)\in C_0^\infty(\boldsymbol{R}^n)$ with supp $u\subset\{|x|<1\}$ and $\int u(x)dx=1$. Put

$$m_1(x,\xi) = \int u(z)m(x-\langle\xi\rangle^{-1}z,\xi)dz = \int u(\langle\xi\rangle(x-z))m(z,\xi)\langle\xi\rangle^n dz\,,$$
$$m_2(x,\xi) = m(x,\xi)-m_1(x,\xi)\,.$$

Then we have

$$|\partial_x^\beta\partial_\xi^\alpha m_1(x,\xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|+|\beta|},\quad |\alpha|\leqq n,\quad \beta\in\boldsymbol{N}^n\,; \tag{5.8}$$

$$|\partial_x^\beta\partial_\xi^\alpha m_1(x,\xi+\eta)-\partial_x^\beta\partial_\xi^\alpha m_1(x,\xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{-n+|\beta|-b}|\eta|^b\,, \quad |\eta|<\langle\xi\rangle/2,\quad |\alpha|=n,\quad \beta\in\boldsymbol{N}^n\,; \tag{5.9}$$

$$|\partial_\xi^\alpha m_2(x,\xi)| \leqq C_\alpha\langle\xi\rangle^{-|\alpha|}\omega(\langle\xi\rangle,1/\langle\xi\rangle),\quad |\alpha|\leqq n\,; \tag{5.10}$$

$$|\partial_\xi^\alpha m_2(x,\xi+\eta)-\partial_\xi^\alpha m_2(x,\xi)| \leqq C_\alpha\langle\xi\rangle^{-n-b}|\eta|^b\omega(\langle\xi\rangle,1/\langle\xi\rangle)\,, \tag{5.11}$$

$$|\eta|<\langle\xi\rangle/2, \ |\alpha|=n,$$

(cf. [10, Section 5, Lemma 5.1] or Nagase [13]).

Then by Theorem 3.2 in [10], we have for any $1<p<\infty$

$$\|m_2(x, D)f\|_p \leqq C_p\|f\|_p . \tag{5.12}$$

From (5.7) and (5.12) it follows that $m_1(x, D)$ is bounded on $L^2(\boldsymbol{R}^n)$. From (5.8) and (5.9), using Lemma 5.1 and the consideration after it, we see that $\hat{m}_1(x, y-x)$ satisfies (5.4), (5.5) and (5.6), since $\omega(s, t)=st$ clearly satisfies the assumptions on ω in Lemma 5.1. Therefore by the arguments in Journé [5. Chapter 4] we see that $m_1(x, D)$ is bounded on $L^p(\boldsymbol{R}^n)$ for any $1<p<\infty$. Combining this with (5.12) we get that $m(x, D)$ is bounded on $L^p(\boldsymbol{R}^n)$, $1<p<\infty$. Hence its adjoint operator $m^*(x, D)$ is bounded on $L^r(\boldsymbol{R}^n)$ for $1<r<\infty$. From (2.3) and (2.4) we have, by Lemma 3.1 ($p=1$), for the kernel $K^*(x, y)$ associated with $m^*(x, D)$

$$|K^*(z, y)-K^*(x, y)| \leqq C|x-y|^{-n}(|x-z|/|x-y|)^b, \ |x-z|<|x-y|/2 .$$

Therefore by Proposition 1.4 we have

$$(m^*(x, D)f)^\sharp(x) \leqq C_r M_r f(x), \ 1<r<\infty .$$

Hence for any $1<p<\infty$ and any $w\in A_p$, taking $1<r<p$ so that $w\in A_{q/r}$ (this can be done from the A_p weight theory, see, for example, Coifman-Fefferman [2]), and applying Lemma 1.3 we have

$$\|m^*(x, D)f\|_{L^p(w)} \leqq C(p, w)\|f\|_{L^p(w)} ,$$

since for $f\in C_0^\infty(\boldsymbol{R}^n)$ we obtain $w(\{M_1(m^*(x, D)f)(x)>a\})<+\infty$ for any $a>0$ as in the proof of Theorem 2.1. Therefore, by duality (using the fact that $w\in A_p$ is equivalent to $w^{-q/p}\in A_q$, if $1/p+1/q=1$) we obtain for any $v\in A_q$

$$\|m(x, D)f\|_{L^q(v)} \leqq C(q, v)\|f\|_{L^q(v)} ,$$

which shows (P–1). Hence the proof of Theorem 2.3 is complete.

6. The case of double symbols $m(x, \xi, y)$

Let us consider pseudo-differential operators of the following type:

$$m(x, D, y)f(x) = (2\pi)^{-n}\iint_{\boldsymbol{R}^n\times\boldsymbol{R}^n} e^{i(x-y)\cdot\xi} m(x, \xi, y)f(y)dyd\xi .$$

The $L^p(\boldsymbol{R}^n)$-boundedness of these operators is studied by Muramatu and Nagase [12] and then by Miyachi and Yabuta [10]. We have also in this case a result similar to Theorem 2.2.

Theorem 6.1. *Let $\omega_1(s, t)$ and $\omega_2(s, t)$ be moduli of continuity such that* *there exist* $B_1, B_2>0$ *and* $0<\delta_1, \delta_2<1$ *satisfying the following conditions* (i), (ii) *or* (i), (iii):

(i) $\sum_{1\leq 2^j\leq 1/R}\omega_l(2^j, R)\leqq B_l \quad (0<R\leqq 1),\ l=1, 2;$

(ii) $\int_0^1[\omega_1(1/t, t^{\delta_1})+\omega_2^2(1/t, t^{\delta_2})]t^{-1}dt<+\infty;$

(iii) $\int_0^1[\omega_1^2(1/t, t^{\delta_1})+\omega_2(1/t, t^{\delta_2})]t^{-1}dt<+\infty.$

Suppose $m(x, \xi, y)$ *satisfies for* $0<b\leqq 1$:

$$|\partial_\xi^\alpha m(x, \xi, y)|\leqq C\langle\xi\rangle^{-|\alpha|}, \quad |\alpha|\leqq n; \tag{6.1}$$

$$|\partial_\xi^\alpha m(x, \xi+\eta, y)-\partial_\xi^\alpha m(x, \xi, y)|\leqq C\langle\xi\rangle^{-n-b}|\eta|^b, \quad |\eta|<\langle\xi\rangle/2,\ |\alpha|=n; \tag{6.2}$$

$$|\partial_\xi^\alpha m(x+h, \xi, y+k)-\partial_\xi^\alpha m(x, \xi, y)| \leqq C\langle\xi\rangle^{-|\alpha|}[\omega_1(\langle\xi\rangle, |h|)+\omega_2(\langle\xi\rangle, |k|)],\quad |\alpha|\leqq n; \tag{6.3}$$

$$|\partial_\xi^\alpha m(x+h, \xi+\eta, y+k)-\partial_\xi^\alpha m(x+h, \xi, y+k)-\partial_\xi^\alpha m(x, \xi+\eta, y) +\partial_\xi^\alpha m(x, \xi, y)|\leqq C\langle\xi\rangle^{-n-b}|\eta|^b[\omega_1(\langle\xi\rangle, |h|)+\omega_2(\langle\xi\rangle, |k|)], \quad |\eta|<\langle\xi\rangle/2,\ |\alpha|=n. \tag{6.4}$$

Then the pseudo-differential operator $m(x,D,y)$ *satisfies the properties* (P–0)–(P–4) *in Theorem* 2.2.

Proof. Let $K(x, y)=(2\pi)^{-n}\hat{m}(x, y-x, y)$ be the distribution kernel associated with $m(x,D,y)$. Using (i), Lemmas 3.1, 3.2, and easy variants of Lemmas 3.3 and 3.4, we see that $K(x,y)$ satisfies (5.4), (5.5) and (5.6). By Theorem 8.1 in [10], using (ii) or (iii), we see that $m(x,D,y)$ is bounded on $L^p(\boldsymbol{R}^n)$ for any $1<p<\infty$ and hence especially on $L^2(\boldsymbol{R}^n)$. Therefore by the arguments in Journé [5, Chapter 4], we see that $m(x,D,y)$ has the desired properties (P–0)–(P–4). This completes the proof.

REMARK 6.2. If one can show by some means the $L^r(\boldsymbol{R}^n)$-boundedness for some $1<r<\infty$, then the conditoin (ii) or (iii) in the above can be removed.

Note also that $\int_0^1\omega_1(1/t, t^{\delta_1})t^{-1}dt<+\infty$ implies (i) with $l=1$. So, as a partial result of Theorem 6.1 we have the following.

Theorem 6.3. *Let* ω_1, ω_2 *be moduli of continuity satisfying* (ii) *in Theorem* 6.1 (*respectively* (iii)). *Then, if* $m(x, \xi, y)$ *fulfills* (6.1)–(6.4), $m(x, D, y)$ *has the properties* (P–0), (P–1) *and* (P–4) (*respectively* (P–1), (P–2) *and* (P–3)).

Since the proof is similar to the above, we omit it.

7. Concluding remarks

a) We note that all theorems and lemmas in this paper, it is sufficient to consider only differential operators of the form $\partial_{\xi_j}^l$ for the assumptions on the derivatives of symbols.

b) Relating to Proposition 2.4 ii), we note that in Theorem 2.1 (2.1), (2.10), (2.11) and (2.3)–(2.6) with $k=n$ do not imply (2.7). So the condition (2.12a) in Proposition 2.4 is in this sense sharp. This can be seen by the following example. Let $h(t)\in C_0^\infty(\boldsymbol{R})$ be an odd function such that $h(t)=0$ ($|t|<1$), $=1$ ($t>2$). Put $G(\xi)=h(\xi)/\log(2+|\xi|)$, $g_b(x)=\min((\log\log|x|^{-1})^{-b}, (\log 2)^{-b})$ $(0<b<1)$ and define $m(x,\xi)=(\operatorname{sgn} x)g_b(x)G(\xi)$. Then by elementary calculations one sees that $\hat{G}(x)=-2ix^{-1}(\log(2+2\pi|x|^{-1})^{-1}+\mathcal{O}(|x|^{-1}(\log(2+|x|^{-1}))^{-2})$ near $x=0$. Hence $m(x,D)\chi_{(0,1)}=C(\operatorname{sgn} x)$ $(\log\log|x|^{-1})^{1-b}$+a bounded function near $x=0$, and so $m(x,D)\chi_{(0,1)}\notin \mathrm{BMO}(\boldsymbol{R})$. On the other hand $|\partial_\xi^k m(x,\xi)|\leqq C\langle\xi\rangle^{-k}(\log(2+|\xi|))^{-1}$, $k\in\boldsymbol{N}$. Note also that $m(x,\xi)$ is continuous on $\boldsymbol{R}\times\boldsymbol{R}$. This example satisfies (2.1), (2.3)–(2.6), (2.10) and (2.11) with $\omega(s,t)=C(\log(2+s))^{-1}$, but $m(x,D)$ do not satisfy (2.7).

c) Let $0<b\leqq 1$ with $[n/2]+b>n/2$. Under the assumption on ω in Theorem 2.1, if $m(x,\xi)$ satisfies (2.3), (2.4) with $k=n$ and (2.5), (2.6) with $k=[n/2]$, then $m(x,D)$ has the properties (P–1)–(P–4) in Theorem 2.2. This can be shown by using Theorem 2.1 with $k=[n/2]$ and the last part of the proof of Theorem 2.3.

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